Article

Not peer-reviewed version

Categorical Join and Generating Families in Diffeological Spaces

Enrique Macías -Virgós * and Reihaneh Mehrabi

Posted Date: 14 September 2023

doi: 10.20944/preprints202309.0918.v1

Keywords: Diffeological space; Generating family; Categorical join

Article

Categorical Join and Generating Families in Diffeological Spaces †

Enrique Macías-Virgós * and Reihaneh Mehrabi

Department of Mathematics, University of Santiago de Compostela, 15782-Spain

* Correspondence: quique.macias@usc.es

† This article belongs to the Special Issue: Differentiable Manifolds and Geometric Structures. Authors received invitation (no APC). Partially supported by Xunta de Galicia ED431C 2019/10 with FEDER funds and MINECO-Spain research project PID2020-MTM-114474GB-I00.

Abstract: We prove that a diffeological space is diffeomorphic to the categorical join of any generating family of plots.

Keywords: diffeological space; generating family; categorical join

MSC: 14F40

---

1. Introduction

Diffeological spaces were introduced by Souriau [5] as a generalization of differentiable manifolds. This setting includes not only finite-dimensional manifolds, but also manifolds with boundary, infinite-dimensional manifolds, leaf spaces of foliations or spaces of differentiable maps. The main reference is Iglesias-Zemmour's book [1]. Many fundamental results are exposed in nLab [3] too.

A diffeological structure on a set X is given by declaring which maps from open subsets of Euclidean spaces *into* X are considered to be smooth. These maps are called the *plots* of the diffeology (Section 2). They can be seen as n-dimensional curves, $n \geq 0$, on X. This contravariant idea differs from the covariant classical one of declaring which real maps *from* a manifold X are smooth. On the other hand, the idea of an atlas on a manifold is generalized by the notion of a *generating family* of plots (Section 4).

Our main result (Theorem 6.1) will be to give a categorical interpretation of a generating family, namely by proving that the diffeological space is the *join* of the family, that is, the push-out of the pull-back.

Most examples will be related to finite-dimensional manifolds.

2. Diffeological Spaces

A diffeology on the set X is a family $\mathcal{D}$ of set maps $\alpha\colon U \to X$ called plots such that:

1. each plot α is defined on an open subset $U \subset \mathbb{R}^n$ of some Euclidean space $\mathbb{R}^n$, $n \geq 0$;
2. any constant map $\alpha\colon U \subset \mathbb{R}^n \to X$ belongs to $\mathcal{D}$;
3. if $\alpha\colon U \subset \mathbb{R}^n \to X$ belongs to $\mathcal{D}$ and if $h\colon V \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$ is a C^∞-map, then the composition $\alpha \circ h$ belongs to $\mathcal{D}$;
4. the map $\alpha\colon U \to X$ belongs to $\mathcal{D}$ if and only if it locally belongs to $\mathcal{D}$, that is, for each $p \in U$ there exists some open subset $p \in V \subset U$ such that $\alpha_{|V}$ belongs to $\mathcal{D}$.

Note that the domain U and the Euclidean space $\mathbb{R}^n$, $n \geq 0$, depend on the plot α.

A diffeological space $(X, \mathcal{D})$ is a set X endowed with a diffeology $\mathcal{D}$.

Remark 2.1. Diffeological spaces are considered to be of class $\mathcal{C}^\infty$. By changing $\mathcal{C}^\infty$ by $\mathcal{C}^r$ in Axiom (3) for some $0 \leq r \leq \omega$ we could obtain a theory for diffeological spaces of class $\mathcal{C}^r$.

Example 2.2. Let M be a finite-dimensional manifold. The *manifold diffeology* $\mathcal{D}_M$ is the collection of all $\mathcal{C}^\infty$-maps $U \subset \mathbb{R}^m \to M$ defined on open subsets of Euclidean spaces with values in M.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 September 2023 doi:10.20944/preprints202309.0918.v1

Example 2.3. Let R be any equivalence relation on the manifold M and let $\pi\colon M \to M/R$ be the quotient map. We endow M/R with the *quotient diffeology* $\mathcal{D}_M/R$ where the map $\alpha\colon U \to M/R$ belongs to $\mathcal{D}_M/R$ if it locally factors through some plot of the manifold diffeology $\mathcal{D}_M$ on M.

