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Posted Date: 13 September 2023

doi: 10.20944/preprints202309.0854.v1

Keywords: Double fuzzy rough set, Optimistic multi-granulation double fuzzy rough set, Pessimistic multi-granulation double fuzzy rough set

*Article*

# Multi-Granulation Double Fuzzy Rough Sets

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**Abstract:** Based on the definition of double fuzzy relations, two types of new rough set models are constructed, which are multi-granulation double fuzzy rough sets called optimistic and pessimistic multi-granulation double fuzzy rough sets, and some of their properties were studied. In addition, we study the relationship between optimistic and pessimistic multi-granulation double fuzzy rough sets and double fuzzy rough sets.

**Keywords:** double fuzzy rough set; optimistic multi-granulation double fuzzy rough set; pessimistic multi-granulation double fuzzy rough set

**MSC:** 47H10; 47H09; 47H04; 46S40; 54H25

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## 1. Introduction and preliminaries

Undoubtedly, the theory of rough sets, proposed by Pawlak [19,20], has become a well-established mathematical tool for the study of uncertainty in a wide variety of applications and intelligent systems characterized by inadequate and incomplete information where the equivalence classes created by the equivalence relation are used to define the lower and upper approximations to approximate an undefinable set. More over, in recent years, it has been widely used in a variety of fields, such as granular computing, graph theory, algebraic systems, partially ordered sets, medical diagnosis, data mining, conflict analysis (see, for example, [4,5,10,21,22,27]).

The generalization and extension of the rough set model is an important direction in the study of set theory. On the one hand, one such trend was introduced by Qian et al. [23,24], which has been called the multi-granulation rough set. It is defined by a family of equivalence relations, while Pawlak's rough set is defined by only one equivalence relation, producing two types of multi-granulation rough sets called the optimistic multi-granulation rough set and the pessimistic multi-granulation rough set. The word "optimistic" is used in the lower and upper approximation to denote the idea that in multi independent granular structures, at least one granular structure must satisfy the inclusion relation between the equivalence class and the undefinable set, whereas the word "pessimistic" denotes the idea that each granular structure must satisfy the inclusion relation between the equivalence class and the undefinable set. Following that, a number of researchers looked into multi-granulation rough set models based on various types of relationships, and they came up with a number of interesting ideas (see, for example, [12–16,25])

On the other hand, one of these trends is to combine other theories that deal with uncertain knowledge, such as fuzzy set and rough set theory. While fuzzy set theory deals with potential uncertainties associated with inaccurate cases, perceptions, and preferences, we find that approximate sets, in turn, deal with uncertainty caused by ambiguity of information. Because the two types of uncertainty can occur in real-world problems, numerous approaches to combining fuzzy set theory with approximation set theory have been proposed. Rough fuzzy sets and fuzzy rough sets were presented by Dubois and Prade based on approximations of fuzzy sets by crisp approximation spaces and crisp sets by fuzzy approximation spaces, respectively [6,7]. In the same framework, the researchers presented an approach to enrich coarse fuzzy rough sets and rough fuzzy sets, (see for example, [2,9,11,17,18,28,29]).

As a result of the intuitionistic fuzzy sets given by Atanassov [3], which give the membership and nonmembership degrees to which an element belongs, dealing with incomplete and inaccurate information is more flexible and effective compared to Zadeh's fuzzy sets [30].

Working under tthe name "intuitionistic" has sparked and doubts debate over the term's applicability, particularly when dealing with of complete lattice $L$. Garcia and Rodabaugh [8] put an end to these doubts in 2005. They proved that in mathematics and applications, this word is inappropriate. They concluded that they work under the name "double".

The main contributions of the present paper are to further investigations into multi-granulation double fuzzy approximation spaces, mainly including double fuzzy upper and lower approximation operators with respect to multi-granulation double fuzzy approximation spaces. By using more than one pair of double fuzzy relations on $U$, two kinds of double fuzzy sets were introduced and the relationship between them was studied.

Throughout this paper, Let $U = \{x_1, x_2, ...x_n\}$ be a nonempty and finite set of objects and $I = [0,1]$. A fuzzy set is a map from $U$ to $I$. The set of all fuzzy sets on $U$ is denoted by $I^U$. $R$ is a fuzzy binary relation on $U$ i.e. $R(x,y) \in [0,1]$ for any $x, y \in U$. The set of all fuzzy binary relation on $U$ is denoted by $I^{U\times U}$.

**Definition 1.1.** [1] Let $U$ and $V$ be two arbitrary sets. A double fuzzy relation on $U \times V$ is a pair $(R, R^*)$ of maps $R, R^* : U \times V \to I$ such that $R(x,y) \le 1 - R^*(x,y)$ for all $(x,y) \in U \times V$. If $R, R^* : U \times U \to I$, $(R, R^*)$ is called a double fuzzy relation on $U$. $R(x,y)$ (resp. $R^*(x,y)$), referred to as the degree of relation (resp. non-relation ) between $x$ and $y$.

**Definition 1.2.** [1] Let $U$ be an arbitrary universal set and $(R, R^*)$ a double fuzzy relation on $U$. Then for each fuzzy set $\lambda$ on $U$, the pairs $(\underline{\mathcal{R}}_R\lambda, \underline{\mathcal{R}}^*_{R^*}\lambda)$, $(\overline{\mathcal{R}}_R\lambda, \overline{\mathcal{R}}^*_{R^*}\lambda)$ of maps $\underline{\mathcal{R}}_R\lambda, \underline{\mathcal{R}}^*_{R^*}\lambda, \overline{\mathcal{R}}_R\lambda, \overline{\mathcal{R}}^*_{R^*}\lambda : U \to I$ are called double fuzzy lower approximation and double fuzzy upper approximation of a fuzzy set $\lambda$, respectively and are defined as follows: For all $x \in U$,

$$(\underline{\mathcal{R}}_R\lambda)(x) = \bigwedge_{y\in U} ((1 - R(x,y)) \vee \lambda(y)),\ (\underline{\mathcal{R}}^*_{R^*}\lambda)(x) = \bigvee_{y\in U} ((1 - R^*(x,y)) \wedge (1 - \lambda(y)))$$

$$(\overline{\mathcal{R}}_R\lambda)(x) = \bigvee_{y\in U} (R(x,y) \wedge \lambda(y)),\ (\overline{\mathcal{R}}^*_{R^*}\lambda)(x) = \bigwedge_{y\in U} (R^*(x,y) \vee (1 - \lambda(y))).$$

The quaternary $(\underline{\mathcal{R}}_R\lambda, \underline{\mathcal{R}}^*_{R^*}\lambda, \overline{\mathcal{R}}_R\lambda, \overline{\mathcal{R}}^*_{R^*}\lambda)$ is called double fuzzy rough set of $\lambda$. The pairs $(\underline{\mathcal{R}}_R, \underline{\mathcal{R}}^*_{R^*})$, $(\overline{\mathcal{R}}_R, \overline{\mathcal{R}}^*_{R^*})$ of operators $\underline{\mathcal{R}}_R, \underline{\mathcal{R}}^*_{R^*}, \overline{\mathcal{R}}_R, \overline{\mathcal{R}}^*_{R^*} : I^U \to I^U$ are called double fuzzy lower approximation and double fuzzy upper approximation operators, respectively.

