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Novel Method for Ranking Generalized Fuzzy Numbers Based on Normalized Height Coefficient and Benefit and Cost Areas

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Abstract: To avoid loss of information and incorrect ranking, this paper proposes a method for ranking generalized fuzzy numbers, which guarantees both horizontal and vertical values are important parameters affecting the final ranking score. In this method, the normalized height coefficient is introduced to evaluate the influence of the height of fuzzy numbers on the final ranking score. The higher the normalized height coefficient of a fuzzy number is, the higher its ranking. The left area and the right area are presented to calculate the impact of vertical value on the final ranking score. The left area is considered the benefit area. The right area is considered the cost area. The fuzzy number A_i is preferred if the benefit area is larger and the cost area is smaller. The proposed method can be employed to rank both normal and non-normal fuzzy numbers without normalization or height minimization. Numerical examples and comparison with other methods highlight the feasibility and robustness of the proposed method, which can overcome the shortcomings of some existing methods and can support decision-makers to select the best alternative.

Keywords: generalized fuzzy numbers; ranking; normalized height coefficient; left area; right area

1. Introduction

Ranking fuzzy numbers is a very important issue in fuzzy sets theory and applications and has been extensively researched (Wang & Luo, 2009). Some ranking methods have been reviewed and compared by Bortolan & Degani (1985), Brunelli & Mezei (2013), and Chu & Kysely (2021). Nevertheless, none of these methods can always guarantee a consistent result in every situation, and some are even unintuitive and indiscriminate (Chou et al., 2011). Especially, when fuzzy numbers are non-normal, some methods used height minimization ($\min w_i$) or normalization, which leads to information loss (Yu et al., 2013; Chi & Yu, 2018). Methods used $\min w_i$ include the maximizing set and minimizing set for ranking fuzzy numbers (Chen, 1985), and the rank and mode approach for ranking generalized fuzzy numbers (Kumar et al., 2011). The method used the normalization is ranking fuzzy numbers with integral value (Liou & Wang, 1992).

It is impossible to define the boundary of the membership function of a fuzzy number in normal form. Thus, most recent studies have focused on taking into account the height of the fuzzy number to avoid loss of information, and incorrect ranking (Chi & Yu, 2018). However, such studies have some limitations. Chen & Chen (2009) pointed out that three factors affect ranking score: the defuzzified value, height, and spread. The defuzzified value and height of a generalized fuzzy number are the major factors determining its ranking score; the spread is only a minor factor. However, Kumar et al., (2011) indicated that the ranking function proposed by Chen & Chen (2009) does not satisfy the reasonable property $A \succ B \not\Rightarrow (A-B) \succ (B-A)$ for the ordering of fuzzy quantities which is a contradiction according to Wang & Kerre (2001) (see Example 1 in Section 2.3). Chen & Sanguansat

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(2011) considered the areas on the positive side, the areas on the negative side and the height of the generalized fuzzy numbers to evaluate the ranking score of the generalized fuzzy numbers. Xu et al. (2012) pointed out that in the situation when the score is zero, the results of the Chen and Sanguansat's ranking method (2011) ranking method are unreasonable. Chi & Yu (2018) proposed ranking generalized fuzzy numbers based on the centroid and rank index, which prevents the truncation of heights during comparison. To avoid information loss, the original height of given fuzzy number is retained and considered an important factor to affect the ranking of the generalized fuzzy numbers. However, this considers three factors, namely, the centroid, rank and mode, and height, as discrete factors, with height being the least influential, leading to incorrect final ranking results (see Example 2 in Section 2.3). De et al. (2020) indicated that the height of fuzzy numbers plays essential role in preventing information loss. This study considers centroid point, rank index, and height for ranking interval type-2 fuzzy numbers. However, this method cannot be used to rank fuzzy numbers with different centroids and heights (see Example 3 in Section 2.3). Revathi & Valliathal (2021) used centroid method for ordering non-normal fuzzy numbers with more parameters is investigated using level analysis, which gives flexibility to the expert's opinion. Nguyen & Chu (2023) proposed a DEMATEL-ANP-Based fuzzy PROMETHEE II to rank startups in which areas based on a subject confidence level was suggested and height was not considered. He et al. (2023) introduced a new fuzzy distance based on a novel interval distance that considers all points within the intervals by using the concept of integration to calculate the average distance between all points belonging to two intervals, respectively.

Jain (1977) proposed maximizing set to rank fuzzy numbers and restricted the membership function $f_A(x)$ to the normal form. Chen (1985) developed the maximizing set and minimizing set for generalized fuzzy numbers. However, this paper chose $w_i = \sup_x f_{A_i}(x), w = \inf w_i$, this method fails to rank the same fuzzy numbers with different heights (see Example 4 in Section 2.3). Wang et al. (2009) based on maximizing set and minimizing set developed the deviation degree method. According to Chutia (2017) the expectation value of the centroid points involved in the epsilon deviation degree method does not influence the heights of fuzzy numbers, which leads to an incorrect ranking of non-normal fuzzy numbers (illustrated in Example 4 in Section 2.3). Furthermore, in the case where $\lambda = 0$ and $1-\lambda = 0$, when the left deviation degree and the right deviation degree are multiplied by these values, they become valueless (Nejad & Mashinchi, 2011b). Wang & Luo (2009) proposed ranking indices based on areas and considered maximizing set and minimizing set as positive ideal point and negative ideal point, respectively. However, this study does not consider the height of fuzzy numbers, therefore it fails to rank non-normal fuzzy numbers (as shown in Example 4 in Section 2.3). Asady (2010) revised the deviation degree method with the new left and right areas. However, Hajjari & Abbasbandy (2011) pointed out that Asady's revision has a shortcoming the same as Wang's (2009) method. Nejad & Mashinchi (2011) proposed ranking fuzzy numbers based on the areas on the left and the right sides. To prevent the values of $\lambda = 0$ and $1-\lambda = 0$, and $S_i^R = 0$ and $S_i^L = 0$, in any collection including the fuzzy number A_i, $i = 1, 2, \ldots, n$, two triangular fuzzy numbers, A_0 and A_{n+1}, are added. Yu et al. (2013) pointed out that Asady (2010) and Nejad & Mashinchi (2011) redefined the deviation degree of a fuzzy number to overcome the shortcomings of Wang et al. (2009). However, most methods based on the deviation degree approach exhibit the same limitations due to neglect of the decision-makers' attitude, incoherent transfer coefficient formulas, and unreliable ranking index computation. Chutia (2017) proposed a method for ranking fuzzy numbers by using value and angle in the epsilon-deviation degree method. This method also has some limitations, which are illustrated in Example 5 in Section 2.3. The historical timeline of the aforementioned research is presented in the following chart.

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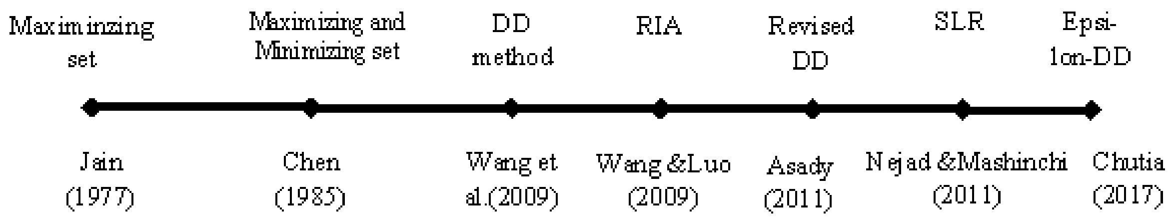

To overcome the aforementioned obstacles, this paper proposes a method for ranking generalized fuzzy numbers based on the left area (benefit area), the right area (cost area) and a normalized height coefficient. In this method, the left area denotes the area from $x_{\min}$ to x^L and is bounded by the maximizing membership function f_M and minimizing membership function f_G. A ranking is higher if the left area is larger; therefore, the left area is considered the benefit area. The right area denotes the area from $x_{\max}$ to x^R and is bounded by the maximizing membership function f_M and minimizing membership function f_G. A ranking is higher if the right area is smaller; therefore, the right area is considered the cost area. The normalized height coefficient reflects the influence of the height of fuzzy numbers on their final ranking scores. The proposed method can rank both normal and non-normal fuzzy numbers without normalization or height minimization, thereby avoiding information loss and incorrect final ranking results.

The main contributions of this study to bridge these gaps are briefly as follows:

(I) This research develops a new coefficient to calculate the impact of the height of fuzzy numbers on the final ranking score.
(II) The new areas considered as benefit and cost are introduced to reflect the influence of vertical values on the final ranking score.
(III) A new index is proposed to guarantee that both vertical and horizontal values of a fuzzy number are important parameters that impact the final ranking score.
(IV) The proposed method can rank both normal and nonnormal fuzzy numbers without normalization or height minimization, thereby avoiding information loss and incorrect final ranking results.
(V) The proposed method can overcome the shortcomings of some existing methods and can be applied to many fuzzy MCDM model to support decision-makers to select the most suitable alternative in the decision-making process.

