Preprints.org

Article

Not peer-reviewed version

Multiple Positive Solutions for a Coupled System of Fractional Order BVP with p-Laplacian Operators and Parameters

Abdullah Ali H. Ahmadini , Mahammad Khuddush , Sabbavarapu Nageswara Rao *

Posted Date: 14 September 2023

doi: 10.20944/preprints202309.0968.v1

Keywords: Fractional derivative; Positive solutions; Boundary value problems; p-Laplacian; parameters

Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Multiple Positive Solutions for a Coupled System of Fractional Order BVP with p-Laplacian Operators and Parameters

Abdullah Ali H. Ahmadini [1], Mahammad Khuddush [2], Sabbavarapu Nageswara Rao [1,*,†]

1 Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia; aahmadini@jazanu.edu.sa; snrao@jazanu.edu.sa
2 Department of Mathematics, Chegg India Pvt., Visakhapatnam, 530002, Andhra Pradesh, India; khuddush89@gmail.com
* Correspondence: snrao@jazanu.edu.sa
† Current address: [1,3] Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia, [2] Department of Mathematics, Chegg India Pvt., Visakhapatnam, 530002, Andhra Pradesh, India
‡ These authors contributed equally to this work.

Abstract: In this study, we investigate the existence of positive solutions within a system of Riemann-Liouville fractional differential equations that incorporate the (r_1, r_2, r_3)-Laplacian operator while being subject to three-point boundary conditions. These equations incorporate various fractional derivatives and are influenced by parameters represented as (ψ_1, ψ_2, ψ_3). Our approach involves employing techniques such as cone expansion and compression of the functional type, in conjunction with the Leggett-Williams fixed point theorem, to establish the existence of positive solutions. To emphasize the practical significance of our findings in the realm of fractional differential equations, we provide two illustrative examples.

Keywords: fractional derivative; positive solutions; boundary value problems; p-Laplacian; parameters

MSC: 34B09; 34B15; 34B18; 34B27

---

1. Introduction

The study of nonlinear fractional systems is currently a topic of significant interest, with researchers dedicating substantial efforts to understanding these systems and their applications in various fields. These endeavors are primarily driven by the desire to utilize the findings regarding the existence of positive solutions in practical contexts. This expansion of differential calculus into the realm of complex real-world dynamics has led to the development and evaluation of novel theories based on empirical data [1].

A wide range of materials and processes that exhibit hereditary and memory characteristics find accurate descriptions through the nonlocal nature of fractional calculus (FC) [2,3]. The implications of these findings span across diverse scientific disciplines, including biomathematics [4], viscoelasticity [5], non-Newtonian fluid mechanics [6], and the characterization of anomalous diffusion [7].

The literature in the field of fractional differential equations (FDEs) predominantly centers on established concepts like Riemann-Liouville or Caputo derivatives, with some emerging alternatives such as conformable fractional derivatives [8]. However, it is worth noting that the body of work on FDEs of the conformable type is still evolving. The concept of conformable fractional derivatives was first introduced in 2014 and has gained attention for its computational advantages in solving differential equations. This approach has led to more efficient numerical solutions compared to Riemann-Liouville or Caputo fractional derivatives. Recently, a new formulation of the conformable fractional derivative has been proposed and extensively discussed in the literature [8–10]. Researchers

© 2023 by the author(s). Distributed under a Creative Commons CC BY license.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

have successfully applied conformable fractional derivatives to a wide range of domains, resulting in the establishment of various replicable methodologies [11].

Beyond the realm of mathematical theory, applications of these findings extend to diverse industries, including telecommunication equipment, synthetic chemicals, automobiles, and pharmaceuticals, where boundary value problems (BVPs) play a significant role. In these industrial processes, positive solutions are often highly desirable for practical reasons, as highlighted in recent research [12,13].

In this study, our primary objective is to investigate the following system of fractional differential equations that incorporate (r_1, r_2, r_3)-Laplacian operators. We aim to provide a comprehensive analysis of these equations, considering their potential implications and applications:

$$\begin{cases} -\mathcal{D}_{h^+}^{p_1}\left(\phi_{r_1}\left(\mathcal{D}_{h^+}^{q_1}\varpi(\xi)\right)\right) = f_1(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi \in (h,k), \\ -\mathcal{D}_{h^+}^{p_2}\left(\phi_{r_2}\left(\mathcal{D}_{h^+}^{q_2}\vartheta(\xi)\right)\right) = f_2(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi \in (h,k), \\ -\mathcal{D}_{h^+}^{p_3}\left(\phi_{r_3}\left(\mathcal{D}_{h^+}^{q_3}\omega(\xi)\right)\right) = f_3(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi \in (h,k), \end{cases} \tag{1}$$

where h and k are real numbers with $h < k$. The operators $\mathcal{D}h^{+q_i}, \mathcal{D}h^{+p_i}, \mathcal{D}h^{+\alpha i}$ correspond to standard Riemann-Liouville fractional order derivatives. Additionally, $q_i \in (1,2]$, $p_i, \alpha_i \in (0,1]$, and $\phi_{r_i}(\zeta) = |\zeta|^{r_i-1}\zeta$, with $r_i > 1$, and $\phi_{r_i}^{-1} = \phi_{\varphi_i}$, where $\frac{1}{\varphi_i} + \frac{1}{r_i} = 1$ for $i = 1,2,3$.

The boundary conditions for this system are given as:

$$\begin{cases} \varpi(h) = 0,\ \ \phi_{r_1}\left(\mathcal{D}_{h^+}^{q_1}\varpi(h)\right) = 0,\ \ \mu_1\, \mathcal{D}_{h^+}^{\alpha_1}\varpi(k) = \psi_1 + \lambda_1\, \mathcal{D}_{h^+}^{\alpha_1}\varpi(\eta_1), \\ \vartheta(h) = 0,\ \ \phi_{r_2}\left(\mathcal{D}_{h^+}^{q_2}\vartheta(h)\right) = 0,\ \ \mu_2\, \mathcal{D}_{h^+}^{\alpha_2}\vartheta(k) = \psi_2 + \lambda_2\, \mathcal{D}_{h^+}^{\alpha_2}\vartheta(\eta_2), \\ \omega(h) = 0,\ \ \phi_{r_3}\left(\mathcal{D}_{h^+}^{q_3}\omega(h)\right) = 0,\ \ \mu_3\, \mathcal{D}_{h^+}^{\alpha_3}\omega(k) = \psi_3 + \lambda_3\, \mathcal{D}_{h^+}^{\alpha_3}\omega(\eta_3), \end{cases} \tag{2}$$

Here, μ_i, λ_i are positive constants, and η_i are real numbers within the interval (h,k). It is essential that the conditions $\mu_i(k-h)^{q_i-\alpha_i-1} > \lambda_i(\eta_i - h)^{q_i-\alpha_i-1}$ hold for all $i = 1,2,3$.

To ensure the existence of positive solutions to the system (1)-(2), we make the following assumptions:

(B1) The functions f_1, f_2, and f_3 are continuous on the specified domains.
(B2) The parameters α_i, q_i, μ_i, λ_i, and η_i satisfy certain inequalities, ensuring the conditions required for the existence of solutions.
(B3) We introduce positive constants Φ_1, Φ_2, Φ_3, Θ_1, Θ_2, and Θ_3 with the constraint that $\frac{1}{\Phi_1} + \frac{1}{\Phi_2} + \frac{1}{\Phi_3} + \frac{1}{\Theta_1} + \frac{1}{\Theta_2} + \frac{1}{\Theta_3} \leq 1$.

The study of fractional differential equations is a rapidly expanding field with numerous applications in various domains. Our paper provides essential conditions for the functions f_1, f_2, and f_3, as well as intervals for the parameters (ψ_1, ψ_2, ψ_3), guaranteeing the existence of at least one and three positive solutions for the specified boundary value problem (1)-(2). A positive solution is defined as a triplet of functions $(\varpi(\xi), \vartheta(\xi), \omega(\xi))$ in the space $(\mathcal{C}[h,k],[0,\infty))^3$ that satisfies (1)-(2) with non-negative values for all $\xi \in [h,k]$, and where $(\varpi(\xi), \vartheta(\xi), \omega(\xi))$ is not equal to $(0,0,0)$.

For further insights into the applications of fractional calculus in various fields and related literature on positive solutions with different boundary conditions, we recommend reading the referenced books [14–16] and exploring the cited papers [17-35].

The subsequent sections of this paper are organized as follows: Section 2 introduces foundational concepts and key lemmas essential for underpinning our central results. Moving on to Section 3, we employ various methodological approaches, including cone expansion and compression of functional type, along with the utilization of the Leggett-Williams fixed point theorem, to expound upon our primary findings. Finally, in Section 4, we provide practical insights by presenting two illustrative examples that effectively demonstrate the application and relevance of our core discoveries.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

2. Preliminaries and lemmas

We refrain from including definitions and valuable lemmas pertaining to fractional calculus theory in this section, as they can be readily referenced in contemporary literature [2,3].

