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Article

Boundedness and Compactness of Weighted Composition Operators from α-Bloch Spaces to Bers-Type Spaces on Generalized Hua Domains of the First Kind †

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Abstract: We address weighted composition operators ψC_ϕ from α-Bloch spaces to Bers-type spaces of bounded holomorphic functions on Y, where Y is a generalized Hua domain of the first kind, and obtain some necessary and sufficient conditions for the boundedness and compactness of those operators.

Keywords: generalized Hua domain of the first type; α-Bloch space; Bers-type space; weighted composition operators; boundedness and compactness

MSC: 32A27; 47B33

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1. Introduction

Let Ω be a bounded domain of $\mathbb{C}^n$ and $H(\Omega)$ the class of all holomorphic functions on Ω . Then consider a holomorphic self-map ϕ of Ω and a function $\psi \in H(\Omega)$. The linear operator

$$(\psi C_\phi f)(z) = \psi(z) f(\phi(z)),$$

is referred to as a weighted composition operator for $f \in H(\Omega)$. If $\psi(z) \equiv 1$, it reduces to the composition operator, whereas for $\phi(z) = z$ it becomes the multiplication operator. For any given holomorphic function f, $(\psi C_\phi f)(z)$ represents a generalised composition/multiplication operator. The reader is referred to book[1] for an extensive introduction to the topic.

In this paper, we study the boundedness and the compactness of weighted composition operators from α-Bloch spaces $\mathcal{B}^\alpha$ to Bers-type spaces built on generalised Hua domains of the first kind. On $\mathrm{GHE_I}$ the α-Bloch space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathrm{GHE_I})$ such that

$$\|f\|_{\mathcal{B}^\alpha} := |f(0,0)| + \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha |\nabla f(Z,\xi)| < \infty,$$

where

$$\nabla f(Z,\xi) = \Big(\frac{\partial f(Z,\xi)}{\partial z_{11}}, \frac{\partial f(Z,\xi)}{\partial z_{12}}, \cdots, \frac{\partial f(Z,\xi)}{\partial z_{mn}}, \frac{\partial f(Z,\xi)}{\partial \xi_1}, \cdots, \frac{\partial f(Z,\xi)}{\partial \xi_r}\Big).$$

It is clear that $\mathcal{B}^\alpha(\mathrm{GHE_I})$ is a Banach space.

In 1930, Cartan [2] was the the first to characterise the six types of irreducible bounded symmetric domains, which consist of four types of bounded symmetric classical domains, also referred to as

Cartan domains, and two exceptional domains whose complex dimension are 16 and 27, respectively. The Cartan domains are defined as follows:

$$\begin{aligned}
&\Re_{\mathrm{I}}(m,n) := \{Z \in \mathbb{C}^{m\times n} : I_m - Z\overline{Z}' > 0\},\\
&\Re_{\mathrm{II}}(p) := \left\{Z \in \mathbb{C}^{\frac{p(p+1)}{2}} : I_m - Z\overline{Z}' > 0, Z = Z^{'}\right\},\\
&\Re_{\mathrm{III}}(q) := \left\{Z \in \mathbb{C}^{\frac{q(q-1)}{2}} : I_m - Z\overline{Z}' > 0, Z = -Z^{'}\right\},\\
&\Re_{\mathrm{IV}}(n) := \left\{z \in \mathbb{C}^{n} : 1 + |zz^{'}|^2 - 2z\overline{z}' > 0, 1 - |zz^{'}|^2 > 0\right\}.
\end{aligned}$$

where Z' denotes the transpose of Z, $\overline{Z}$ denotes the conjugate of Z, and m, n, p, q are positive integers. In 1998, building on the notion of bounded symmetric domains, Yin and Roos constructed a new type of domain called the Cartan-Hartogs domain[3], and Yin introduced the so-called Hua domains[4], which include the Cartan-Hartogs domains, the Cartan-Egg domains, the Hua domains, the generalized Hua domains, and the Hua construction. The generalized Hua domains are defined as follows:

$$\begin{aligned}
&\mathrm{GHE}_{\mathrm{I}}(N_1, N_2, \cdots, N_r; m, n; p_1, p_2, \cdots, p_r; k)\\
&= \left\{\xi_j \in \mathbb{C}^{N_j}, Z \in \Re_{\mathrm{I}}(m,n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z}^{'})^k, j = 1, 2, \cdots, r\right\}\\
&\mathrm{GHE}_{\mathrm{II}}(N_1, N_2, \cdots, N_r; p; p_1, p_2, \cdots, p_r; k)\\
&= \left\{\xi_j \in \mathbb{C}^{N_j}, Z \in \Re_{\mathrm{II}}(p) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z})^k, j = 1, 2, \cdots, r\right\}\\
&\mathrm{GHE}_{\mathrm{III}}(N_1, N_2, \cdots, N_r; q; p_1, p_2, \cdots, p_r; k)\\
&= \left\{\xi_j \in \mathbb{C}^{N_j}, Z \in \Re_{\mathrm{III}}(q) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I + Z\overline{Z})^k, j = 1, 2, \cdots, r\right\}\\
&\mathrm{GHE}_{\mathrm{IV}}(N_1, N_2, \cdots, N_r; n; p_1, p_2, \cdots, p_r; k)\\
&= \left\{\xi_j \in \mathbb{C}^{N_j}, z \in \Re_{\mathrm{IV}}(n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < (1 + |zz^{'}|^2 - 2z\overline{z}')^k, j = 1, 2, \cdots, r\right\}
\end{aligned}$$

where $\xi_j = (\xi_{j1}, \cdots, \xi_{jN_j})$, $j = 1, \cdots, r, \Re_{\mathrm{I}}(m,n), \Re_{\mathrm{II}}(p), \Re_{\mathrm{III}}(q), \Re_{\mathrm{IV}}(n)$ denote respectively the Cartan domains of the first type, second type, third type and fourth type, Z' denotes the transpose of Z, $\overline{Z}$ denotes the conjugate of Z , $N_1, \cdots, N_r, m, n, p, q$ are positive integers , and $p_1, \cdots, p_r$ are positive real numbers. For $k = 1, m = 1, p_1 = \cdots = p_r = 1$, the generalized Hua domain of the first kind reduces to the unit ball. Without loss of generality, we may assume that $N_j = 1$, then $\xi_j \in \mathbb{C}$, $j = 1, \cdots, r, \xi = (\xi_1, \cdots, \xi_r)$ and $\|\xi\|_p^2 = \sum_{j=1}^{r} |\xi_j|^{2p_j}$. We define

$$\langle \xi, t\rangle_p = \langle \xi_1, t_1\rangle^{p_1} + \langle \xi_2, t_2\rangle^{p_2} + \cdots + \langle \xi_r, t_r\rangle^{p_r}.$$

We also write

$$\begin{aligned}
|\langle \xi, t\rangle_p| &\le |\langle \xi_1, t_1\rangle^{p_1}| + |\langle \xi_2, t_2\rangle^{p_2}| + \cdots + |\langle \xi_r, t_r\rangle^{p_r}|\\
&\le |\xi_1|^{p_1}|t_1|^{p_1} + \cdots + |\xi_r|^{p_r}|t_r|^{p_r}\\
&= |\langle \alpha, \beta\rangle| \le |\alpha||\beta| = \|\xi\|_p\|t\|_p,
\end{aligned}$$

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where $|\xi_i|^{p_i} = \alpha_i, |t_i|^{p_i} = \beta_i (i = 1, \cdots, r), \alpha = (\alpha_1, \cdots, \alpha_r), \beta = (\beta_1, \cdots, \beta_r)$.

For the sake of convenience, the four types of generalized Hua domains will be referred to as $\mathrm{GHE_I}, \mathrm{GHE_{II}}, \mathrm{GHE_{III}}, \mathrm{GHE_{IV}}$.

On $\mathrm{GHE_I}$, a Bers-type space $\mathcal{A}_\beta$ consists of all $f \in H(\mathrm{GHE_I})$ such that

$$\|f\|_{\mathcal{A}_\beta} := \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |f(Z,\xi)| < \infty.$$

It is easy to see that $\mathcal{A}_\beta(\mathrm{GHE_I})$ is a Banach space with norm $\|\cdot\|$.