Example 2.4. Let $N \subset M$ be any subset of the manifold M. We endow N with the *subspace diffeology* formed by the plots $\alpha\colon U \to M$ in the manifold diffeology $\mathcal{D}_M$ such that $\alpha(U) \subset N$.

The last two examples show that diffeology is a much more flexible setting than the classical one.

3. Smooth Maps

Definition 3.1. Let $(X, \mathcal{D}_X)$, $(Y, \mathcal{D}_Y)$ be two diffeological spaces. A set map $f\colon X \to Y$ is *smooth* when $\alpha \in \mathcal{D}_X$ implies $f \circ \alpha \in \mathcal{D}_Y$.

Example 3.2. If M, N are $\mathcal{C}^\infty$-manifolds endowed with the manifold diffeologies $\mathcal{D}_M$, $\mathcal{D}_N$, respectively, then the smooth maps between M and N as diffeological spaces are the $\mathcal{C}^\infty$-maps as differentiable manifolds.

Proposition 3.3. *The composition of smooth maps is a smooth map.*

Proposition 3.4. *The quotient map $\pi\colon M \to M/R$ of Example 2.3 is smooth. Moreover, a map $f\colon M/R \to N$ is smooth if and only if the composition $f \circ \pi\colon M \to N$ is smooth.*

Corollary 3.5. *A quotient map $\alpha\colon U \to X$ is a diffeomorphism if and only if it is bijective.*

Proof. Let $\alpha^{-1}\colon X \to U$ be the inverse map. Since $\alpha^{-1} \circ \alpha = \mathrm{id}_X$ is smooth, α^{-1} is smooth. □

Example 3.6. If $N \subset M$ is a subset endowed with the subspace diffeology of Example 2.4 then the inclusion map $i_N\colon N \subset M$ is smooth. Moreover, a map $f\colon P \to N$ is smooth if and only if $i_N \circ f\colon P \to M$ is smooth.

Example 3.7. By Axiom 3 of diffeology, the plots $\alpha\colon U \to X$ of a diffeology are smooth maps.

Definition 3.8. A diffeomorphism is a smooth map with a smooth inverse.

4. Generating Families

Definition 4.1. It is easy to check that the intersection of diffeologies on a set X is a diffeology on X. Then if $\mathcal{F}$ is any family of set maps $\alpha_i\colon U_i \subset \mathbb{R}^{n_i} \to X$ we can consider the smallest diffeology $\mathcal{D} = \langle \mathcal{F} \rangle$ containing $\mathcal{F}$. We will say that $\mathcal{F}$ is a *generating family* for the diffeology $\mathcal{D} = \langle \mathcal{F} \rangle$.

We will always assume that the family $\mathcal{F}$ contains all the constant plots on X.

Example 4.2. Any atlas on a manifold M is a generating family of the manifold diffeology.

Example 4.3. Let $(X, \mathcal{D}_X)$ be a diffeological space and let $f\colon X \to Y$ be a set map. The diffeology on Y generated by the maps $f \circ \alpha$, with $\alpha \in \mathcal{D}_X$, is called the *final diffeology*. It is formed by the maps that locally are of the form $f \circ \alpha$ for some $\alpha \in \mathcal{D}_X$.

Example 4.4. Let $(Y, \mathcal{D}_Y)$ be a diffeological space and let $f\colon X \to Y$ be a set map. The i*nitial diffeology* on X is the diffeology generated by the set maps $\alpha\colon U \subset \mathbb{R}^n \to X$ such that $f \circ \alpha \in \mathcal{D}_Y$.

Example 4.5. Let $(X, \mathcal{D}_X)$, $(Y, \mathcal{D}_Y)$ be two diffeological spaces. The *product diffeology* on $X \times Y$ is the intersection of the initial diffeologies for the projections $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$.

Example 4.6. The *coproduct diffeology* on the disjoint union $X \sqcup Y$ is the intersection of the final diffeologies for the inclusions $i_X\colon X \hookrightarrow X \sqcup Y$ and $i_Y\colon Y \hookrightarrow X \sqcup Y$.

