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Article

Fractional Integro-Differential Diffusion Equations with Nonlocal Initial Conditions via the Resolvent Family

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Abstract: The fractional diffusion equations with integrals play a significant role in describing anomalous diffusion phenomena. We first construct an appropriate resolvent family, and use the convolution theorem of Laplace transform, the probability density function, Cauchy integral formula and Fubini theorem to study its related equicontinuity, strongly continuity, compactness, etc. Then, a reasonable mild solution is defined. Finally, by combining some fixed point theorems, we obtain some conclusions on the existence and uniqueness of mild solutions.

Keywords: fractional integro-differential diffusion equations; nonlocal initial conditions; resolvent family; probability density function

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1. Introduction

In this paper, we consider the existence and uniqueness of mild solutions to the following nonlinear fractional integro-differential diffusion equations with nonlocal initial conditions:

$$\begin{cases} \partial_t^\alpha u(x,t) = Au(x,t) + I_t^\beta f(x,t,u(x,t)), \ (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, \ (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) + g(u) = u_0(x), \ (x,t) \in \Omega \times (0,T], \end{cases} \tag{1}$$

where $\alpha \in (0,1)$, $\beta \in (0,\infty)$, ∂_t^α and I_t^β are the α order partial Caputo derivative and β order partial Riemann-Liouville integral with respect to t. $u_0(x) \in H^{-1}(\Omega)$, which is given in Section 2, $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, $T > 0$, the nonlinear term f and the nonlocal term g are given functions. The coefficient linear operator A is defined by:

$$Au(x,t) = \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i} \left[a_{ij}(x)\partial_{x_j} u(x,t)\right] - p(x)u(x,t), \tag{2}$$

where $a_{ij}(x)(1 \le i,j \le n)$ are some real valued functions in Ω satisfying conditions:

$$a_{ij}(x) \in L^\infty(\Omega), 1 \le i,j \le n,$$

$$\sum_{i,j=1}^n a_{ij}(x)\vartheta_i\vartheta_j \ge \varsigma|\vartheta|^2, \vartheta \in \mathbb{R}^n, \text{ a.e. } x \in \Omega,$$

with some constants $\varsigma > 0$, $p(x) \in \Omega$ is also a real valued function satisfying

$$p(x) \in L^\infty(\Omega), \ p(x) \ge p_0 > 0, \ \text{a.e. } x \in \Omega,$$

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where $L^{\infty}(\Omega)$ denotes the space composed of measurable functions defined on Ω and almost bounded everywhere.

Fractional differential equations originated in 1695, see [1–3]. As we have known that they can provide such excellent descriptive models to resolve various problems in reality, to the extent that they are applied in various fields, such as control engineering [4,5], viscoelastic materials [6], fluid mechanics [7], electrochemistry [8], the analysis of epidemic [9] and complex networks [10], and statistical mechanics [11], etc. The relevant problems for the diffusion equations have been studied by many scholars, see [12–14] and recent relevant references [15–17].

The existence results to nonlocal initial problems in Banach space was initiated by Byszewski and Lakshmikantham [18]. The motivation for these studies is that the nonlocal condition is better to describe the diffusion phenomenon than using the usual local Cauchy problem $u(0) = u_0$. For example, $g(u)$ can be given by

$$g(u) = \sum_{i=1}^{l} c_i u(\tau_i),$$

where c_i $(i = 1, 2, \ldots, l)$ are given constants and $0 < \tau_1 < \tau_2 < \cdots < \tau_l < T$. In addition, for some applications of nonlocal conditions, please refer to [19–23].

Fractional equations containing only differential terms have been studied widely. For instance, in [24], Mu et al. considered the initial-boundary value problem of fractional diffusion equations in Caputo sense:

$$\begin{cases} \partial_t^{\alpha} u(x,t) = Au(x,t) + f(x,t) & \text{in } \Omega \times (0,b), \\ u = 0 & \text{on } \partial\Omega \times (0,b), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

where f is weighted Hölder continuous. The existence, uniqueness and regularity of solutions of (3) be established in $C([0,b], H^{-1}(\Omega)) \cap C((0,b], D(-A)) \cap C^{\alpha}((0,b], H^{-1}(\Omega))$ with some assumptions. Nevertheless, a lot of practical phenomena can be depicted via appropriate models, which include differential and integral terms. Hence the appearance of integro-differential equations shows that their excellent applicability in some physical or engineering areas. In [25], Amin et al. obtained the solutions of a integro-differential equation with initial condition

$$\begin{cases} \partial_t^{\xi} w(t) = r(t) + b(t)w(t) + \int_0^t W(t,\tau)w(\tau)d\tau, & t \in [0,1], \\ w(0) = w_0, \end{cases}$$

where $\xi \in (0,1]$, W is the kernel of integral, $r(t)$ and $b(t)$ are known. This integral term has certain limitations when describing nonlocal diffusion phenomena, and the applicability of the initial condition is also relatively weak.

According to our knowledge, the mild solutions of fractional differential equations are usually represented by the Mittag-Leffler function or probability density function, see [26–32]. In [32], Zhou et al. obtained a mild solution, where

$$\lambda^{q-1}(\lambda^q I - A)^{-1} = \int_0^{\infty} e^{-\lambda t} \left[\int_0^{\infty} \psi_q(\theta) T\left(\frac{t^q}{\theta^q}\right) d\theta \right] dt, \; q \in (0,1), \tag{4}$$

$\psi_q(\theta)$ is the probability density function defined on $(0,\infty)$ and $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup generated by A. On the other hand, if $f(x,t,u(x,t)) = g(u(x,t)) = 0$, then a formal solution of (1) is

$$u(x,t) = \sum_{i=1}^{\infty} (u_0, X_i) E_{\alpha}(-\lambda_i t^{\alpha}) X_i(x).$$

Where X_i is the eigenfunction related to the eigenvalue λ_i of the corresponding eigenvalue problem, that is, $-A(X_i) = \lambda_i X_i$, $i = 1, 2, \ldots$, see further details in [29]. Obviously, using these technique directly to solve the problem (1) is quite difficult.

