

Communication

Not peer-reviewed version

Impedance Variation of a Coaxial Coil Encircling a Metal Tube Adapter

[Yao Luo](#) * and Xinyi Yang

Posted Date: 11 September 2023

doi: 10.20944/preprints202309.0629.v1

Keywords: Tube adapter; eddy current; numerical eigenfunctions; Clenshaw-Curtis quadrature



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Communication

Impedance Variation of a Coaxial Coil Encircling a Metal Tube Adapter

Yao Luo ^{1,2,*}, Xinyi Yang ^{1,2}

¹ State Key Laboratory of Power Grid Environmental Protection, School of Electrical Engineering and Automation, Wuhan University, Wuhan 430072, China

² School of Electrical Engineering and Automation, Wuhan University, Wuhan 430072, China

* Correspondence: luoyao@whu.edu.cn

Abstract: The impedance change of an induction coil surrounding a metal tube adapter is investigated by the truncated region eigenfunction expansion (TREE) method. Conventional TREE method is inapplicable to this problem as a consequence of the numerical overflow of eigenfunctions of the air-metal multi-subdomain regions. The difficulty is surmounted by a normalization procedure for the numerical eigenfunctions obtained from the finite element method (FEM). Efficient algorithm is devised by the Clenshaw-Curtis quadrature rule for integrals involving the numerical eigenfunctions. Numerical results of the TREE and FEM simulation coincide very well in all cases, and the efficiency of the proposed method is also confirmed.

Keywords: tube adapter; eddy current; numerical eigenfunctions; Clenshaw-Curtis quadrature

1. Introduction

A tube adapter is a component connecting two tubes of different diameters. The standard analytical method of Dodd and Deeds [1] is unable to investigate the interaction of an induction coil with a metal tube adapter, due to the end effects involved in this problem. The truncated region eigenfunction expansion (TREE) method, pioneered by Hannakam and Tepe [2], and developed by Theodoulidis, Kriezis and Bowler [3-9] for modeling of the eddy current nondestructive testing (EC NDT), is capable of analyzing the end effects and establishing analytical models. However, successful implementation of TREE for the model of end effects depends on the solution of relevant eigenvalue equations, which are transcendental and complex roots should be determined. Conventionally, the Newton-Raphson algorithm [10-13] or contour integral based on the Cauchy's theorem [14-17] are applied to solve the eigenvalue equations. A novel method based on the Sturm-Liouville theory and Galerkin approach has been proposed recently [18, 19], which greatly simplifies the process of locating the complex eigenvalues.

Unfortunately, the TREE cannot be used immediately for the problem of this work, even with the novel approach. The bottleneck lies in the numerical issues for the regions composed of multiple air-metal subdomains, more specifically, the numerical overflow occurs constantly when the symbolic eigenfunctions are employed for the n -subdomain () regions. Admittedly, for certain three-subdomain problems, the overflow can be suppressed by splitting the model into even and odd parts [6, 8, 20-22], but the technique is merely available for the problems with additional symmetry.

More recently, the difficulties of overflow have been overcome in [23] by an innovative approach based on the FEM solution of the Sturm-Liouville equation. By this method, no numerical overflow arises in the solving process, therefore, many new models can be dealt with the TREE. According to [23], the Clenshaw-Curtis quadrature rule is appropriate for the evaluations of the integrals involving the numerical eigenfunctions, to obtain an efficient algorithm. The investigation of this work will be based on the approach.

In Section 2, the TREE solution is given for the metal tube adapter surrounding by a coaxial coil. The permeability of the metal is not restricted to μ_0 in comparison with [23]. In Section 3, the numerical eigenfunctions are obtained by the FEM solution of the Sturm-Liouville equations and normalized, and the Clenshaw-Curtis quadrature rule is applied to the computation of the integrals

involving the numerical eigenfunctions. In Section 4, the TREE results are compared with those from the FEM simulation.

2. Formulation

A metal tube adapter of conductivity σ and permeability $\mu = \mu_r \mu_0$ (μ_r is supposed to be constant), is encircled by a coaxial induction coil excited by a time harmonic current of frequency ω and amplitude I (See Figure 1). The geometry of the coil and tube adapter is shown in Figure 2. Perfect electric boundary is imposed at $z=0$ and $z=b$ to discretize the eigenvalues of this boundary value problem (BVP).

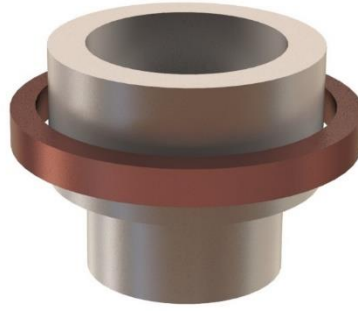


Figure 1. A metal tube adapter encircled by a coaxial coil.

The solution domain is divided into 5 regions along the r -axis (See Figure 2). The vector potentials A_1 to A_5 satisfy the Laplace or Helmholtz equations in the corresponding regions:

$$\nabla^2 A_{1,5} = 0 \quad (1)$$

$$\nabla^2 A_{2,3,4} = k^2 A_{2,3,4} \quad (2)$$

where $k = \sqrt{i\omega\sigma\mu_0\mu_r}$ is the wavenumber of the metal.

Only the φ -component of the vector potential exists, due to the axisymmetry of the BVP, i.e. $A = A e_\varphi$, and the vector Laplacian of equations (1) and (2) is reduced to

$$\nabla_\varphi^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \quad (3)$$

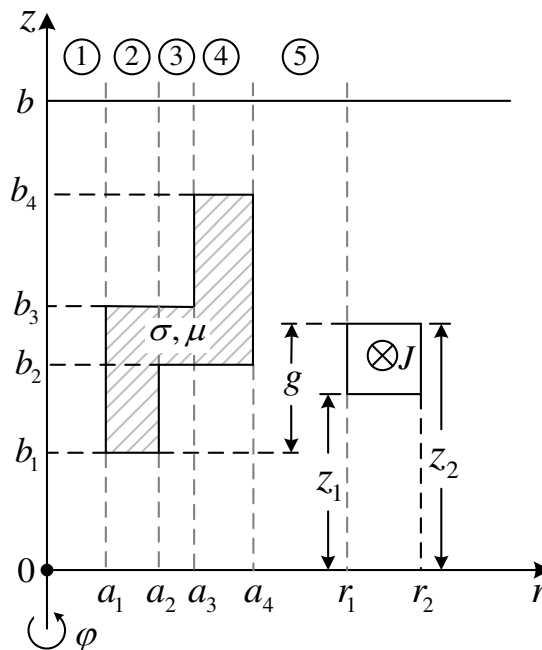


Figure 2. Side view of a metal tube adapter encircled by a coaxial coil.