Definition 1. *For a function f given on the interval $[h,k]$, the α^{th} Riemann-Liouville fractional order derivative of f is defined by*

$$(D^{\alpha}_{h^+}f)(\xi) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\xi}\right)^{\alpha}\int_h^{\xi}(\xi-s)^{n-\alpha-1}f(s)ds,$$

here $n=[\alpha]+1$ and $[\alpha]$ denotes the integral part of α.

Definition 2. *The functional (arbitrary) order integral of the function $f \in L^1([h,k],R_+)$ of order $\alpha \in R_+$ is defined by*

$$(I^{\alpha}_{h^+}f)(\xi) = \frac{1}{\Gamma(\alpha)}\int_h^{\xi}(\xi-s)^{\alpha-1}f(s)ds,$$

where Γ is the Gamma function.

Lemma 1. *Assume that $\mathcal{D}^{\sigma}_{h^+} \in L^1[h,k]$ with a fractional derivative of order $\sigma>0$ then*

$$\mathcal{I}^{\sigma}_{h^+}\mathcal{D}^{\sigma}_{h^+}u(t) = u(\xi)+c_1(\xi-h)^{\sigma-1}+c_2(\xi-h)^{\sigma-2}+\cdots\cdots+c_n(\xi-h)^{\sigma-n}$$

for some $c_i \in R, i=1,2,3,\cdots,n$ where n is the smallest integer greater than or equal to σ.

Definition 3. *Let φ be a cone in the real Banach space $\mathcal{S}$. A map $\sigma:\varphi\to[0,\infty)$ is said to be nonnegative continuous concave functional on φ if σ is continuous and $\sigma(\lambda u+(1-\lambda)v) \geq \lambda\sigma(u)+(1-\lambda)\sigma(v)$ for all $u,v\in\varphi$ and $\lambda\in[0,1]$.*

Definition 4. *Let φ be a cone in the real Banach space $\mathcal{S}$. A map $\rho:\varphi\to[0,\infty)$ is said to be nonnegative continuous convex functional on φ if ρ is continuous and $\rho(\lambda u+(1-\lambda)v) \leq \lambda\rho(u)+(1-\lambda)\rho(v)$ for all $u,v\in\varphi$ and $\lambda\in[0,1]$.*

Rule S_1: Let κ be a cone in a Banach space $\mathcal{D}$ and x be a bounded open subset of $\mathcal{D}$ and $0\in x$. Then a continuous functional $\sigma:\kappa\to[0,\infty)$ is said to satisfy Rule S_1 if one of the following conditions holds:

(i) σ is convex, $\sigma(0)=0$, $\sigma(t)\neq0$ if $t\neq0$ and $\inf_{t\in\kappa\cap\partial x}\sigma(t)>0$,
(ii) σ is sublinear, $\sigma(0)=0$, $\sigma(t)\neq0$ if $t\neq0$ and $\inf_{t\in\kappa\cap\partial x}\sigma(t)>0$,
(iii) σ is concave and unbounded.

Rule S_2: Let κ be a cone in a Banach space $\mathcal{D}$ and x be a bounded open subset of $\mathcal{D}$ and $0\in x$. Then a continuous functional $\rho:\kappa\to[0,\infty)$ is said to satisfy Rule $\mathcal{S}_2$ if one of the following conditions holds:

(i) ρ is convex, $\rho(0)=0$, $\rho(t)\neq0$ if $t\neq0$,
(ii) ρ is sublinear, $\rho(0)=0$, $\rho(t)\neq0$ if $t\neq0$,
(iii) $\rho(t+s)\geq\rho(t)+\rho(s)$ for all $t,s\in\kappa$, $\rho(0)=0$, $\rho(t)\neq0$ if $t\neq0$.

Theorem 1. *[17] Consider two bounded open subsets, Ω_1 and Ω_2, within a Banach space denoted as $\mathcal{D}$. It is assumed that 0 belongs to Ω_1, and Ω_1 is a subset of Ω_2. Furthermore, let κ represent a cone within the same Banach space $\mathcal{D}$. We introduce an operator $\mathcal{L}$, which maps from $\kappa\cap(\overline{\Omega}_2\backslash\Omega_1)$ to κ and is characterized as completely continuous. Alongside this, two non-negative continuous functionals, σ and ρ, are defined on κ. The main result is contingent upon one of the following two conditions being satisfied:*

(a) σ adheres to Rule $\mathcal{S}1$ with $\sigma(\mathcal{L}t)\geq\sigma(t)$ for all t belonging to $\kappa\cap\partial\Omega1$, and ρ adheres to Rule $\mathcal{S}_2$ with $\rho(\mathcal{L}t)\leq\rho(t)$ for all t in $\kappa\cap\partial\Omega2$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

(b) Conversely, ρ follows Rule $\mathcal{S}_2$ with $\rho(\mathcal{L}t) \leq \rho(t)$ for all t in $\kappa \cap \partial\Omega 1$, and σ conforms to Rule $\mathcal{S}_1$ with $\sigma(\mathcal{L}t) \geq \sigma(t)$ for all t in $\kappa \cap \partial\Omega 2$.

In either case, the conclusion is that the operator $\mathcal{L}$ possesses at least one fixed point within the set $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 2. *(Leggett-Williams [18]) Let p, q, r and s be positive real numbers, let κ be a cone in a real Banach space $\mathcal{D}$, $\kappa_s = \{t \in \kappa : \|t\| < s\}$, ψ be a nonnegative continuous concave functional on κ such that $\psi(t) \leq \|t\|$, $\forall t \in \overline{\kappa_s}$ and $\kappa(\psi, q, r) = \{t \in \kappa; q \leq \psi(t), \|t\| \leq r\}$. Suppose $\mathcal{L} : \overline{\kappa_s} \to \overline{\kappa_s}$ be a completely continuous operator and there exist constants $0 < p < q < r \leq s$ such that*

(i) $\{t \in \kappa(\psi, q, r) \mid \psi(t) > q\} \neq \emptyset$ and $\psi(\mathcal{L}t) > q$ for $t \in \kappa(\psi, q, r)$,
(ii) $\|\mathcal{L}t\| < p$ for $\|t\| \leq p$,
(iii) $\psi(\mathcal{L}t) > q$ for $t \in \kappa(\psi, q, s)$ with $\|\mathcal{L}t\| > r$.

Then $\mathcal{L}$ has at least three fixed points t_1, t_2 and t_3 in $\overline{\kappa_s}$ satisfying $\|t_1\| < p, q < \psi(t_2), p < \|t_3\|$ and $\psi(t_3) < q$.

In what follows, we calculate the Green's function associate with (1)-(2). Consider the homogeneous boundary value problem:

$$-\mathcal{D}_{h^+}^{q_1}\varpi(\xi) = 0, \;\; \xi \in (h, k), \tag{3}$$

$$\varpi(h) = 0; \;\; \mu_1\, \mathcal{D}_{h^+}^{\alpha_1}\varpi(k) = \psi_1 + \lambda_1\, \mathcal{D}_{h^+}^{\alpha_1}\varpi(\eta_1). \tag{4}$$

Lemma 2. *Let $\Delta_1 \neq 0$. If $x(\xi) \in \mathcal{C}[h, k]$ and $1 < q_1 \leq 2$, then the boundary value problem:*

$$\mathcal{D}_{h^+}^{q_1}\varpi(\xi) + x(\xi) = 0, \;\; h < \xi < k, \tag{5}$$

satisfying the boundary condition (4), has a unique solution

$$\varpi(\xi) = \int_h^k \mathcal{H}_1(\xi, \zeta)x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1 - \alpha_1)(\xi - h)^{q_1 - 1}}{\Delta_1}, \; \xi \in [h, k],$$

where $\mathcal{H}_1(\xi, \zeta)$ is the Green's function for the BVP (5)-(4) and is given by

$$\mathcal{H}_1(\xi, \zeta) = h_1(\xi, \zeta) + \frac{\lambda_1(\xi - h)^{q_1 - 1}}{\mathcal{N}_1} h_2(\eta_1, \zeta).$$

Here $\Delta_1 = \Gamma(q_1)\mathcal{N}_1 \neq 0$; $\mathcal{N}_1 = \mu_1(k - h)^{q_1 - \alpha_1 - 1} - \lambda_1(\eta_1 - h)^{q_1 - \alpha_1 - 1}$ and

$$\begin{aligned} h_1(\xi, \zeta) &= \frac{1}{\Gamma(q_1)} \begin{cases} \frac{(\xi - h)^{q_1 - 1}(k - \zeta)^{q_1 - \alpha_1 - 1}}{(k - h)^{q_1 - \alpha_1 - 1}} - (\xi - \zeta)^{q_1 - 1}, & h \leq \zeta \leq \xi \leq k, \\ \frac{(\xi - h)^{q_1 - 1}(k - \zeta)^{q_1 - \alpha_1 - 1}}{(k - h)^{q_1 - \alpha_1 - 1}}, & h \leq \xi \leq \zeta \leq k, \end{cases} \\ h_2(\xi, \zeta) &= \frac{1}{\Gamma(q_1)} \begin{cases} \frac{(\xi - h)^{q_1 - \alpha_1 - 1}(k - \zeta)^{q_1 - \alpha_1 - 1}}{(k - h)^{q_1 - \alpha_1 - 1}} - (\xi - \zeta)^{q_1 - \alpha_1 - 1}, & h \leq \zeta \leq \xi \leq k, \\ \frac{(\xi - h)^{q_1 - \alpha_1 - 1}(k - \zeta)^{q_1 - \alpha_1 - 1}}{(k - h)^{q_1 - \alpha_1 - 1}}, & h \leq \xi \leq \zeta \leq k. \end{cases} \end{aligned} \tag{6}$$