Based on the above, in this paper we apply the (α, β)-resolvent family to discuss the mild solutions. Resolvent families are powerful for studying solutions to fractional diffusion equations. Chen et al. [33] established the existence and controllability estimation of mild solutions for a class of evolution equations with nonlocal conditions through a resolvent family. Ponce [23] obtained properties on the behavior of mild solutions for fraction Cauchy problems by a resolvent family. Later, Chang et al. [19] proved that if the source function of a diffusion equation has vector value periodicity or almost periodicity or almost automorphism, then the diffusion equation has a mild solution through a resolvent family. Although there is also an integral of f in [19], some of the proof techniques in which are not applicable to this article due to the derivative order of u being different.

This paper is organized as follows. In Section 2, by selecting the appropriate space, we transform (1) into an abstract Cauchy problem and provide some necessary definitions and preliminary results that will be used in the sequel. Afterwards, we define the mild solutions for (12) by Laplace transform. In Section 3, the existence and uniqueness of the mild solutions be established by several fixed point theorems under some assumptions.

2. Preliminaries

In this section, we provide some definitions and lemmas about fractional calculus and the (α, β)-resolvent family that will be used in this paper.

Definition 1 ([2])**.** The Riemann-Liouville fractional integral of order $\alpha \in (0, 1)$ with respect to t for an integrable function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 ([2])**.** The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ with respect to t for an absolutely continuous function $f : [0, \infty) \to \mathbb{R}$ is defined as

$${}^R D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \quad t > 0. \tag{6}$$

Definition 3 ([2])**.** The Caputo fractional derivative of order $\alpha \in (0, 1)$ with respect to t for an absolutely continuous function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = {}^R D_t^\alpha (f(t) - f(0)), \quad t > 0. \tag{7}$$

If $0 < \alpha < 1$ and $f : [0, \infty) \to \mathbb{R}$ is continuously differentiable, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha} ds = I_t^{1-\alpha} f'(t). \tag{8}$$

If f is an abstract function with values in a Banach space, then the integrals and derivatives appearing in (5) and (7) are understood in Bochner's sense.

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Definition 4 ([34]). Let P be a metric space, and let $S \subset P$ be a bounded set. The Kuratowski measure of noncompactness is defined by:

$$\nu(S) = \inf\{\delta > 0 | S = \bigcup_{i=1}^{m} S_i, \mathrm{diam} S_i \leq \delta\}.$$

Lemma 5 ([35]). *Let E be a Banach space, and S, S_1, S_2 be some subsets of E. Then we have the following properties :*

(i) $\nu(S) = 0 \Leftrightarrow S$ *is relatively compact;*
(ii) $\nu(S) = \nu(\overline{S})$*;*
(iii) $S_1 \subset S_2 \Rightarrow \nu(S_1) \leq \nu(S_2)$*;*
(iv) $\nu(S_1 + S_2) \leq \nu(S_1) + \nu(S_2)$*, where* $S_1 + S_2 = \{x | x = y + z,\ y \in S_1,\ z \in S_2\}$*;*
(v) $\nu(cS) = |c|\nu(S)$*,* $c \in \mathbb{R}$*, where* $cS = \{x | x = cz,\ z \in S\}$*;*
(vi) $0 \leq \nu(S) < +\infty$.

Let $X = H^{-1}(\Omega)$ with the norm $\|\cdot\|$ and $J = [0, T]$. Here $H^{-1}(\Omega)$ represents the dual space of $H_0^1(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, $C_c^\infty(\Omega)$ consists of the functions in $C^\infty(\Omega)$ with compact support, $H^1(\Omega)$ is the Sobolev space (see [36]). $C(J, X)$ denotes the Banach space of all continuous functions from J into X with the norm

$$\|\cdot\|_\infty = \sup_{t \in J}\{\|\cdot(t)\|\}.$$

Similarly, $C(J, \mathbb{R}^+)$ denotes all continuous functions from J into $\mathbb{R}^+$ with the norm

$$\|\cdot\|_\infty = \sup_{t \in J}\{\cdot(t)\}.$$

We define $A : D(A) \subset X \to X$ with $D(A) = H_0^1(\Omega)$ and $(Au)(t)x = Au(x, t)$. Then according to [37, Theorem 2.3], A generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ on X. Without loss of generality, we can assume that $0 \in \rho(A)$, $\{T(t)\}_{t \geq 0}$ is uniformly bounded, and there exists a constant $N > 0$ such that

$$\|T(t)\| \leq N, \quad \text{for } t > 0. \tag{9}$$

Denote $\rho(A)$ and $R(\lambda, A) = (\lambda I - A)^{-1}$ are the resolvent set and resolvent operator of A, respectively, where I is the identity operator. By [38, Theorem 5.2, p. 61], there exist $\delta \in (0, \frac{\pi}{2})$ and $C > 0$ such that

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{\lambda \middle| |\arg\lambda| < \frac{\pi}{2} + \delta\right\} \subset \rho(A) \tag{10}$$

and

$$||R(\lambda, A)|| \leq \frac{C}{|\lambda|}, \lambda \in \Sigma_{\frac{\pi}{2}+\delta} \backslash \{0\}. \tag{11}$$

Set $u(t)(x) = u(x, t)$, $f(t, u(t))(x) = f(x, t, u(x, t))$, where $x \in X$, then (1) can be formulated as an abstract problem with nonlocal initial conditions:

$$\begin{cases} D_t^\alpha u(t) = Au(t) + I_t^\beta f(t, u(t)),\ t \in (0, T], \\ u(0) + g(u) = u_0, \end{cases} \tag{12}$$

where D_t^α and I_t^β denote the α order Caputo derivative and β order Riemann-Liouville integral, respectively. $g : C(J, X) \to X$, $f : J \times X \to X$ and $u_0 \in X$.