This paper is organized as follows. In Section 2, some basic definitions are introduced. Section 2 also provides an overview of the deviation degree method and explores the shortcomings of recent methods. In Section 3, the proposed method ranking of generalized fuzzy numbers based on the normalized height coefficient and benefit and cost areas is presented. In Section 4, numerical examples and comparisons are presented. Finally, we provide concluding remarks in Section 5.

2. Preliminary

2.1. Definitions and notions

Definition 1. *Fuzzy Sets*

$A=\{(x, f_A(x)) \mid x \in U\}$, where *U* is the universe of discourse, *x* is an element in *U*, *A* is a fuzzy set in *U*, $f_A(x)$ is the membership function of *A* at *x* (Kaufmann and Gupta, 1991). The larger $f_A(x)$, the stronger the grade of membership for *x* in *A*.

Definition 2. *Fuzzy Numbers*

A real fuzzy number *A* is described as any fuzzy subset of the real line *R* with membership function f_A which possesses the following properties (Dubois and Prade, 1978):

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(a) f_A is a continuous mapping from R to [0,1];

(b) $f_A(x)=0,\ \forall x\in(-\infty,\ a]$;

(c) f_A is strictly increasing on $[a,b]$;

(d) $f_A(x)=1,\ x\in[b,c]$; (e) f_A is strictly decreasing on $[c,d]$;

(f) $f_A(x)=0,\ \forall x\in[d,\infty)$, where $a\le b\le c\le d$, A can be denoted as $[a,b,c,d;w]$. The membership function f_A of the fuzzy number A can also be expressed as follows:

$$f_A(x)=\begin{cases}0, & x<a;\\ \dfrac{w(x-a)}{b-a}, & a\le x\le b;\\ w, & b\le x\le c;\\ \dfrac{w(x-d)}{c-d}, & c\le x\le d;\\ 0, & x>d\end{cases} \tag{1}$$

This trapezoidal fuzzy set, $A=(a,b,c,d;w)$, $0\le w\le 1$, as shown in Figure 1.

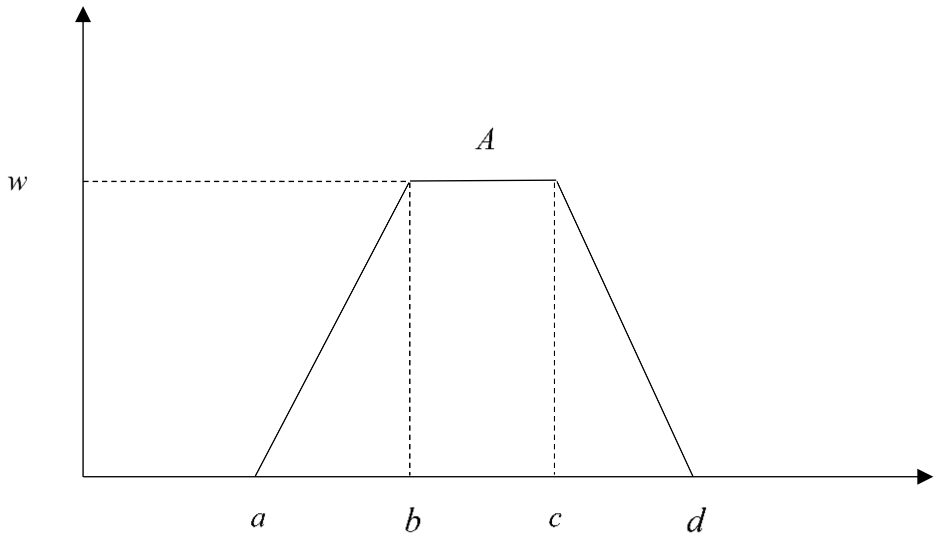

Figure 1. A trapezoidal fuzzy set.

Definition 3. *The arithmetic operations between two generalized trapezoidal fuzzy numbers* $A_1=(a_1,b_1,c_1,d_1;w_1)$ and $A_2=(a_2,b_2,c_2,d_2;w_2)$ are defined as below:

$$A_1\oplus A_2=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2;\min(w_1,w_2)) \tag{2}$$

$$A_1\ominus A_2=(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2;\min(w_1,w_2)) \tag{3}$$

$$A_1\otimes A_2=(a_1\times a_2,b_1\times b_2,c_1\times c_2,d_1\times d_2;\min(w_1,w_2)) \tag{4}$$

2.2. A review of the deviation degree method

In this section, firstly the minimal and maximal reference sets are reviewed. Then, based on the minimal and maximal reference sets, the left and right deviation degree (L–R deviation degree) is defined. Moreover, the transfer coefficient which measures the relative variation of L–R deviation degree of fuzzy numbers is quoted.

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Definition 4. *For any group of fuzzy numbers* $A_1, A_2, \ldots, A_n$*, let* $x_{\min}$ and $x_{\max}$ *be the infimum and supremum of the support set of these fuzzy numbers. Then,* $A_{\min}$ and $A_{\max}$ *are the minimal reference set and maximal reference set of these fuzzy numbers, respectively, and their membership functions are given by*

$$f_G = \begin{cases} \dfrac{x_{\max} - x}{x_{\max} - x_{\min}}, & \text{if } x \in S, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$f_M = \begin{cases} \dfrac{x - x_{\min}}{x_{\max} - x_{\min}}, & \text{if } x \in S, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

Where S is the support set of these fuzzy numbers, i.e., $S = \bigcup_{i=1}^{n} S(A_i)$.

Definition 5. *For any group of fuzzy numbers* $A_1, A_2, \ldots, A_n$*, let* $A_{\min}$ and $A_{\max}$ *be the minimal reference set and maximal reference set of these fuzzy numbers, respectively. Then, the left deviation degree and right deviation degree of* $A_i, i = 1, 2, \ldots, n.$ *are defined as follows:*

$$d_i^L = \int_{x_{\min}}^{x_{A_i}^L} (f_G - f_{A_i}^L) dx \tag{7}$$

$$d_i^R = \int_{x_{\min}}^{x_{A_i}^R} (f_M - f_{A_i}^R) dx \tag{8}$$

Where $x_{A_i}^L$ and $x_{A_i}^R, i = 1, 2, \ldots, n,$ are the abscissas of the crossover points of $f_{A_i}^L$ and f_G, and $f_{A_i}^R$ and f_M respectively.

Definition 6. *For any fuzzy number* $A_i = (a_i, b_i, c_i, d_i, w_i)$*, its expectation value of centroid is defined as follows:*

$$M_i = \frac{\int_{a_i}^{d_i} x f_{A_i}(x) dx}{\int_{a_i}^{d_i} f_{A_i}(x) dx} \tag{9}$$

Definition 7. *For any fuzzy numbers* $A_i = (a_i, b_i, c_i, d_i, w_i)$*, the transfer coefficient of* $A_i, i = 1, 2, \ldots, n,$ *is given by*

$$\lambda_i = \frac{M_i - M_{\min}}{M_{\max} - M_{\min}} \tag{10}$$

Where $M_{\max} = \max\{M_1, M_2, \ldots, M_n\}$ and $M_{\min} = \min\{M_1, M_2, \ldots, M_n\}$.

Definition 8. *The ranking index value of fuzzy number* $A_i, i = 1, 2, \ldots, n,$ *is given by:*

$$d_i = \begin{cases} \dfrac{d_i^L \lambda_i}{1 + d_i^R (1 - \lambda_i)}, & M_{\max} \neq M_{\min} \\ \dfrac{d_i^L}{1 + d_i^R}, & M_{\max} = M_{\min} \end{cases} \tag{11}$$

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Now, by using d_i given in Eq.(11), for any two fuzzy numbers A_i, and A_j, their orders are determined based on the following rules:

$(1) A_i \succ A_j$, if and only if $d_i \succ d_j$

$(2) A_i \prec A_j$, if and only if $d_i \prec d_j$

$(3) A_i \sim A_j$, if and only if $d_i = d_j$

2.3. Limitations and shortcomings of existing methods

Example 1. *Consider two generalized trapezoidal fuzzy numbers* $A_1 = (0.2, 0.4, 0.6, 0.8; 0.35)$ *and* $A_2 = (0.1, 0.2, 0.3, 0.4; 0.7)$ *adopted from Kumar et al. (2011). According to Chen & Chen, (2009) approach* $A_2 \succ A_1$. *However, Kumar et al., (2011) noted that* $A_2 - A_1 \prec A_1 - A_1$, *which is unreasonable and a contradiction, according to Wang & Kerre (2001).*

Example 2. *Consider two sets; each set comprises two type-1 trapezoidal fuzzy numbers (Figure 2 and Figure 3) as follows:*

Set 1 comprises $A_1 = (0, 0.2, 0.5, 0.7; 1)$ and $A_2 = (0.1, 0.2, 0.6, 0.8; 1)$.