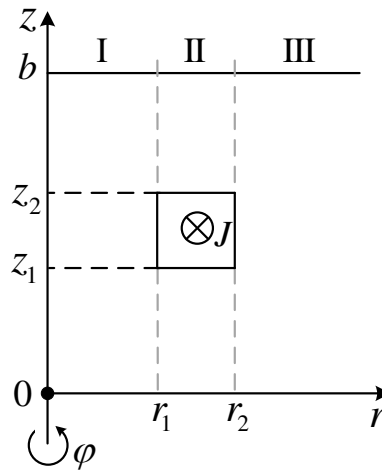
2.1. Vector Potential of the Source Coil

The formulation of source vector potential can be obtained by the source expansion of the Poisson equation [24, 25]. The vector potential of the coil can be written in the form (See Figure 3)

$$A_I(r, z) = \mathbf{S}^T(z) \mathbf{I}_1(\alpha r) \mathbf{C}_1^{(e)} \quad (4a)$$

$$A_{II}(r, z) = \mathbf{S}^T(z) \left[\mathbf{I}_1(\alpha r) \mathbf{C}_2^{(e)} + \mathbf{K}_1(\alpha r) \mathbf{D}_2^{(e)} + \mathbf{V}(r) \right] \quad (4b)$$

$$A_{III}(r, z) = \mathbf{S}^T(z) \mathbf{K}_1(\alpha r) \mathbf{D}_3^{(e)} \quad (4c)$$

**Figure 3.** Side view of an isolated coil with truncation boundary.

where the source vector $\mathbf{V}(r)$ is

$$\mathbf{V}(r) = \begin{bmatrix} v_1(r) \\ v_2(r) \\ \vdots \end{bmatrix}$$

with the elements

$$v_i(r) = \kappa_i \mathbf{L}_1(\alpha_i r) \quad (5)$$

where $\alpha_i = i\pi/b$, and $\text{Ln}(x)$ is the modified Struve function of order n , and

$$\kappa_i = \frac{2\mu_0 J}{i\alpha_i^2} \sin\left[\frac{\alpha_i}{2}(z_1 - z_2)\right] \sin\left[\frac{\alpha_i}{2}(z_1 + z_2)\right] \quad (6)$$

Other matrices and vectors in (4a)-(4c) are

$$\alpha = \begin{bmatrix} \alpha_1 & 0 & \dots \\ 0 & \alpha_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathbf{I}_1(\alpha r) = \begin{bmatrix} I_1(\alpha_1 r) & 0 & \dots \\ 0 & I_1(\alpha_2 r) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathbf{K}_1(\alpha r) = \begin{bmatrix} K_1(\alpha_1 r) & 0 & \dots \\ 0 & K_1(\alpha_2 r) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathbf{S}(z) = \begin{bmatrix} \sin(\alpha_1 z) \\ \sin(\alpha_2 z) \\ \vdots \end{bmatrix},$$

where $\text{I}_n(x)$ and $\text{K}_n(x)$ are the modified Bessel functions of the first and second kinds of order n , respectively, and $\mathbf{C}_1^{(e)}$, $\mathbf{C}_2^{(e)}$, $\mathbf{D}_2^{(e)}$, $\mathbf{D}_3^{(e)}$ are coefficients to be determined. With the interface conditions of B_r and H_z at $r=r_1$ and $r=r_2$, the coefficients can be found:

$$C_{1,i}^{(e)} = \kappa_i [\chi(\alpha_i r_1) - \chi(\alpha_i r_2)] \quad (7a)$$

$$C_{2,i}^{(e)} = -\kappa_i \chi(\alpha_i r_2) \quad (7b)$$

$$D_{2,i}^{(e)} = \kappa_i \eta(\alpha_i r_1) \quad (7c)$$

$$D_{3,i}^{(e)} = \kappa_i [\eta(\alpha_i r_1) - \eta(\alpha_i r_2)] \quad (7d)$$

where

$$\chi(x) = x [K_1(x) \mathbf{L}_0(x) + K_0(x) \mathbf{L}_1(x)] \quad (8a)$$

$$\eta(x) = x [I_1(x) \mathbf{L}_0(x) - I_0(x) \mathbf{L}_1(x)] \quad (8b)$$

For the function $\chi(x)$ used for the subsequent analysis, it is advisable to adopt an alternative form for the practical evaluations, namely

$$\chi(x) = \begin{cases} \frac{x^2}{2} \sum_{m=0}^{m_0} \frac{(x/2)^{2m}}{\Gamma(m+3/2)} \left[\frac{K_1(x)}{\Gamma(m+3/2)} + \frac{xK_0(x)}{2\Gamma(m+5/2)} \right], & x < 15 \\ 1 + \frac{x}{\pi^2} \sum_{m=0}^{m_1} \frac{\Gamma^2(m+1/2)}{(x/2)^{2m}} \left[\frac{K_0(x)}{m-1/2} - \frac{K_1(x)}{x/2} \right], & x \geq 15 \end{cases} \quad (9)$$

Expression (9) is obtained by the Maclaurin and asymptotic expansions of $\mathbf{L}_n(x)$ [26], and high accuracy can be achieved by setting $m_0=23$ and $m_1=10$, respectively.

2.2. Impedance Change of the Coil Encircling the Metal Tube Adapter

The vector potentials in the 5 regions of Figure 2 are expansible by the separation of variables

$$A_1(r, z) = \mathbf{S}^T(z) \mathbf{I}_1(\alpha r) \mathbf{C}_1 \quad (10a)$$

$$A_2(r, z) = \mathbf{F}^T(z) [\mathbf{I}_1(\mathbf{P}_1 r) \mathbf{C}_2 + \mathbf{K}_1(\mathbf{P}_1 r) \mathbf{D}_2] \quad (10b)$$

$$A_3(r, z) = \mathbf{G}^T(z) [\mathbf{I}_1(\mathbf{P}_2 r) \mathbf{C}_3 + \mathbf{K}_1(\mathbf{P}_2 r) \mathbf{D}_3] \quad (10c)$$

$$A_4(r, z) = \mathbf{H}^T(z) [\mathbf{I}_1(\mathbf{P}_3 r) \mathbf{C}_4 + \mathbf{K}_1(\mathbf{P}_3 r) \mathbf{D}_4] \quad (10d)$$

$$A_5(r, z) = \mathbf{S}^T(z) [\mathbf{I}_1(\alpha r) \mathbf{C}_1^{(e)} + \mathbf{K}_1(\alpha r) \mathbf{D}^{(s)}] \quad (10e)$$

where $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are the eigenvalue matrices of the Regions 2, 3 and 4, respectively,