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Definition 6. If A generates a uniformly bounded and analytic semigroup, which satisfies (10) and (11), and for operator-valued function $S_{\alpha,\beta}(t) : X \to X$ we have

$$\lambda^{-\beta}(\lambda^{\alpha} I - A)^{-1}x = \int_0^{\infty} e^{-\lambda t} S_{\alpha,\beta}(t)x dt, \ t \geq 0, \ x \in X, \tag{13}$$

then $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α, β) resolvent family generated by A.

It can be seen from the reference [24,31] that the following results can be obtained.

Remark 1. *Let $\Phi(t)$ and $\Lambda(t)$ satisfy*

$$\lambda^{\alpha-1}(\lambda^{\alpha} I - A)^{-1}x = \int_0^{\infty} e^{-\lambda t} \Phi(t)x dt, \ t \geq 0, \ x \in X,$$

and

$$(\lambda^{\alpha} I - A)^{-1}x = \int_0^{\infty} e^{-\lambda t} \Lambda(t)x dt, \ t \geq 0, \ x \in X. \tag{14}$$

Then

$$\Lambda(t) = t^{\alpha-1}\Psi(t), \quad \Psi(t) = \alpha \int_0^{\infty} \theta \zeta_{\alpha}(\theta) T(t^{\alpha}\theta) d\theta,$$

$$S_{\alpha,1-\alpha}(t) = \Phi(t) = \int_0^{\infty} \zeta_{\alpha}(\theta) T(t^{\alpha}\theta) d\theta,$$

where $\zeta_{\alpha}(\theta)$ is a probability density function defined on $(0, \infty)$ which satisfies

$$\zeta_{\alpha}(\theta) \geq 0, \theta \in (0, \infty),$$

$$\int_0^{\infty} \theta^{v} \zeta_{\alpha}(\theta) d\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)} \text{ for } v \in (-1, \infty),$$

$$\int_0^{\infty} e^{-z\theta} \zeta_{\alpha}(\theta) d\theta = E_{\alpha}(-z), \ \alpha \int_0^{\infty} \theta e^{-z\theta} \zeta_{\alpha}(\theta) d\theta = E_{\alpha,\alpha}(-z), \text{ for } z \in \mathbb{C}, \tag{15}$$

E_{α}, $E_{\alpha,\alpha}$ are the Mittag-Leffler functions [2].

If $u : J \to X$ is a solution of

$$\begin{cases} D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \ t \in (0, T], \\ u(0) = u_0, u_0 \in X, \end{cases}$$

then

$$u(t) = S_{\alpha,1-\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1} \Psi(t-s) f(s, u(s)) ds.$$

Remark 2. *Assume that $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is the (α, β) resolvent family generated by A. Because the inverse Laplace transform of $\lambda^{-\beta}$ is $\frac{t^{\beta-1}}{\Gamma(\beta)}$, by (14) and the convolution theorem of Laplace transform, we have*

$$S_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\alpha-1} \Psi(s) ds.$$

Lemma 7. *If A generates a uniformly bounded and analytic semigroup, which satisfies (10) and (11), then*

(i) There exist $\delta \in (0, \frac{\pi}{2})$ and a constant $C > 0$ such that

$$\left\{ \lambda^{\alpha} \middle| |arg\lambda| < \frac{\pi}{2} + \delta \right\} \subset \rho(A)$$

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and

$$\|\lambda^{-\beta}R(\lambda^{\alpha},A)\| \le \frac{C}{|\lambda|^{\alpha+\beta}}. \tag{16}$$

(ii) A generates a (α,β) resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$, and

$$S_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\lambda^{-\beta}R(\lambda^{\alpha},A)d\lambda,$$

where

$$\Gamma = \{re^{-i(\frac{\pi}{2}+\delta_1)}|\ r\in[\rho,\infty)\}\cup\{\rho e^{i\delta_2}|\ |\delta_2|\le\frac{\pi}{2}+\delta_1\}\cup\{re^{i(\frac{\pi}{2}+\delta_1)}|\ r\in[\rho,\infty)\},$$

$\delta_1 \in (0,\delta)$, $\delta\in(0,\frac{\pi}{2})$ *and* $\rho>0$. *Moreover,*

$$||S_{\alpha,\beta}(t)|| \le Mt^{\alpha+\beta-1}, \quad t>0, \tag{17}$$

$M>0$ *is a constant.*

Proof. (i) Let $K(\lambda)=\lambda^{\alpha}$, $\lambda = re^{i(\frac{\pi}{2}+\delta)}$ with $\delta\in(0,\frac{\pi}{2})$ and $r>0$, then

$$\begin{aligned}|\arg(K(\lambda))| &= |\mathrm{Im}(\ln(K(re^{i(\frac{\pi}{2}+\delta)})))|\\ &= \left|\mathrm{Im}\int_0^{\frac{\pi}{2}+\delta}\frac{d}{dt}\ln(K(re^{it}))dt\right|\\ &= \left|\mathrm{Im}\int_0^{\frac{\pi}{2}+\delta}\frac{K'(re^{it})ire^{it}}{K(re^{it})}dt\right|\\ &\le \alpha(\frac{\pi}{2}+\delta)\\ &< \frac{\pi}{2}+\delta.\end{aligned}$$

Thus $\lambda^{\alpha}\in\rho(A)$, and

$$\|\lambda^{-\beta}R(\lambda^{\alpha},A)\| \le \frac{C}{|\lambda|^{\alpha+\beta}},$$

where $C>0$ is a constant.