Set 2 comprises $A_1 = (0, 0.2, 0.5, 0.7; 1)$ and $A_2 = (0.1, 0.2, 0.6, 0.8; 0.1)$.

Based on Chi & Yu (2018) the final ranking of Set 1 and Set 2 are the same: $A_2 \succ A_1$, which shows that the height does not affect the final ranking. These two sets have fuzzy numbers with the same support but different heights; in Set 2, the height of A_2 is only 0.1.

Figure 2. Fuzzy number A_1 and A_2 of set 1 in example 2.

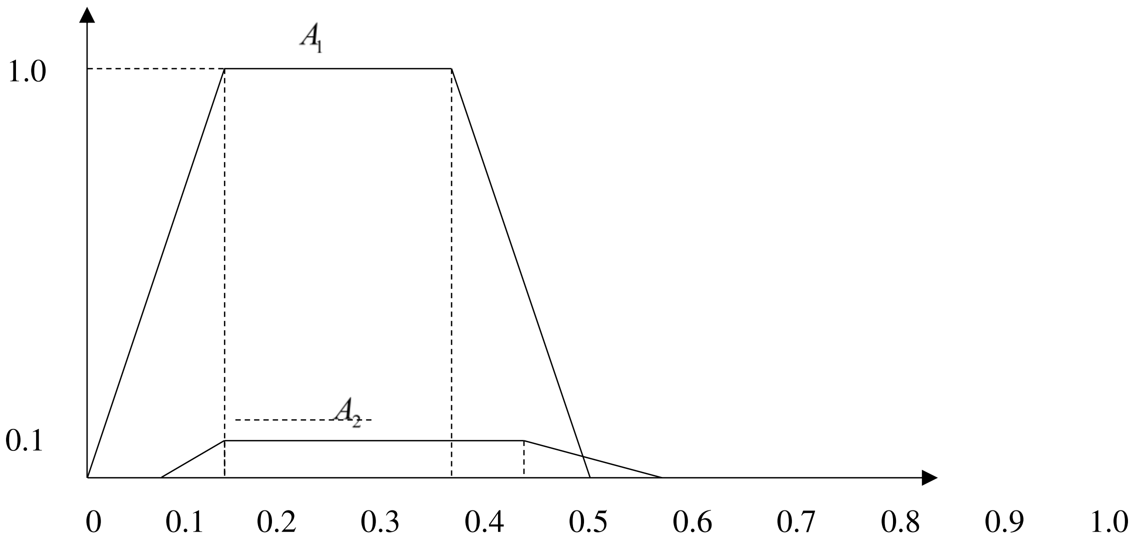

Figure 3. Fuzzy number A_1 and A_2 of set 2 in example 2.

Example 3. *Consider two sets, each comprising two type-2 trapezoidal fuzzy numbers (Figure 4 and Figure 5) as follows:*

Set 3 comprises

$A_1 = ((0, 0.3, 0.5, 0.6; 1); (0.1, 0.3, 0.4, 0.5; 0.7))$ and

$A_2 = ((0.1, 0.2, 0.4, 0.8; 1); (0.2, 0.3, 0.4, 0.6; 0.8))$.

Set 4 comprises

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$$A_1 = ((0, 0.3, 0.5, 0.6; 1); (0.1, 0.3, 0.4, 0.5; 0.7)) \text{ and}$$
$$A_2 = ((0.1, 0.2, 0.4, 0.8; 0.3); (0.2, 0.3, 0.4, 0.6; 0.1)).$$

Based on De et al., (2020) the final rankings of Set 3 and Set 4 are the same: $A_2 \succ A_1$. Therefore, height does not affect the final ranking. These two sets have fuzzy numbers with the same support but different heights; in Set 4, the heights of the upper and lower trapezoidal fuzzy numbers of A_2 are only 0.3 and 0.1, respectively.

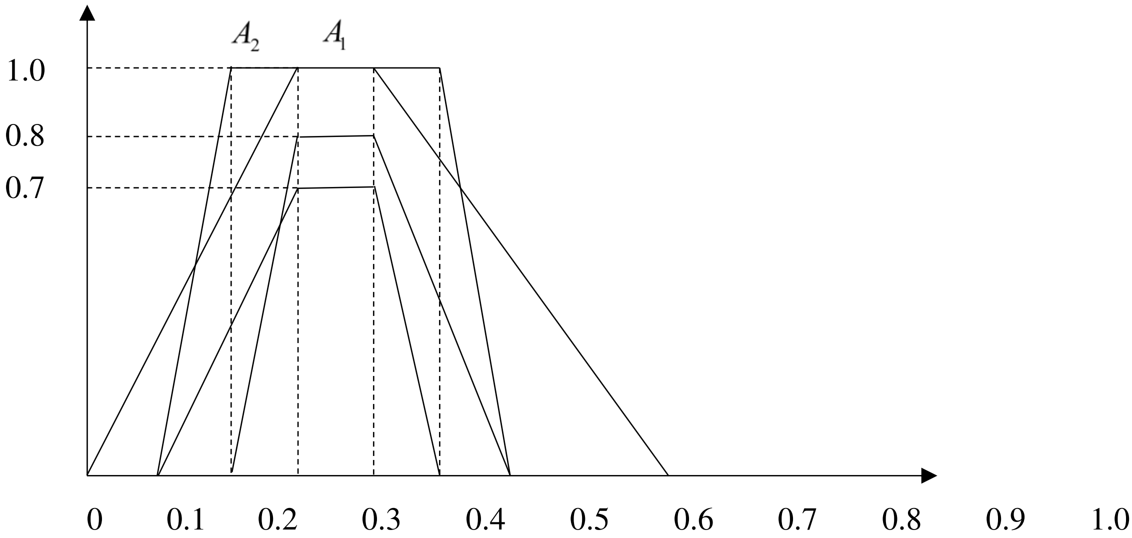

Figure 4. Fuzzy number A_1 and A_2 of set 3 in example 3.

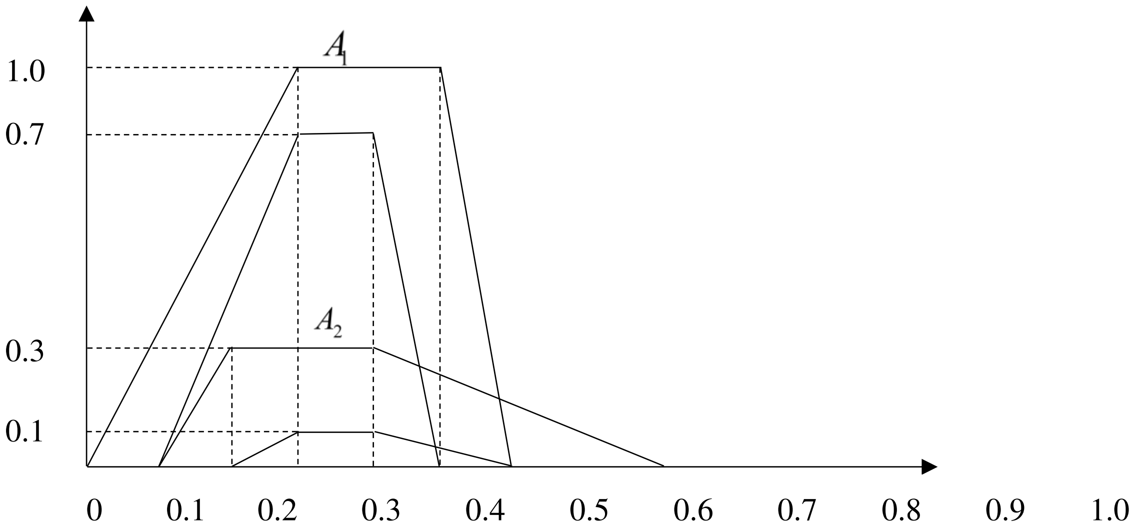

Figure 5. Fuzzy number A_1 and A_2 of set 4 in example 3.

Example 4. *Consider two trapezoidal fuzzy numbers (Figure 6) as follows:*

$$A_1 = (0.1, 0.3, 0.3, 0.5; 1) \text{ and } A_2 = (0.1, 0.3, 0.3, 0.5; 0.3).$$

These two fuzzy numbers have the same support, but the height of A_2 is lower than that of A_1. However, the final ranking according to Chen (1985); Wang & Luo (2009) is $A_1 \approx A_2$, which is counterintuitive, thus illustrating a shortcoming in ranking nonnormal fuzzy numbers. According to Wang et al., (2009) the final ranking result $A_1 \prec A_2$ which is inconsistent with human intuition.