It is easy to check that the product and coproduct diffeologies verify the usual universal properties. The following *criterion of generation* is very useful.

Theorem 4.7 ([1] Art. 1.68)**.** *Let the diffeology $\mathcal{D} = \langle \mathcal{F} \rangle$ be generated by the family $\mathcal{F}$. A set map $\alpha\colon U \subset \mathbb{R}^n \to X$ belongs to $\mathcal{D}$ if and only if it locally factors through some element of $\mathcal{F}$ (we assume that constant plots are all contained in $\mathcal{F}$).*
Moreover, a map $f\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is smooth if and only if $f \circ \alpha \in \mathcal{D}_Y$ for any $\alpha \in \mathcal{F}$ in a generating family $\mathcal{F}$ of $\mathcal{D}_X$.

5. Categorical Constructions

The category of diffeological spaces and smooth maps has limits and colimits. Here, we will show how to construct the pull-back, the push-out and the join of two smooth maps. Most times these maps will be plots of some diffeology on X.

5.1. Pull-Back

Let $\alpha\colon U \to X$, $\beta\colon V \to X$ be two smooth maps with the same codomain. The pull-back of α and β is the set

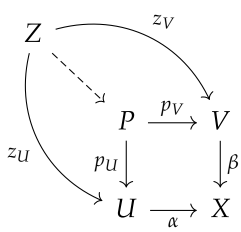
$$P = \{(u, v) \in U \times V \colon \alpha(u) = \beta(v)\}$$

endowed with the subspace diffeology of the product diffeology on $U \times V$.

The projections $p_1\colon U \times V \to U$ and $p_2\colon U \times V \to V$ induce smooth maps

$$p_U\colon P \to U, \quad p_V\colon P \to V,$$

which verify $\alpha \circ p_U = \beta \circ p_V$ and the universal property of a pull-back:

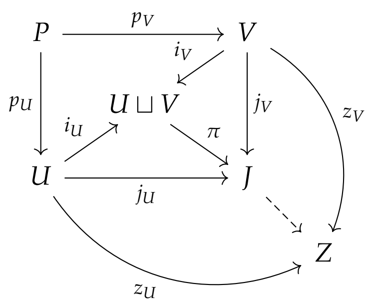


5.2. Push-Out

Let $p_U\colon P \to U$, $p_V\colon P \to V$ be two smooth maps with the same domain. The push-out of p_U and p_V is the quotient $J = (U \sqcup V)/R$ of the coproduct $U \sqcup V$ by the equivalence relation R generated by

$$u \in U, v \in V, \quad u\, R\, v \text{ iff } \alpha(u) = \beta(v).$$

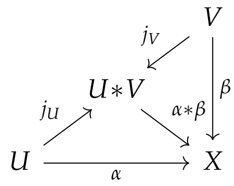
The maps $j_U = \pi \circ i_U$ and $j_V = \pi \circ i_V$ verify $j_U \circ p_U = j_V \circ p_V$ and the universal property of a push-out:



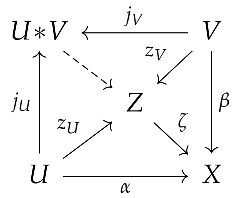
5.3. *Join*

Given two smooth maps $\alpha\colon U \to X$ and $\beta\colon V \to X$, it is an exercise to check that the push-out of the pull-back of α and β is diffeomorphic to the join $U{*}V$ of α and β ([4]), defined as follows: there is a well-defined smooth map $\alpha{*}\beta\colon U{*}V \to X$ and maps $j_U\colon U \to U{*}V$ and $j_V\colon V \to U{*}V$ such that

1. $(\alpha{*}\beta)\circ j_U = \alpha, \quad (\alpha{*}\beta)\circ j_V = \beta,$



2. they verify the universal property



Analogously, given k maps $\alpha_i\colon U_i \to X$, the join $U = U_1{*}\cdots{*}U_k$ and the universal map $\alpha = \alpha_1{*}\cdots{*}\alpha_k\colon U = U_1{*}\cdots{*}U_k \to X$ are defined by induction.