$$\mathbf{P}_1 = \begin{bmatrix} p_{1,1} & 0 & \cdots \\ 0 & p_{1,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} p_{2,1} & 0 & \cdots \\ 0 & p_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathbf{P}_3 = \begin{bmatrix} p_{3,1} & 0 & \cdots \\ 0 & p_{3,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$\mathbf{F}(z) = \begin{bmatrix} f_1(p_{1,1}, z) \\ f_2(p_{1,2}, z) \\ \vdots \end{bmatrix}, \mathbf{G}(z) = \begin{bmatrix} g_1(p_{2,1}, z) \\ g_2(p_{2,2}, z) \\ \vdots \end{bmatrix},$$

$$\mathbf{H}(z) = \begin{bmatrix} h_1(p_{3,1}, z) \\ h_2(p_{3,2}, z) \\ \vdots \end{bmatrix}$$

are the axial eigenfunctions satisfying the relevant Sturm-Liouville equations:

$$\frac{d^2 f_i(z)}{dz^2} - k_1^2(z) f_i(z) = -p_{1,i}^2 f_i(z), f_i(0) = f_i(b) = 0 \quad (11a)$$

$$\frac{d^2 g_i(z)}{dz^2} - k_2^2(z) g_i(z) = -p_{2,i}^2 g_i(z), g_i(0) = g_i(b) = 0 \quad (11b)$$

and

$$\frac{d^2 h_i(z)}{dz^2} - k_3^2(z) h_i(z) = -p_{3,i}^2 h_i(z), h_i(0) = h_i(b) = 0 \quad (11c)$$

with

$$k_1(z) = \begin{cases} k, & b_1 \leq z \leq b_3 \\ 0, & \text{others} \end{cases} \quad (12a)$$

$$k_2(z) = \begin{cases} k, & b_2 \leq z \leq b_3 \\ 0, & \text{others} \end{cases} \quad (12b)$$

and

$$k_3(z) = \begin{cases} k, & b_2 \leq z \leq b_4 \\ 0, & \text{others} \end{cases} \quad (12c)$$

Taking account of the interface conditions of Br and Hz at $r=a_1$, $r=a_2$, $r=a_3$ and $r=a_4$, the following equations for the coefficients $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{D}_1$, $\mathbf{D}_2$ and $\mathbf{D}_3$ can be derived

$$\frac{b}{2} \mathbf{I}_1(\alpha a_1) \mathbf{C}_1 = \mathbf{T}_1 [\mathbf{I}_1(\mathbf{P}_1 a_1) \mathbf{C}_2 + \mathbf{K}_1(\mathbf{P}_1 a_1) \mathbf{D}_1] \quad (13a)$$

$$\mathbf{T}_1^T \alpha \mathbf{I}_0(\alpha a_1) \mathbf{C}_1 = \mathbf{P}_1 [\mathbf{I}_0(\mathbf{P}_1 a_1) \mathbf{C}_2 - \mathbf{K}_0(\mathbf{P}_1 a_1) \mathbf{D}_2] \quad (13b)$$

$$\mathbf{I}_1(\mathbf{P}_1 a_2) \mathbf{C}_2 + \mathbf{K}_1(\mathbf{P}_1 a_2) \mathbf{D}_2 = \mathbf{T}_2 [\mathbf{I}_1(\mathbf{P}_2 a_2) \mathbf{C}_3 + \mathbf{K}_1(\mathbf{P}_2 a_2) \mathbf{D}_3] \quad (13c)$$

$$\mathbf{T}_2^T \mathbf{P}_1 [\mathbf{I}_0(\mathbf{P}_1 a_2) \mathbf{C}_2 - \mathbf{K}_0(\mathbf{P}_1 a_2) \mathbf{D}_2] = \mathbf{P}_2 [\mathbf{I}_0(\mathbf{P}_2 a_2) \mathbf{C}_3 - \mathbf{K}_0(\mathbf{P}_2 a_2) \mathbf{D}_3] \quad (13d)$$

$$\mathbf{I}_1(\mathbf{P}_2 a_3) \mathbf{C}_3 + \mathbf{K}_1(\mathbf{P}_2 a_3) \mathbf{D}_3 = \mathbf{T}_3 [\mathbf{I}_1(\mathbf{P}_3 a_3) \mathbf{C}_4 + \mathbf{K}_1(\mathbf{P}_3 a_3) \mathbf{D}_4] \quad (13e)$$

$$\mathbf{T}_3^T \mathbf{P}_2 [\mathbf{I}_0(\mathbf{P}_2 a_3) \mathbf{C}_3 - \mathbf{K}_0(\mathbf{P}_2 a_3) \mathbf{D}_3] = \mathbf{P}_3 [\mathbf{I}_0(\mathbf{P}_3 a_3) \mathbf{C}_4 - \mathbf{K}_0(\mathbf{P}_3 a_3) \mathbf{D}_4] \quad (13f)$$

$$\mathbf{T}_4 [\mathbf{I}_1(\mathbf{P}_3 a_4) \mathbf{C}_4 + \mathbf{K}_1(\mathbf{P}_3 a_4) \mathbf{D}_4] = \frac{b}{2} [\mathbf{I}_1(\alpha a_4) \mathbf{C}_1^{(e)} + \mathbf{K}_1(\alpha a_4) \mathbf{D}^{(s)}] \quad (13g)$$

$$\mathbf{P}_3 [\mathbf{I}_0(\mathbf{P}_3 a_4) \mathbf{C}_4 - \mathbf{K}_0(\mathbf{P}_3 a_4) \mathbf{D}_4] = \mathbf{T}_4^T \alpha [\mathbf{I}_0(\alpha a_4) \mathbf{C}_1^{(e)} - \mathbf{K}_0(\alpha a_4) \mathbf{D}^{(s)}] \quad (13h)$$

where

$$\mathbf{T}_1 = \int_0^b \mathbf{S}(z) \mathbf{F}^T(z) dz \quad (14)$$

$$\mathbf{T}_2 = \int_0^b \frac{1}{\mu_r^{(1)}(z)} \mathbf{F}(z) \mathbf{G}^T(z) dz \quad (15)$$

$$\mathbf{T}_3 = \int_0^b \frac{1}{\mu_r^{(2)}(z)} \mathbf{G}(z) \mathbf{H}^T(z) dz \quad (16)$$

$$\mathbf{T}_4 = \int_0^b \mathbf{S}(z) \mathbf{H}^T(z) dz \quad (17)$$

In (13a)-(13h), the orthogonalities of the eigenfunctions

$$\int_0^b \mathbf{S}(z) \mathbf{S}^T(z) dz = \frac{b}{2} \mathbf{I} \quad (18a)$$

$$\int_0^b \frac{1}{\mu_r^{(1)}(z)} \mathbf{F}(z) \mathbf{F}^T(z) dz = \mathbf{I} \quad (18b)$$

$$\int_0^b \frac{1}{\mu_r^{(2)}(z)} \mathbf{G}(z) \mathbf{G}^T(z) dz = \mathbf{I} \quad (18c)$$

$$\int_0^b \frac{1}{\mu_r^{(3)}(z)} \mathbf{H}(z) \mathbf{H}^T(z) dz = \mathbf{I} \quad (18d)$$

have been adopted, where $\mathbf{I}$ is the identity matrix, and

$$\mu_r^{(1)}(z) = \begin{cases} \mu_r, & b_1 \leq z \leq b_3 \\ 1, & \text{others} \end{cases} \quad (19a)$$

$$\mu_r^{(2)}(z) = \begin{cases} \mu_r, & b_2 \leq z \leq b_3 \\ 1, & \text{others} \end{cases} \quad (19b)$$

$$\mu_r^{(3)}(z) = \begin{cases} \mu_r, & b_2 \leq z \leq b_4 \\ 1, & \text{others} \end{cases} \quad (19c)$$

The orthonormalization relations of (18b)-(18d) will be expounded in Section III.