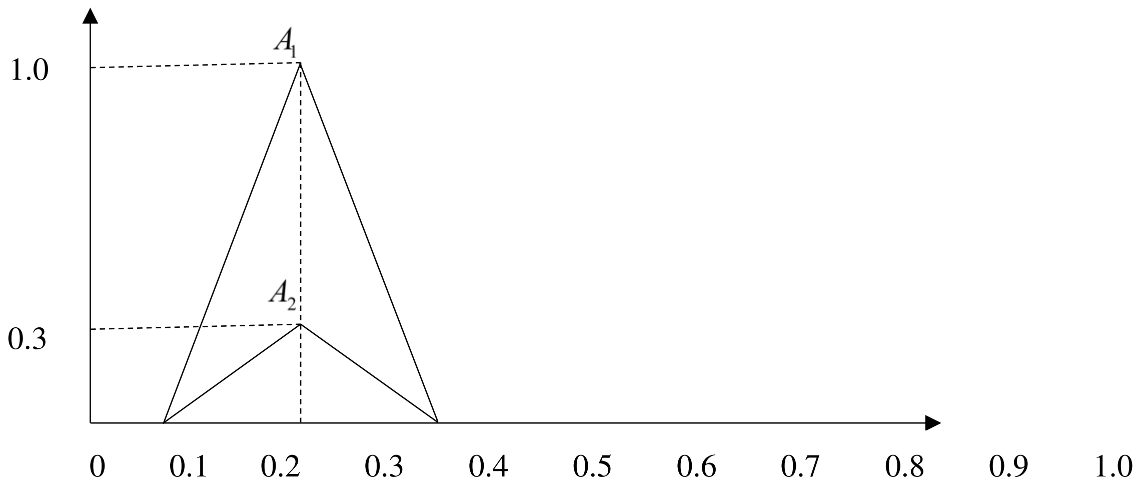

Figure 6. Fuzzy number A_1 and A_2 in example 4.

Example 5. *Consider three fuzzy numbers,* $A_1 = (0.3, 0.5, 0.5, 0.7; 1)$, $A_2 = (0.3, 0.5, 0.5, 0.9; 1)$, *and* $A_3 = (0.3, 0.5, 0.8, 0.9; 1)$. *Seghir (2021) pointed out that all the compared and proposed methods provide the correct ranking* $A_3 \succ A_2 \succ A_1$, *which is intuitive. However, the ranking* $A_3 \succ A_1 \succ A_2$ *of Chutia (2017) is incorrect and counterintuitive.*

3. Proposed method

This study proposes a method that considers maximizing and minimizing sets to be reference sets, the left area to be the benefit area, and the right area to be the cost area. Additionally, the normalized height coefficient is used to determine the influence of height on the final ranking score, thus enabling the ranking of both normal and nonnormal fuzzy numbers without normalization or height minimization, which avoids information loss and incorrect final rankings.

To guarantee that the vertical value is considered an important parameter that impacts the final ranking score, the left area and the right area are evaluated. Assume there are n fuzzy numbers $A_i = (a_i, b_i, c_i, d_i; w_i), i = 1, 2, ..., n$. The left area denotes the area from $x_{\min}$ to $x_{A_i}^L$ and is bounded by the maximizing membership function f_M and minimizing membership function f_G. Where $x_{A_i}^L$ is the intersection of the crossover point of the minimizing membership function f_G and the left membership function $f_{A_i}^L x$. The left area is shown in Figure 7 and is described by Eqs. (12) and (13).

$$S_{A_i}^L = \int_{x_{\min}}^{x^L} (f_G - f_M) dx \quad \text{if } x^L \le x_I \tag{12}$$

$$S_{A_i}^L = \int_{x_{\min}}^{x_I} (f_G - f_M) dx + \int_{x_I}^{x^L} (f_M - f_G) dx \quad \text{if } x^L \succ x_I \tag{13}$$

$$x_{A_i}^L = \frac{w_i a_i (x_{\max} - x_{\min}) + w x_{\max} (b_i - a_i)}{w_i (x_{\max} - x_{\min}) + w(b_i - a_i)} \tag{14}$$

$$x_I = \frac{(x_{\min} + x_{\max})}{2} \tag{15}$$

$$x_{\min} = \inf a_i \tag{16}$$

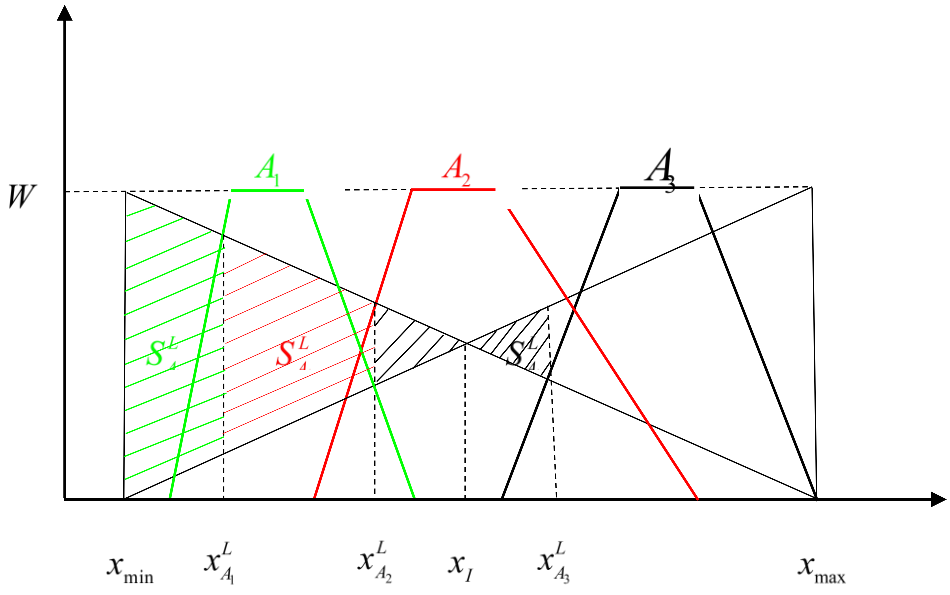

Figure 7. The left area.

In Figure 7 the left areas of fuzzy number A_1 and fuzzy number A_2 are in the case of $x^L \leq x_I$. Therefore, applying Eq. (12), the left area of fuzzy number A_1 is the green area, the left area of fuzzy number A_2 is the green area adding the red area. The left area of fuzzy number A_3 is in the case of $x^L \succ x_I$, applying Eq. (13) the left area of fuzzy number A_3 is the green area adding the red area adding the black area.

The right area denotes the area from $x_{\max}$ to $x^R_{A_i}$ and is bounded by the maximizing membership function f_M and minimizing membership function f_G. Where $x^R_{A_i}$ is the intersection of the crossover point of the maximizing membership function f_M and the right membership function $f^R_{A_i} x$. The right area is shown in Figure 8 and is described by Eqs. (17) and (18).

$$S^R_{A_i} = \int_{x^R}^{x_{\max}} (f_M - f_G) dx \quad \text{if } x^R \geq x_I \tag{17}$$

$$S^R_{A_i} = \int_{x^R}^{x_I} (f_G - f_M) dx + \int_{x_I}^{x_{\max}} (f_M - f_G) dx \quad \text{if } x^R \prec x_I \tag{18}$$

$$x^R_{A_i} = \frac{w_i d_i (x_{\max} - x_{\min}) - w x_{\min} (c_i - d_i)}{w_i (x_{\max} - x_{\min}) - w (c_i - d_i)} \tag{19}$$

$$x_{\max} = \sup d_i \tag{20}$$

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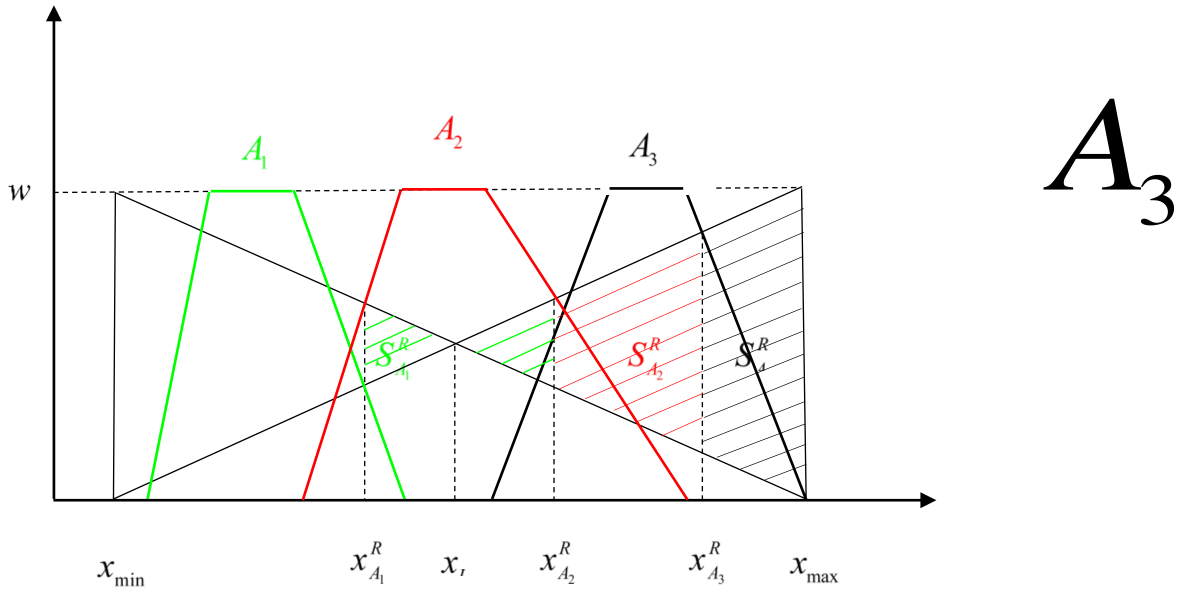

Figure 8. The right area.