(ii) For $t>0$, and $\delta_1\in(0,\delta)$, we set

$$S_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\lambda^{-\beta}R(\lambda^{\alpha},A)d\lambda, \tag{18}$$

where

$$\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3,$$

$$\Gamma_1 = \{re^{-i(\frac{\pi}{2}+\delta_1)}|\ r\in[\frac{1}{t},\infty)\},$$

$$\Gamma_2 = \{\frac{1}{t}e^{i\theta}|\ |\theta|\le\frac{\pi}{2}+\delta_1\},$$

$$\Gamma_3 = \{re^{i(\frac{\pi}{2}+\delta_1)}|\ r\in[\frac{1}{t},\infty)\}$$

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are oriented counterclockwise. From (16) it is easy to see that for $t > 0$ the integral in (18) converges in the uniform topology. Moreover,

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_{\Gamma_3} e^{\lambda t}\lambda^{-\beta}R(\lambda^\alpha, A)d\lambda\right\| &\leq \frac{C}{2\pi}\int_{\frac{1}{t}}^{\infty}\frac{e^{-rt\sin\delta_1}}{r^{\alpha+\beta}}dr\\ &\leq \frac{C(\sin\delta_1)^{\alpha+\beta-1}}{2\pi}\int_{\sin\delta_1}^{\infty}\frac{e^{-\tau}}{\tau^{\alpha+\beta}}d\tau t^{\alpha+\beta-1}\\ &\leq C_1 t^{\alpha+\beta-1}.\end{aligned}$$

Similarly, the integral on Γ_1 has the same estimation and on Γ_2 we get

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_{\Gamma_2} e^{\lambda t}\lambda^{-\beta}R(\lambda^\alpha, A)d\lambda\right\| &\leq \frac{C}{2\pi}\int_{-(\frac{\pi}{2}+\delta)}^{\frac{\pi}{2}+\delta} e^{\cos\theta}d\theta t^{\alpha+\beta-1}\\ &\leq C_2 t^{\alpha+\beta-1}.\end{aligned}$$

Hence, (17) holds.

Next we fix $\lambda > 0$, we have

$$\begin{aligned}\int_0^T e^{-\lambda t}S_{\alpha,\beta}(t)dt &= \frac{1}{2\pi i}\int_\Gamma \mu^{-\beta}R(\mu^\alpha, A)\int_0^T e^{-(\lambda-\mu)t}dtd\mu\\ &= \lambda^{-\beta}R(\lambda^\alpha, A) + \frac{1}{2\pi i}\int_\Gamma e^{-(\lambda-\mu)T}\frac{\mu^{-\beta}R(\mu^\alpha, A)}{\mu-\lambda}d\mu,\end{aligned} \tag{19}$$

where the Cauchy integral formula and Fubini theorem are also used. Due to

$$\left\|\int_\Gamma e^{-(\lambda-\mu)T}\frac{\mu^{-\beta}R(\mu^\alpha, A)}{\mu-\lambda}d\mu\right\| \leq Me^{-T\lambda}\int_\Gamma \frac{|d\mu|}{|\mu|^{\alpha+\beta}|\lambda-\mu|} \to 0, \quad T\to\infty.$$

Therefore, by taking the limit as $T \to \infty$ in (19), we have

$$\int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)dt = \lambda^{-\beta}R(\lambda^\alpha, A).$$

That is, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by A. □

Remark 3. *Due to different parameters, the method in [19, Theorem 5 and Theorem 6] cannot be directly applied to this paper. By comparing the forms of mild solutions to the studied equations, we find that (17) agrees well with $\|\Psi(t)\| \leq \frac{M_\omega}{\Gamma(\alpha)}$ in [24].*

Lemma 8. *For $t > 0$, $R_{\alpha,\beta}(t) = t^{1-\alpha-\beta}S_{\alpha,\beta}(t)$ is continuous in the uniform operator topology, where $S_{\alpha,\beta}(t)$ is the (α,β) resolvent family generated by A.*

Proof. Let $\epsilon > 0$ be fixed. Due to the fact that $\Psi(t)$ is continuous in the uniform operator topology for $t > 0$ [39], for arbitrary $t_0 > 0$, there exists $\delta > 0$ such that

$$\|\Psi(t_2) - \Psi(t_1)\| \leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\epsilon, \tag{20}$$

for $t_2 > t_1 \geq t_0$, and $|t_2 - t_1| < \delta$.

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Then, owing to

$$\begin{aligned}\|R_{\alpha,\beta}(t_2)x - R_{\alpha,\beta}(t_1)x\| &= \frac{1}{\Gamma(\beta)}\left\| t_2^{1-\alpha-\beta}\int_0^{t_2}(t_2-s)^{\beta-1}s^{\alpha-1}\Psi(s)ds \right.\\ &\quad \left. -t_1^{1-\alpha-\beta}\int_0^{t_1}(t_1-s)^{\beta-1}s^{\alpha-1}\Psi(s)ds\right\| \\ &= \frac{1}{\Gamma(\beta)}\int_0^1(1-\tau)^{\beta-1}\tau^{\alpha-1}\|\Psi(t_2\tau)-\Psi(t_1\tau)\|d\tau,\end{aligned}$$

and

$$\int_0^1(1-\tau)^{\beta-1}\tau^{\alpha-1}d\tau = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where $B(\cdot,\cdot)$ is the Beta function [2], we conclude that

$$\|R_{\alpha,\beta}(t_2) - R_{\alpha,\beta}(t_1)\| \leq \epsilon.$$

That is, by the arbitrariness of t_0, $R_{\alpha,\beta}(t)$ is continuous in the uniform operator topology for $t > 0$. □

Remark 4. *$\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is strongly continuous. That is, for arbitrary $x \in X$ and $0 \leq t_1 < t_2 \leq T$, we have*

$$\|R_{\alpha,\beta}(t_2)x - R_{\alpha,\beta}(t_1)x\| \to 0 \quad \text{as } t_1 \to t_2.$$

Proof. Since $\{\Psi(t)\}_{t\geq 0}$ is strongly continuous [40], there exist $\delta_1 > 0$ such that

$$\|\Psi(t_2)x - \Psi(t_1)x\| \leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\epsilon, \tag{21}$$

for $t_2 > t_1 \geq 0$, and $|t_2 - t_1| < \delta_1$.