Matrix algebra of (13a)-(13h) yields the equation system

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{D}_2 \\ \mathbf{C}_4 \\ \mathbf{D}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{E} \end{bmatrix} \quad (20)$$

where

$$\mathbf{A}_{11} = \mathbf{U}_1 \mathbf{I}_1 (\mathbf{P}_1 a_1) - \mathbf{P}_1 \mathbf{I}_0 (\mathbf{P}_1 a_1) \quad (21a)$$

$$\mathbf{A}_{12} = \mathbf{U}_1 \mathbf{K}_1 (\mathbf{P}_1 a_1) + \mathbf{P}_1 \mathbf{K}_0 (\mathbf{P}_1 a_1) \quad (21b)$$

$$\mathbf{A}_{21} = \mathbf{I}_1 (\mathbf{P}_2 a_3) \mathbf{M}_3 + \mathbf{K}_1 (\mathbf{P}_2 a_3) \mathbf{M}_1 \quad (21c)$$

$$\mathbf{A}_{22} = \mathbf{I}_1 (\mathbf{P}_2 a_3) \mathbf{M}_4 + \mathbf{K}_1 (\mathbf{P}_2 a_3) \mathbf{M}_2 \quad (21d)$$

$$\mathbf{A}_{23} = -\mathbf{T}_3 \mathbf{I}_1 (\mathbf{P}_3 a_3) \quad (21e)$$

$$\mathbf{A}_{24} = -\mathbf{T}_3 \mathbf{K}_1 (\mathbf{P}_3 a_3) \quad (21f)$$

$$\mathbf{A}_{31} = \mathbf{T}_3^T \mathbf{P}_2 [\mathbf{I}_0 (\mathbf{P}_2 a_3) \mathbf{M}_3 - \mathbf{K}_0 (\mathbf{P}_2 a_3) \mathbf{M}_1] \quad (21g)$$

$$\mathbf{A}_{32} = \mathbf{T}_3^T \mathbf{P}_2 [\mathbf{I}_0 (\mathbf{P}_2 a_3) \mathbf{M}_4 - \mathbf{K}_0 (\mathbf{P}_2 a_3) \mathbf{M}_2] \quad (21h)$$

$$\mathbf{A}_{33} = -\mathbf{P}_3 \mathbf{I}_0 (\mathbf{P}_3 a_3) \quad (21i)$$

$$\mathbf{A}_{34} = \mathbf{P}_3 \mathbf{K}_0 (\mathbf{P}_3 a_3) \quad (21j)$$

$$\mathbf{A}_{43} = \mathbf{U}_2 \mathbf{I}_1 (\mathbf{P}_3 a_4) + \mathbf{P}_3 \mathbf{I}_0 (\mathbf{P}_3 a_4) \quad (21k)$$

$$\mathbf{A}_{44} = \mathbf{U}_2 \mathbf{K}_1 (\mathbf{P}_3 a_4) - \mathbf{P}_3 \mathbf{K}_0 (\mathbf{P}_3 a_4) \quad (21l)$$

$$\mathbf{E} = \mathbf{T}_4^T \boldsymbol{\alpha} [\mathbf{I}_0 (\boldsymbol{\alpha} a_4) + \mathbf{K}_0 (\boldsymbol{\alpha} a_4) \mathbf{K}_1^{-1} (\boldsymbol{\alpha} a_4) \mathbf{I}_1 (\boldsymbol{\alpha} a_4)] \mathbf{C}_1^{(e)} \quad (21m)$$

with

$$\mathbf{U}_1 = \frac{2}{b} \mathbf{T}_1^T \boldsymbol{\alpha} \mathbf{I}_0 (\boldsymbol{\alpha} a_1) \mathbf{I}_1^{-1} (\boldsymbol{\alpha} a_1) \mathbf{T}_1 \quad (22a)$$

$$\mathbf{U}_2 = \frac{2}{b} \mathbf{T}_4^T \boldsymbol{\alpha} \mathbf{K}_0 (\boldsymbol{\alpha} a_4) \mathbf{K}_1^{-1} (\boldsymbol{\alpha} a_4) \mathbf{T}_4 \quad (22b)$$

$$\mathbf{M}_1 = a_2 [\mathbf{X}_1 \mathbf{I}_1 (\mathbf{P}_1 a_2) - \mathbf{X}_2 \mathbf{I}_0 (\mathbf{P}_1 a_2)] \quad (22c)$$

$$\mathbf{M}_2 = a_2 [\mathbf{X}_1 \mathbf{K}_1 (\mathbf{P}_1 a_2) + \mathbf{X}_2 \mathbf{K}_0 (\mathbf{P}_1 a_2)] \quad (22d)$$

$$\mathbf{M}_3 = a_2 [\mathbf{X}_3 \mathbf{I}_1 (\mathbf{P}_1 a_2) + \mathbf{X}_4 \mathbf{I}_0 (\mathbf{P}_1 a_2)] \quad (22e)$$

$$\mathbf{M}_4 = a_2 [\mathbf{X}_3 \mathbf{K}_1 (\mathbf{P}_1 a_2) - \mathbf{X}_4 \mathbf{K}_0 (\mathbf{P}_1 a_2)] \quad (22f)$$

$$\mathbf{X}_1 = \mathbf{P}_2 \mathbf{I}_0 (\mathbf{P}_2 a_2) \mathbf{T}_2^{-1} \quad (22g)$$

$$\mathbf{X}_2 = \mathbf{I}_1 (\mathbf{P}_2 a_2) \mathbf{T}_2^T \mathbf{P}_1 \quad (22h)$$

$$\mathbf{X}_3 = \mathbf{P}_2 \mathbf{K}_0 (\mathbf{P}_2 a_2) \mathbf{T}_2^{-1} \quad (22i)$$

$$\mathbf{X}_4 = \mathbf{K}_1 (\mathbf{P}_2 a_2) \mathbf{T}_2^T \mathbf{P}_1 \quad (22j)$$