Proof. Assume that $\varpi \in C^{[q_1]+1}[h, k]$ is a solution of fractional order boundary value problem (5)-(4) and is uniquely expressed by

$$\varpi(\xi) = -\int_h^\xi \frac{(\xi - \zeta)^{q_1 - 1}}{\Gamma(q_1)} x(\zeta)d\zeta + c_1(\xi - h)^{q_1 - 1} + c_2(\xi - h)^{q_1 - 2}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

In view of conditions (4), one can get $c_2 = 0$ and

$$c_1 = \frac{1}{\Delta_1}\Big[\mu_1 \int_h^k (k-h)^{q_1-\alpha_1-1} x(\zeta)d\zeta - \lambda_1 \int_h^{\eta_1} (\eta_1-\zeta)^{q_1-\alpha_1-1} x(\zeta)d\zeta\Big] + \frac{\psi_1\Gamma(q_1-\alpha_1)}{\Delta_1}.$$

Hence, we have

$$\begin{aligned}
\varpi(\xi) &= \frac{-1}{\Gamma(\alpha_1)}\int_h^\xi (\xi-\zeta)^{q_1-1}x(\zeta)d\zeta + \frac{(\xi-h)^{q_1-1}}{\Delta_1}\int_h^k (k-\zeta)^{q_1-\mu_1-1}x(\zeta)d\zeta \\
&\quad - \frac{\lambda_1(\xi-h)^{q_1-1}}{\Delta_1}\int_h^{\eta_1}(\eta_1-\zeta)^{q_1-\alpha_1-1}x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1} \\
&= \frac{-1}{\Gamma(q_1)}\int_h^\xi (\xi-\zeta)^{q_1-1}x(\zeta)d\zeta + \frac{(\xi-h)^{q_1-1}}{\Gamma(q_1)}\int_h^k \frac{(k-\zeta)^{q_1-\alpha_1-1}}{(k-h)^{q_1-\alpha_1-1}}x(\zeta)d\zeta \\
&\quad + \frac{\lambda_1(\xi-h)^{q_1-1}}{N}\int_h^k \frac{(\eta_1-h)^{q_1-\alpha_1-1}(k-\zeta)^{q-1-\alpha_1-1}}{\Gamma(q_1)(k-h)^{q_1-\alpha_1-1}}x(\zeta)d\zeta \\
&\quad - \frac{\lambda_1(\xi-h)^{q_1-1}}{N}\int_h^{\eta_1}\frac{(\eta_1-\zeta)^{q_1-\alpha_1-1}}{\Gamma(q_1)}x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1} \\
&= \int_h^k \mathcal{H}_1(\xi,\zeta)x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1}.
\end{aligned}$$

□

Lemma 3. *Let $1 < q_1 \le 2$, $0 < p_1 \le 1$. Then the FBVP*

$$\begin{cases} \mathcal{D}_{h^+}^{p_1}\Big(\phi_{r_1}(\mathcal{D}_{h^+}^{q_1}\varpi(\xi))\Big) + f_1(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)) = 0,\ h<\xi<k, \\ \varpi(h)=0,\ \ \mathcal{D}_{h^+}^{q_1}\varpi(h)=0,\ \ \mu_1\,\mathcal{D}_{h^+}^{\alpha_1}\varpi(k) = \psi_1 + \lambda_1\,\mathcal{D}_{h^+}^{\alpha_1}\varpi(\eta_1), \end{cases} \tag{7}$$

has a unique solution,

$$\begin{aligned}
\varpi(\xi) &= \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\quad + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi\in[h,k].
\end{aligned}$$

Proof. It follows from Lemma 1 and $0 < p_1 \le 1$ that

$$\phi_{r_1}(\mathcal{D}_{h^+}^{q_1}\varpi(\xi)) = -\int_h^\xi \frac{(\xi-\tau)^{p_1-1}}{\Gamma(p_1)}f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau + c_1(\xi-h)^{p_1-1}.$$

By $\mathcal{D}_{h^+}^{q_1}\varpi(h)=0$, we have $c_1 = 0$. So,

$$\mathcal{D}_{h^+}^{q_1}\varpi(\xi) + \phi_{\varphi_1}\Big(\int_h^t \frac{(\xi-\tau)^{p_1-1}}{\Gamma(p_1)}f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big) = 0.$$

Thus, the BVP (7) is equal to the following problem:

$$\mathcal{D}_{h^+}^{q_1}\varpi(\xi) + \phi_{\varphi_1}\Big(\int_h^\xi \frac{(\xi-\tau)^{p_1-1}}{\Gamma(p_1)}f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big) = 0;\ \ h<\xi<k;$$

$$\varpi(h)=0;\ \ \mu_1\,\mathcal{D}_{h^+}^{\alpha_1}\varpi(k) = \psi_1 + \lambda_1\,\mathcal{D}_{h^+}^{\alpha_1}\varpi(\eta_1).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

By Lemma 2, that boundary value problem (7) has a unique solution

$$\varpi(\xi) = \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi\in[h,k].$$

□

Lemma 4. *[25] Suppose that condition (B2) hold, then Green's function $\mathcal{H}_1$ have the following properties:*
(i) $\mathcal{H}_1(\xi,\zeta)\geq 0$ for all $(\xi,\zeta)\in(h,k)\times(h,k)$,
(ii) $\mathcal{H}_1(\xi,\zeta)\leq \mathcal{H}_1(k,\zeta)$, for all $(\xi,\zeta)\in[h,k]\times[h,k]$,
(iii) $\mathcal{H}_1(\xi,\zeta)\geq \left(\frac{1}{4}\right)^{q_1-1}\mathcal{H}_1(k,\zeta)$, for all $(\xi,\zeta)\in\mathcal{I}\times(h,k)$, where $\mathcal{I}=\left[\frac{3h+k}{4},\frac{h+3k}{4}\right]$.

Remark 1. *In a similar manner, the results of the Green's function $\mathcal{H}_2(\xi,\zeta)$ and $\mathcal{H}_3(\xi,\zeta)$ for the homogeneous BVP corresponding to the fractional differential equation are obtained. Consider the following condition:*

$$\mathcal{H}_i(\xi,\zeta)\geq \aleph\mathcal{H}_i(k,\zeta)\ \text{ for all } (\xi,\zeta)\in\mathcal{I}\times(h,k); i=1,2,3,$$

where $\mathcal{I}=\left[\frac{3h+k}{4},\frac{h+3k}{4}\right]$ and $\aleph=\min\left\{\left(\frac{1}{4}\right)^{q_1-1},\left(\frac{1}{4}\right)^{q_2-1},\left(\frac{1}{4}\right)^{q_3-1}\right\}$.

We consider the Banach space $\mathcal{X}=\mathcal{C}[h,k]$ with the supremum norm $\|\cdot\|$ and the Banach space $\mathcal{Y}=\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ with the norm $\|(\varpi,\vartheta,\omega)\|=\|\varpi\|+\|\vartheta\|+\|\omega\|$. We define the cone

$$\mathcal{P}=\big\{(\varpi,\vartheta,\omega)\in\mathcal{Y}:\varpi(\xi),\vartheta(\xi),\omega(\xi)\geq 0,\forall\xi\in[h,k],\text{ and } \min_{\xi\in\mathcal{I}}[\varpi(\xi)+\vartheta(\xi)+\omega(\xi)]\geq\aleph\,\|\,(\varpi,\vartheta,\omega)\,\|\,\big\},$$

where $\mathcal{I}=\left[\frac{3h+k}{4},\frac{h+3k}{4}\right]$ and $\aleph=\min\left\{\left(\frac{1}{4}\right)^{q_1-1},\left(\frac{1}{4}\right)^{q_2-1},\left(\frac{1}{4}\right)^{q_3-1}\right\}$.