**Definition 1.3.** [1] For all $x, y \in U$ a double fuzzy relation $(R, R^*)$ on $U$ is called:

(1) Double fuzzy reflexive if $R(x,x) = 1$ and $R^*(x,x) = 0$.

(2) Double fuzzy transitive if $R(x,z) \ge \bigvee_{y\in U}(R(x,y) \wedge R(y,z))$ and $R^*(x,z) \le \bigwedge_{y\in U}(R^*(x,y) \vee R^*(y,z))\quad \forall z \in U$.

(3) Double fuzzy symmetric if $R(x,y) = R(y,x)$ and $R^*(x,y) = R^*(y,x)$.

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## 2. Optimistic Multi-granulation double fuzzy rough sets

**Definition 2.1.** Let $U$ be an arbitrary universal set and $(R_1, R_1^*)$ and $(R_2, R_2^*)$ are double fuzzy relations on $U$. Then for each fuzzy set $\lambda$ on $U$, the pairs $(\underline{\mathcal{OR}_{R_1+R_2}}\lambda, \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)$ and $(\overline{\mathcal{OR}_{R_1+R_2}}\lambda, \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)$ of maps $\underline{\mathcal{OR}_{R_1+R_2}}\lambda$, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$, $\overline{\mathcal{OR}_{R_1+R_2}}\lambda$, $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda : U \to I$ are called optimistic two-granulation double fuzzy lower approximation and optimistic two-granulation double fuzzy upper approximation of a fuzzy set $\lambda$, respectively and are defined as follows: For all $x \in U$,

$$(\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) = \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee \lambda(y)) \right\} \vee \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee \lambda(y)) \right\};$$

$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) = \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge 1 - \lambda(y)) \right\} \wedge \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge 1 - \lambda(y)) \right\};$$

$$(\overline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) = \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y)) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y)) \right\};$$

$$(\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) = \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \vee \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\}.$$

The quaternary $(\underline{\mathcal{OR}_{R_1+R_2}}\lambda, \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda, \overline{\mathcal{OR}_{R_1+R_2}}\lambda, \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)$ is called optimistic two-granulation double fuzzy rough set of $\lambda$ (in short, OTGDFRS). The pairs $(\underline{\mathcal{OR}_{R_1+R_2}}, \underline{\mathcal{OR}^*_{R_1^*+R_2^*}})$ and $(\overline{\mathcal{OR}_{R_1+R_2}}, \overline{\mathcal{OR}^*_{R_1^*+R_2^*}})$ of operators $\underline{\mathcal{OR}_{R_1+R_2}}$, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}$, $\overline{\mathcal{OR}_{R_1+R_2}}$, $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}} : U \to I$ are called optimistic two-granulation double fuzzy lower approximation and optimistic two-granulation double fuzzy upper approximation operators, respectively.

The OTGDFRS approximations are defined by many separate pairs of double fuzzy relations, whereas the normal double fuzzy rough approximations are represented by those produced by only one pair of double fuzzy relation, as can be seen from the preceding definition. In fact, when $(R_1, R_1^*) = (R_2, R_2^*)$, the OTGDFRS degenerates into a double fuzzy rough set. To put it another way, a double fuzzy rough set model is a subset of the OTGDFRS.

**Proposition 2.2.** Let $U$ be an arbitrary universal set and $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on $U$. Then for each $\lambda \in I^U$ we obtain the following:

(1) $\underline{\mathcal{OR}_{R_1+R_2}}\lambda = \underline{\mathcal{R}_{R_1}}\lambda \vee \underline{\mathcal{R}_{R_2}}\lambda$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda = \underline{\mathcal{R}^*_{R_1^*}}\lambda \wedge \underline{\mathcal{R}^*_{R_2^*}}\lambda$.

(2) $\overline{\mathcal{OR}_{R_1+R_2}}\lambda = \overline{\mathcal{R}_{R_1}}\lambda \wedge \overline{\mathcal{R}_{R_2}}\lambda$ and $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda = \overline{\mathcal{R}^*_{R_1^*}}\lambda \vee \overline{\mathcal{R}^*_{R_2^*}}\lambda$.

**Proof.** The proofs follow directly from Definition 1.2 and Definition 2.1.

**Theorem 2.3.** Let $U$ be an arbitrary universal set and $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on $U$. Then for each $\lambda \in I^U$ we obtain the following:

(1) $\overline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$ and $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(2) $\underline{\mathcal{OR}_{R_1+R_2}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{1} = \tilde{0}$.

(3) $\overline{\mathcal{OR}_{R_1+R_2}}\tilde{0} = \tilde{0}$ and $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{OR}_{R_1+R_2}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}_{R_1+R_2}}\lambda$ and $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(5) $\underline{\mathcal{OR}_{R_1+R_2}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}_{R_1+R_2}}\lambda$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

**Proof.** (1) For each $x \in U,\ \lambda \in I^U$ we have

$$\begin{aligned}
&(\tilde{1} - (\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda))(x) \\
=\ & 1 - \left\{\left\{\bigwedge_{y\in U}(R_1^*(x,y) \vee 1 - \lambda(y))\right\} \vee \left\{\bigwedge_{y\in U}(R_2^*(x,y) \vee 1 - \lambda(y))\right\}\right\} \\
=\ & \left\{1 - \left\{\bigwedge_{y\in U}(R_1^*(x,y) \vee 1 - \lambda(y))\right\}\right\} \wedge \left\{1 - \left\{\bigwedge_{y\in U}(R_2^*(x,y) \vee 1 - \lambda(y))\right\}\right\} \\
=\ & \left\{\bigvee_{y\in U} 1 - \{R_1^*(x,y) \vee 1 - \lambda(y)\}\right\} \wedge \left\{\bigvee_{y\in U} 1 - \{R_2^*(x,y) \vee 1 - \lambda(y)\}\right\} \\
=\ & \left\{\bigvee_{y\in U} \{1 - R_1^*(x,y) \wedge \lambda(y)\}\right\} \wedge \left\{\bigvee_{y\in U} \{1 - R_2^*(x,y) \wedge \lambda(y)\}\right\} \\
\geq\ & \left\{\bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y))\right\} \wedge \left\{\bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y))\right\} \\
=\ & (\overline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) \text{ for all } x \in U.
\end{aligned}$$

Hence, $\overline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(2) Since, for each $x \in U$, $\tilde{1}(x) = 1$, we obtain

$$\begin{aligned}
(\underline{\mathcal{OR}_{R_1+R_2}}\tilde{1})(x) &= \left\{\bigwedge_{y\in U}((1 - R_1(x,y)) \vee \tilde{1}(y))\right\} \vee \left\{\bigwedge_{y\in U}((1 - R_2(x,y)) \vee \tilde{1}(y))\right\} \\
&= 1 = \tilde{1}(x),
\end{aligned}$$

and

$$\begin{aligned}
(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{1})(x) &= \left\{\bigvee_{y\in U}((1 - R_i(x,y)) \wedge 1 - \tilde{1}(y))\right\} \\
&\wedge \left\{\bigvee_{y\in U}((1 - R_i(x,y)) \wedge 1 - \tilde{1}(y))\right\} \\
&= 0 = \tilde{0}(x).
\end{aligned}$$

Therefore, we obtain $\underline{\mathcal{OR}_{R_1+R_2}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{1} = \tilde{0}$.