Solving equation (20) will give the coefficients $\mathbf{C}_2$, $\mathbf{D}_2$, $\mathbf{C}_4$ and $\mathbf{D}_4$, and other coefficients can be found by

$$\mathbf{C}_1 = \frac{2}{b} \mathbf{I}_1^{-1} (\alpha a_1) \mathbf{T}_1 [\mathbf{I}_1 (\mathbf{P}_1 a_1) \mathbf{C}_2 + \mathbf{K}_1 (\mathbf{P}_1 a_1) \mathbf{D}_2] \quad (23)$$

$$\mathbf{C}_3 = \mathbf{M}_3 \mathbf{C}_2 + \mathbf{M}_4 \mathbf{D}_2 \quad (24)$$

$$\mathbf{D}_3 = \mathbf{M}_1 \mathbf{C}_2 + \mathbf{M}_2 \mathbf{D}_2 \quad (25)$$

The coefficient required for calculation of ΔZ is

$$\mathbf{D}^{(s)} = \mathbf{K}_1^{-1} (\alpha a_4) \left\{ -\mathbf{I}_1 (\alpha a_4) \mathbf{C}_1^{(e)} + \frac{2}{b} \mathbf{T}_4 [\mathbf{I}_1 (\mathbf{P}_3 a_4) \mathbf{C}_4 + \mathbf{K}_1 (\mathbf{P}_3 a_4) \mathbf{D}_4] \right\} \quad (26)$$

Accordingly, the coil impedance variation is given by

$$\begin{aligned} \Delta Z &= \frac{i\omega}{I^2} \int_V \mathbf{A}^{(s)} \cdot \mathbf{J} dV \\ &= \frac{\pi^2 i\omega N^2}{(r_2 - r_1)^2 (z_2 - z_1)^2} \sum_{n=1}^{\infty} \frac{\cos(\alpha_n z_1) - \cos(\alpha_n z_2)}{\alpha_n^3} [\chi(\alpha_n r_2) - \chi(\alpha_n r_1)] d_n^{(s)} \end{aligned} \quad (27)$$

where the current density $\mathbf{J}$ has been omitted (letting $\mathbf{J}=1$) to simplify the expression.

3. Eigenfunctions and the Associated Integrals of the Multi-Subdomain Regions

The conventional TREE method is limited to the two-subdomain problems, apart from certain three-subdomain problems which can be split in the even and odd parts. In general, it is impossible to evade the numerical overflow of the symbolic eigenfunctions of Regions 2, 3 and 4. Accordingly, the approach of [23] will be applied to cope with $\mathbf{F}(z)$, $\mathbf{G}(z)$ and $\mathbf{H}(z)$. In accord with [18, 19], the eigenvalues of (11a) can be obtained by the solution of a generalized eigenvalue equation

$$\mathbf{K} \mathbf{U}_i = p_{1,i}^2 \mathbf{W} \mathbf{U}_i \quad (28)$$

where $\mathbf{K}$ is the stiffness matrix with the elements

$$\mathbf{K}_{mn} = \int_0^b \frac{1}{\mu_r^{(1)}(z)} \left[\frac{d\varphi_m(z)}{dz} \frac{d\varphi_n(z)}{dz} + k_1^2(z) \varphi_m(z) \varphi_n(z) \right] dz \quad (29)$$

and $\mathbf{W}$ is the damping matrix of the elements

$$\mathbf{W}_{mn} = \int_0^b \frac{\varphi_m(z) \varphi_n(z)}{\mu_r^{(1)}(z)} dz \quad (30)$$

where φ_m and φ_n are FEM functions consisting of the Lagrange polynomials defined on the reference interval $-1 \leq \xi \leq 1$ (the shape functions).

Sparse matrix $\mathbf{K}$ will be generated from the FEM basis. Hence, equation (28) can be solved by the efficient algorithm such as Arnoldi iteration [27]. The solution provides both the eigenvalues $p_{1,i}$ and the eigenvectors $\mathbf{U}_i$, which are the discrete eigenfunctions $f_i(z)$. Moreover, denoting

$$\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \dots]^T \quad (31)$$

and by virtue of the vector normalization

$$\mathbf{U}' = \frac{\mathbf{U}}{\sqrt{\text{diag}(\mathbf{U}\mathbf{W}\mathbf{U}^T)}} \quad (32)$$

the eigenfunction normalization

$$\int_0^b \frac{1}{\mu_r^{(1)}} f_i^2(z) dz = 1 \quad (33)$$

can be established automatically. Equations (32) and (33) can be validated by inspecting the diagonal entries of $\mathbf{U}\mathbf{W}\mathbf{U}^T$, and taking equation (30) into account. Consequently, the orthonormality of (18b)-(18d) can be established.

The requirement of the accurate and efficient algorithm leads to the choice of high order Lagrange polynomials for the FEM basis. Here, we choose the cubic Lagrange polynomials

$$\begin{cases} N_0(\xi) = -\frac{1}{16}(\xi-1)(3\xi-1)(3\xi+1) \\ N_1(\xi) = \frac{9}{16}(\xi-1)(\xi+1)(3\xi-1) \\ N_2(\xi) = -\frac{9}{16}(\xi-1)(\xi+1)(3\xi+1) \\ N_3(\xi) = \frac{1}{16}(\xi+1)(3\xi-1)(3\xi+1) \end{cases} \quad (34)$$

The cubic interpolation of the eigenfunction is

$$f_i(z) = \sum_{e=0}^3 u'_{l+e} N_e(z) \quad (35)$$

where u'_{l+e} are successive 4 entries of $\mathbf{U}'_i$, $N_e(z)$ is obtained by (34) with the change of variable

$$\xi = \frac{2z - z_a - z_b}{z_b - z_a} \quad (36)$$

where z_a and z_b are the mesh nodes corresponding to the reference interval. The numerical overflow of $f_i(z)$ is eliminated by this procedure. They are consequently well adapted for the subsequent integral computation. Furthermore, it appears to be very effective to evaluate directly the integrals (14)-(17) with the Clenshaw-Curtis quadrature rule, which is quoted here for completeness [28-30]

$$\int_{-1}^1 f(x) dx = \sum_{k=0}^n w_k f(x_k) \quad (37)$$

where the weights w_k are given by

$$w_k = \frac{g_k}{n} \left(1 - \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{b_j}{4j^2 - 1} \cos(2jk\pi/n) \right) \quad (38)$$

and the quadrature nodes are

$$x_k = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, 1, \dots, n \quad (39)$$

with

$$g_k = \begin{cases} 1, & k = 0, n \\ 2, & \text{otherwise} \end{cases} \quad (40)$$

$$b_j = \begin{cases} 1, & j = \frac{1}{2}n \\ 2, & \text{otherwise} \end{cases} \quad (41)$$