Consider the coupled system of integral equations

$$\begin{aligned}
\varpi(\xi) &= \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi\in[h,k],\\
\vartheta(\xi) &= \int_h^k \mathcal{H}_2(\xi,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)} f_2\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_2\Gamma(q_2-\alpha_2)(\xi-h)^{q_2-1}}{\Delta_2},\ \xi\in[h,k],\\
\omega(\xi) &= \int_h^k \mathcal{H}_3(\xi,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)} f_3\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_3\Gamma(q_3-\alpha_3)(\xi-h)^{q_3-1}}{\Delta_3},\ \xi\in[h,k].
\end{aligned}$$

By Lemma 2, $(\varpi,\vartheta,\omega)\in\mathcal{P}$ is a solution of boundary value problems (1)-(2) if and only if it is a solution of the system of integral equations.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Define the operators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{P} \to \mathcal{X}$ by

$$\begin{aligned}
\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi) &= \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\quad + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1}, \ \xi \in [h,k], \\
\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) &= \int_h^k \mathcal{H}_2(\xi,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)} f_2(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\quad + \frac{\psi_2\Gamma(q_2-\alpha_2)(\xi-h)^{q_2-1}}{\Delta_2}, \ \xi \in [h,k], \\
\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi) &= \int_h^k \mathcal{H}_3(\xi,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)} f_3(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\quad + \frac{\psi_3\Gamma(q_3-\alpha_3)(\xi-h)^{q_3-1}}{\Delta_3}, \ \xi \in [h,k],
\end{aligned}$$

and an operator $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ as

$$\mathcal{T}(\varpi,\vartheta,\omega) = \big(\mathcal{T}_1(\varpi,\vartheta,\omega), \mathcal{T}_2(\varpi,\vartheta,\omega), \mathcal{T}_3(\varpi,\vartheta,\omega)\big), \ (\varpi,\vartheta,\omega) \in \mathcal{Y}.$$

It is clear that the existence of a positive solution to the system (1)-(2) is equivalent to the existence of a fixed points of the operator $\mathcal{T}$.

3. Main results

We denote the following notations for our convenience:

$$\begin{aligned}
\mathcal{D} = \max\Bigg\{&\phi_{\varphi_1}\Big(\frac{4^{p_1}\Gamma(p_1+1)}{(k-h)^{p_1}}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_1(k,\zeta)d\zeta, \ \phi_{\varphi_2}\Big(\frac{4^{p_2}\Gamma(p_2+1)}{(k-h)^{p_2}}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_2(k,\zeta)d\zeta, \\
&\phi_{\varphi_3}\Big(\frac{4^{p_3}\Gamma(p_3+1)}{(k-h)^{p_3}}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_3(k,\zeta)d\zeta\Bigg\}, \\
\mathcal{C} = \min\Bigg\{&\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta, \quad \int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta, \\
&\int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta\Bigg\}.
\end{aligned}$$

Let us define two continuous functionals α and β on the cone $\mathcal{P}$ by

$$\begin{aligned}
\alpha(\varpi,\vartheta,\omega) &= \min_{\xi\in\mathcal{I}}\{|\varpi|+|\vartheta|+|\omega|\} \text{and} \\
\beta(\varpi,\vartheta,\omega) &= \max_{\xi\in[h,k]}\{|\varpi|+|\vartheta|+|\omega|\} = \varpi(k)+\vartheta(k)+\omega(k) = \|(\varpi,\vartheta,\omega)\|.
\end{aligned}$$

It is clear that $\alpha(\varpi,\vartheta,\omega) \le \beta(\varpi,\vartheta,\omega)$, for all $(\varpi,\vartheta,\omega) \in \mathcal{P}$.

Lemma 5. *$\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Proof. By using standard arguments, we can easily show that, the operator $\mathcal{T}$ is completely continuous and we need only to prove $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$. Let $(\varpi, \vartheta, \omega) \in \mathcal{P}$, by Lemma 3, we have

$$\begin{aligned}
\|\mathcal{T}_1(\varpi,\vartheta,\omega)\| \leq & \int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(s-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
& + \frac{\psi_1\Gamma(q_1-\alpha_1)(k-h)^{q_1-1}}{\Delta_1}, \\
\|\mathcal{T}_2(\varpi,\vartheta,\omega)\| \leq & \int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)} f_2(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)))d\tau\Big)d\zeta \\
& + \frac{\psi_2\Gamma(q_2-\alpha_2)(k-h)^{q_2-1}}{\Delta_2}, \\
\|\mathcal{T}_3(\varpi,\vartheta,\omega)\| \leq & \int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_a^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)} f_3(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
& + \frac{\psi_3\Gamma(q_3-\alpha_3)(k-h)^{q_3-1}}{\Delta_3},
\end{aligned}$$

and

$$\begin{aligned}
\min_{\xi\in\mathcal{I}}\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi) = & \min_{\xi\in\mathcal{I}}\Big[\int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
& + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1}\Big], \\
\geq & \Big(\frac{1}{4}\Big)^{q_1-1}\Big[\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
& + \frac{\psi_1\Gamma(q_1-\alpha_1)(k-h)^{q_1-1}}{\Delta_1}\Big], \\
\geq & \aleph\|\mathcal{T}_1(\varpi,\vartheta,\omega)\|.
\end{aligned}$$

Similarly, $\min_{\xi\in\mathcal{I}}\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) \geq \aleph\|\mathcal{T}_2(\varpi,\vartheta,\omega)\|$ and $\min_{\xi\in\mathcal{I}}\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi) \geq \aleph\|\mathcal{T}_3(\varpi,\vartheta,\omega)\|$. Therefore

$$\begin{aligned}
\min_{\xi\in I}\{ & \mathcal{T}_1(\varpi,\vartheta,\omega)(\xi) + \mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) + \mathcal{T}_3(\varpi,\vartheta,\omega)(\xi)\} \\
& \geq \aleph\|\mathcal{T}_1(\varpi,\vartheta,\omega)\| + \aleph\|\mathcal{T}_2(\varpi,\vartheta,\omega)\| + \aleph\|\mathcal{T}_3(\varpi,\vartheta,\omega)\| \\
& = \aleph\|(\mathcal{T}_1(\varpi,\vartheta,\omega),\mathcal{T}_2(\varpi,\vartheta,\omega),\mathcal{T}_3(\varpi,\vartheta,\omega))\| \\
& = \aleph\|\mathcal{T}(\varpi,\vartheta,\omega)\|.
\end{aligned}$$

Hence, we get $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$. By using standard arguments involving the Arzela-Ascoli theorem, we can easily show that $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are completely continuous, and then $\mathcal{T}$ is a completely continuous operator from $\mathcal{P}$ to $\mathcal{P}$. □

Theorem 3. *Assume that conditions $(B1)-(B3)$ holds and suppose that there exist positive real numbers $r, \mathcal{R}$ with $r < \eta\mathcal{R}$ and $\psi_j < \frac{r\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}} \leq \frac{\mathcal{R}\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}$ such that $f_j; j = 1,2,3$ satisfying the following conditions:*

(C1) $f_j(\xi,\varpi,\vartheta,\omega) \geq \phi_{r_j}(\frac{1}{3}\frac{r}{\aleph\mathcal{D}})$, $\xi \in \mathcal{I}$ and $(\varpi,\vartheta,\omega) \in [r,\mathcal{R}]$,
(C2) $f_j(\xi,\varpi,\vartheta,\omega) \leq \phi_{r_j}(\frac{1}{\Phi_j}\frac{\mathcal{R}}{\mathcal{C}})$, $\xi \in [h,k]$ and $(\varpi,\vartheta,\omega) \in [0,\mathcal{R}]$.

Then the system of fractional order boundary value problem (1)-(2) has at least one positive and nondecreasing solution $(\varpi^\star,\vartheta^\star,\omega^\star)$ satisfying $r \leq \alpha(\varpi^\star,\vartheta^\star,\omega^\star)$ with $\beta(\varpi^\star,\vartheta^\star,\omega^\star) \leq \mathcal{R}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Proof. Let $\Omega_1 = \{(\varpi, \vartheta, \omega); \alpha(\varpi, \vartheta, \omega) < r\}$ and $\Omega_2 = \{(\varpi, \vartheta, \omega); \beta(\varpi, \vartheta, \omega) < \mathcal{R}\}$. It is easy to see that $0 \subset \Omega_1$, set Ω_1, Ω_2 are bounded open subsets of $\mathcal{E}$. Letting $(\varpi, \vartheta, \omega) \in \Omega$, we have

$$\begin{aligned} r > \alpha(\varpi, \vartheta, \omega) &= \min_{\xi \in I}\{\varpi(\xi) + \vartheta(\xi) + \omega(\xi)\} \\ &\geq \aleph\{\|\varpi\| + \|\vartheta\| + \|\omega\|\} \\ &= \aleph\beta(\varpi, \vartheta, \omega). \end{aligned}$$