Due to

$$\|R_{\alpha,\beta}(t_2)x - R_{\alpha,\beta}(t_1)x\| = \frac{1}{\Gamma(\beta)}\int_0^1(1-\tau)^{\beta-1}\tau^{\alpha-1}\|\Psi(t_2\tau)x-\Psi(t_1\tau)x\|d\tau \leq \epsilon,$$

we obtain that $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is strongly continuous. □

It is noteworthy that the strongly continuity of $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ can not be obtained immediately by Lemma 8, in which $t > 0$, not $t \geq 0$.

Lemma 9. *If the analytic semigroup $\{T(t)\}_{t\geq 0}$ generated by A is compact, then $R_{\alpha,\beta}(t)$ is compact for $t > 0$.*

Proof. Set $B_k = \{x \in X | \|x\| \leq k\}$. In order to show $R_{\alpha,\beta}(t)$ is compact for $t > 0$, we need to show that

$$\{R_{\alpha,\beta}(t)x | x \in B_k\}$$

is relatively compact in X, for any $k > 0$ and $t > 0$.

Let $t > 0$ be fixed. For any $\epsilon > 0$, define

$$R_\epsilon(t) := \frac{\alpha}{\Gamma(\beta)}t^{1-\alpha-\beta}\int_0^t(t-s)^{\beta-1}s^{\alpha-1}\int_\epsilon^\infty \theta\zeta_\alpha(\theta)T(t^\alpha\theta)d\theta ds.$$

Then

$$R_\epsilon(t) = T(t^\alpha\epsilon)\frac{\alpha}{\Gamma(\beta)}t^{1-\alpha-\beta}\int_0^t(t-s)^{\beta-1}s^{\alpha-1}\int_\epsilon^\infty \theta\zeta_\alpha(\theta)T(t^\alpha\theta - t^\alpha\epsilon)d\theta ds$$

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and

$$\begin{aligned}&\left\|\frac{\alpha}{\Gamma(\beta)}t^{1-\alpha-\beta}\int_0^t(t-s)^{\beta-1}s^{\alpha-1}\int_\epsilon^\infty\theta\zeta_\alpha(\theta)T(t^\alpha\theta-t^\alpha\epsilon)d\theta ds\right\|\\&\leq\frac{N}{\Gamma(\beta)}\int_0^t(1-\tau)^{\beta-1}\tau^\alpha d\tau\cdot\alpha\int_0^\infty\theta\zeta_\alpha(\theta)d\theta\\&=\frac{N}{\Gamma(\beta)}B(\alpha,\beta)\frac{1}{\Gamma(\alpha)}=\frac{N}{\Gamma(\alpha+\beta)},\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function. Due to the compactness of $T(t^\alpha\epsilon)$ $(t^\alpha\epsilon>0)$, we obtain that $\{R_\epsilon(t)x|x\in B_k\}$ is relatively compact in X for arbitrary $\epsilon>0$.

In addition, for any $x\in B_k$, we have

$$\|R_{\alpha,\beta}(t)x-R_\epsilon(t)x\|\leq\frac{NK}{\Gamma(\alpha+\beta)}\alpha\int_0^\epsilon\theta\zeta_\alpha(\theta)d\theta.$$

Therefore, we obtain that $\{R_{\alpha,\beta}(t)x|x\in B_k\}$ is relatively compact in X. □

Remark 5. *If B is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq0}$ on a Banach space, then $S(t)$ is compact if and only if $S(t)$ is continuous in the uniform operator topology for $t>0$ and $R(\lambda,A)$ is compact for $\lambda\in\rho(A)$ [38]. If $\alpha\in(0,1)$ and $\beta=1-\alpha$, $S_{\alpha,\beta}$ is compact if and only if A generates a compact C_0 semigroup, which is also obtained in [40].*

Lemma 10. *Assume that $u\in C(J,X)$, $D_t^\alpha u\in C((0,T],X)$, $u(t)\in D(A)$ for $t\in(0,T]$, and u satisfies (12). Then u satisfies the formal integral equation*

$$u(t)=S_{\alpha,1-\alpha}(t)(u_0-g(u))+\int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds.$$

Proof. By the definitions of Caputo derivative and Riemann-Liouville integral [2], one can rewrite (12) as in the equivalent integral equation

$$\begin{aligned}u(t)=u_0&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Au(s)ds\\&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}f(s,u(s))ds.\end{aligned}\tag{22}$$

For $\lambda\in\Sigma_{\frac{\pi}{2}+\delta}$, using the Laplace transform

$$\hat{u}(\lambda)=\int_0^\infty e^{-\lambda s}u(s)ds\quad\text{and}\quad\hat{f}(\lambda,u(\lambda))=\int_0^\infty e^{-\lambda s}f(s,u(s))ds$$

to (12), we have

$$\lambda^\alpha\hat{u}(\lambda)-\lambda^{\alpha-1}u(0)=A\hat{u}(\lambda)+\lambda^{-\beta}\hat{f}(\lambda,u(\lambda)).\tag{23}$$

Then (23) is equivalent to

$$\hat{u}(\lambda)=\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}u(0)+\lambda^{-\beta}(\lambda^\alpha I-A)^{-1}\hat{f}(\lambda,u(\lambda)).$$

By the inverse Laplace transform and Definition 6, we have

$$u(t)=S_{\alpha,1-\alpha}(t)(u_0-g(u))+\int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds,$$

where $t\in J$. □

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Consequently, we give the definition of a mild solution to (12) as follows.

Definition 11. The function $u \in C(J, X)$ is called a mild solution of equation (12) if

$$u(t) = S_{\alpha,1-\alpha}(t)(u_0 - g(u)) + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds.$$

Lemma 12. *([41])The convex closure $\overline{conv(Z)}$ is compact provided Z is a compact subset of a Banach space.*

Lemma 13. *([42])Assume that $H : Y \to Y$ is completely continuous, where Y is a convex subset of a Banach space and $0 \in Y$. Then either there is a fixed point of H or the set $\{y \in Y | y = \mu H(y)\}$ is unbounded, where $\mu \in (0, 1)$.*

Lemma 14. *([43]) Suppose that D is a bounded, convex and closed subset of a Banach space, $0 \in D$, $N : D \to D$ is continuous. If $V = \overline{conv}N(V)$ or $V = N(V) \cup \{0\}$ can obtain $\nu(V) = 0$ for every subset V of D, then N has a fixed point.*

3. Main results

In order to obtain the main results, we should introduce some hypotheses.