(3) It is similar to the proof of (2).

(4) For each $x \in U$, we have

$$\begin{aligned}
& \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda)(x) \\
= & \left\{\bigwedge_{y\in U}(R_1^*(x,y) \vee 1-(1-\lambda(y)))\right\} \vee \left\{\bigwedge_{y\in U}(R_2^*(x,y) \vee 1-(1-\lambda(y)))\right\} \\
= & \left\{1-\left\{\bigvee_{y\in U}(1-R_1^*(x,y) \wedge 1-\lambda(y))\right\}\right\} \vee \left\{1-\left\{\bigvee_{y\in U}(1-R_2^*(x,y) \wedge 1-\lambda(y))\right\}\right\} \\
= & 1-\left\{\left\{\bigvee_{y\in U}(1-R_1^*(x,y) \wedge 1-\lambda(y))\right\} \wedge \left\{\bigvee_{y\in U}(1-R_2^*(x,y) \wedge 1-\lambda(y))\right\}\right\} \\
= & 1-\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda(x).
\end{aligned}$$

Thus, we obtain $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda) = \tilde{1}-\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, we can prove $\overline{\mathcal{OR}_{R_1+R_2}}(\tilde{1}-\lambda) = \tilde{1}-\underline{\mathcal{OR}_{R_1+R_2}}\lambda$

(5) Similarly to that of (4).

**Theorem 2.4.** Let $U$ be an arbitrary universal set and $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on $U$. Then for each $\lambda, \mu \in I^U$:

(1) $\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu) \leq \underline{\mathcal{OR}_{R_1+R_2}}\lambda \wedge \underline{\mathcal{OR}_{R_1+R_2}}\mu$ and

$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu) \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \vee \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(2) $\overline{\mathcal{OR}_{R_1+R_2}}(\lambda \vee \mu) \geq \overline{\mathcal{OR}_{R_1+R_2}}\lambda \vee \overline{\mathcal{OR}_{R_1+R_2}}\mu$ and

$\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu) \leq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \wedge \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(3) If $\lambda \leq \mu$, then $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \underline{\mathcal{OR}_{R_1+R_2}}\mu$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(4) If $\lambda \leq \mu$, then $\overline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \overline{\mathcal{OR}_{R_1+R_2}}\mu$ and $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \geq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(5) $\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \vee \mu) \geq \underline{\mathcal{OR}_{R_1+R_2}}\lambda \vee \underline{\mathcal{OR}_{R_1+R_2}}\mu$ and

$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu) \leq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \wedge \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(6) $\overline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu) \leq \overline{\mathcal{OR}_{R_1+R_2}}\lambda \wedge \overline{\mathcal{OR}_{R_1+R_2}}\mu$ and

$\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu) \geq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \vee \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

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**Proof.** (1) For each $x \in U$ and $\lambda, \mu \in I^U$, we have

$$
\begin{aligned}
& (\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(x) \\
= & \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee (\lambda \wedge \mu)(y)) \right\} \vee \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee (\lambda \wedge \mu)(y)) \right\} \\
= & \left\{ \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee (\lambda)(y) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee (\mu)(y) \right\} \right\} \\
\vee & \left\{ \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee (\lambda)(y) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee (\mu)(y) \right\} \right\} \\
= & \left\{ (\underline{\mathcal{R}_{R_1}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_1}}\mu)(x) \right\} \vee \left\{ (\underline{\mathcal{R}_{R_2}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_2}}\mu)(x) \right\} \\
\leq & \left\{ (\underline{\mathcal{R}_{R_1}}\lambda)(x) \bigvee (\underline{\mathcal{R}_{R_2}}\lambda)(x) \right\} \wedge \left\{ (\underline{\mathcal{R}_{R_1}}\mu)(x) \bigvee (\underline{\mathcal{R}_{R_2}}\mu)(x) \right\} \\
= & (\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) \wedge (\underline{\mathcal{OR}_{R_1+R_2}}\mu)(x).
\end{aligned}
$$

Also, for each $x \in U$, we have

$$
\begin{aligned}
& (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) \\
= & \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge 1 - (\lambda \wedge \mu)(y)) \right\} \wedge \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge 1 - (\lambda \wedge \mu)(y)) \right\} \\
= & \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge (1 - \lambda(y) \vee 1 - \mu(y)) \right\} \wedge \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge (1 - \lambda(y) \vee 1 - \mu(y)) \right\} \\
= & \left\{ \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge (1 - \lambda(y) \right\} \vee \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge (1 - \mu(y) \right\} \right\} \\
\wedge & \left\{ \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge (1 - \lambda(y) \right\} \vee \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge (1 - \mu(y) \right\} \right\} \\
= & \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \right\} \wedge \left\{ (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x) \right\} \\
\geq & \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \wedge (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \right\} \vee \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \wedge (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x) \right\} \\
= & (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) \vee (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(x)
\end{aligned}
$$

(2) Similar to (1).

(3) If $\lambda \leq \mu$, then for all $y \in U$, $\lambda(y) \leq \mu(y)$ we have

$$
\bigwedge_{y\in U} (1 - R_1(x,y) \vee \lambda(y)) \leq \bigwedge_{y\in U} (1 - R_1(x,y) \vee \mu(y)) \tag{1}
$$

and

$$
\bigwedge_{y\in U} (1 - R_2(x,y) \vee \lambda(y)) \leq \bigwedge_{y\in U} (1 - R_2(x,y) \vee \mu(y)). \tag{2}
$$

 doi:10.20944/preprints202309.0854.v1



Form Equations (2.1) and (2.2) we have

$$\left\{\bigwedge_{y\in U}(1-R_1(x,y)\vee\lambda(y))\right\}\vee\left\{\bigwedge_{y\in U}(1-R_2(x,y)\vee\lambda(y))\right\}$$
$$\leq\left\{\bigwedge_{y\in U}(1-R_1(x,y)\vee\mu(y))\right\}\vee\left\{\bigwedge_{y\in U}(1-R_2(x,y)\vee\mu(y))\right\}.$$

Therefore, $\underline{\mathcal{OR}_{R_1+R_2}}\lambda\leq\underline{\mathcal{OR}_{R_1+R_2}}\mu$. Also,

$$\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\geq\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\mu(y)) \tag{3}$$

and

$$\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\geq\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\mu(y)). \tag{4}$$

Form Equations (2.3) and (2.4) we have

$$\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\wedge\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}$$
$$\geq\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\mu(y))\right\}\wedge\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\mu(y))\right\}$$

Hence $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\geq\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(4) Similar to (3).