Thus $\mathcal{R} > \frac{r}{\aleph} > \beta(\varpi, \vartheta, \omega)$, i.e $(\varpi, \vartheta, \omega) \in \Omega_2$, so $\Omega_1 \subseteq \Omega_2$.
Claim 1: If $(\varpi, \vartheta, \omega) \in \mathcal{P} \cap \partial\Omega_1$, then $\alpha(\mathcal{T}(\varpi, \vartheta, \omega)) \geq \alpha(\varpi, \vartheta, \omega) = r$, for $\zeta \in \mathcal{I}$. It follows $(C1)$ Lemma 4 that

$$\begin{aligned} \alpha(\mathcal{T}(\varpi, \vartheta, \omega)) &= \min_{\xi \in \mathcal{I}} \sum_{j=1}^{3} \Big[\int_h^k \mathcal{H}_j(\xi, \zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j(\tau, \varpi(\tau), \vartheta(\tau), \omega(\tau))d\tau\Big)d\zeta \\ &\quad + \frac{\psi_j \Gamma(q_j - \alpha_j)(\xi - h)^{q_j-1}}{\Delta_j}, \Big] \\ &\geq \sum_{j=1}^{3} \Big[\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \Big(\frac{1}{4}\Big)^{q_j-1} \mathcal{H}_j(k, \zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} \phi_{r_j}\Big(\frac{1}{3}\frac{r}{\aleph\mathcal{D}}\Big)d\zeta \\ &\quad + \frac{\psi_j \Gamma(q_j - \alpha_j)\big(\frac{1}{4}\big)^{q_j-1}(k-h)^{q_j-1}}{\Delta_j}\Big] \\ &\geq \sum_{j=1}^{3} \phi_{\varphi_j}\Big(\frac{(k-h)^{p_j}}{4^{p_j}\Gamma(p_j+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \aleph\mathcal{H}_j(k, \zeta)\Big(\frac{1}{3}\frac{r}{\aleph\mathcal{D}}\Big)d\zeta \\ &\geq \frac{r}{3\mathcal{D}}\phi_{\varphi_1}\Big(\frac{(k-h)^{p_1}}{4^{p_1}\Gamma(p_1+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_1(k, \zeta)d\zeta + \frac{r}{3\mathcal{D}}\phi_{\varphi_2}\Big(\frac{(k-h)^{p_2}}{4^{p_2}\Gamma(p_2+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_2(k, \zeta)d\zeta \\ &\quad + \frac{r}{3\mathcal{D}}\phi_{\varphi_3}\Big(\frac{(k-h)^{p_3}}{4^{p_3}\Gamma(p_3+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_3(k, \zeta)d\zeta \\ &= \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r = \alpha(\varpi, \vartheta, \omega). \end{aligned}$$

Claim 2: If $(\varpi, \vartheta, \omega) \in \mathcal{P} \cap \partial\Omega_2$, then $\beta(\mathcal{T}(\varpi, \vartheta, \omega)) \leq \beta(\varpi, \vartheta, \omega)$. Then

$$\Big[\varpi(\zeta) + \vartheta(\zeta) + \omega(\zeta)\Big] \leq \beta(\varpi, \vartheta, \omega) = \mathcal{R},$$

for $\zeta \in [h, k]$. It follows from $(C2)$ and Lemma 4 that

$$\begin{aligned} \beta(\mathcal{T}(\varpi, \vartheta, \omega)) &= \max_{\xi \in [h,k]} \sum_{j=1}^{3} \Big[\int_h^k \mathcal{H}_j(\xi, \zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j(\tau, \varpi(\tau), \vartheta(\tau), \omega(\tau))d\tau\Big)d\zeta \\ &\quad + \frac{\psi_j \Gamma(q_j - \alpha_j)(\xi - h)^{q_j-1}}{\Delta_j}\Big] \\ &\leq \sum_{j=1}^{3} \Big[\int_h^k \mathcal{H}_j(k, \zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j(\tau, \varpi(\tau), \vartheta(\tau), \omega(\tau))d\tau\Big)d\zeta \\ &\quad + \frac{\psi_j \Gamma(q_j - \alpha_j)(k - h)^{q_j-1}}{\Delta_j}\Big] \end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

$$
\begin{aligned}
&< \frac{\mathcal{R}}{\Phi_1\mathcal{C}}\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta + \frac{\mathcal{R}}{\Phi_2\mathcal{C}}\int_h^k \mathcal{H}_2(\xi,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta \\
&\quad + \frac{\mathcal{R}}{\Phi_3\mathcal{C}}\int_h^\zeta \mathcal{H}_3(\xi,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta + \frac{\mathcal{R}}{\Theta_1}+\frac{\mathcal{R}}{\Theta_2}+\frac{\mathcal{R}}{\Theta_3} \\
&= \frac{\mathcal{R}}{\Phi_1}+\frac{\mathcal{R}}{\Phi_2}+\frac{\mathcal{R}}{\Phi_3}+\frac{\mathcal{R}}{\Theta_1}+\frac{\mathcal{R}}{\Theta_2}+\frac{\mathcal{R}}{\Theta_3} \\
&= \mathcal{R}\Big[\frac{1}{\Phi_1}+\frac{1}{\Phi_2}+\frac{1}{\Phi_3}+\frac{1}{\Theta_1}+\frac{1}{\Theta_2}+\frac{1}{\Theta_3}\Big] \le \mathcal{R} = \beta(\varpi,\vartheta,\omega).
\end{aligned}
$$

Clearly, α satisfies Rule $(S_1)(iii)$ and β satisfies Rule $(S_2)(i)$. Therefore the condition (a) of Theorem 1 is satisfied and hence $\mathcal{T}$ has at least one fixed point $(\varpi^\star,\vartheta^\star,\omega^\star)\in\mathcal{P}\cap(\overline{\Omega_2}\setminus\Omega_1)$,i.e, the system of fractional order boundary value problem (1) – (2) has at least one positive and nondecreasing solution $(\varpi^\star,\vartheta^\star,\omega^\star)$ satisfying $r\le\alpha(\varpi^\star,\vartheta^\star,\omega^\star)$ with $\beta(\varpi^\star,\vartheta^\star,\omega^\star)\le\mathcal{R}$. □

Theorem 4. *Assume that conditions $(B1)-(B3)$ holds and suppose that there exist positive real numbers $r,\mathcal{R}$ with $r<\mathcal{R}$ and $\psi_j<\frac{r\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}\le\frac{\mathcal{R}\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}$ such that $f_j;j=1,2,3$ satisfying the following conditions:*

(C3) $f_j(\xi,\varpi,\vartheta,\omega)\le\phi_{r_j}\big(\frac{1}{\Phi_j}\frac{r}{\mathcal{D}}\big)$, $\xi\in[h,k]$ and $(\varpi,\vartheta,\omega)\in[0,r]$,
(C4) $f_j(\xi,\varpi,\vartheta,\omega)\ge\phi_{r_j}\big(\frac{1}{\Phi_j}\frac{\mathcal{R}}{\aleph\mathcal{C}}\big)$, $\xi\in\mathcal{I}$ and $(\varpi,\vartheta,\omega)\in[\mathcal{R},\frac{\mathcal{R}}{\aleph}]$.

Then the system of fractional order boundary value problem (1) – (2) has at least one positive and nondecreasing solution $(\varpi^\star,\vartheta^\star,\omega^\star)$ satisfying $r\le\beta(\varpi^\star,\vartheta^\star,\omega^\star)$ with $\alpha(\varpi^\star,\vartheta^\star,\omega^\star)\le\mathcal{R}$.

Proof. Let $\Omega_3=\{(\varpi,\vartheta,\omega);\beta(\varpi,\vartheta,\omega)<r\}$ and $\Omega_4=\{(\varpi,\vartheta,\omega);\alpha(\varpi,\vartheta,\omega)<\mathcal{R}\}$. We have $0\in\Omega_3$, set $\Omega_3\subset\Omega_4$ with Ω_3 and Ω_4 are bounded open subsets of $\mathcal{E}$.
Claim 1: If $(\varpi,\vartheta,\omega)\in\mathcal{P}\cap\partial\Omega_3$, then $\beta(\mathcal{T}(\varpi,\vartheta,\omega))\le\beta(\varpi,\vartheta,\omega)$. To see this, let $(\varpi,\vartheta,\omega)\in\mathcal{P}\cap\partial\Omega_3$. Then $[\varpi(\zeta)+\vartheta(\zeta)+\omega(\zeta)]\le\beta(\varpi,\vartheta,\omega)=r$, for $\zeta\in[h,k]$. It follows from (C3) and Lemma 4 that

$$
\begin{aligned}
\beta(\mathcal{T}(\varpi,\vartheta,\omega)) &= \max_{\xi\in[h,k]}\sum_{j=1}^{3}\Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\qquad + \frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j}\Big] \\
&\le \sum_{j=1}^{3}\Big[\int_h^k \mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\qquad + \frac{\psi_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}{\Delta_j}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{r}{\Phi_1\mathcal{D}}\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta + \frac{r}{\Phi_2\mathcal{D}}\int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta \\
&\quad + \frac{r}{\Phi_3\mathcal{D}}\int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta + \frac{r}{\Theta_1}+\frac{r}{\Theta_2}+\frac{r}{\Theta_3} \\
&= \frac{r}{\Phi_1}+\frac{r}{\Phi_2}+\frac{r}{\Phi_3}+\frac{r}{\Theta_1}+\frac{r}{\Theta_2}+\frac{r}{\Theta_3} \\
&= r\Big[\frac{1}{\Phi_1}+\frac{1}{\Phi_2}+\frac{1}{\Phi_3}+\frac{1}{\Theta_1}+\frac{1}{\Theta_2}+\frac{1}{\Theta_3}\Big] \le r = \beta(\varpi,\vartheta,\omega).
\end{aligned}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Claim 2: If $(\varpi,\vartheta,\omega) \in \mathcal{P} \cap \partial\Omega_4$, then $\alpha(\mathcal{T}(\varpi,\vartheta,\omega)) \geq \alpha(\varpi,\vartheta,\omega)$. $\mathcal{T}$ see this, let $(\varpi,\vartheta,\omega) \in \mathcal{P} \cap \partial\Omega_4$. Then $\frac{\mathcal{R}}{\aleph} = \frac{\alpha(\varpi,\vartheta,\omega)}{\eta} \geq \beta(\varpi,\vartheta,\omega) \geq [\varpi(\zeta)+\vartheta(\zeta)+\omega(\zeta)] \geq \alpha(\varpi,\vartheta,\omega) = \mathcal{R}$ for $\zeta \in \mathcal{I}$. It follows from $(C4)$, and Lemma 4, we have