The boundedness and the compactness of weighted composition operators on (or between) spaces of holomorphic functions on various domains received a large attention. Wang and Liu [5] studied the boundedness and the compactness of the weighted composition operators on the Bers-type space on the open unit disc, whereas Zhou and Xu [6] characterised the boundedness and the compactness of the weighted composition operators between α-Bloch space and β-Bloch space, and Li [11] investigated the boundedness and the compactness of the weighted composition operators from Hardy space to Bers-type space, Zhu [19] characterised the boundedness and compactness of $D^n_{\phi,u} : \mathcal{B} \to H^\infty_\alpha$. For the unit poly-disk, Li and Stević [7][8] presented some necessary and sufficient conditions for the boundedness and the compactness of the weighted composition operators between H^∞ and α-Bloch space, whereas for the open unit ball, Li and Stević [9] studied the boundedness and the compactness of the weighted composition operators between H^∞ and Bloch space [see also [14]-[17]].

Jiang[10] has charaterised the boundedness and the compactness of the weighted composition operators on the Bers-type space on the Hua domains. On the other hand, the boundedness and the compactness of the weighted composition operators from α-Bloch to $\mathcal{A}_\beta$ have not been studied in details. In this paper, we obtain some necessary conditions and sufficient conditions for the boundedness and the compactness of the weighted composition operators from α-Bloch to $\mathcal{A}_\beta$ on generalised Hua domain of the first kind by using a generalisation of Hua's inequalities.

2. Preliminaries

Lemma 2.1 *Let $\beta > 0$, then*

$$|f(Z,\xi)| \leq \frac{\|f\|_{\mathcal{A}_\beta}}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}. \tag{2.1}$$

for all $(Z,\xi) \in \mathrm{GHE_I}$ and $f \in \mathcal{A}_\beta(\mathrm{GHE_I})$.

Proof. By the very definition of Bers-type space $\mathcal{A}_\beta$, we know that

$$\|f\|_{\mathcal{A}_\beta} = \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |f(Z,\xi)| < \infty,$$

and so,

$$|f(Z,\xi)| \leq \frac{\|f\|_{\mathcal{A}_\beta}}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}.$$

□

Lemma 2.2 *Let $0 < a \leq 1, 0 < b \leq 1$ and $b \leq a$, q is a positive integer, then*

$$a - b \leq q(a^{\frac{1}{q}} - b^{\frac{1}{q}}) \tag{2.2}$$

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Proof.

$$\begin{aligned} a-b &= (a^{\frac{1}{q}})^q - (b^{\frac{1}{q}})^q \\ &= (a^{\frac{1}{q}} - b^{\frac{1}{q}})(a^{\frac{1}{q}\times(q-1)} + a^{\frac{1}{q}\times(q-2)}b^{\frac{1}{q}} + \cdots + b^{\frac{1}{q}\times(q-1)}) \\ &\leq q(a^{\frac{1}{q}} - b^{\frac{1}{q}}). \end{aligned}$$

□

Lemma 2.3 (see[12]) *Let $x \geq -1$, if $0 < \alpha \leq 1$, then*

$$(1+x)^{\alpha} \leq 1 + \alpha x, \tag{2.3}$$

if $\alpha < 0$ or $\alpha > 1$, then

$$(1+x)^{\alpha} \geq 1 + \alpha x, \tag{2.4}$$

and " = " holds if and only if $x = 0$ or $\alpha = 1$.

Lemma 2.4 (see[12]) *Let $a_k \geq 0, k = 1, 2, \cdots, m$, then*

$$(a_1 \cdot a_2 \cdots a_m)^{\frac{1}{m}} \leq \frac{a_1 + a_2 + \cdots + a_m}{m}, \tag{2.5}$$

where the equality holds if and only if $a_1 = a_2 = \cdots = a_m$.

Lemma 2.5 (see[12]) *Let $a_k \in \mathbb{C}$, if $p \geq 1$, then*

$$\sum_{k=1}^{n} |a_k|^p \leq \left[\sum_{k=1}^{n} |a_k|\right]^p \leq n^{p-1} \sum_{k=1}^{n} |a_k|^p. \tag{2.6}$$

If $0 < p < 1$, then

$$\sum_{k=1}^{n} |a_k|^p \geq \left[\sum_{k=1}^{n} |a_k|\right]^p \geq n^{p-1} \sum_{k=1}^{n} |a_k|^p, \tag{2.7}$$

where the equality holds if and only if $p > 1$, then $|a_1| = \cdots = |a_n|$. If $p = 1$, the equality always holds. If $0 < p < 1$, then at most one of the $a_1, \cdots, a_n$ is not zero.

