In[ ]:=

Supplementary Information

(Mathematica v13.2.0 code of TraditionalForm)

In[ ]:=

Biased Random Process of Randomly Moving Particles with Fixed Mean and Curl

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NOTE:
1. The "Euclid Math One" regular and bold fonts are needed to display the contents correctly in this Notebook.
2. If there is no special case, the Mathematica code starts with gray "*In*[∘]:=" and is bold by default according to Mathematica's rules.

Part 1. Angular Speed Distribution of Random Spin Particles

In[∘]:=

```
Clear["Global`*"];
X = TransformedDistribution[N1/Sqrt[N1^2 + N2^2 + N3^2],
    {N1, N2, N3} ≈ ProductDistribution[{NormalDistribution[], 3}]];

PDF[X, x]
```

Out[∘]=

$$\begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{True} \end{cases}$$

The expression of $\|\boldsymbol{\Omega}'\|$ in the main text is calculated as follows. NOTE: $\|V_S\|$ in the main text is substituted by **Vs** here.

In[∘]:=

```
X = Vs Sin[ArcCos[Θ]] Cos[H];
Y = Vs Sin[ArcCos[Θ]] Sin[H];
Z = 0;
FullSimplify[Sqrt[X^2 + Y^2 + Z^2], Assumptions → Vs > 0]
```

Out[∘]=

$$\sqrt{1-\Theta^2}\ \text{Vs}$$

To obtain the probability density of the aforementioned function, we approach the problem step by step. Initially, we compute the probability density of $\sqrt{1-\Theta^2}$.

In[∘]:=

```
FullSimplify[PDF[TransformedDistribution[Sqrt[1 - Θ^2], Θ ≈ UniformDistribution[{-1, 1}]], x]]
```

Out[∘]=

$$\begin{cases} \frac{x}{\sqrt{1-x^2}} & 0 < x < 1 \\ 0 & x > 1 \vee x \le 0 \\ \text{Indeterminate} & \text{True} \end{cases}$$

Then, we calculate the probability density of $\|V_S\|\ \sqrt{1-\Theta^2}$, that is, $\|\omega'\|\ (x)$. NOTE: $\|\omega'\|\ (x)$ in the main text is substituted by ω' as the probability density expression here.

In[∘]:=

```
ω' = FullSimplify[PDF[TransformedDistribution[Vs fΘ,
    {Vs ≈ MaxwellDistribution[1/2 Sqrt[π/2] c], fΘ ≈ ProbabilityDistribution[x/Sqrt[1 - x^2], {x, 0, 1}]}], x]]
```

Out[∘]=

$$\begin{cases} \frac{8\,x\,e^{-\frac{4x^2}{\pi c^2}}}{\pi c^2} & x > 0 \\ 0 & \text{True} \end{cases}$$

NOTE: $\omega_{S,X}(x)$ in the main text is substituted by ωsx here. Then, the probability density of X, which represents one of the three equivalent coordinates of the angular velocity $\boldsymbol{\Omega}_{\mathrm{S}}$ contributed by the random vector $\boldsymbol{V}_{\mathrm{S}}$, can be obtained:

In[ ]:= ωsx = FullSimplify[PDF[TransformedDistribution[F1 F2, {F1 ≈ ProbabilityDistribution[$\frac{1}{2}$, {x, −1, 1}], F2 ≈ ProbabilityDistribution[$\frac{8\,x\,e^{-\frac{4x^2}{\pi c^2}}}{\pi c^2}$, {x, 0, +∞}]}], x], Assumptions → c > 0]

Out[ ]=

$$\begin{cases} \frac{\mathrm{erfc}\left(\frac{2x}{\sqrt{\pi}\,c}\right)}{c} & x \geq 0 \\ \frac{\mathrm{erf}\left(\frac{2x}{\sqrt{\pi}\,c}\right)+1}{c} & \text{True} \end{cases}$$

NOTE: $\omega_{\mathrm{B},X}(x,r)$ in the main text is substituted by ωbx here. Then, the probability density of the norm r of the radius $\boldsymbol{r}$ at which the starting point of the vector $\boldsymbol{V}_{\mathrm{B}}$ is located within the ball can be obtained:

In[ ]:= ωbx = PDF[TransformedDistribution[$\frac{1}{r}$ x, x ≈ ProbabilityDistribution[$\begin{cases} \frac{\mathrm{Erfc}\left[\frac{2x}{\sqrt{\pi}\,c}\right]}{c} & x \geq 0 \\ \frac{\mathrm{Erf}\left[\frac{2x}{\sqrt{\pi}\,c}\right]+1}{c} & x < 0 \end{cases}$, {x, −∞, ∞}], Assumptions → r > 0 ∧ c > 0], x]

Out[ ]=

$$\begin{cases} \frac{r\left(\mathrm{erf}\left(\frac{2rx}{\sqrt{\pi}\,c}\right)+1\right)}{c} & r\,x < 0 \\ \frac{r\,\mathrm{erfc}\left(\frac{2rx}{\sqrt{\pi}\,c}\right)}{c} & \text{True} \end{cases}$$

The distribution function of the contribution of $\boldsymbol{V}_{\mathrm{B}}$ to the equivalent coordinate $\boldsymbol{\Omega}_{\mathrm{B},X}$ of $\boldsymbol{\Omega}_{\mathrm{B}}$ is calculated as follows:

In[ ]:= FullSimplify[CDF[ProbabilityDistribution[$\begin{cases} \frac{r\left(\mathrm{Erf}\left(\frac{2rx}{\sqrt{\pi}\,c}\right)+1\right)}{c} & x < 0 \\ \frac{r\,\mathrm{Erfc}\left(\frac{2rx}{\sqrt{\pi}\,c}\right)}{c} & x \geq 0 \end{cases}$, {x, −∞, ∞}, Assumptions → r > 0 ∧ c > 0], x]]

Out[ ]=

$$\begin{cases} \frac{1}{2}\,e^{-\frac{4r^2x^2}{\pi c^2}} + \frac{r\,x\left(\mathrm{erf}\left(\frac{2rx}{\sqrt{\pi}\,c}\right)+1\right)}{c} & x \leq 0 \\ -\frac{1}{2}\,e^{-\frac{4r^2x^2}{\pi c^2}} + \frac{r\,x\,\mathrm{erfc}\left(\frac{2rx}{\sqrt{\pi}\,c}\right)}{c} + 1 & \text{True} \end{cases}$$

We integrate the function in the whole unit ball:

In[∘]:= FullSimplify$\left[\int_0^1 4\pi r^2 \left(\begin{cases} \frac{1}{2} e^{-\frac{4r^2x^2}{\pi c^2}} + \frac{r x \left(\text{Erf}\left(\frac{2 r x}{\sqrt{\pi} c}\right)+1\right)}{c} & x \le 0 \\ -\frac{1}{2} e^{-\frac{4r^2x^2}{\pi c^2}} + \frac{r x\, \text{Erfc}\left(\frac{2 r x}{\sqrt{\pi} c}\right)}{c} + 1 & x > 0 \end{cases}\right) dr\right]$

Out[∘]=

$$\begin{cases} \frac{1}{64}\pi\left(\left(\frac{\pi^2 c^3}{x^3}+\frac{64x}{c}\right)\text{erf}\left(\frac{2x}{\sqrt{\pi}c}\right)+e^{-\frac{4x^2}{\pi c^2}}\left(32-\frac{4\pi c^2}{x^2}\right)+\frac{64x}{c}\right) & x \le 0 \\ \frac{1}{192}\pi\left(\left(\frac{3\pi^2c^3}{x^3}+\frac{192x}{c}\right)\text{erfc}\left(\frac{2x}{\sqrt{\pi}c}\right)-\frac{3\pi^2c^3}{x^3}+e^{-\frac{4x^2}{\pi c^2}}\left(\frac{12\pi c^2}{x^2}-96\right)+256\right) & \text{True} \end{cases}$$

We find the first derivative of the above result with respect to x:

In[∘]:= FullSimplify$\left[D\left[\begin{cases} \frac{1}{64}\pi\left(\left(\frac{\pi^2 c^3}{x^3}+\frac{64x}{c}\right)\text{Erf}\left(\frac{2x}{\sqrt{\pi}c}\right)+e^{-\frac{4x^2}{\pi c^2}}\left(32-\frac{4\pi c^2}{x^2}\right)+\frac{64x}{c}\right) & x \le 0 \\ \frac{1}{192}\pi\left(\left(\frac{3\pi^2c^3}{x^3}+\frac{192x}{c}\right)\text{Erfc}\left(\frac{2x}{\sqrt{\pi}c}\right)-\frac{3\pi^2c^3}{x^3}+e^{-\frac{4x^2}{\pi c^2}}\left(\frac{12\pi c^2}{x^2}-96\right)+256\right) & x > 0 \end{cases}, x\right]\right]$

Out[∘]=

$$\begin{cases} \frac{\pi\left(\left(64x^4-3\pi^2c^4\right)\text{erf}\left(\frac{2x}{\sqrt{\pi}c}\right)+4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{64cx^4} & x < 0 \\ \frac{\pi\left(\left(3\pi^2c^4-64x^4\right)\text{erf}\left(\frac{2x}{\sqrt{\pi}c}\right)-4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{64cx^4} & x > 0 \\ \text{Indeterminate} & \text{True} \end{cases}$$

We integrate the function in the whole interval (-∞, ∞).

In[∘]:= Integrate$\left[\begin{cases} \frac{\pi\left(\left(64x^4-3\pi^2c^4\right)\text{Erf}\left[\frac{2x}{\sqrt{\pi}c}\right]+4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{64cx^4} & x < 0 \\ \frac{\pi\left(\left(3\pi^2c^4-64x^4\right)\text{Erf}\left[\frac{2x}{\sqrt{\pi}c}\right]-4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{64cx^4} & x > 0 \end{cases}, \{x, -\infty, \infty\}, \text{Assumptions} \to c > 0\right]$

Out[∘]=

$$\frac{4\pi}{3}$$

The above function is normalized according to the integration results:

In[∘]:= FullSimplify$\left[\frac{1}{\frac{4\pi}{3}}\begin{cases} \frac{\pi\left(\left(64x^4-3\pi^2c^4\right)\text{Erf}\left[\frac{2x}{\sqrt{\pi}c}\right]+4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{64cx^4} & x < 0 \\ \frac{\pi\left(\left(3\pi^2c^4-64x^4\right)\text{Erf}\left[\frac{2x}{\sqrt{\pi}c}\right]-4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{64cx^4} & x > 0 \end{cases}\right]$

Out[∘]=

$$\begin{cases} \frac{3\left(\left(64x^4-3\pi^2c^4\right)\text{erf}\left(\frac{2x}{\sqrt{\pi}c}\right)+4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{256cx^4} & x < 0 \\ \frac{3\left(\left(3\pi^2c^4-64x^4\right)\text{erf}\left(\frac{2x}{\sqrt{\pi}c}\right)-4cxe^{-\frac{4x^2}{\pi c^2}}\left(3\pi c^2+8x^2\right)+64x^4\right)}{256cx^4} & x > 0 \end{cases}$$

Further more, when the radius of the ball is R, the above situation scales to $\frac{\Omega_{B,X}(x)}{R}$. Accordingly, the probability density of the contribution of the random vector $\boldsymbol{V}$ to the single equivalent coordinate X of angular velocity Ω is (Note: $\omega_X(x)$ in the main text is substituted by ωx here):

In[∘]:=
```
ωxd = TransformedDistribution[x/R,
  x ≈ ProbabilityDistribution[Piecewise[{
     {3 ((64 x^4 - 3 π^2 c^4) Erf[2 x/(Sqrt[π] c)] + 4 c x E^(-4 x^2/(π c^2)) (3 π c^2 + 8 x^2) + 64 x^4)/(256 c x^4), x < 0},
     {3 ((3 π^2 c^4 - 64 x^4) Erf[2 x/(Sqrt[π] c)] - 4 c x E^(-4 x^2/(π c^2)) (3 π c^2 + 8 x^2) + 64 x^4)/(256 c x^4), x > 0}}], {x, -∞, ∞}],
  Assumptions → R > 0 ∧ c > 0];
ωx = FullSimplify[PDF[ωxd, x]]
```