(5) Since $\lambda\leq\lambda\vee\mu$ and $\mu\leq\lambda\vee\mu$, by (3) we have

$$\underline{\mathcal{OR}_{R_1+R_2}}\lambda\leq\underline{\mathcal{OR}_{R_1+R_2}}(\lambda\vee\mu)\text{ and }\underline{\mathcal{OR}_{R_1+R_2}}\mu\leq\underline{\mathcal{OR}_{R_1+R_2}}(\lambda\vee\mu).$$

Therefore $\underline{\mathcal{OR}_{R_1+R_2}}\lambda\vee\underline{\mathcal{OR}_{R_1+R_2}}\mu\leq\underline{\mathcal{OR}_{R_1+R_2}}(\lambda\vee\mu)$. Also, we have

$$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\geq\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)\text{ and }\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu\geq\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu).$$

This implies that $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\wedge\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu\geq\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)$.

(6) Similar to (5).

**Example 2.5.** Let $U=\{x,y,z\}$. Define $R_1,R_1^*,R_2,R_2^*:U\times U\to I$ as follows:

$$R_1=\begin{pmatrix}0.2&0.4&0.7\\0.9&0.3&0.7\\0.6&0.3&0.2\end{pmatrix}\qquad R_1^*=\begin{pmatrix}0.1&0.6&0.2\\0.0&0.6&0.2\\0.3&0.0&0.7\end{pmatrix}$$

$$R_2=\begin{pmatrix}0.1&0.5&0.6\\0.4&0.3&0.8\\0.7&0.2&0.3\end{pmatrix}\qquad R_2^*=\begin{pmatrix}0.2&0.3&0.0\\0.1&0.6&0.1\\0.3&0.6&0.6\end{pmatrix}$$

 doi:10.20944/preprints202309.0854.v1



Define $\lambda, \mu \in I^U$ as follows:

$$\lambda = \{(x, 0.5), (y, 0.7), (z, 0.1)\},$$
$$\mu = \{(x, 0.4), (y, 0.2), (z, 0.8)\},$$
$$\lambda \wedge \mu = \{(x, 0.4), (y, 0.2), (z, 0.1)\}.$$

Then,

$$(\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) = 0.4, (\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(y) = 0.4, (\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(z) = 0.5$$
$$(\underline{\mathcal{OR}_{R_1+R_2}}\mu)(x) = 0.6, (\underline{\mathcal{OR}_{R_1+R_2}}\mu)(y) = 0.7, (\underline{\mathcal{OR}_{R_1+R_2}}\mu)(z) = 0.4$$
$$(\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(x) = 0.2, (\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(y) = 0.4, (\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(z) = 0.4$$

Therefore, $\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu) \neq \underline{\mathcal{OR}_{R_1+R_2}}\lambda \wedge \underline{\mathcal{OR}_{R_1+R_2}}\mu$.

$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) = 0.5, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(y) = 0.5, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(z) = 0.5$$
$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(x) = 0.2, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(y) = 0.2, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(z) = 0.6$$
$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) = 0.6, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(y) = 0.6, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(z) = 0.6.$$

Therefore, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu) \neq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \vee \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

**Theorem 2.6.** Let $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on an universal set $U$. Then the following statements are equivalent:

(1) $(R_1, R_1^*)$ and $(R_2, R_2^*)$ are a double fuzzy reflexive relations.

(2) $\lambda \leq \overline{\mathcal{OR}_{R_1+R_2}}\lambda$ and $\tilde{1} - \lambda \geq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(3) $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \lambda$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \geq \tilde{1} - \lambda$.

**Proof.** (1) $\Rightarrow$ (2) Let $(R_1, R_1^*)$ and $(R_2, R_2^*)$ are a double fuzzy reflexive relations. Then $R_i(x, x) = 1$ and $R_i^*(x, x) = 0$ for all $i \in \{1, 2\}$ and $x \in U$. Therefore,

$$\begin{aligned}
\lambda(x) &= 1 \wedge \lambda(x) \\
&= \{R_1(x, x) \wedge \lambda(x)\} \wedge \{R_2(x, x) \wedge \lambda(x)\} \\
&\leq \left\{\bigvee_{y \in U} (R_1(x, y) \wedge \lambda(y))\right\} \wedge \left\{\bigvee_{y \in U} (R_2(x, y) \wedge \lambda(y))\right\} \\
&= \overline{\mathcal{OR}_{R_1+R_2}}\lambda
\end{aligned}$$

and

$$\begin{aligned}
\tilde{1} - \lambda(x) &= 0 \vee \tilde{1} - \lambda(x) \\
&= \{R_1^*(x, x) \vee \tilde{1} - \lambda(x)\} \vee \{R_2^*(x, x) \vee \tilde{1} - \lambda(x)\} \\
&\geq \left\{\bigwedge_{y \in U} (R_1^*(x, y) \vee \tilde{1} - \lambda(y))\right\} \vee \left\{\bigwedge_{y \in U} (R_2^*(x, y) \vee \tilde{1} - \lambda(y))\right\} \\
&= \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda.
\end{aligned}$$

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(2) $\Rightarrow$ (1) Suppose that there exists some $x \in U$ such that $R_i(x,x) = a_i \neq 1$ and $R_i^*(x,x) = b_i \neq 0$ for all $i \in \{1,2\}$, then we can define fuzzy set $\delta_x : U \longrightarrow I$ as:

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{if } y \neq x. \end{cases}$$

Then

$$\begin{aligned} \overline{\mathcal{OR}_{R_1+R_2}}\delta_x(x) &= \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \delta_x(y)) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \delta_x(y)) \right\} \\ &= R_1(x,x) \wedge R_2(x,x) \\ &= a_1 \wedge a_2 \neq 1 = \delta_x(x) \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\delta_x(x) &= \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \delta_x(y)) \right\} \vee \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \delta_x(y)) \right\} \\ &= R_1^*(x,x) \vee R_2^*(x,x) \\ &= b_1 \vee b_2 \neq 0 = 1 - \delta_x(x). \end{aligned}$$

Therefore $\delta_x \not\leq \overline{\mathcal{OR}_{R_1+R_2}}\delta_x$ and $\tilde{1} - \delta_x \not\geq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\delta_x$. This is a contradiction. Hence, $R_i(x,x) = 1$ and $R_i^*(x,x) = 0$ for all $i \in \{1,2\}$ and $x \in U$.

(2) $\Leftrightarrow$ (3) It is easy from Theorem 2.3 (4), (5).

**Theorem 2.7.** Let $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on an universal set $U$. Then the following statements are equivalent:

(1) $(R_1, R_1^*)$ and $(R_2, R_2^*)$ are a double fuzzy transitive relations.