$$
\begin{aligned}
\alpha(\mathcal{T}(\varpi,\vartheta,\omega)) &= \min_{\xi\in\mathcal{I}} \sum_{j=1}^{3} \Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta \\
&\quad + \frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j}\Big] \\
&\geq \sum_{j=1}^{3} \Big[\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \Big(\frac{1}{4}\Big)^{q_j-1} \mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} \phi_{r_j}\Big(\frac{1}{3}\frac{\mathcal{R}}{\aleph\mathcal{C}}\Big)d\zeta \\
&\quad + \frac{\psi_j\Gamma(q_j-\alpha_j)\big(\frac{1}{4}\big)^{q_j-1}(k-h)^{q_j-1}}{\Delta_j}\Big] \\
&\geq \sum_{j=1}^{3} \phi_{\varphi_j}\Big(\frac{(k-h)^{p_j}}{4^{p_j}\Gamma(p_j+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \aleph\mathcal{H}_j(k,\zeta)\Big(\frac{1}{3}\frac{\mathcal{R}}{\aleph\mathcal{C}}\Big)d\zeta \\
&\geq \frac{\mathcal{R}}{3\,\mathcal{C}}\phi_{\varphi_1}\Big(\frac{(k-h)^{p_1}}{4^{p_1}\Gamma(p_1+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_1(k,\zeta)d\zeta + \frac{\mathcal{R}}{3\,\mathcal{C}}\phi_{\varphi_2}\Big(\frac{(k-h)^{p_2}}{4^{p_2}\Gamma(p_2+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_2(k,\zeta)d\zeta \\
&\quad + \frac{\mathcal{R}}{3\,\mathcal{C}}\phi_{\varphi_3}\Big(\frac{(k-h)^{p_3}}{4^{p_3}\Gamma(p_3+1)}\Big) \int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_3(k,\zeta)d\zeta \\
&= \frac{\mathcal{R}}{3} + \frac{\mathcal{R}}{3} + \frac{\mathcal{R}}{3} = \mathcal{R} = \alpha(\varpi,\vartheta,\omega).
\end{aligned}
$$

Clearly, α satisfies Rule $(S1)(iii)$ and β satisfies Rule $(S2)(i)$. Therefore the condition (a) of Theorem 1 is satisfied and hence $\mathcal{T}$ has at least one fixed point $(\varpi^\star,\vartheta^\star,\omega^\star) \in \mathcal{P} \cap \Big(\overline{\Omega_4} \setminus \Omega_2\Big)$,i.e, the system of fractional order boundary value problem (1) $-$ (2) has at least one positive and nondecreasing solution $(\varpi^\star,\vartheta^\star,\omega^\star)$ satisfying $r \leq \beta((\varpi^\star,\vartheta^\star,\omega^\star))$ with $\alpha(\varpi^\star,\vartheta^\star,\omega^\star) \leq \mathcal{R}$. □

Theorem 5. *Assume that $(B1)-(B3)$ holds. Suppose that there exist $0 < k < l < \aleph d$ and $0 < \psi_j < \frac{k\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}} \leq \frac{d\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}$ such that $f_j (j = 1,2,3)$ satisfies the following conditions:*

(C5) $f_j(\xi,\varpi,\vartheta,\omega) < \phi_{r_j}\big(\frac{d}{\Phi_j\mathcal{C}}\big)$, *for all* $\xi \in [h,k]$, $(\varpi,\vartheta,\omega) \in [0,d]$,
(C6) $f_j(\xi,\varpi,\vartheta,\omega) > \phi_{r_j}\big(\frac{l}{3\aleph\mathcal{D}}\big)$, *for all* $\xi \in \mathcal{I}$, $(\varpi,\vartheta,\omega) \in [l,\frac{l}{\aleph}]$,
(C7) $f_j(\xi,\varpi,\vartheta,\omega) < \phi_{r_j}\big(\frac{k}{\Phi_j\mathcal{C}}\big)$, *for all* $\xi \in [h,k]$, $(\varpi,\vartheta,\omega) \in [0,k]$.

Then the system (1)-(2) has at least three positive solution $(\varpi_1,\vartheta_1,\omega_1), (\varpi_2,\vartheta_2,\omega_2)$ and $(\varpi_3,\vartheta_3,\omega_3)$ with $\varphi(\varpi_1,\vartheta_1,\omega_1) < k$, $l < \psi(\varpi_2,\vartheta_2,\omega_2) < \varphi(\varpi_2,\vartheta_2,\omega_2) < d$, $k < \varphi(\varpi_3,\vartheta_3,\omega_3) < d$ with $\psi(\varpi_3,\vartheta_3,\omega_3) < l$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Proof. Firstly, if $(\varpi,\vartheta,\omega)\in\overline{\mathcal{P}_d}$, then we may assert that $\mathcal{T}:\overline{\mathcal{P}_d}\to\overline{\mathcal{P}_d}$ is a completely continuous operator. To see this, suppose $(\varpi,\vartheta,\omega)\in\overline{\mathcal{P}_d}$, then $\|(\varpi,\vartheta,\omega)\|\le d$. It follows from Lemma 4 and $(C5)$, that

$$\begin{aligned}
&\|\mathcal{T}(\varpi,\vartheta,\omega)\| = \max_{\xi\in[h,k]}\{\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi)\}\\
&= \max_{\xi\in[h,k]}\sum_{i=1}^{3}\Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta\\
&\quad+\frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j}\Big]\\
&\le \sum_{j=1}^{3}\Big[\int_h^k \mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta\\
&\qquad\qquad+\frac{\psi_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}{\Delta_j}\Big]\\
&< \frac{1}{\Phi_1}\frac{d}{C}\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta+\frac{1}{\Phi_2}\frac{d}{C}\int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta\\
&\quad+\frac{1}{\Phi_3}\frac{d}{C}\int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta+\frac{d}{\Theta_j}+\frac{d}{\Theta_j}+\frac{d}{\Theta_j}\\
&= d\Big[\frac{1}{\Phi_1}+\frac{1}{\Phi_2}+\frac{1}{\Phi_3}+\frac{1}{\Theta_1}+\frac{1}{\Theta_2}+\frac{1}{\Theta_3}\Big]\le d.
\end{aligned}$$

Therefore, $\mathcal{T}:\overline{\mathcal{P}_d}\to\overline{\mathcal{P}_d}$. This together with Lemma 5 implies that $\mathcal{T}:\overline{\mathcal{P}_d}\to\overline{\mathcal{P}_d}$ is a completely continuous operator. In the similarly way, if $(\varpi,\vartheta,\omega)\in\overline{\mathcal{P}_k}$, then from $(C7)$ yields $\|\mathcal{T}(\varpi,\vartheta,\omega)\|<k$. This shows that condition (ii) of Theorem 2 is fulfilled.