Out[∘]=

$$\begin{cases} \frac{3}{256}\left(\left(\frac{64 R}{c}-\frac{3\pi^2 c^3}{R^3 x^4}\right)\operatorname{erf}\left(\frac{2 R x}{\sqrt{\pi}\, c}\right)+\frac{4 e^{-\frac{4 R^2 x^2}{\pi c^2}}\left(3\pi c^2+8 R^2 x^2\right)}{R^2 x^3}+\frac{64 R}{c}\right) & R x<0 \\ \frac{3}{256}\left(\left(\frac{3\pi^2 c^3}{R^3 x^4}-\frac{64 R}{c}\right)\operatorname{erf}\left(\frac{2 R x}{\sqrt{\pi}\, c}\right)-\frac{4 e^{-\frac{4 R^2 x^2}{\pi c^2}}\left(3\pi c^2+8 R^2 x^2\right)}{R^2 x^3}+\frac{64 R}{c}\right) & R x>0 \end{cases}$$

The standard deviation of ωxd is:

In[∘]:=
```
StandardDeviation[ωxd]
```

Out[∘]=

$$\frac{\sqrt{\pi}\, c}{2 R}$$

This is the end of the whole proof.

Part 2. Figures Used in This Study

NOTE: To run these codes correctly, the contents in "**MyDirection = ∗∗**" in the next cell should be modified. It is similar to **MyDirection = "/Users/yourdirection/"**. Then, run it (Shift+Enter) beforehand.

```
MyDirection = ∗∗;
Protect[MyDirection];
Off[General::wrsym];

## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##

Clear["Global`∗"];
ℛ1 = Sphere[{0, 0, 0}, 1];
ℛS1 = Point[{0, 0, 0}];
A = Line[{{1/2, -Sqrt[3]/4, 0}, {1/2, -Sqrt[3]/4, 1}}];
pts1 = Solve[{x, y, z} ∈ ℛ1 && {x, y, z} ∈ A, {x, y, z}];
ℛ2 = Sphere[{1/2, -Sqrt[3]/4, z} /. Association[pts1], 6/5];
ℛ3 = Arrow[{{0, 0, 0}, {1/2, -Sqrt[3]/4, z} /. Association[pts1]}];
```

```
{u, v, w} = Normalize[{0, 0, 1} × {1/2, -√3/4, z} /. Association[pts1]];

θ = VectorAngle[{1/2, -√3/4, z} /. Association[pts1], {0, 0, 1}];

M = ( u^2 + (1 - u^2) cos(θ)        u v (1 - cos(θ)) - w sin(θ)   u w (1 - cos(θ)) + v sin(θ)   0
      u v (1 - cos(θ)) + w sin(θ)   v^2 + (1 - v^2) cos(θ)        v w (1 - cos(θ)) - u sin(θ)   0
      u w (1 - cos(θ)) - v sin(θ)   v w (1 - cos(θ)) + u sin(θ)   w^2 + (1 - w^2) cos(θ)        0
      0                             0                             0                             1 );

ℛ4 = ParametricPlot3D[Take[M.{r Cos[t], r Sin[t], 1, 1}, 3], {r, 0, 6/5},
     {t, 0, 2 Pi}, PlotStyle → {Orange, Opacity[0.4]}, PlotPoints → 300, Mesh → None];

ℛ5 = Arrow[{{1/2, -√3/4, z} /. Association[pts1], Take[M.{-1/2, 1/4, 1, 1}, 3]}];

ℛS2 = Point[{1/2, -√3/4, z} /. Association[pts1]];

A = Line[{Take[M.{-1/2, 1/4, 1, 1}, 3], Take[M.{-1/2, 1/4, 3, 1}, 3]}];
pts2 = Solve[{x, y, z} ∈ ℛ2 && {x, y, z} ∈ A, {x, y, z}];

ℛ6 = Line[{Take[M.{-1/2, 1/4, 1, 1}, 3], {x, y, z} /. Association[pts2]}];

ℛ7 = Arrow[{{1/2, -√3/4, z} /. Association[pts1], {x, y, z} /. Association[pts2]}];
ℛ8 = Point[{x, y, z} /. Association[pts2]];
ℛ9 = Line[{Take[M.{-1/2, 1/4, (20 + √5)/20, 1}, 3], Take[M.{-1/2 + 1/10, 1/4 - 1/20, (20 + √5)/20, 1}, 3]}];

ℛ10 = Line[{Take[M.{-1/2 + 1/10, 1/4 - 1/20, 1, 1}, 3], Take[M.{-1/2 + 1/10, 1/4 - 1/20, (20 + √5)/20, 1}, 3]}];
figure1 =
  Show[{Graphics3D[{{Blue, Specularity[White, 30], Opacity[0.3], ℛ1}, {PointSize[Large], Blue, ℛS1},
       {Orange, Specularity[White, 20], Opacity[0.3], ℛ2}, {PointSize[Large], Blue, ℛS2},
       {Blue, Thickness[0.0085], ℛ3}, {Thickness[0.0085], Orange, ℛ5},
       {Thickness[0.005], Dashed, Orange, ℛ6}, {Thickness[0.0085], Orange, ℛ7},
       {PointSize[Large], Orange, ℛ8}, {Thickness[0.0085], Orange, ℛ9}, {Thickness[0.0085],
         Orange, ℛ10}, {Text[Style["S", 24, FontFamily → "Arial", Blue], {0.14, 0, 0}]},
       {Text[Style["r", 24, FontFamily → "Arial", Bold, Italic, Blue], Take[M.{0.11, 0, 1/2, 1}, 3]]},
       {Text[Style["S'", 24, FontFamily → "Arial", Orange], Take[M.{0.14, 0, 1, 1}, 3]]},
       {Text[Style["D'", 24, FontFamily → "Arial", Orange], Take[M.{1/2, 0.12, 1.1, 1}, 3]]},
       {Text[Style["v", 24, FontFamily → "Arial", Bold, Italic, Orange],
          Take[M.{-1/4 + 0.14, 0.23, 1 + √3/4, 1}, 3]]}, {Text[Style["S", 13,
```

```
            FontFamily → "Arial", Orange], Take[M.{-1/4 + 0.216, 0.23, 0.98 + √3/4, 1}, 3]]},
        {Text[Style["ω'", 24, FontFamily → "Arial", Bold, Orange], Take[M.{-1/4, 0, 0.91, 1}, 3]]}},
      Method → {"SpherePoints" → 300}], ℛ4},
    Lighting → Automatic, Boxed → False, ViewPoint → {0, -∞, 0}];
Export[MyDirection <> "figure1.png", figure1, Background → None, ImageResolution → 600];
```