(H_1) $f(t, z)$ is continuous with respect to $z \in X$ for almost all $t \in J$, and strongly measurable with respect to $t \in J$ for any $z \in X$;

(H_1') $f(t, z)$ is strongly measurable with respect to any $z \in X$ and almost all $t \in J$;

(H_2) $g : C(J, X) \to X$ is completely continuous, and there exists a constant $L > 0$ such that

$$\|g(u)\| \le L\|u\|_\infty$$

for any $u \in C(J, X)$;

(H_2') For $g : C(J, X) \to X$, there exists a constant $L > 0$ such that

$$\|g(u)\| \le L\|u\|_\infty$$

for any $u \in C(J, X)$;

(H_3) There exists a continuous function $\varphi : J \to \mathbb{R}^+$ such that

$$\|f(t, z)\| \le \varphi(t)\|z\|,$$

for a.e. $t \in J$ and each $z \in X$;

(H_4) $T(t)$ is compact for $t > 0$;

(H_5) For any bounded subset X_1 of X and each $t \in J$, there exists $a_1 > 0$ such that

$$\nu(g(X_1)) \le a_1\nu(X_1),$$

and $\nu(f(t, X_1)) \le \varphi(t)\nu(X_1)$, where $\varphi(t)$ is defined as in (H_3);

(H_6) There exist $L_g > 0$ such that

$$\|g(y_1) - g(y_2)\| \le L_g\|y_1 - y_2\|_\infty,$$

and $\|f(t, y_1(t)) - f(t, y_2(t))\| \le \varphi(t)(t)\|y_1 - y_2\|_\infty$, where $\varphi(t)$ is defined as in (H_3), $t \in J$, $y_1, y_2 \in B_{k_2}$, and

$$B_{k_2} = \left\{u \in C(J, X) |\ \|u\| \le k_2,\ k_2 = \frac{M(\|u_0\| + \|g(0)\|)(\alpha+\beta)}{(\alpha+\beta)(1 - ML_g) - MT^{\alpha+\beta}\|\varphi\|_\infty}.\right\} \quad (24)$$

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Theorem 15. *If* (H$_1$)-(H$_4$) *are satisfied, then* (12) *has a mild solution provided that*

$$(\alpha+\beta)(1-ML) > MT^{\alpha+\beta}\|\varphi\|_\infty.$$

Proof. Let

$$k_1 = \frac{M\|u_0\|(\alpha+\beta)}{(\alpha+\beta)(1-ML) - MT^{\alpha+\beta}\|\varphi\|_\infty}.$$

For any $u \in B_{k_1}$, by Lemma 7 and (H$_2$), we have

$$\|S_{\alpha,1-\alpha}(t)(u_0 - g(u))\| \leq M(\|u_0\| + k_1 L). \tag{25}$$

Furthermore, due to (H$_1$), $f(t,u(t))$ is a measurable function on J. By Lemma 7 and (H$_3$), we get

$$\begin{aligned}
\int_0^t \|S_{\alpha,\beta}(t-s)f(s,u(s))\|ds &\leq M\int_0^t (t-s)^{\alpha+\beta-1}\varphi(s)\|u(s)\|ds \\
&\leq Mk_1 \int_0^t (t-s)^{\alpha+\beta-1}\varphi(s)ds \\
&\leq \frac{MT^{\alpha+\beta}k_1}{\alpha+\beta}\|\varphi\|_\infty,
\end{aligned} \tag{26}$$

Then, $\|S_{\alpha,\beta}(t-s)f(s,u(s))\|$ is Lebesgue integrable with respect to $s \in J$ and $t \in J$, which implies that $S_{\alpha,\beta}(t-s)f(s,u(s))$ is Bochner integrable with respect to $s \in J$ and $t \in J$ because of Bochner's theorem.

Now we can define an operator Q on B_{k_1} as follows:

$$(Qu)(t) = S_{\alpha,1-\alpha}(t)(u_0 - g(u)) + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds.$$

We firstly prove that Q is a completely continuous operator. Suppose that

$$u_n, u \in B_{k_1} \text{ such that } u_n \to u \text{ as } n \to \infty, \tag{27}$$

then

$$\begin{aligned}
\|Qu_n(t) - Qu(t)\| &\leq M\|g(u) - g(u_n)\| \\
&\quad + M\int_0^t (t-s)^{\alpha+\beta-1}\|f(s,u_n(s)) - f(s,u(s))\|ds \\
&=: I_1 + I_2.
\end{aligned}$$

Obviously, $I_1 \to 0$ as $n \to \infty$ by (27) and (H$_2$). Since (H$_3$) we have

$$\begin{aligned}
I_2 &\leq M\int_0^t (t-s)^{\alpha+\beta-1}(\|f(s,u_n(s))\| + \|f(s,u(s))\|)ds \\
&\leq 2Mk_1 \int_0^t (t-s)^{\alpha+\beta-1}\varphi(s)ds \\
&\leq \frac{2Mk_1T^{\alpha+\beta}}{\alpha+\beta}\|\varphi\|_\infty.
\end{aligned}$$

Then by Lebesgue dominated convergence theorem and (H$_1$), we get $I_2 \to 0$ as $n \to \infty$. Therefore,

$$\|Qu_n(t) - Qu(t)\| \to 0 \text{ as } n \to \infty.$$

That is, Q is continuous on B_{k_1}.

Next, we prove that $\{Qu | u \in B_{k_1}\}$ is relatively compact. It suffices to show that $\{Qu | u \in B_{k_1}\}$ is uniformly bounded and equicontinuous, and $\{Qu(t) | u \in B_{k_1}\}$ is relatively compact in X for any $t \in J$.