(2) $\overline{\mathcal{OR}_{R_1+R_2}}(\overline{\mathcal{OR}_{R_1+R_2}}\lambda) \leq \overline{\mathcal{OR}_{R_1+R_2}}\lambda$ and

$\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1} - \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda) \geq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda.$

(3) $\underline{\mathcal{OR}_{R_1+R_2}}(\underline{\mathcal{OR}_{R_1+R_2}}\lambda) \geq \underline{\mathcal{OR}_{R_1+R_2}}\lambda$ and

$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1} - \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}) \leq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda.$

**Proof.** (1) $\Leftrightarrow$ (2) For each $\lambda \in I^U$,

$$\begin{aligned} & \overline{\mathcal{OR}_{R_1+R_2}}(\overline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) \\ = & \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge (\overline{\mathcal{OR}_{R_1+R_2}})(y) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge (\overline{\mathcal{OR}_{R_1+R_2}})(y) \right\} \\ & = b_1 \vee b_2 \neq 0 = 1 - \delta_x(x). \end{aligned}$$

In the following, by extending the optimistic two-granulation double fuzzy rough set, we will introduce the optimistic multi-granulation double fuzzy rough set (in short, OMGDFRS) and its accompanying properties.

 doi:10.20944/preprints202309.0854.v1



**Definition 2.6.** Let $U$ be arbitrary sets and the pairs $(R_i, R_i^*)$ such that $1 \leq i \leq m$ a double fuzzy relations on $U$. Then $(U, \mathcal{R}, \mathcal{R}^*)$ is called the multi-granulation double fuzzy approximation space (in short, MGDFAS), where $\mathcal{R} = \{R_1, R_2, ...R_i\}$ and $\mathcal{R}^* = \{R_1^*, R_2^*, ...R_i^*\}$.

**Definition 2.7.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each fuzzy set $\lambda$ on $U$, the pairs $(\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ and $(\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ of maps $\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda$, $\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda : U \to I$ are called optimistic multi-granulation double fuzzy lower approximation and optimistic multi-granulation double fuzzy upper approximation of a fuzzy set $\lambda$, respectively are defined as follows: For all $x \in U$,

$$(\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda)(x) = \bigvee_{i=1}^{m} \bigwedge_{y \in U} ((1 - R_i(x,y)) \vee \lambda(y)),$$

$$(\underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x) = \bigwedge_{i=1}^{m} \bigvee_{y \in U} ((1 - R_i^*(x,y)) \wedge 1 - \lambda(y)),$$

$$(\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda)(x) = \bigwedge_{i=1}^{m} \bigvee_{y \in U} (R_i(x,y) \wedge \lambda(y)),$$

$$(\overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x) = \bigvee_{i=1}^{m} \bigwedge_{y \in U} (R_i^*(x,y) \vee 1 - \lambda(y)).$$

The quaternary $(\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda, \overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ is called optimistic multi-granulation double fuzzy rough set of $\lambda$ (in short, OMGDFRS).
The pairs $(\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}, \underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}})$ and $(\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}, \overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}})$ of operators $\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}, \underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}$, $\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}, \overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}} : I^U \to I^U$ are called optimistic multi-granulation double fuzzy lower approximation and optimistic multi-granulation double fuzzy upper approximation operators, respectively.

**Proposition 2.8.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda \in I^U$ we obtain the following:

(1) $\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda = \bigvee_{i=1}^{m} \underline{\mathcal{R}_{R_i}}\lambda$ and $\underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda = \bigwedge_{i=1}^{m} \underline{\mathcal{R}^*_{R_i^*}}\lambda$.

(2) $\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda = \bigwedge_{i=1}^{m} \overline{\mathcal{R}_{R_i}}\lambda$ and $\overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda = \bigvee_{i=1}^{m} \overline{\mathcal{R}^*_{R_i^*}}\lambda$.

**Proof.** Similarly to Proposition 2.2.

**Theorem 2.9.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda \in I^U$ we obtain the following:

(1) $\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda \leq \tilde{1} - \overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda$ and $\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda \geq \tilde{1} - \underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda$.

(2) $\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\tilde{1} = \tilde{0}$.

 doi:10.20944/preprints202309.0854.v1



(3) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\tilde{0} = \tilde{0}$ and $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda$ and $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda$.

(5) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda$ and $\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda$.

**Proof.** It is similar to the Proof of Theorem 2.3.

**Theorem 2.10.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda, \mu \in I^U$ we obtain the following:

(1) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\lambda \wedge \mu) \leq \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \wedge \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$ and

$\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\lambda \wedge \mu) \geq \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \vee \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(2) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\lambda \vee \mu) \geq \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \vee \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$ and

$\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\lambda \vee \mu) \leq \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \wedge \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(3) If $\lambda \leq \mu$, then $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \leq \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$ and $\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \geq \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(4) If $\lambda \leq \mu$, then $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \leq \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$ and $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \geq \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(5) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\lambda \vee \mu) \geq \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \vee \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$ and

$\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\lambda \vee \mu) \leq \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \wedge \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(6) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\lambda \wedge \mu) \leq \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \wedge \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$ and

$\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\lambda \wedge \mu) \geq \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \vee \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

**Proof.** It is similarly to the Proof of Theorem 2.4.

## 3. Pessimistic Multi-granulation double fuzzy rough sets

**Definition 3.1.** Let $U$ be an arbitrary universal set and $(R_1, R_1^*)$ and $(R_2, R_2^*)$ a double fuzzy relations on $U$. Then for each fuzzy set $\lambda$ on $U$, the pairs $(\underline{\mathcal{PR}}_{R_1+R_2}\lambda, \underline{\mathcal{PR}}^*_{R_1^*+R_2^*}\lambda)$ and $(\overline{\mathcal{PR}}_{R_1+R_2}\lambda, \overline{\mathcal{PR}}^*_{R_1^*+R_2^*}\lambda)$ of maps $\underline{\mathcal{PR}}_{R_1+R_2}\lambda, \underline{\mathcal{PR}}^*_{R_1^*+R_2^*}\lambda, \overline{\mathcal{PR}}_{R_1+R_2}\lambda, \overline{\mathcal{PR}}^*_{R_1^*+R_2^*}\lambda : U \to I$ are called pessimistic two-granulation double fuzzy lower approximation and pessimistic two-granulation double fuzzy upper approximation of a fuzzy set $\lambda$, respectively and are defined as follows: For all $x \in U$,

$$(\underline{\mathcal{PR}}_{R_1+R_2}\lambda)(x) = \left\{ \bigwedge_{y \in U} ((1 - R_1(x,y)) \vee \lambda(y)) \right\} \wedge \left\{ \bigwedge_{y \in U} ((1 - R_2(x,y)) \vee \lambda(y)) \right\};$$

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$$(\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda)(x)=\left\{\bigvee_{y\in U}((1-R_1^*(x,y))\wedge 1-\lambda(y))\right\}\vee\left\{\bigvee_{y\in U}((1-R_2^*(x,y))\wedge 1-\lambda(y))\right\};$$

$$(\overline{\mathcal{PR}_{R_1+R_2}}\lambda)(x)=\left\{\bigvee_{y\in U}(R_1(x,y)\wedge\lambda(y))\right\}\vee\left\{\bigvee_{y\in U}(R_2(x,y)\wedge\lambda(y))\right\};$$

$$(\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda)(x)=\left\{\bigwedge_{y\in U}(R_1^*(x,y)\vee 1-\lambda(y))\right\}\wedge\left\{\bigwedge_{y\in U}(R_2^*(x,y)\vee 1-\lambda(y))\right\}.$$