It follows from equations (37)-(41) that the matrix elements of $\mathbf{T}_1$ can be computed by

$$\begin{aligned}
\mathbf{T}_{1,ij} &= \int_0^b \sin(\alpha_i z) f_j(z) dz \\
&= \frac{b}{2} \int_{-1}^1 \sin\left[\frac{b}{2} \alpha_i (1+x)\right] f_j\left[\frac{b}{2}(1+x)\right] dx \\
&= \frac{b}{2} \sum_{k=0}^n w_k \sin(\alpha_i z_k) f_j(z_k)
\end{aligned} \tag{42}$$

where

$$z_k = \frac{b}{2}(1+x_k) \tag{43}$$

The matrix elements of $\mathbf{T}_2$ are likewise given by

$$\begin{aligned}
\mathbf{T}_{2,ij} &= \int_0^b \frac{1}{\mu_r} f_i(z) g_j(z) dz \\
&= \int_0^{b_1} f_i(z) g_j(z) dz + \frac{1}{\mu_r} \int_{b_1}^{b_3} f_i(z) g_j(z) dz + \int_{b_3}^b f_i(z) g_j(z) dz \\
&= \frac{b_1}{2} \sum_{k=0}^n w_k f_i(z_k^{(1)}) g_j(z_k^{(1)}) + \frac{b_3-b_1}{2} \sum_{k=0}^n w_k f_i(z_k^{(2)}) g_j(z_k^{(2)}) + \frac{b-b_3}{2} \sum_{k=0}^n w_k f_i(z_k^{(3)}) g_j(z_k^{(3)})
\end{aligned} \tag{44}$$

where

$$z_k^{(1)} = \frac{b_1}{2}(1+x_k), z_k^{(2)} = \frac{b_3+b_1}{2} + \frac{b_3-b_1}{2} x_k, z_k^{(3)} = \frac{b+b_3}{2} + \frac{b-b_3}{2} x_k \tag{45}$$

The same analysis is also applicable to the matrix elements of $\mathbf{T}_3$ and $\mathbf{T}_4$.

4. Numerical Validation

The proposed method will be verified with the parameters of the metal tube adapter and the induction coil given in Tables 1, 2 and 3. The nonmagnetic alloy UNS (Unified Numbering System) C96400 (70-30 Copper-Nickel), and the magnetic stainless steels S31600 (austenitic) and S32760 (super duplex) [31] are used for the numerical validation. The coil impedance variations are calculated and plotted for these metal materials, with different coil positions. The TREE results are compared with those from the FEM simulation of Comsol Multiphysics®, shown in Figure 4, where the theoretical and FEM data are denoted by solid lines and circles, respectively. The reactance of the isolated induction coil is $X_0 = \omega L_0$, with $L_0 = 4.104132 \text{ mH}$, which can be found by the method such as in [32, 33].

Table 1. Metals used for the tube adapter.

Metal(UNS)	Conductivity σ (MS/m)	Relative Permeability μ_r
C96400	2.9	1
S31600	1.33	1.02
S32760	1.25	29

Table 2. Geometry of the metal tube adapter.

Parameter	Parameter
$a_1(\text{mm})$	5
$a_2(\text{mm})$	8
$a_3(\text{mm})$	11
$a_4(\text{mm})$	14
$b(\text{mm})$	100
$b_1(\text{mm})$	40
$b_2(\text{mm})$	48
$b_3(\text{mm})$	51
$b_4(\text{mm})$	59

Table 3. Parameters of the induction coil.

Parameter	
Inner radius r_1 (mm)	5
Outer radius r_2 (mm)	8
Axial length z_2-z_1 (mm)	11
Number of turns	14

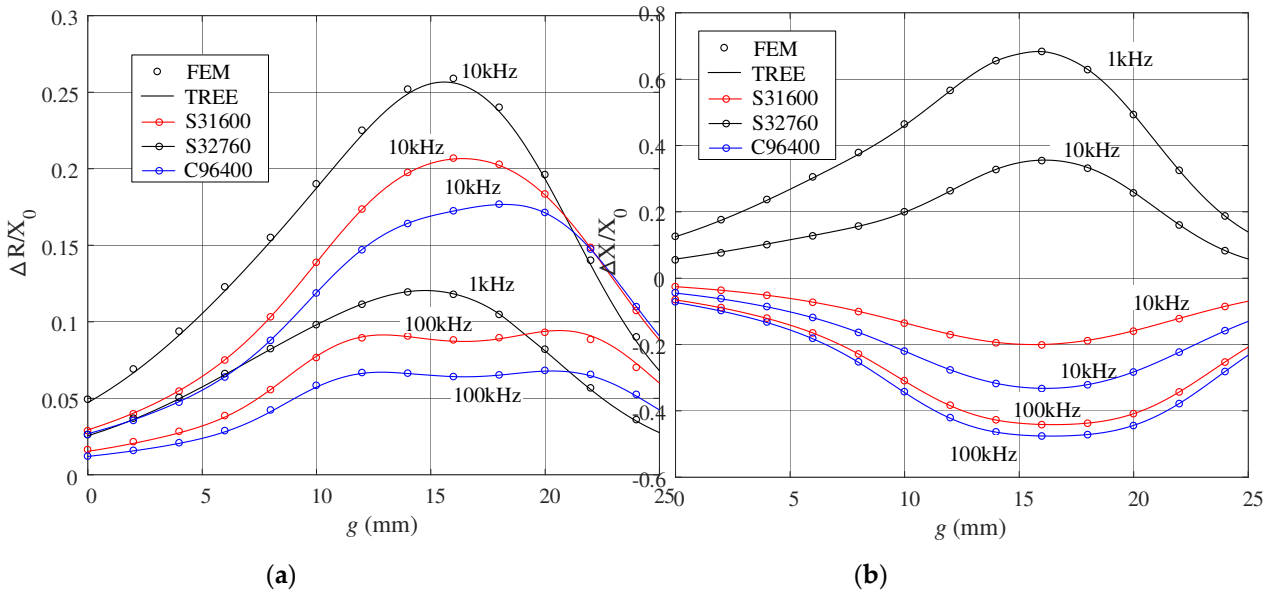


Figure 4. Normalized impedance variations with the abscissa representing the parameter g . (a) The resistance variation. (b) The reactance variation.

Further calculations are carried out for the coil impedance variation with respect to the frequencies. For the alloys of lower μ_r (C96400 and S31600), the calculation frequency ranges from 1kHz to 100kHz, for higher μ_r (S32760) the frequency interval [100Hz, 10kHz] is chosen. The results are shown in Figure 5 and Figure 6, where the TREE data are plotted by solid lines, in connection with the circles representing the data of FEM simulation. Other parameters are referred to the Table 2 and 3.