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(25) and (26) imply
$$\|(Qu)(t)\| \le k_1,$$
which means $\{Qu|u \in B_{k_1}\}$ is uniformly bounded. Take $u \in B_{k_1}$ and $0 \le t_1 < t_2 \le T$, then $\|Qu(t_2) - Qu(t_1)\| \le I_3 + I_4 + I_5$, where

$$\begin{aligned} I_3 &= \|[S_{\alpha,1-\alpha}(t_2) - S_{\alpha,1-\alpha}(t_1)][u_0 - g(u)]\|, \\ I_4 &= \int_{t_1}^{t_2} \|S_{\alpha,\beta}(t_2 - s)f(s,u(s))\|ds, \\ I_5 &= \int_0^{t_1} \|[S_{\alpha,\beta}(t_2 - s) - S_{\alpha,\beta}(t_1 - s)]f(s,u(s))\|ds. \end{aligned}$$

It is easy to see that $I_3 \to 0$ as $t_1 \to t_2$ by Remark 4. By (H_3) and (17), we have

$$I_4 \le M\int_{t_1}^{t_2}(t_2 - s)^{\alpha+\beta-1}\varphi(s)\|u(s)\|ds \le \frac{Mk_1\|\varphi\|_\infty}{\alpha+\beta}(t_2 - t_1)^{\alpha+\beta},$$

which implies that $I_4 \to 0$ as $t_1 \to t_2$.

$$\begin{aligned} I_5 \le{}& k_1\|\varphi\|_\infty \int_0^{t_1}(t_2 - s)^{\alpha+\beta-1}\|R_{\alpha,\beta}(t_2 - s) - R_{\alpha,\beta}(t_1 - s)\|ds \\ &+ Mk_1\|\varphi\|_\infty \int_0^{t_1}\left[(t_1 - s)^{\alpha+\beta-1} - (t_2 - s)^{\alpha+\beta-1}\right]ds \\ \le{}& k_1\varphi\|_\infty \int_0^{t_1-\epsilon}(t_2 - s)^{\alpha+\beta-1}ds \sup_{s\in[0,t_1-\epsilon]}\|R_{\alpha,\beta}(t_2 - s) - R_{\alpha,\beta}(t_1 - s)\| \\ &+ 2Mk_1\varphi\|_\infty \int_{t_1-\epsilon}^{t_1}(t_2 - s)^{\alpha+\beta-1}ds \\ &+ \frac{Mk_1\|\varphi\|_\infty}{\alpha+\beta}\left[t_1^{\alpha+\beta} - t_2^{\alpha+\beta} + (t_2 - t_1)^{\alpha+\beta}\right] \\ \le{}& \frac{k_1\varphi\|_\infty}{\alpha+\beta}\left[t_2^{\alpha+\beta} - (t_2 - t_1 + \epsilon)^{\alpha+\beta}\right] \sup_{s\in[0,t_1-\epsilon]}\|R_{\alpha,\beta}(t_2 - s) - R_{\alpha,\beta}(t_1 - s)\| \\ &+ \frac{2Mk_1\varphi\|_\infty}{\alpha+\beta}\left[(t_2 - t_1 + \epsilon)^{\alpha+\beta} - (t_2 - t_1)^{\alpha+\beta}\right] + \frac{Mk_1\|\varphi\|_\infty}{\alpha+\beta}\left[(t_2 - t_1)^{\alpha+\beta}\right] \end{aligned}$$

Then $I_5 \to 0$ as $t_1 \to t_2$ and $\epsilon \to 0$ by Lemma 8 and (17). Now, we can concluded that $\{Qu|u \in B_{k_1}\}$ is equicontinuous.

Obviously, due to (H_2), $\{Qu(0)|u \in B_{k_1}\}$ is relatively compact. By (H_2), (H_4) and Remark 5, we can similarly prove the compactness of $\{S_{\alpha,1-\alpha}(t)(u_0 - g(u))|u \in B_{k_1}\}$ for $t \in (0,T]$. Due to (H_1), (H_4) and Lemma 9, U is compact for $t \in (0,T]$, then $\overline{\mathrm{conv}(U)}$ is compact for $t \in (0,T]$ by Lemma 12, where

$$U = \{R_{\alpha,\beta}(t - s)f(s,u(s))|s \in [0,t),\ u \in B_{k_1}\}.$$

By Mean-Value Theorem for the Bochner integral [44, Corollary 8, p. 48],

$$\int_0^t S_{\alpha,\beta}(t - s)f(s,u(s))ds \in \frac{t^{\alpha+\beta}}{\alpha+\beta}\overline{\mathrm{conv}(U)},\quad t \in (0,T],$$

As a consequence, $\overline{\{Qu(t)|u \in B_{k_1}\}}$ is compact in X for all $t \in (0,T]$. Then $\{Qu(t)|u \in B_{k_1}\}$ is relatively compact in X for any $t \in J$.

By the Arzela-Ascoli theorem, $\{Qu|u \in B_{k_1}\}$ is relatively compact. By combining with the continuity of Q, we conclude that $Q : B_{k_1} \to B_{k_1}$ is completely continuous.

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Set
$$M_1 = \{u \in B_{k_1}, u = \eta Q u, \eta \in (0,1)\},$$
let us prove the boundedness of M_1. Apparently, $0 \in M_1$. For $u \in M_1$, we have
$$\begin{aligned}\|u(t)\| &\le \eta[M(\|u_0\| + \|g(u)\|) + M\int_0^t (t-s)^{\alpha+\beta-1}\varphi(s)\|u(s)\|ds \\ &\le \eta\big[M(\|u_0\| + k_1 L) + \frac{MT^{\alpha+\beta}k_1\|\varphi\|_\infty}{\alpha+\beta}\big] \\ &< M(\|u_0\| + k_1 L) + \frac{MT^{\alpha+\beta}k_1\|\varphi\|_\infty}{\alpha+\beta}\end{aligned}$$
for $t \in J$. Therefore, Q has a fixed point by Lemma 13. That is, (12) has a mild solution. □

Theorem 16. *Under the assumptions (H_1), (H_2'), (H_3) and (H_5), (12) has a mild solution provided* $M(a_1 + \frac{T^{\alpha+\beta}}{\alpha+\beta}\|\varphi\|_\infty) < 1$ *and* $(\alpha+\beta)(1-ML) > MT^{\alpha+\beta}\|\varphi\|_\infty$.