The quaternary $(\underline{\mathcal{PR}_{R_1+R_2}}\lambda,\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda,\overline{\mathcal{PR}_{R_1+R_2}}\lambda,\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda)$ is called pessimistic two-granulation double fuzzy rough set of $\lambda$ (in short, PTGDFRS). The pairs $(\underline{\mathcal{PR}_{R_1+R_2}},\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}})$ and $(\overline{\mathcal{PR}_{R_1+R_2}},\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}})$ of operators $\underline{\mathcal{PR}_{R_1+R_2}}$, $\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}$, $\overline{\mathcal{PR}_{R_1+R_2}}$, $\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}$ $: U\to I$ are called pessimistic two-granulation double fuzzy lower approximation and pessimistic two-granulation double fuzzy upper approximation operators, respectively.

The PTGDFRS approximations are defined by many separate pairs of double fuzzy relations, whereas the normal double fuzzy rough approximations are represented by those produced by only one pair of double fuzzy relation, as can be seen from the preceding definition. In fact, when $(R_1,R_1^*)=(R_2,R_2^*)$, the PTGDFRS degenerates into a double fuzzy rough set. To put it another way, a double fuzzy rough set model is a subset of the PTGDFRS.

**Proposition 3.2.** Let $U$ be an arbitrary universal set, $(R_1,R_1^*)$ and $(R_2,R_2^*)$ be a double fuzzy relations on $U$. Then for each $\lambda\in I^U$ we have the following:

(1) $\underline{\mathcal{PR}_{R_1+R_2}}\lambda=\underline{\mathcal{R}_{R_1}}\lambda\wedge\underline{\mathcal{R}_{R_2}}\lambda$ and $\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda=\underline{\mathcal{R}^{*}_{R_1^*}}\lambda\vee\underline{\mathcal{R}^{*}_{R_2^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{R_1+R_2}}\lambda=\overline{\mathcal{R}_{R_1}}\lambda\vee\overline{\mathcal{R}_{R_2}}\lambda$ and $\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda=\overline{\mathcal{R}^{*}_{R_1^*}}\lambda\wedge\overline{\mathcal{R}^{*}_{R_2^*}}\lambda$.

**Proof.** They can be proved by Definition 1.2 and Definition 3.1.

**Theorem 3.3.** Let $U$ be an arbitrary universal set, $(R_1,R_1^*)$ and $(R_2,R_2^*)$ be a double fuzzy relations on $U$. Then for each $\lambda\in I^U$ we have the following:

(1) $\overline{\mathcal{PR}_{R_1+R_2}}\lambda\leq\tilde{1}-\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda$ and $\underline{\mathcal{PR}_{R_1+R_2}}\lambda\geq\tilde{1}-\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda$.

(2) $\underline{\mathcal{PR}_{R_1+R_2}}\tilde{1}=\tilde{1}$ and $\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\tilde{1}=\tilde{0}$.

(3) $\overline{\mathcal{PR}_{R_1+R_2}}\tilde{0}=\tilde{0}$ and $\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\tilde{0}=\tilde{1}$.

(4) $\overline{\mathcal{PR}_{R_1+R_2}}(\tilde{1}-\lambda)=\tilde{1}-\underline{\mathcal{PR}_{R_1+R_2}}\lambda$ and $\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}(\tilde{1}-\lambda)=\tilde{1}-\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda$.

(5) $\underline{\mathcal{PR}_{R_1+R_2}}(\tilde{1}-\lambda)=\tilde{1}-\overline{\mathcal{PR}_{R_1+R_2}}\lambda$ and $\underline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}(\tilde{1}-\lambda)=\tilde{1}-\overline{\mathcal{PR}^{*}_{R_1^*+R_2^*}}\lambda$.

**Proof.** (1) For each $x\in U,\ \lambda\in I^U$ we have

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$$
\begin{aligned}
& (\tilde{1} - (\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda))(x) \\
= \;& 1 - \left\{ \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \wedge \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \\
= \;& \left\{ 1 - \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \vee \left\{ 1 - \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \\
= \;& \left\{ \bigvee_{y\in U} 1 - \{R_1^*(x,y) \vee 1 - \lambda(y)\} \right\} \vee \left\{ \bigvee_{y\in U} 1 - \{R_2^*(x,y) \vee 1 - \lambda(y)\} \right\} \\
= \;& \left\{ \bigvee_{y\in U} \{1 - R_1^*(x,y) \wedge \lambda(y)\} \right\} \vee \left\{ \bigvee_{y\in U} \{1 - R_2^*(x,y) \wedge \lambda(y)\} \right\} \\
\geq \;& \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y)) \right\} \vee \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y)) \right\} \\
= \;& (\overline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) \text{ for all } x \in U.
\end{aligned}
$$

Hence, $\overline{\mathcal{PR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, $\underline{\mathcal{PR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$.

(2) Since, for each $x \in U$, $\tilde{1}(x) = 1$, we obtain

$$
\begin{aligned}
(\underline{\mathcal{PR}_{R_1+R_2}}\tilde{1})(x) &= \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee \tilde{1}(y)) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee \tilde{1}(y)) \right\} \\
&= 1 = \tilde{1}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\tilde{1})(x) &= \left\{ \bigvee_{y\in U} ((1 - R_i(x,y)) \wedge 1 - \tilde{1}(y)) \right\} \vee \left\{ \bigvee_{y\in U} ((1 - R_i(x,y)) \wedge 1 - \tilde{1}(y)) \right\} \\
&= 0 = \tilde{0}(x).
\end{aligned}
$$

Therefore, we obtain $\underline{\mathcal{PR}_{R_1+R_2}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\tilde{1} = \tilde{0}$.

(3) Similar to (2).

(4) For each $x \in U$, we have

$$\begin{aligned}
&\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda)(x) \\
= &\left\{\bigwedge_{y\in U}(R_1^*(x,y)\vee 1-(1-\lambda(y)))\right\}\wedge\left\{\bigwedge_{y\in U}(R_2^*(x,y)\vee 1-(1-\lambda(y)))\right\} \\
= &\left\{1-\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\right\}\wedge\left\{1-\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \\
= &\ 1-\left\{\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\vee\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \\
= &\ 1-\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda(x).
\end{aligned}$$

Thus, we obtain $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda) = \tilde{1}-\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$. Similarly,we can prove $\overline{\mathcal{PR}_{R_1+R_2}}(\tilde{1}-\lambda) = \tilde{1}-\underline{\mathcal{PR}_{R_1+R_2}}\lambda$.

(5) It is similar to the proof of (4).