Remark 5.1. It is possible to define the join of an infinite number of plots, but we will not develop this idea, for the sake of simplicity.

Lemma 5.2. *Let $\mathcal{F} = \{\alpha_i\colon U_i \to X\}$ be a generating family. Then the universal map $\alpha = *_i\alpha_i\colon U = *_iU_i \to X$ of the join is a quotient map.*

Proof. Let $\gamma\colon W \to X$ be a plot on X. By Criterion 4.7, for each $p \in W$ there exists a neighborhood $W_p \subset W$ such that γ factors through some α_i. Hence it factors through the disjoint union and consequently through the join. □

6. Main Result

Next theorem is our main result.

Theorem 6.1. *Let $(X, \mathcal{D})$ be a diffeological space. A family $\mathcal{F} = \{\alpha_i\colon U_i \to X\}$ of plots is a generating family for $\mathcal{D}$ if and only if there is a diffeomorhism $\alpha = \alpha_1{*}\cdots{*}\alpha_n\colon U = U_1{*}\cdots{*}U_n \to X$ commuting with the maps α_i and the natural maps $j_i\colon U_i \to U$, that is $\alpha\circ j_i = \alpha_i$ for all i.*

Proof. Let $\mathcal{F}$ be a generating family. The map α is surjective because generating families include all constant plots.

It is injective because $\alpha([u_i]) = \alpha([u_j])$ means that either $u_i = u_j$ or $u_i \in U_i$, $u_j \in U_j$ and $\alpha_i(u_i) = \alpha_j(u_j)$, that is $[u_i] = [u_j]$. Then α is bijective and Corollary 3.5 applies. Hence α is a diffeomorphism.

The converse statement follows from Lemma 5.2. □

Example 6.2. Let X be the disjoint union of a line $X_1 = \mathbb{R}$ and a point $X_0 = \{0\}$, endowed with the diffeology generated by the constant plot $\alpha_0\colon \mathbb{R}^0 \to X$, $\alpha_0(0) = 0$, and the identity plot $\alpha_1\colon \mathbb{R}^1 \to X_1 \subset X$, $\alpha_1(t) = t$, respectively. Clearly, $X \cong U_0 * U_1 = U_0 \sqcup U_1$.

 doi:10.20944/preprints202309.0918.v1

Example 6.3. Let X be the set
$$X = \{(x, y) \in \mathbb{R}^2 \colon xy = 0\}.$$
We consider the diffeology on X generated by the two plots

$$\alpha_1 \colon U_1 = \mathbb{R} \to X, \quad \alpha_1(t) = (t, 0),$$
$$\alpha_2 \colon U_2 = \mathbb{R} \to X, \quad \alpha_2(t) = (0, t).$$

Then $X \cong U_1 * U_2$.

As we pointed out in [2], this is not the *cross* diffeology on X induced by the manifold diffeology of $\mathbb{R}^2$.

Example 6.4. The manifold diffeology on a manifold M is the join of any atlas. More precisely, let $\{\alpha_i \subset \mathbb{R}^m \colon U_i \to M\}$ be an atlas on the manifold M^m. Then

$$\bigcup_i \alpha_i(U_i) = M \cong *_i U_i,$$

where the open subsets $U_i \cong \alpha(U_i) \subset M$ are glued by inclusions.

References

1. Iglesias-Zemmour, Patrick. *Diffeology*. (Mathematical Surveys and Monographs 185. Providence, RI: American Mathematical Society (AMS) xxiii, 439 p. (2013).
2. Macías-Virgós, Enrique; Mehrabi, Reihaneh. Mayer-Vietoris sequence for generating families in diffeological spaces. *Indag. Math., New Ser.* 34, No. 4, 661-672 (2023).
3. nLab. *Diffeological space*. https://ncatlab.org/nlab/show/diffeological+space (viewed on August 28, 2023)
4. Rijke, Egbert. The join construction, arXiv:1701.07538 (2017).
5. Souriau, Jean-Marie. Groupes différentiels, in *Differential Geometrical Methods in Mathematical Physics* (Proc. Conf., Aix-en-Provence/Salamanca, 1979), Lecture Notes in Math. 836, Springer, Berlin, (1980), pp. 91–128.