Now, we let $\varpi(\xi)+\vartheta(\xi)+\omega(\xi)=\frac{l}{\aleph}$ for $\xi\in[h,k]$. It is easy to verify that $\varpi(\xi)+\vartheta(\xi)+\omega(\xi)=\frac{l}{\aleph}\in\mathcal{P}(\psi,l,\frac{l}{\aleph})$ and $\psi(\varpi,\vartheta,\omega)=\frac{l}{\aleph}>l$, and so $\{(\varpi,\vartheta,\omega)\in\mathcal{P}(\psi,l,\frac{l}{\aleph});\psi(\varpi,\vartheta,\omega)>l\}\neq\emptyset$. Thus, for all $(\varpi,\vartheta,\omega)\in\mathcal{P}(\psi,l,\frac{l}{\aleph})$, we have that $l\le\varpi(\xi)+\vartheta(\xi)+\omega(\xi)\le\frac{l}{\aleph}$ for $\xi\in\mathcal{I}$ and $\mathcal{T}(\varpi,\vartheta,\omega)\in\mathcal{P}$, from $(C6)$, we have

$$\begin{aligned}
&\psi(\mathcal{T}(\varpi,\vartheta,\omega)(\xi)) = \min_{\xi\in\mathcal{I}}\{\mathcal{T}_1(\varpi,\vartheta,\omega)(t)+\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi)\}\\
&= \min_{\xi\in I}\sum_{i=1}^{3}\Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta\\
&\quad+\frac{\psi_j\Gamma(q_j-\alpha_j)(t-h)^{q_j-1}}{\Delta_j}\Big]\\
&\ge \aleph\sum_{i=1}^{3}\Big[\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\Big(\frac{1}{4}\Big)^{q_j-1}\mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}\phi_{r_j}\Big(\frac{l}{3\aleph\mathcal{D}}\Big)\Big)d\zeta\\
&\qquad\qquad+\frac{\psi_j\Gamma(q_j-\alpha_j)\big(\frac{1}{4}\big)^{q_j-1}(k-h)^{q_j-1}}{\Delta_j}\Big]\\
&\ge \sum_{j=1}^{3}\phi_{\varphi_j}\Big(\frac{(k-h)^{p_j}}{4^{p_j}\Gamma(p_j+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\aleph\mathcal{H}_j(k,\zeta)\Big(\frac{l}{3\aleph\mathcal{D}}\Big)d\zeta\\
&\ge \frac{l}{3\,\mathcal{D}}\phi_{\varphi_1}\Big(\frac{(k-h)^{p_1}}{4^{p_1}\Gamma(p_1+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_1(k,\zeta)d\zeta+\frac{l}{3\,\mathcal{D}}\phi_{\varphi_2}\Big(\frac{(k-h)^{p_2}}{4^{p_2}\Gamma(p_2+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_2(k,\zeta)d\zeta\\
&\quad+\frac{l}{3\,\mathcal{D}}\phi_{\varphi_3}\Big(\frac{(k-h)^{p_3}}{4^{p_3}\Gamma(p_3+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_3(k,\zeta)d\zeta\\
&= \frac{l}{3}+\frac{l}{3}+\frac{l}{3}=l.
\end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Hence the condition (i) of Theorem 2 is verified. Next, we prove that (iii) of Theorem 2 is satisfied. By Lemma 5, we have $\min_{\xi\in I}|\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_3(\varpi,\vartheta,\omega)(t)| > \aleph\|\mathcal{T}(\varpi,\vartheta,\omega)\| > d$ for $(\varpi,\vartheta,\omega)\in\mathcal{P}(\psi,l,d)$ with $\|\mathcal{T}(\varpi,\vartheta,\omega)\| > \frac{l}{\aleph}$. To sum up, all the conditions of Theorem 2 are satisfied, then there exist three positive solutions $(\varpi_1,\vartheta_1,\omega_1),(\varpi_2,\vartheta_2,\omega_2)$ and $(\varpi_3,\vartheta_3,\omega_3)$ with $\varphi(\varpi_1,\vartheta_1,\omega_1) < k$, $l < \psi(\varpi_2,\vartheta_2,\omega_2) < \varphi(\varpi_2,\vartheta_2,\omega_2) < d$, $k < \varphi(\varpi_3,\vartheta_3,\omega_3) < d$, with $\psi(\varpi_3,\vartheta_3,\omega_3) < l$. $\square$

4. Examples

Let $h=1,\ k=2,\ p_1=0.5,\ p_2=0.6,\ p_3=0.7,\ q_1=1.5,\ q_2=1.6,\ q_3=1.7,\ \alpha_1=0.5,\ \alpha_2=0.6,\ \alpha_3=0.7,\ \eta_1=1.5,\ \eta_2=1.6,\ \eta_3=1.7, \mu_1=2,\ \mu_2=3,\ \mu_3=4,\ \lambda_1=1,\ \lambda_2=2,\ \lambda_3=3,\ r_1=2,\ r_2=2,\ r_3=2.$

We consider the system of fractional differential equations

$$\begin{cases} -\mathcal{D}^{0.5}_{1^+}\left(\phi_2\left(\mathcal{D}^{1.5}_{1^+}\varpi(\xi)\right)\right)=f_1(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi\in(1,2),\\ -\mathcal{D}^{0.6}_{1^+}\left(\phi_2\left(\mathcal{D}^{1.6}_{1^+}\vartheta(\xi)\right)\right)=f_2(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi\in(1,2),\\ -\mathcal{D}^{0.7}_{1^+}\left(\phi_2\left(\mathcal{D}^{1.7}_{1^+}\omega(\xi)\right)\right)=f_3(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi\in(1,2), \end{cases} \tag{8}$$

$$\begin{cases} \varpi(1)=0;\ \ \phi_2\left(\mathcal{D}^{1.5}_{1^+}\varpi(1)\right)=0;\ \ 2\,\mathcal{D}^{0.5}_{1^+}\varpi(2)=\psi_1+1\,\mathcal{D}^{0.5}_{1^+}\varpi(1.5),\\ \vartheta(1)=0;\ \ \phi_2\left(\mathcal{D}^{1.6}_{1^+}\vartheta(1)\right)=0;\ \ 3\,\mathcal{D}^{0.6}_{1^+}\vartheta(2)=\psi_2+2\,\mathcal{D}^{0.6}_{1^+}\vartheta(1.6),\\ \omega(1)=0;\ \ \phi_2\left(\mathcal{D}^{1.7}_{1^+}\omega(1)\right)=0;\ \ 4\,\mathcal{D}^{0.7}_{1^+}\omega(2)=\psi_3+3\,\mathcal{D}^{0.7}_{1^+}\omega(1.7), \end{cases} \tag{9}$$

where $\psi_1,\ \psi_2,\ \psi_3$ are parameters. We have $\aleph=0.378929$; $\Delta_1=0.886227>0$; $\Delta_2=0.893515$; $\Delta_3=0.908639$, so assumption $(A2)$ satisfied. Beside we found $\mathcal{D}=\max\{2.0,\ 3.446,\ 5.278\}=5.278$; $\mathcal{C}=\min\{1.858929,\ 2.620029,\ 3.237839\}=1.858929$.

Example 1. *We consider the functions*

$$f_1(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{18}e^{-(\varpi+\vartheta+\omega)}+\sin\xi, & 0\le\varpi,\vartheta,\omega<5,\\ \frac{1}{2}(e^{-(\varpi+\vartheta+\omega)}+1)+\frac{1}{3}\log\xi, & 5\le\varpi,\vartheta,\omega\le10,\end{cases}$$

$$f_2(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{3}(e^{-(\varpi+\vartheta+\omega)}+1)+\frac{t}{7}\sin\xi, & 0\le\varpi,\vartheta,\omega<5,\\ \frac{1}{2}(e^{-\xi}+1)+\frac{1}{\xi+1}(e^{-(\varpi+\vartheta+\omega)}\log\xi), & 5\le\varpi,\vartheta,\omega\le10,\end{cases}$$

$$f_3(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{\xi+1}+\log\xi(e^{-(\varpi+\vartheta+\omega)}+2)^{-1}, & 0\le\varpi,\vartheta,\omega<5,\\ \frac{1}{\xi+1}(\log\xi+1)+\frac{\xi e^{-\xi}}{\varpi+\vartheta+\omega}+\frac{10}{119}, & 5\le\varpi,\vartheta,\omega\le10.\end{cases}$$

Choosing $r=1$, $\mathcal{R}=10$, with $\frac{1}{\Phi_1}=\frac{1}{\Phi_2}=\frac{1}{\Phi_3}=\frac{1}{\Theta_1}=\frac{1}{\Theta_2}=\frac{1}{\Theta_3}=\frac{1}{6}$ then $r<\aleph\mathcal{R}$ and $f_i(i=1,2,3)$ fulfilling the following conditions:

(C1) $f_i(\xi,\varpi,\vartheta,\omega)\ge0.166668=\phi_{r_i}\left(\frac{1}{3}\frac{r}{\aleph\mathcal{D}}\right)$, $\xi\in[2.25,2.75]$ and $(\varpi,\vartheta,\omega)\in[1,10]$
(C2) $f_i(\xi,\varpi,\vartheta,\omega)\le0.896574=\phi_{r_i}\left(\frac{1}{\Phi_j}\frac{\mathcal{R}}{\mathcal{C}}\right)$, $\xi\in[1,2]$ and $(\varpi,\vartheta,\omega)\in[0,10]$.