In Figure 8 the right areas of fuzzy number A_3 and fuzzy number A_2 are in the case of $x^R \geq x_I$. Therefore, applying Eq. (17), the right area of fuzzy number A_3 is the black area, and the left area of fuzzy number A_2 is the black area adding the red area. The right area of fuzzy number A_3 is in the case of $x^R \prec x_I$, applying Eq. (18) the right area of fuzzy number A_1 is the black area adding the red area adding the green area.

Herein, $x_{\min} = \inf a_i$ is considered the negative ideal solution, and $x_{\max} = \sup d_i$ is considered the positive ideal solution, x_I is the intersection of the maximizing membership function f_M and minimizing membership function f_G. In the proposed method, the left and right areas are new areas that are simple to calculate and provide greater consistency and robustness in comparison with other methods.

The fuzzy number A_i is preferred if it is the farthest from the negative ideal solution $x_{\min}$ and closest to the positive ideal solution $x_{\max}$. If $S^L_{A_i}$ is larger, the fuzzy number A_i is farther from the negative ideal solution and closer to the positive ideal solution. Therefore, $S^L_{A_i}$ is considered a benefit; thus, larger $S^L_{A_i}$ is better. Conversely, if $S^R_{A_i}$ is smaller, A_i is farther from the negative ideal solution and closer to the positive ideal solution. Therefore, $S^R_{A_i}$ is considered a cost; thus, smaller $S^R_{A_i}$ is better. In other words, larger $S^L_{A_i}$ and smaller $S^R_{A_i}$ indicate a larger fuzzy number A_i.

To guarantee that horizontal value is also considered an important parameter to influence the final ranking, the normalized height coefficient is defined in Eq. (21) to reflect the influence of the height of a fuzzy number on the final ranking score. The higher the normalized height coefficient of fuzzy number A_i, the higher the ranking of A_i is.

$$\varsigma_{A_i} = \frac{h_{A_i}}{\sum_{i=1}^{n} h_{A_i}} \tag{21}$$

The final ranking score (RS) for fuzzy number A_i is defined as in Eq. (22):

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$$RS_{A_i} = \frac{S_{A_i}^{L} \varsigma_{A_i}}{S_{A_i}^{L} \varsigma_{A_i} + S_{A_i}^{R} (1 - \varsigma_{A_i})} \tag{22}$$

If A_i and A_j are two fuzzy numbers, then the ranking score leads to the following decisions:

$$\begin{aligned} &\text{If } RS_{A_i} \succ RS_{A_j}, \text{then } A_i \succ A_j. \\ &\text{If } RS_{A_i} \prec RS_{A_j}, \text{then } A_i \prec A_j. \\ &\text{If } RS_{A_i} = RS_{A_j}, \text{then } A_i \approx A_j. \end{aligned} \tag{23}$$

A flowchart in Figure 9 as below is used to present the procedure of the proposed method.

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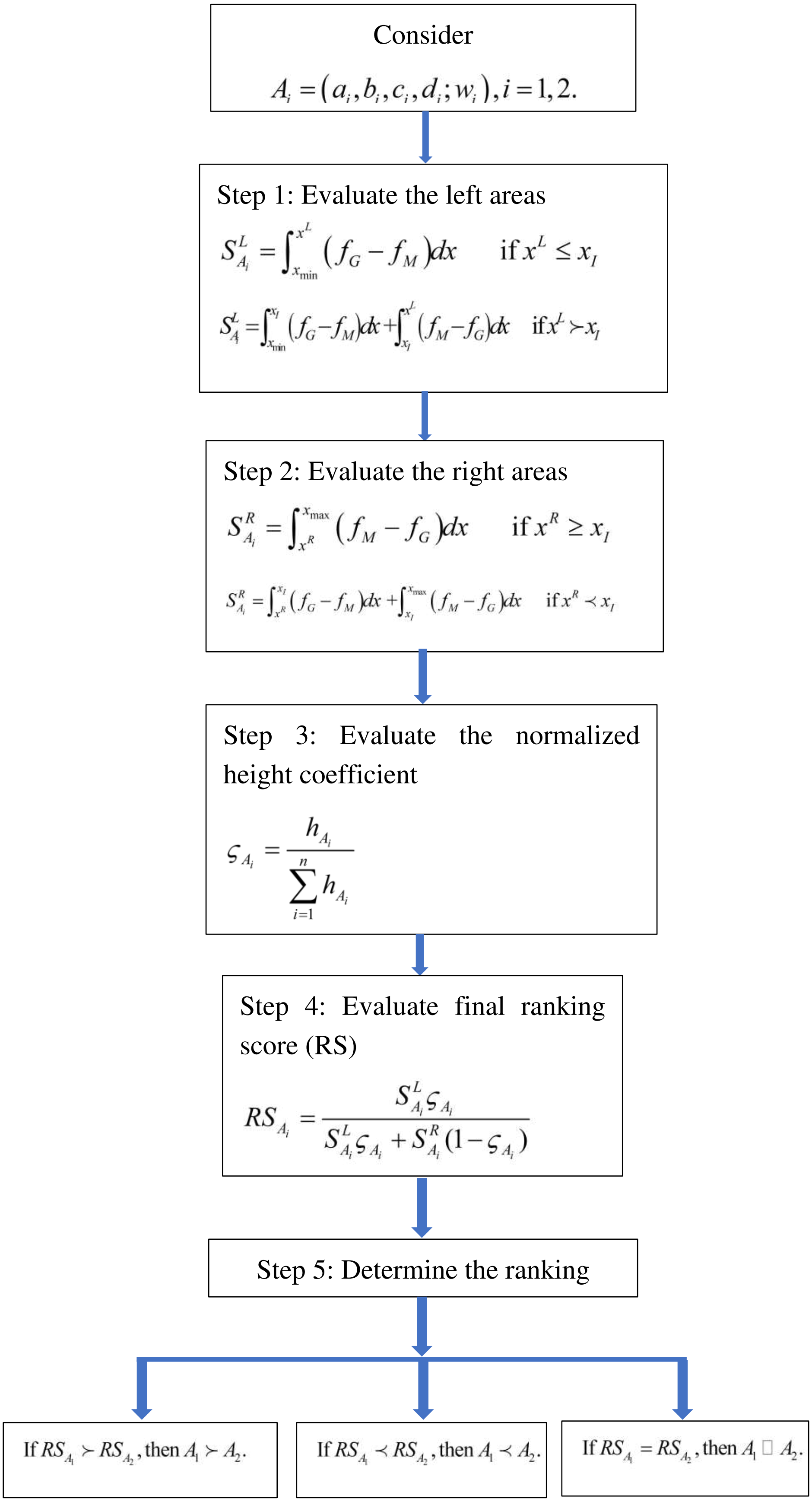

Figure 9. The graphical abstract of the proposed method.

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4. Numerical example and comparative study

4.1. Examples

To highlight the advantages, consistency, and robustness of this method, numerical examples are used. Step-by-step, this example demonstrates the simple computation and application of the proposed method.

Example 6. *Consider two trapezoidal fuzzy numbers* $A_1=(0.1,0.2,0.3,0.5;1)$ *and* $A_2=(0.1,0.3,0.4,0.6;1)$ *(Figure 10). According to the proposed method, the final ranking is determined to be* $A_1 \prec A_2$ *as follows:*

Step 1: Based on Eqs. (16) and (20), $x_{\min}=0.1, x_{\max}=0.6$

Step 2: Based on Eqs. (14) and (19), the $x_{A_i}^L$ and $x_{A_i}^R$ of fuzzy number A_1 are $x_{A_1}^L=0.18333$ and $x_{A_1}^R=0.38571$. The $x_{A_i}^L$ and $x_{A_i}^R$ of fuzzy number A_2 are $x_{A_2}^L=0.24286$ and $x_{A_2}^R=0.45714$.