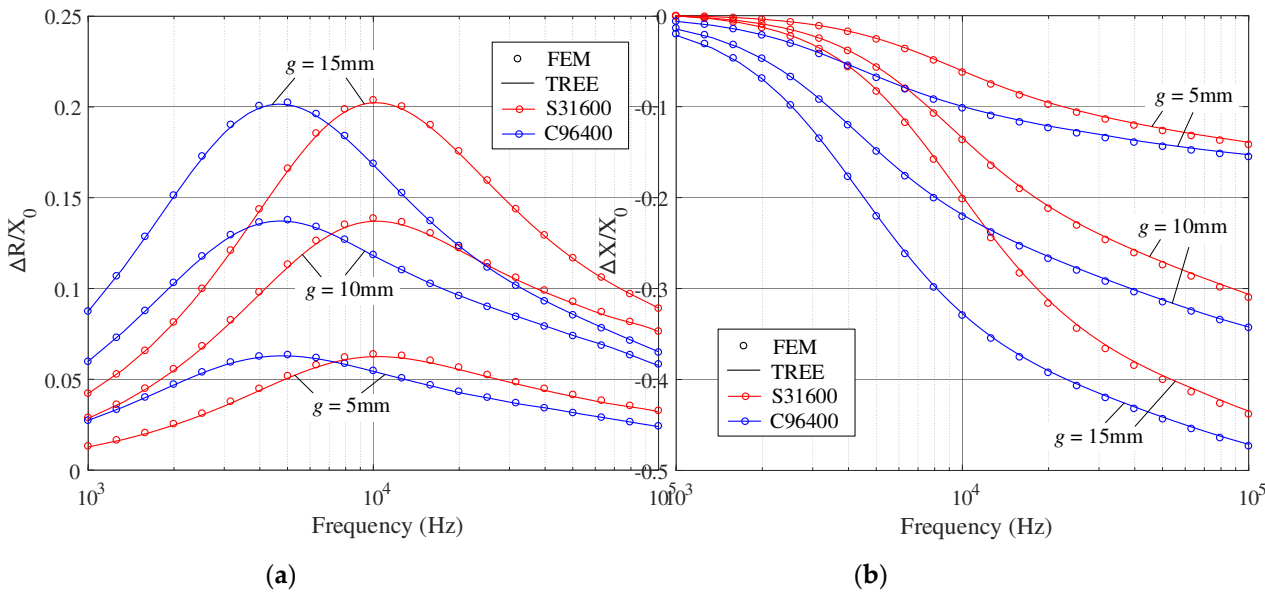


Figure 5. Normalized impedance variations with the abscissa representing the frequency. The alloys are C96400 and S31600. (a) The resistance variation. (b) The reactance variation.

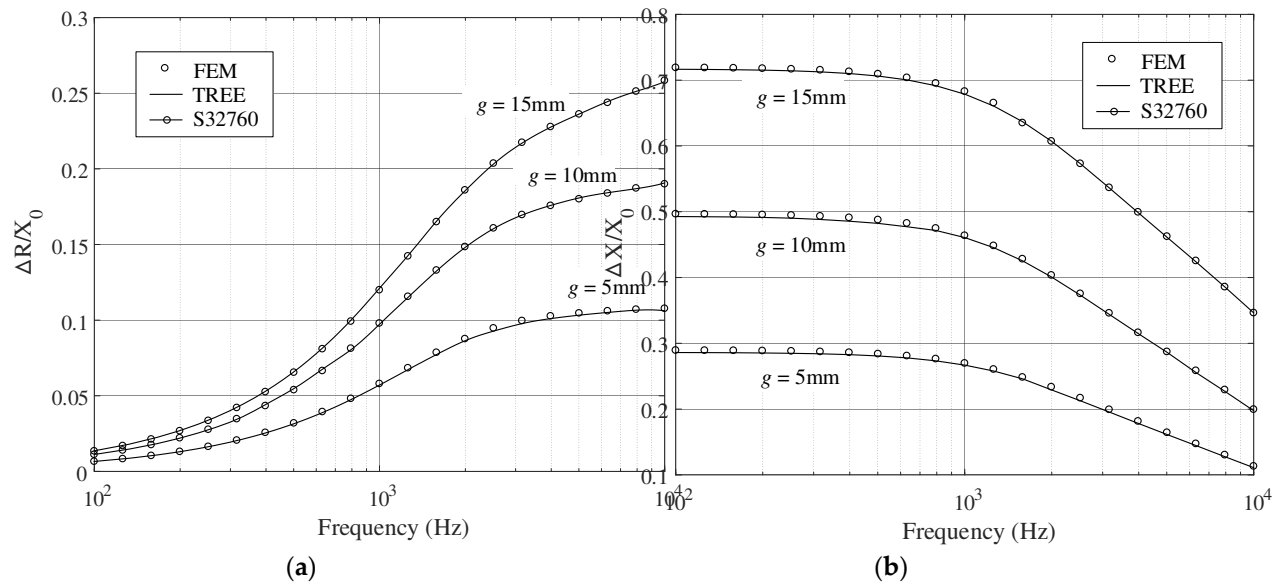


Figure 6. Normalized impedance variations with the abscissa representing the frequency. The alloy is S32760. (a) The resistance variation. (b) The reactance variation.

Very good agreement is obtained between the TREE and FEM results in the numerical comparisons. The calculations were implemented on a personal computer of a 4.2 GHz processor (Intel® Core i7-7700K) and 16 GB RAM. Additional algorithm details are shown in Table 4, where the frequencies, summation terms (matrix size), mesh elements and quadrature nodes used in the computation are listed. The execution time of the eigenvalue and eigenfunction computation, and the total execution time of TREE evaluation are also provided. No more than 1.5 seconds (including the time consumed by the calculation of eigenvalues and eigenfunctions) are consumed for a TREE evaluation. The satisfactory algorithm efficiency is in evidence.

Table 4. Computation configuration and execution time of TREE method.

Metal (UNS)	Frequency	Summation terms	Quadrature nodes	Mesh elements	Execution time of eigenvalue and eigenfunction computation	Total execution time
S31600	10kHz	30	80	510	0.19s	0.55s
	100kHz	40	80	510	0.26s	0.73s
S32760	1kHz	55	80	510	0.36s	1.00s
	10kHz	70	90	510	0.54s	1.30s
C96400	10kHz	30	80	510	0.19s	0.56s
	100kHz	50	80	510	0.34s	0.90s

5. Conclusions

The interaction of an eddy-current coil with a metal tube adapter has been investigated by the TREE method. The numerical overflow for symbolic eigenfunctions of air-metal multi-subdomain regions has been removed by the normalization of the eigenvectors, and satisfactory computational speed is achieved by Clenshaw-Curtis quadrature rule applied to the integrals associated with the numerical eigenfunctions. The calculation accuracy has been verified by the numerical comparisons, and the efficiency of our approach has also been confirmed. Considerable potential has been shown for the developing of new analytical models with the aid of the proposed approach.