Thus, all conditions of Theorem 3 are fulfilled. Hence, for $\psi_1\le1.47705$, $\psi_2\le1.489192$, $\psi_3\le1.514398$, the system of (8) and (9) has at least three positive solutions.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

Example 2. *We consider the functions*

$$f_1(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{\xi+25}\log(\varpi+\vartheta+\omega+1)+\frac{e^{-\xi}}{12}, & 0\le\varpi,\vartheta,\omega\le 1,\\ \frac{2}{\xi^2+1}+\frac{\log\xi+1}{e^{-\xi}+5}, & 1<\varpi,\vartheta,\omega\le 10,\\ \frac{1}{5}(\xi+e^{-(\varpi+\vartheta+\omega)})+\frac{1}{9}(\xi+\sin\xi), & 10<\varpi,\vartheta,\omega\le 20,\end{cases}$$

$$f_2(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{40+\xi^5}e^{(\varpi+\vartheta+\omega)}+\frac{\log\xi}{10}, & 0\le\varpi,\vartheta,\omega\le 1,\\ \frac{1}{\xi+1}[\log\xi+e^{-(\varpi+\vartheta+\omega)}]+\frac{e^{-\xi}}{u+v+w}+\frac{9}{24}, & 1<\varpi,\vartheta,\omega\le 10,\\ \frac{\log\xi+1}{\varpi+\vartheta+\omega}+\frac{e^{\xi}}{\xi^2+1}-\sin\xi, & 10<\varpi,\vartheta,\omega\le 20,\end{cases}$$

$$f_3(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{\xi+6}[e^{-(\varpi+\vartheta+\omega)}\log\xi], & 0\le\varpi,\vartheta,\omega\le 1,\\ \frac{1}{2}[e^{-(\varpi+\vartheta+\omega)}+1]+\frac{2}{\xi+1}[e^{-\xi}\sin\xi], & 1<\varpi,\vartheta,\omega\le 10,\\ \log\xi+\frac{2}{5}(1+\xi)-\frac{2}{\varpi+\vartheta+\omega}, & 10<\varpi,\vartheta,\omega\le 20.\end{cases}$$

Choosing $k=4,\ l=5,\ d=727.55,\ \frac{1}{\Im_1}=\frac{1}{\Im_2}=\frac{1}{\Im_3}=\frac{1}{\Re_1}=\frac{1}{\Re_2}=\frac{1}{\Re_3}=\frac{1}{6}$ *then* $0<k<l<\aleph d$ *and* $f_i(i=1,2,3)$ *fulfilling the following conditions:*

(C5) $f_i(\xi,\varpi,\vartheta,\omega)<1.793147=\phi_{r_j}\left(\frac{d}{\Phi_j\mathcal{C}}\right)$, *for all* $\xi\in[1,2]$, $(\varpi,\vartheta,\omega)\in[0,20]$,
(C6) $f_i(\xi,\varpi,\vartheta,\omega)>0.333335=\phi_{r_j}\left(\frac{l}{3\aleph\mathcal{D}}\right)$, *for all* $\xi\in[2.25,2.75]$, $(\varpi,\vartheta,\omega)\in[2,5.278034]$,
(C7) $f_i(\xi,\varpi,\vartheta,\omega)<0.089657=\phi_{r_j}\left(\frac{k}{\Phi_j\mathcal{C}}\right)$, *for all* $\xi\in[1,2]$, $(\varpi,\vartheta,\omega)\in[0,1]$.

Thus, all conditions of Theorem 5 are fulfilled. Hence, for $\sigma_1\le 2.95409$, $\sigma_2\le 2.978384$, $\sigma_3\le 3.028796$, the system of (8) and (9) has at least three positive solutions.

Author Contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Funding: The authors extend their appreciation to deanship of scientific research, Jazan University, for supporting this research work through the research units support program support number: RUP2-02.

Acknowledgments: A. A. H. Ahmadini, S. Nageswara Rao are thankful to Jazan University and Mahammad Khuddush is thankful to Chegg India Pvt., Visakhapatnam for the support given throughout the writing of this paper

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. H.G. Sun, Y. Zhang, D. Baleanu, W. Chen and Y.Q. Chen, A new collection of real world applications of fractional calculus in science and engineering, Commun. Nonlinear Sci. Numer. Simul., **64**(2018), 213–231.
2. I. Podulbny, Fractional Differential Equations, Academic Press, San Diego, 1999.
3. A.A. Kilbas, H.M. Srivasthava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.
4. E.F.D. Goufo, A biomathematical view on the fractional dynamics of cellulose degradation, Fract. Calc. Appl. Anal., **18(3)**(2015), 554–564.
5. H.H. Sherief and M.A. el-Hagary, Fractional order theory of thermo-viscoelasticity and application, Mech Time-Depend Mater., **24**(2020), 179–195.
6. G. Alotta, E. Bologna, G. Failla and M. Zingales, A Fractional Approach to Non-Newtonian Blood Rheology in Capillary Vessels, J. Peridyn Nonlocal Model, **1**(2019), 88–96.
7. H. Sun, W. Chen, C. Li and Y.Q. Chen, Fractional differential models for anomalous diffusion, Physica A: Statistical mechanics and its applications, **389**(2010), 2719–2724.
8. R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., **264**(2014), 65–70.
9. M.A. Hammad and R. Khalil, Abel's formula and Wronskian for conformable fractional differential equations, International J. Differential Equations Appl., **13**(2014), 177–183.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

10. U. Katugampola, A new fractional derivative with classical properties, J. American Math. Soc., arXiv:1410.6535v2.
11. D.R. Anderson and R.I. Avery, Fractional order boundary value problem with Sturm–Liouville boundary conditions, Electron. J. Differential Equations, **2015**(2015), 1–10.
12. K.R. Prasad and B.M.B. Krushna, Multiple positive solutions for a coupled system of Riemann–Liouville fractional order two-point boundary value problems, Nonlinear Studies, **20**(2013), 501–511.
13. K.R. Prasad and B.M.B. Krushna, Eigenvalues for iterative systems of Sturm–Liouville fractional order two-point boundary value problems, Fract. Calc. Appl. Anal., **17**(2014), 638–653.
14. K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).
15. D. Jiang, C. Yuan, *The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its applications,* Nonlinear Analysis., **72** (2010), 710–719.
16. D. J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones.* Academic Press, New York 1988.
17. R.I. Avery, J. Henderson, D. O'Regan, *Functional compression expansion fixed point theorem,* Electron. J. Differential Equations., **22** (2008), 1–12.
18. R. Leggett, L. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces,* Indian Univ. Math. J., **28**, 673–688, (1976)
19. X. Dong, Z. Bai and S. Zhang, *Positive solutions to boundary value problems of p-Laplacian with fractional derivative,* Bound. Value Probl., **2017**, 5 (2017).
20. J. Xu, O'Regan, *Positive solutions for a fractional p-Laplacian boundary value problem,* Filomat., **31**, (6) (2017), 1549–1558.
21. Y. Tian, S. Sun, Z. Bai, *Positive solutions of fractional differential equations with p-Laplacian.* J. Funct. Spaces., **2017,** 2017, 3187492.
22. L. Zhang, W. Zhang, X. Liu, M. Jia, *Positive solutions of fractional p-Laplacian equations with integral boundary value and two parameters.* J. Inequal. Appl., **2020,** 2 (2020).
23. Y. Liu, S. Chunfang, Z. Hul, *Existence of triple positive solutions to a four-point boundary value problem for a fractional differential equations*, Wuhan Univ. J. Nat. Sci., **27**(4), 287-295, (2022)
24. D. Li, Y. Liu, C. Wang, *Multiple positive solutions for fractional three-point boundary value problem with p-Laplacian operator*, Math. Probl. Eng., **2020**, 2327580 (2020).
25. SN. Rao, *Multiple positive solutions for a coupled system of p-Laplacian fractional order three-point boundary value problems.* Rocky Mountain J. Math., **49** (2), (2019).
26. R. Luca, *Positive solutions for a system of Riemann-Liouville fractional differential equations with multi-point fractional boundary conditions.* Bound. Value Probl., **2017** 2017:102.
27. Y. Liu, D. Xie, C. Bai and D. Yang, *Multiple positive solutions for a coupled system of fractional multi-point BVP with p-Laplacian operator,* Adv. Difference Equ., **2017**, 168 (2017).
28. S.N.Rao, *Multiplicity of positive solutions for coupled system of fractional differential equation with p-Laplacian two-point BVPs.* J. Appl. Math. Comput., **55** (2017), 41–58.
29. X. Hao, H. Wang, L.Liu and Y. Cui, *Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator,* Bound. Value Probl., **2017** 2017:182.
30. Luca, R.*Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions., Nonlinear Anal., Model. Control.*, **23** (5), 771–801 (2018).
31. Luca, R. *On a system of fractional boundary value problems with p-Laplacian operator., Dynam. Systems Appl.*, **28**, 2019, 691–713.
32. Tudorache, A.; Luca, R. *Positive solutions for a system Riemann-Liouville fractional boundary value problem with p-Laplacian operators.. Adv. Differ. Equ.*, **2020,** 292 (2020).
33. Luca, R. *On a system of fractional differential equations with p-Laplacian operators and integral boundary conditions. Revue Roum. Math. Pures Appl.*, **66,** 749766, (2021).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0968.v1

34. Wang, H.; Jiang, J. *Existence and multiplicity of positive solutions for a system of nonlinear fractional multi-point boundary value problems with p-Laplacian operator.*, *J. Appl. Anal. Comput.*, **11,** 351–366, (2021).
35. Henderson, J. Luca, R,; Tudorache, A. *On a system of Riemann-Liouville fractional boundary value problems with* ϱ*-Laplacian operators and positive parameters.*, *Fractal Fract.*, **6,** 299, (2022).

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.