Step 3: Based on Eqs. (12) and (17), the $S_{A_i}^L$ and $S_{A_i}^R$ of fuzzy number A_1 are $S_{A_1}^L=0.06944$ and $S_{A_1}^R=0.12245$. The $S_{A_i}^L$ and $S_{A_i}^R$ of fuzzy number A_2 are $S_{A_2}^L=0.10204$ and $S_{A_2}^R=0.10204$.

Step 4: Based on Eq. (21), the normalized height coefficient of fuzzy number A_1 and fuzzy number A_2 are $\varsigma_{A_1}=0.5$ and $\varsigma_{A_2}=0.5$, respectively.

Step 5: Based on Eq. (22), the ranking score (RS) of fuzzy number A_1 and fuzzy number A_2 are $RS_{A_1}=0.36189$ and $RS_{A_2}=0.5000$.

Step 6: Based on Eq. (23) the final ranking is $A_1 \prec A_2$.

The following numerical examples (Figures 11–17) are calculated step-by-step as in Example 6; the results are shown in Table 1.

Table 1. Numerical Examples.

	Fuzzy numbers	S^L	S^R	3	*RS*	Rank
Ex.6	A_1 (0.1,0.2,0.3,0.5;1)	0.069	0.122	0.500	*0.362*	2
	A_2(0.1,0.3,0.4,0.6;1)	0.102	0.102	0.500	*0.500*	1
Ex.7	A_1(0.1,0.2,0.3,0.5;1)	0.064	0.089	0.500	0.419	1
	A_2(0.1,0.2,0.3,0.5;1)	0.064	0.089	0.500	0.419	1
Ex.8	A_1 (0.1,0.2,0.3,0.5;1)	0.044	0.065	0.556	0.460	1
	A_2(0.1,0.2,0.3,0.5;0.8)	0.051	0.071	0.444	0.365	2
Ex.9	A_1(0.1,0.3,0.5,0.6;1)	0.113	0.122	0.500	0.479	2
	A_2(0.2,0.3,0.6,0.7;1)	0.122	0.073	0.500	0.625	1
Ex.10	A_1(0.1,0.3,0.5,0.6;1)	0.050	0.066	0.625	0.558	1
	A_2(0.2,0.3,0.6,0.7;0.6)	0.073	0.044	0.375	0.500	2
Ex.11	A_1(0.1,0.3,0.5,0.6;0.9)	0.085	0.096	0.529	0.499	2
	A_2(0.2,0.3,0.6,0.7;0.8)	0.098	0.059	0.471	0.597	1
Ex.12	A_1(0.1,0.2,0.3,0.5;1)	0.231	0.139	0.500	0.625	1
	A_2(-0.5,-0.3,-0.2,-0.1;1)	0.139	0.231	0.500	0.375	2
Ex.13	A_1(0.1,0.2,0.3,0.5;1)	0.073	0.150	0.333	0.197	3
	A_2(0.1,0.3,0.5,0.6;1)	0.113	0.122	0.333	0.315	2
	A_3(0.2,0.3,0.6,0.7;1)	0.122	0.073	0.333	0.455	1

Example 7. *Consider two normal trapezoidal fuzzy numbers* $A_1=(0.1,0.2,0.3,0.5;1)$ *and* $A_2=(0.1,0.2,0.3,0.5;1)$ *(Figure 11). These fuzzy numbers are the same, so the ranking scores are equal and the final ranking is* $A_1 \sim A_2$.

Example 8. *Consider the normal trapezoidal fuzzy number* $A_1=(0.1,0.2,0.3,0.5;1)$ *and non-normal trapezoidal fuzzy number* $A_2=(0.1,0.2,0.3,0.5;0.8)$ *(Figure 12). These fuzzy numbers share the same support, but the ranking scores are different because of different heights. The final ranking result is* $A_1 \succ A_2$. *This example indicates that the proposed method can rank both normal and non-normal fuzzy numbers.*

Example 9. *Consider two normal trapezoidal fuzzy numbers* $A_1=(0.1,0.3,0.5,0.6;1)$ *and* $A_2=(0.2,0.3,0.6,0.7;1)$ *(Figure 13). The final ranking result is* $A_1 \prec A_2$.

Example 10. *Consider the normal trapezoidal fuzzy number* $A_1=(0.1,0.3,0.5,0.6;1)$ *and nonnormal trapezoidal fuzzy number* $A_2=(0.2,0.3,0.6,0.7;0.6)$ *(Figure 14). These two fuzzy numbers share the same support as in Example 9, but the height of* A_2 *is 0.6. Therefore, the final ranking is* $A_1 \succ A_2$. *This example demonstrates that height is an important parameter affecting the final ranking score.*

Example 11. *Consider two nonnormal trapezoidal fuzzy numbers,* $A_1=(0.1,0.3,0.5,0.6;0.9)$ *and* $A_2=(0.2,0.3,0.6,0.7;0.8)$ *(Figure 15). These two fuzzy numbers have the same support as in Example 9; however, the height of* A_1 *is 0.9, and the height of* A_2 *is 0.8. Therefore, the final ranking is* $A_1 \prec A_2$. *This example also indicates that the final ranking is sensitive to height*

Example 12. *Consider two normal trapezoidal fuzzy numbers,* $A_1=(0.1,0.2,0.3,0.5;1)$ *and* $A_2=(-0.5,-0.3,-0.2,-0.1;1)$ *(Figure 16). The final ranking is* $A_1 \succ A_2$. *This example shows that the proposed method can be used to rank positive and negative fuzzy numbers.*

Example 13. *Consider three normal trapezoidal fuzzy numbers,* $A_1=(0.1,0.2,0.3,0.5;1)$, $A_2=(0.1,0.3,0.5,0.6;1)$, *and* $A_3=(0.2,0.3,0.6,0.7;1)$ *(Figure 17). The final ranking is* $A_1 \prec A_2 \prec A_3$. *This example reveals that the proposed method be used to rank sets comprising more than two fuzzy numbers.*

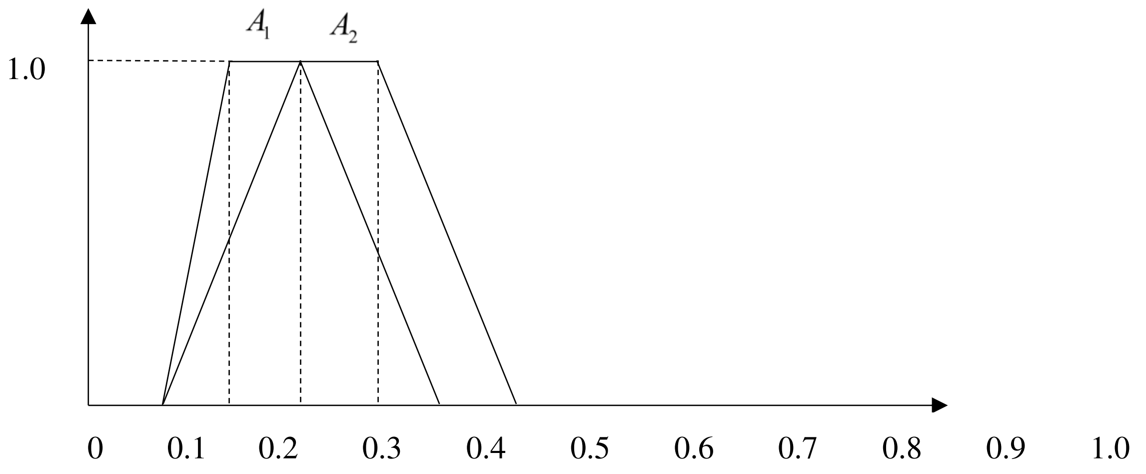

Figure 10. Fuzzy number A_1 and A_2 in example 6.

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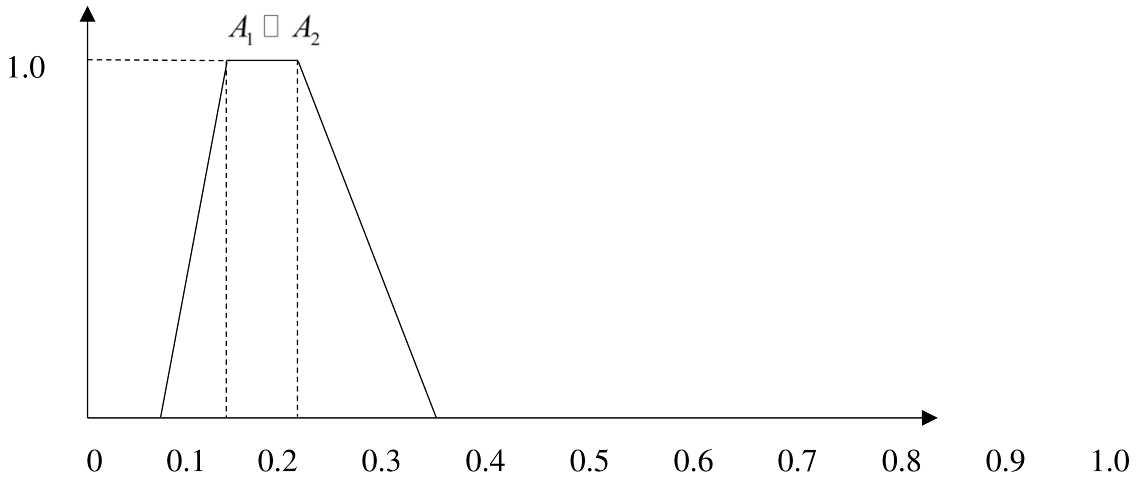

Figure 11. Fuzzy number A_1 and A_2 in example 7.