Figure1

Figure2

```
Clear["Global`*"];
c = 100;
u = 60;
σ = 10;
ω = 0.1;
```

$$\mathcal{P} = \text{ItoProcess}\Big[\Big\{d\,\text{x1}[t] == -\sqrt{u^2 - \omega^2\,\text{x1}[t]^2}\;dt + \frac{\sqrt{c^2-u^2}}{c}\,\sigma\,d\,\text{w1}[t],$$

$$d\,\text{x2}[t] == \omega\,\text{x1}[t]\,dt + \frac{\sqrt{c^2-u^2}}{c}\,\sigma\,d\,\text{w2}[t],\ d\,\text{x3}[t] == \frac{\sqrt{c^2-u^2}}{c}\,\sigma\,d\,\text{w3}[t]\Big\},\ \{\text{x1}[t], \text{x2}[t], \text{x3}[t]\},$$

$$\{\{\text{x1}, \text{x2}, \text{x3}\}, \{0, 0, 0\}\}, \{t, 0\}, \{\text{w1} \approx \text{WienerProcess}[], \text{w2} \approx \text{WienerProcess}[], \text{w3} \approx \text{WienerProcess}[]\}\Big];$$

```
SeedRandom[125];
sample = Table[First[Transpose[RandomFunction[𝒫, {0, 10, 0.01}]["ValueList"]ᵀ, {2, 1, 3}]], {100}];
figure2 = Graphics3D[Table[{ColorData["SolarColors"][RandomReal[]], Line@sample[[i]]}, {i, 100}],
    BoxStyle → Directive[Black, Thickness → 0.001], ViewPoint → {-3, 7, -30}];
Export[MyDirection <> "figure2.png", figure2, Background → None, ImageResolution → 600];
```

Figure2