Author Contributions: Conceptualization, Y. L. and X.Y.; methodology, Y.L.; software, Y.L. and X.Y.; data curation, X.Y.; writing—original draft preparation, Y.L.; writing—review and editing, Y.L. and X.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Data sharing not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dodd, C. V.; Deeds, W. E. Analytical solutions to eddy current probe-coil problems. *J. Appl. Phys.* 1968, 39, 2829-2838.
2. Hannakam, L.; Tepe, R. Feldschwächung durch leitende Rechteckzylinder im Luftspalt. *Archiv für Elektrotechnik.* 1979, 61, 137-144.
3. Theodoulidis, T.; Kriezis, E. Eddy Current Canonical Problems (With Applications to Nondestructive Evaluation). Forsyth, GA: Tech Science Press, 2006.
4. Theodoulidis, T. Model of ferrite-cored probes for eddy current nondestructive evaluation. *J. Appl. Phys.* 2003, 93, 3071-3078.
5. Theodoulidis, T.; Bowler, J. Eddy current coil interaction with a right-angled conductive wedge. *Proc. R. Soc. A.* 2005, 461, 3123-3139.
6. Bowler, J.; Theodoulidis, T. Eddy currents induced in a conducting rod of finite length by a coaxial encircling coil. *J. Phys. D: Appl. Phys.* 2005, 38, 2861-2868.
7. Theodoulidis, T.; Kriezis, E. Series expansions in eddy current nondestructive evaluation models. *J. Mater. Process. Technol.* 2005, 161, 343-347.
8. Theodoulidis, T.; Bowler, J. Eddy-current interaction of a long coil with a slot in a conductive plate. *IEEE Trans. Magn.* 2005, 41, 1238-1247.
9. Bowler, J.; Theodoulidis, T. Coil impedance variation due to induced current at the edge of a conductive plate. *J. Phys. D: Appl. Phys.* 2006, 39, 2862-2868.
10. Hannakam, L.; Kost, A. Leitender Rechteckkeil im Felde einer Doppelleitung. *Archiv für Elektrotechnik.* 1982, 65, 363-368.
11. Hannakam, L.; Orglmeister, R. Induzierte Wirbelstrombelag an ausgeprägten Massivpolen hoher Permeabilität bei Wanderfelderregung. *Archiv für Elektrotechnik.* 1984, 67, 49-55.
12. Filtz, M.; Nethe, A. Bemerkung zur Lösung des dreidimensionalen Wirbelstromproblems in Kreiszylindern endlicher Länge, *Archiv für Elektrotechnik.* 1993, 76, 195-200.
13. Filtz, M.; Nethe, A. Anregung dreidimensionaler Wirbelströme in massiven Kreiszylindern endlicher Länge durch ein homogenes magnetisches Wechselfeld beliebiger Ausrichtung. *Archiv für Elektrotechnik.* 1990, 73, 227-237.
14. Theodoulidis, T.; Bowler, J. Interaction of an eddy-current coil with a right-angled conductive wedge. *IEEE Trans. Magn.* 2010, 46, 1034-1042.
15. Delves, L.; Lyness, J. A numerical method for locating the zeros of an analytic function. *Math. Comput.* 1967, 21, 543-560.
16. Tytko, G.; Dawidowski, L. Locating complex eigenvalues for analytical eddy-current models used to detect flaws. *COMPEL.* 2019, 36 1800-1809.
17. Vasic, D.; Ambrus, D.; Bilas, V. Computation of the eigenvalues for bounded domain. *IEEE Trans. Magn.* 2016, 52, 7004310.
18. Yang, X.; Luo, Y.; Kyrgiazoglou, A.; Tytko, G.; Theodoulidis, T. An analytical model of an eddy-current coil near the edge of a conductive plate. *IET Electr. Power Appl.* 2022, 16, 1017-1029.
19. Theodoulidis, T.; Skarlatos, A.; Tytko, G. Computation of eigenvalues and eigenfunctions in the solution of eddy current problems, *Sensors.* 2023, 23, 3055.
20. Stahlmann, H. D. Der Differentialtransformator als induktiver Stellungsmelder. *Archiv für Elektrotechnik.* 1983, 66, 277-281.
21. Sun, H.; Bowler, J.; Theodoulidis, T. Eddy currents induced in a finite length layered rod by a coaxial coil, *IEEE Trans. Magn.* 2005, 41, 2455-2461.
22. Skarlatos, A.; Theodoulidis, T. Calculation of the eddy-current flow around a cylindrical through-hole in a finite-thickness plate. *IEEE Trans. Magn.* 2015, 51, 6201507.
23. Luo, Y.; Yang, X. Impedance variation of an induction coil above a metal disk with a borehole and an annular slot. *IEEE Sens. J.*, in preparation.
24. Luo, Y. Field and inductance calculations for coaxial circular coils with magnetic cores of finite length and constant permeability. *IET Electr. Power Appl.* 2017, 11, 1254-1264.

25. Courant, R.; Hilbert, D. *Methoden der Mathematischen Physik I*. New York: Springer-Verlag, 1968.
26. Abramowitz, M.; Stegun, A. *Handbook of Mathematical Functions*. New York: Dover, 1972.
27. Arnoldi, W. E. The principle of minimized iterations in the solution of the matrix eigenvalue problem, *Q. Appl. Math.* 1951, 9, 17-29.
28. Olver, F.; Lozier, D.; Boisvert, R.; Clark, C. *NIST Handbook of Mathematical Functions*. New York: Cambridge University Press, 2010.
29. Clenshaw, C.; Curtis, A. A method for numerical integration on a automatic computer. *Numer. Math. (Heidelb.)* 1960, 2, 197-205.
30. Hildebrand, F. B. *Introduction to Numerical Analysis*. New York: Dover Publications, 1987.
31. Boniardi, M.; Casaroli, A. Rostfreie Edelstähle. [Online]. Available: http://www.fafe.com/files/pdf/libri_articoli/de/1-Rostfreie_Edelstahle.pdf.
32. Luo, Y.; Chen, B. Improvement of self-inductance calculations for circular coils of rectangular cross section. *IEEE Trans. Magn.* 2013, 49, 1249-1255.
33. Luo, Y.; Wang, X.; Zhou, X. Inductance calculations for circular coils with rectangular cross section and parallel axes using inverse Mellin transform and generalized hypergeometric functions. *IEEE Trans. Power Electron.* 2017, 32, 1367-1374.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.