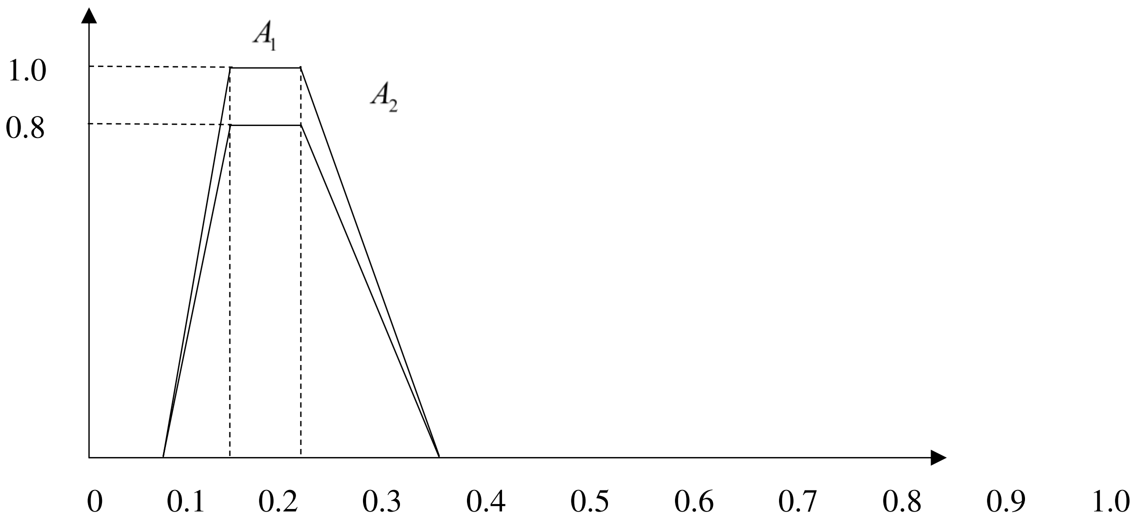

Figure 12. Fuzzy number A_1 and A_2 in example 8.

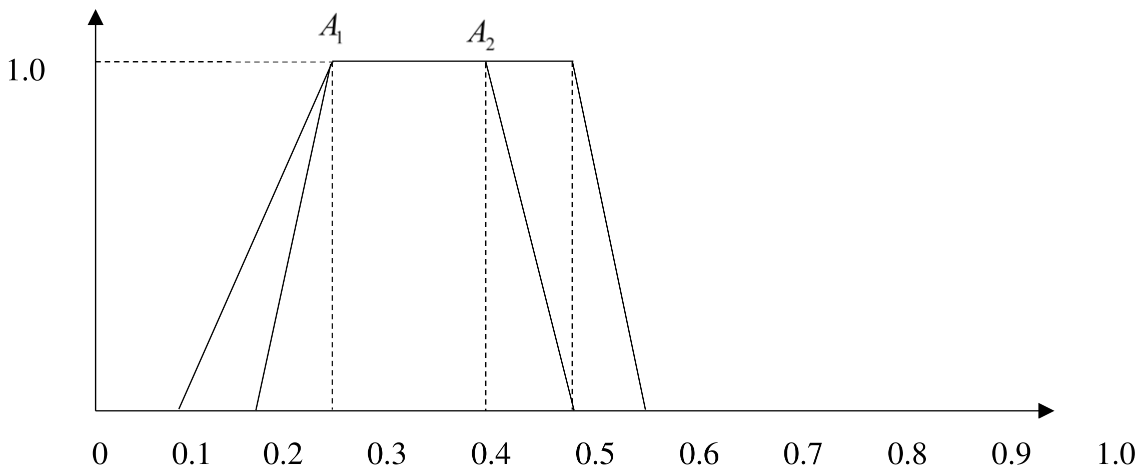

Figure 13. Fuzzy number A_1 and A_2 in example 9.

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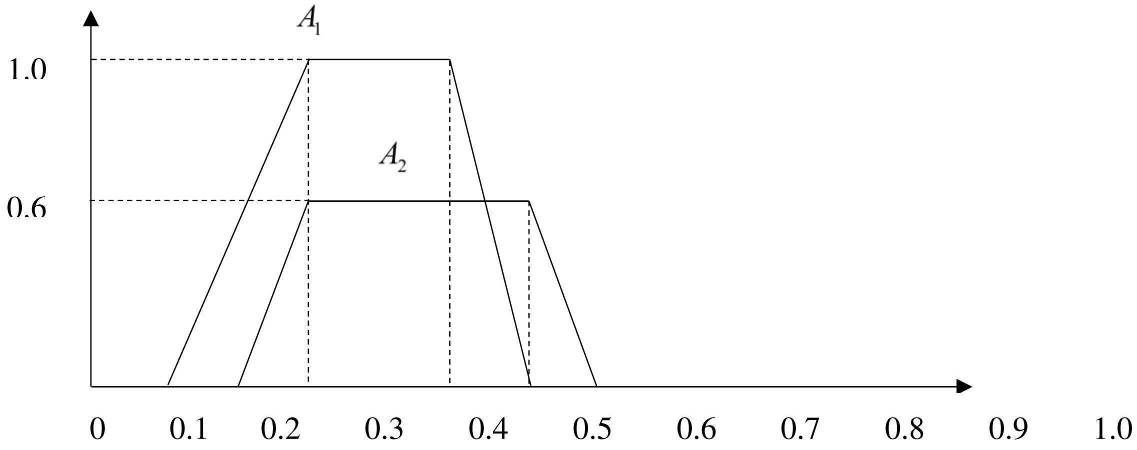

Figure 14. Fuzzy number A_1 and A_2 in example 10.

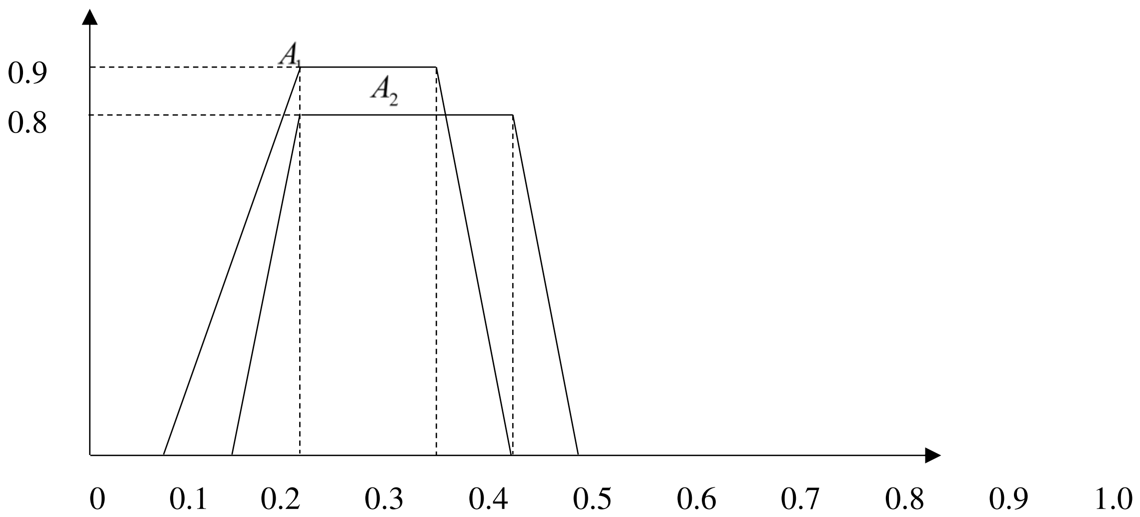

Figure 15. Fuzzy number A_1 and A_2 in example 11.