**Theorem 3.4.** Let $U$ be an arbitrary universal set, $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on $U$. Then for each $\lambda, \mu \in I^U$ we obtain the following:

(1) $\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\wedge\mu) = \underline{\mathcal{PR}_{R_1+R_2}}\lambda\wedge\underline{\mathcal{PR}_{R_1+R_2}}\mu$ and

$\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\wedge\mu) = \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\vee\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(2) $\overline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu) = \overline{\mathcal{PR}_{R_1+R_2}}\lambda\vee\overline{\mathcal{PR}_{R_1+R_2}}\mu$ and

$\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu) = \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\wedge\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(3) If $\lambda\le\mu$, then $\underline{\mathcal{PR}_{R_1+R_2}}\lambda\le\underline{\mathcal{PR}_{R_1+R_2}}\mu$ and $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\ge\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(4) If $\lambda\le\mu$, then $\overline{\mathcal{PR}_{R_1+R_2}}\lambda\le\overline{\mathcal{PR}_{R_1+R_2}}\mu$ and $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\ge\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(5) $\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu)\ge\underline{\mathcal{PR}_{R_1+R_2}}\lambda\vee\underline{\mathcal{PR}_{R_1+R_2}}\mu$ and

$\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)\le\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\wedge\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(6) $\overline{\mathcal{PR}_{R_1+R_2}}(\lambda\wedge\mu)\le\overline{\mathcal{PR}_{R_1+R_2}}\lambda\wedge\overline{\mathcal{PR}_{R_1+R_2}}\mu$ and

$\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\wedge\mu)\ge\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\vee\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

**Proof.** (1) For each $x \in U,\ \lambda, \mu \in I^U$,

$$
\begin{aligned}
& (\underline{\mathcal{PR}_{R_1+R_2}}(\lambda \wedge \mu))(x) \\
= & \left\{ \bigwedge_{y \in U} ((1 - R_1(x,y)) \vee (\lambda \wedge \mu)(y)) \right\} \wedge \left\{ \bigwedge_{y \in U} ((1 - R_2(x,y)) \vee (\lambda \wedge \mu)(y)) \right\} \\
= & \left\{ \left\{ \bigwedge_{y \in U} ((1 - R_1(x,y)) \vee (\lambda)(y) \right\} \wedge \left\{ \bigwedge_{y \in U} ((1 - R_1(x,y)) \vee (\mu)(y) \right\} \right\} \\
\wedge & \left\{ \left\{ \bigwedge_{y \in U} ((1 - R_2(x,y)) \vee (\lambda)(y) \right\} \wedge \left\{ \bigwedge_{y \in U} ((1 - R_2(x,y)) \vee (\mu)(y) \right\} \right\} \\
= & \left\{ (\underline{\mathcal{R}_{R_1}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_1}}\mu)(x) \right\} \wedge \left\{ (\underline{\mathcal{R}_{R_2}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_2}}\mu)(x) \right\} \\
= & \left\{ (\underline{\mathcal{R}_{R_1}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_2}}\lambda)(x) \right\} \wedge \left\{ (\underline{\mathcal{R}_{R_1}}\mu)(x) \wedge (\underline{\mathcal{R}_{R_2}}\mu)(x) \right\} \\
= & (\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) \wedge (\underline{\mathcal{PR}_{R_1+R_2}}\mu)(x).
\end{aligned}
$$

Also, for each $x \in U$,

$$
\begin{aligned}
& (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) \\
= & \left\{ \bigvee_{y \in U} ((1 - R_1^*(x,y)) \wedge 1 - (\lambda \wedge \mu)(y)) \right\} \vee \left\{ \bigvee_{y \in U} ((1 - R_2^*(x,y)) \wedge 1 - (\lambda \wedge \mu)(y)) \right\} \\
= & \left\{ \bigvee_{y \in U} ((1 - R_1^*(x,y)) \wedge (1 - \lambda(y) \vee 1 - \mu(y)) \right\} \\
\vee & \left\{ \bigvee_{y \in U} ((1 - R_2^*(x,y)) \wedge (1 - \lambda(y) \vee 1 - \mu(y)) \right\} \\
= & \left\{ \left\{ \bigvee_{y \in U} ((1 - R_1^*(x,y)) \wedge (1 - \lambda(y) \right\} \vee \left\{ \bigvee_{y \in U} ((1 - R_1^*(x,y)) \wedge (1 - \mu(y) \right\} \right\} \\
\vee & \left\{ \left\{ \bigvee_{y \in U} ((1 - R_2^*(x,y)) \wedge (1 - \lambda(y) \right\} \vee \left\{ \bigvee_{y \in U} ((1 - R_2^*(x,y)) \wedge (1 - \mu(y) \right\} \right\} \\
= & \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \right\} \vee \left\{ (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x) \right\} \\
= & \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \right\} \vee \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x) \right\} \\
= & (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) \vee (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(x)
\end{aligned}
$$

(2) It is similar to the proof of (1).

(3) If $\lambda \leq \mu$, then for all $y \in U,\ \lambda(y) \leq \mu(y)$. Therefore,

$$
\bigwedge_{y \in U} (1 - R_1(x,y) \vee \lambda(y)) \leq \bigwedge_{y \in U} (1 - R_1(x,y) \vee \mu(y)) \qquad (5)
$$

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and

$$\bigwedge_{y\in U}(1-R_2(x,y)\vee\lambda(y))\leq\bigwedge_{y\in U}(1-R_2(x,y)\vee\mu(y)). \tag{6}$$

Form equations (3.1) and (3.2) we have

$$\begin{aligned}&\left\{\bigwedge_{y\in U}(1-R_1(x,y)\vee\lambda(y))\right\}\wedge\left\{\bigwedge_{y\in U}(1-R_2(x,y)\vee\lambda(y))\right\}\\ \leq\ &\left\{\bigwedge_{y\in U}(1-R_1(x,y)\vee\mu(y))\right\}\wedge\left\{\bigwedge_{y\in U}(1-R_2(x,y)\vee\mu(y))\right\}\end{aligned}$$

Thus, $\underline{\mathcal{PR}_{R_1+R_2}}\lambda\leq\underline{\mathcal{PR}_{R_1+R_2}}\mu$, also,

$$\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\geq\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\mu(y)) \tag{7}$$

and

$$\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\geq\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\mu(y)). \tag{8}$$

Form equations (3.3) and (3.4) we have

$$\begin{aligned}&\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\vee\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\\ \geq\ &\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\mu(y))\right\}\vee\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\mu(y))\right\}\end{aligned}$$

Thus, $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(4) It is similar to the proof of (3).

(5) Since $\lambda\leq\lambda\vee\mu$ and $\mu\leq\lambda\vee\mu$, by (3) we have

$$\underline{\mathcal{PR}_{R_1+R_2}}\lambda\leq\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu)\text{ and }\underline{\mathcal{PR}_{R_1+R_2}}\mu\leq\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu).$$

Therefore $\underline{\mathcal{PR}_{R_1+R_2}}\lambda\vee\underline{\mathcal{PR}_{R_1+R_2}}\mu\leq\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu)$. Also, we have

$$\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)\text{ and }\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu).$$

This implies that $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\wedge\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)$.