Proof. Similar to Theorem 15, $Q : B_{k_1} \to B_{k_1}$ is continuous, $\{Qu | u \in B_{k_1}\}$ is uniformly bounded and equicontinuous, where k_1 is defined as in Theorem 15. Let $V \subset B_{k_1}$ such that $V \subset \overline{Q(V)} \cup \{0\}$, then $v(t) = \nu(V(t))$ is continuous for any $t \in J$ because of the boundness and equicontinuity of V. Then by Lemma 5, (H_5) and (17), we have
$$\begin{aligned}\|v\|_\infty &\le \sup_{t\in J}\|\nu((QV)(t) \cup \{0\})\| \\ &\le \sup_{t\in J}\|\nu((QV)(t))\| \\ &\le \nu(g(V))\sup_{t\in J}\|S_{\alpha,1-\alpha}(t)\| \\ &\quad + \sup_{t\in J}\int_0^t \|S_{\alpha,\beta}(t-s)\|\varphi(s)\nu(V(s))ds \\ &\le Ma_1\|v\|_\infty + M\sup_{t\in J}\int_0^t (t-s)^{\alpha+\beta-1}\varphi(s)v(s)ds \\ &\le M(a_1 + \frac{T^{\alpha+\beta}}{\alpha+\beta}\|\varphi\|_\infty)\|v\|_\infty.\end{aligned}$$
Since $M(a_1 + \frac{T^{\alpha+\beta}}{\alpha+\beta}\|\varphi\|_\infty) < 1$, we have $\|v\|_\infty = 0$, namely, $v(t) = \nu(V(t)) = 0$. Consequently, $V(t)$ is relatively compact in X. By the Arzela-Ascoli theorem, $\{Qu | u \in B_{k_1}\}$ is relatively compact, which means $\nu(V) = 0$. Thus the proof is completed by Lemma 14. □

Theorem 17. *If (H_1'), (H_3) and (H_6) are satisfied, then (12) has a unique mild solution provided* $(\alpha+\beta)(1 - ML_g) > MT^{\alpha+\beta}\|\varphi\|_\infty$.

Proof. Let
$$k_2 = \frac{M(\|u_0\| + \|g(0)\|)(\alpha+\beta)}{(\alpha+\beta)(1-ML_g) - MT^{\alpha+\beta}\|\varphi\|_\infty}.$$
By (H_6), we have
$$\|S_{\alpha,1-\alpha}(t)(u_0 - g(u))\| \le M(\|u_0\| + \|g(0)\| + k_2 L_g),$$
and g is continuous. Similar to Theorem 15, for $u \in B_{k_2}$, $S_{\alpha,1-\alpha}(t)(u_0 - g(u))$ exists, and $S_{\alpha,\beta}(t-s)f(s,u(s))$ is Bochner integrable with respect to $s \in J$ and $t \in J$. We also have that Q maps B_{k_2} into

itself. Apparently, we only need prove that Q has a unique fixed point on B_{k_2}. For any $u, v \in B_{k_2}$ and $t \in J$, according to (H_6) we have

$$\begin{aligned}\|(Qu)(t) - (Qv)(t)\| &\leq \|S_{\alpha,1-\alpha}(t)[g(v) - g(u)]\| \\ &\quad + \int_0^t \|S_{\alpha,\beta}(t-s)\| \|f(s,u(s)) - f(s,v(s))\| ds \\ &\leq ML_g\|u - v\|_\infty + M\|u - v\|_\infty \int_0^t (t-s)^{\alpha+\beta-1}\varphi(s)ds \\ &\leq (ML_g + \frac{MT^{\alpha+\beta}\|\varphi\|_\infty}{\alpha+\beta})\|u - v\|_\infty.\end{aligned}$$

Since $ML_g + \frac{MT^{\alpha+\beta}\|\varphi\|_\infty}{\alpha+\beta} < 1$, by Banach contraction principle, we conclude that Q has a unique fixed point on B_{k_2}, the proof is complete. □

4. Conclusions

This article investigates the nonlocal initial value problem for a class of fractional order integro-differential diffusion equations with Dirichlet boundary conditions. By selecting appropriate Banach spaces, we transform the original problem into an abstract Cauchy problem. At the same time, we prove that operator A can generate a resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, and provide proof that it is continuous in the uniform topology and strongly continuous. If the analytical semigroup $\{T(t)\}_{t\geq 0}$ generated by A is compact, then $\{S_{\alpha,\beta}(t)\}_{t>0}$ is also compact. Through the relationship between Laplace transform and convolution, we also establish the relationship between $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ and the probability density function $\zeta_\alpha(\theta)$. It is worth noting that since A in [23] is ω-sector (ω<0) and the order of the equation is different from that in this paper, the proof of $\|S_{\alpha,\beta}(t)\|$ is bounded cannot be directly applied in this paper. By the Laplace transform, the definition of a mild solution for (12) be given, and finally prove the existence and uniqueness of the mild solution through several fixed point theorems. In the future, we may focus on the following meaningful topics:

- The condition (H_4) is strict, we hope to obtain the existence of the solution in this paper without requiring $T(t)$ to be compact;
- For $\alpha \in (0,1)$ and $\beta \in (0,\infty)$ in this paper, the stability and periodicity of $S_{\alpha,\beta}(t)$;
- By Remark 2, we have known $S_{\alpha,\beta}(t)$ can establisn a relationship with $\zeta_\alpha(\theta)$ and $T(t)$, then we expect to investigate the regularity of solutions for (12) by $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ if f satisfies a certain degree of continuity.

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