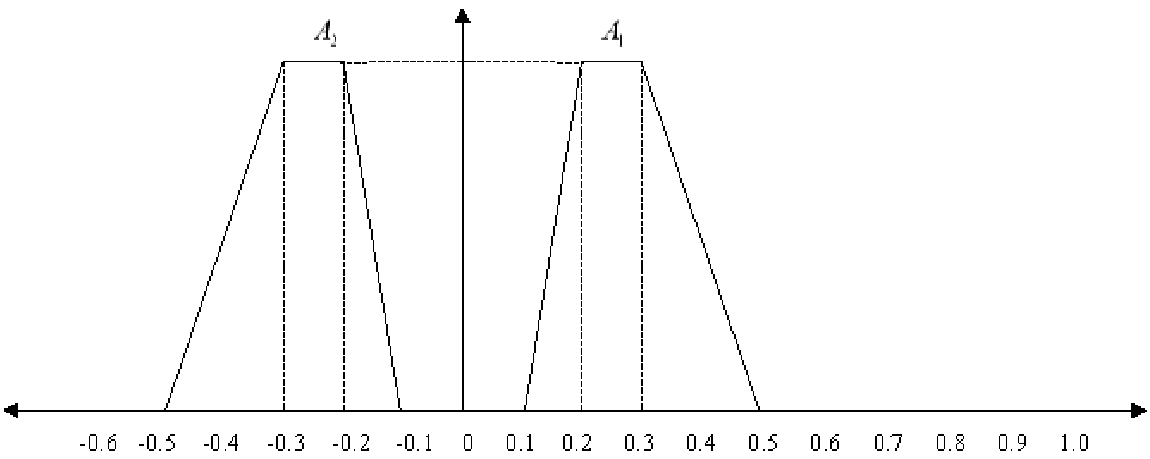

Figure 16. Fuzzy number A_1 and A_2 in example 12.

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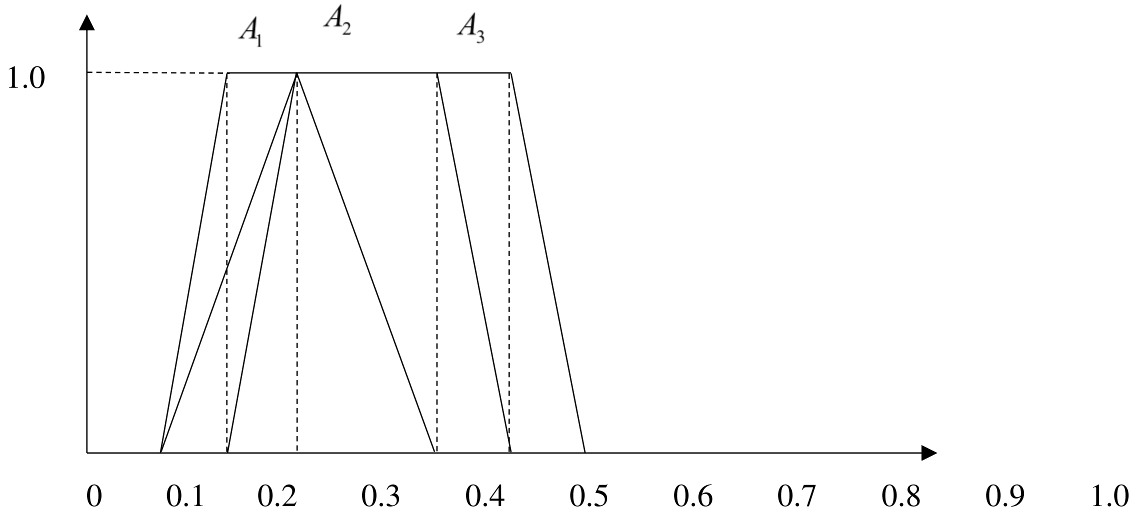

Figure 17. Fuzzy number A_1 and A_2 in example 13.

4.2. Comparison

For objective comparison, fuzzy sets are adopted from Chen & Chen (2007). This section presents a comparison of the proposed method based on ranking score (RS) with the methods based on maximizing and minimizing set method (U_T) Chen (1985); deviation degree (DD) Wang et al., (2009); area ranking based on positive and negative ideal points (RIA) Wang & Luo (2009); revised method of deviation degree (RDD) Asady (2010); areas on the left and right sides of fuzzy number (SLR) Nejad & Mashinchi (2011); and the value and angle in the epsilon-deviation degree $(MEDD)$ Chutia (2017), the final ranking results and comparison are presented in Table 2 where R is final ranking.

Table 2. Comparison of the proposed method with other methods.

Set	FNs	Ut	R	DD	R	RIA	R	RDD	R	SLR	R	MEDD	R	RS	R
Set 5	A_1(0.1,0.3,0.3,0.5;1)	0.375	2	0.000	2	0.250	2	0.222	2	0.075	2	0.041	2	0.429	2
	A_2(0.3,0.5,0.5,0.7;1)	0.625	1	0.300	1	0.750	1	0.571	1	0.303	1	24.462	1	0.571	1
Set 6	A1(0.1,0.3,0.3,0.5;1)	0.500	1	0.063	1	0.500	1	0.286	1	0.130	1	1.000	1	0.500	1
	A_2(0.1,0.3,0.3,0.5;1)	0.500	1	0.063	1	0.500	1	0.286	1	0.130	1	1.000	1	0.500	1
Set 7	A_1(0.1,0.3,0.3,0.5;0.8)	0.400	1	0.063	1	0.500	1	0.242	2	0.242	2	0.126	2	0.444	2
	A_2(0.1,0.3,0.3,0.5;1)	0.400	1	0.061	2	0.500	1	0.286	1	0.286	1	9.488	1	0.556	1
Set 8	A_1(-0.5,-0.3,-0.3,-0.1;1)	0.250	2	0.000	2	0.125	2	0.154	2	0.035	2	0.015	2	0.333	2
	A_2(0.1,0.3,0.3,0.5;1)	0.750	1	1.333	1	0.875	1	1.143	1	0.679	1	65.091	1	0.667	1
Set 9	A_1(0.3,0.5,0.5,1.0;1)	0.503	1	0.327	1	0.545	1	0.514	1	0.285	1	1.185	1	0.502	1
	A_2(0.1,0.6,0.6,0.8;1)	0.497	2	0.000	2	0.455	2	0.436	2	0.196	2	0.844	2	0.498	2
Set 10	A_1(0.0,0.4,0.6,0.8;1)	0.517	3	0.000	3	0.500	3	0.474	3	0.229	3	0.087	3	0.349	3
	A_2(0.2,0.5,0.5,0.9;1)	0.554	2	0.313	1	0.636	2	0.600	2	0.363	1	0.498	2	0.363	2
	A_3(0.1,0.6,0.7,0.8;1)	0.614	1	0.207	2	0.700	1	0.647	1	0.362	2	2.176	1	0.434	1

Table 2 illustrates that the final ranking of Set 7 by the proposed method is consistent with the rankings generated by the methods based on revised method of deviation degree (RDD), the areas on the left and right sides of fuzzy number (SLR), and the value and angle in the epsilon-deviation degree $(MEDD)$; thus, our method is intuitive for ranking nonnormal fuzzy numbers. The methods based on maximizing set and minimizing set (U_T), the area ranking based on positive and negative ideal points (RIA) equally rank the fuzzy numbers, which is counterintuitive because the two fuzzy numbers share the same score support but differ in height. Furthermore, the final ranking based on deviation degree (DD) is unreasonable because the fuzzy number with lower height has a higher ranking, making it counterintuitive. These final rankings of Sets 5, 6, 8, and 9 generated by proposed method are consistent with those by the other methods. The final ranking of Set 10 is the same as that generated by most other methods, namely, maximizing set and minimizing set (U_T), the area ranking based on positive and negative ideal points (RIA), the revised method of deviation degree (RDD), and the value and angle in the epsilon-deviation degree $(MEDD)$. Thus, the final ranking generated by proposed method is consistent with those of other methods for normal fuzzy numbers. Additionally, the proposed method can be used to rank the nonnormal fuzzy numbers described in Set 7 without normalization or height minimization ($\min w_i$).

5. Conclusion

This paper proposes an approach for ranking generalized fuzzy numbers on the basis of a normalized height coefficient and benefit and cost areas. In this method, the left area denotes the area from $x_{\min}$ to $x_{A_i}^L$ and is bounded by the maximizing membership function f_M and minimizing membership function f_G. The right area denotes the area from $x_{\max}$ to $x_{A_i}^R$ and is bounded by the maximizing membership function f_M and minimizing membership function f_G. $S_{A_i}^L$ is considered as the benefit, larger is better. $S_{A_i}^R$ is considered as the cost, smaller is better. In other words, larger $S_{A_i}^L$ and smaller $S_{A_i}^R$ mean bigger fuzzy number A_i. The normalized height coefficient is designed to reflect the influence of the height of fuzzy numbers on the final ranking score. The higher the normalized height coefficient of a fuzzy number, the higher its ranking is. The numerical example and comparison presented herein demonstrate the feasibility and robustness of the proposed method.

The proposed ranking method can be applied to fuzzy multicriteria decision-making MCDM to support decision-makers to select the best alternative. Future research can extend this ranking method to develop other ranking methods for fuzzy numbers, including interval type-2 fuzzy numbers, intuitionistic fuzzy numbers and hesitant fuzzy numbers etc., to solve more complex decision-making problems in practice.

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