(6) It is similar to the proof of (5).

**Example 3.5.** Let $U=\{x,y,z\}$. Define $R_1,R_1^*,R_2,R_2^*:U\times U\to I$ as in the example 2.5 and $\lambda,\mu\in I^U$ as in the example 2.5. Then

$$(\underline{\mathcal{PR}_{R_1+R_2}}\mu)(x)=0.7,(\underline{\mathcal{PR}_{R_1+R_2}}\mu)(y)=0.8,(\underline{\mathcal{PR}_{R_1+R_2}}\mu)(z)=0.4,$$

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$$(\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) = 0.5, (\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(y) = 0.5, (\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(z) = 0.5,$$

$$(\underline{\mathcal{PR}_{R_1+R_2}}(\lambda \wedge \mu))(x) = 0.2, (\underline{\mathcal{PR}_{R_1+R_2}}(\lambda \wedge \mu))(y) = 0.4, (\underline{\mathcal{PR}_{R_1+R_2}}(\lambda \wedge \mu))(z) = 0.4,$$

Therefore, $\underline{\mathcal{PR}_{R_1+R_2}}(\lambda \wedge \mu) \neq \underline{\mathcal{PR}_{R_1+R_2}}\lambda \wedge \underline{\mathcal{PR}_{R_1+R_2}}\mu$.

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(x) = 0.2, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(y) = 0.2, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(z) = 0.6,$$

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(x) = 0.3, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(y) = 0.5, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(z) = 0.3,$$

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) = 0.6, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(y) = 0.6, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(z) = 0.6.$$

Therefore, $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu) \neq \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \vee \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

In the following, by extending the pessmistic two-granulation double fuzzy rough set, we will introduce the pessmistic multi-granulation double fuzzy rough set (in short, PMGDFRS) and its accompanying properties.

**Definition 3.6.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS such that $1 \leq i \leq m$. Then for each fuzzy set $\lambda$ on $U$, the pairs $(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ and $(\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ of maps $\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda$, $\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda : U \to I$ are called pessimistic multi-granulation double fuzzy lower approximation and pessimistic multi-granulation double fuzzy upper approximation of a fuzzy set $\lambda$, respectively and are defined as follows: For all $x \in U$,

$$(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda)(x) = \bigwedge_{i=1}^m \bigwedge_{y \in U} ((1 - R_i(x, y)) \vee \lambda(y))$$

$$(\underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x) = \bigvee_{i=1}^m \bigvee_{y \in U} ((1 - R_i^*(x, y)) \wedge 1 - \lambda(y))$$

$$(\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda)(x) = \bigvee_{i=1}^m \bigvee_{y \in U} (R_i(x, y) \wedge \lambda(y))$$

$$(\overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x) = \bigwedge_{i=1}^m \bigwedge_{y \in U} (R_i^*(x, y) \vee 1 - \lambda(y)).$$

The quaternary $(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda, \overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ is called pessimistic multi-granulation double fuzzy rough set of $\lambda$ (in short, PMGDFRS).
The pairs $(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}})$ and $(\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}})$ of operators $\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}$, $\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}} : I^U \to I^U$ are called pessimistic multi-granulation double fuzzy lower approximation and pessimistic multi-granulation double fuzzy upper approximation operators, respectively.

**Proposition 3.7.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda \in I^U$ we obtain the following:

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(1) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda = \bigwedge_{i=1}^{m} \underline{\mathcal{R}_{R_i}}\lambda$ and $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda = \bigvee_{i=1}^{m} \underline{\mathcal{R}^*_{R_i^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda = \bigvee_{i=1}^{m} \overline{\mathcal{R}_{R_i}}\lambda$ and $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda = \bigwedge_{i=1}^{m} \overline{\mathcal{R}^*_{R_i^*}}\lambda$.

**Proof.** It is similar to the proof of Proposition 3.2.

**Theorem 3.8.**Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda \in I^U$ we have

(1) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \leq \tilde{1} - \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$ and $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \geq \tilde{1} - \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

(2) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\tilde{1} = \tilde{0}$.

(3) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\tilde{0} = \tilde{0}$ and $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda$ and $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

(5) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda$ and $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

**Proof.** It is similar to the proof of Theorem 3.3.

**Theorem 3.9.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda, \mu \in I^U$ we obtain the following:

(1) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \wedge \mu) = \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \wedge \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$ and

$\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \wedge \mu) = \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \vee \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(2) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \vee \mu) = \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \vee \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$ and

$\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \vee \mu) = \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \wedge \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(3) If $\lambda \leq \mu$, then $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \leq \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$ and $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \geq \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(4) If $\lambda \leq \mu$, then $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \leq \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$ and $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \geq \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(5) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \vee \mu) \geq \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \vee \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$ and

$\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \vee \mu) \leq \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \wedge \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(6) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \wedge \mu) \leq \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \wedge \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$ and

$\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \wedge \mu) \geq \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \vee \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

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**Proof.** It is similar to the proof of Theorem 3.4.

**Proposition 3.10.** Let $U$ be an arbitrary universal set, $(R_1, R_1^*)$ and $(R_2, R_2^*)$ be a double fuzzy relations on $U$. Then for each $i \in \{1,2\}$ and $\lambda \in I^U$ we obtain the following:

(1) $\underline{\mathcal{PR}_{R_1+R_2}}\lambda \leq \underline{\mathcal{R}_{R_i}}\lambda \leq \underline{\mathcal{OR}_{R_1+R_2}}\lambda$ and $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \geq \underline{\mathcal{R}^*_{R_i^*}}\lambda \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{R_1+R_2}}\lambda \geq \overline{\mathcal{R}_{R_i}}\lambda \geq \overline{\mathcal{OR}_{R_1+R_2}}\lambda$ and $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \leq \overline{\mathcal{R}^*_{R_i^*}}\lambda \leq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

**Proof.** It can be proven by, Propositions 2.2 and 3.2.

**Proposition 3.11.** Let $(U, \mathcal{R}, \mathcal{R}^*)$ be a MGDFAS. Then for each $\lambda \in I^U$ and $1 \leq i \leq m$ we obtain the following:

(1) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \leq \underline{\mathcal{R}_{R_i}}\lambda \leq \underline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}\lambda$ and $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \geq \underline{\mathcal{R}^*_{R_i^*}}\lambda \geq \underline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \geq \overline{\mathcal{R}_{R_i}}\lambda \geq \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda$ and $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \leq \overline{\mathcal{R}^*_{R_i^*}}\lambda \leq \overline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

**Proof.** It can be proven by, Propositions 2.8 and 3.7.

## 4. Conclusion

It has been discovered that the rough set theory is a potent theory with numerous applications in the artificial intelligence fields of pattern recognition, machine learning, and automated knowledge acquisition. In this study, the idea of double fuzzy rough sets was introduced, which was seen as a generalization of fuzzy rough sets.

**Conflicts of Interest:** The authors declare that there is no conflicts of interest regarding the publication of this paper.

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