Lemma 2.6 (see[13]) *Let*

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix}$$

be an $m \times n$ matrix $(m \leq n)$. Then, there exist an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_m & 0 & \dots & 0 \end{pmatrix} V \ \ (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0),$$

and

$$Z\overline{Z}' = U \begin{pmatrix} {\lambda_1}^2 & 0 & \dots & 0 \\ 0 & {\lambda_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {\lambda_m}^2 \end{pmatrix} \overline{U}',$$

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where ${\lambda_1}^2, \cdots, {\lambda_m}^2$ *are the characteristic values of* $Z\overline{Z}'$. $I - Z\overline{Z}' > 0 \Longleftrightarrow \lambda_1 < 1$.

Lemma 2.7 (see[13]) *Let*

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_m \end{pmatrix} \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0),$$

$$\Lambda_2 = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \mu_m \end{pmatrix} \quad (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0),$$

satisfying

$$\lambda_j \mu_k < 1 \; (j, k = 1, \cdots, m),$$

Then, there exists a square matrix P such that

$$\inf_{U\overline{U}'=I,\, V\overline{V}'=I} |\det(I - \Lambda_1 U \Lambda_2 \overline{U}' V)| = |\det(I - \Lambda_1 P \Lambda_2 P')|,$$

and the minimum value is obtained for $U = \Theta P$ *and* $V = I$, *where*

$$\Theta = \begin{pmatrix} e^{i\theta_1} & 0 & \dots & 0 \\ 0 & e^{i\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & e^{i\theta_m} \end{pmatrix}$$

.

Lemma 2.8 (see[12] Minkowski inequality of integration formula) *Let* $a_k, b_k \geq 0, \quad k = 1, 2 \cdots, n$, *then*

$$\left[\prod_{k=1}^{n} (a_k + b_k)\right]^{\frac{1}{n}} \geq \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}}, \tag{2.8}$$

where the equal sign holds if and only if $a_k = cb_k, k = 1, 2, \cdots, n$.

Lemma 2.9 *Let* p_i $(i = 1, 2, \cdots, r)$ *be positive integers,* $0 < km \leq 1$, *and* $t \in [0, 1]$, *then*

$$1 - \det(I - t^2 Z\overline{Z}')^k + \|t\xi\|_p^2 \leq t^2 \left[1 - \det(I - Z\overline{Z}')^k + \|\xi\|_p^2\right],$$

for $(Z, \xi) \in \mathrm{GHE_I}$.

Proof. Decomposition in polar coordinates gives

$$\det(I - tZ\overline{Z}')^k = \prod_{i=1}^{m} (1 - t\lambda_i^2)^k.$$

Given $\lambda_i^2 = \hbar_i, i = 1, 2, \cdots, m$, we may consider the function

$$f(t) = \prod_{i=1}^{m} (1 - t\hbar_i)^k, \quad t \in [0, 1]$$

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$$\ln f(t) = k\sum_{i=1}^{m}\ln(1-t\hbar_i),$$

Upon differentiating with respect to t, we obtain

$$\begin{aligned} f'(t) &= f(t)k\sum_{i=1}^{m}\frac{-\hbar_i}{1-t\hbar_i} \le 0, \\ f''(t) &= f'(t)k\sum_{i=1}^{m}\frac{-\hbar_i}{1-t\hbar_i} - f(t)k\sum_{i=1}^{m}\frac{\hbar_i^2}{(1-t\hbar_i)^2} \\ &= f(t)k^2\left(\sum_{i=1}^{m}\frac{\hbar_i}{1-t\hbar_i}\right)^2 - f(t)k\sum_{i=1}^{m}\frac{\hbar_i^2}{(1-t\hbar_i)^2} \\ &= f(t)k\left[k\left(\sum_{i=1}^{m}\frac{\hbar_i}{1-t\hbar_i}\right)^2 - \sum_{i=1}^{m}\frac{\hbar_i^2}{(1-t\hbar_i)^2}\right]. \end{aligned}$$

An application of (2.6) then gives

$$\begin{aligned} f''(t) &= f(t)k\left[k\left(\sum_{i=1}^{m}\frac{\hbar_i}{1-t\hbar_i}\right)^2 - \sum_{i=1}^{m}\frac{\hbar_i^2}{(1-t\hbar_i)^2}\right] \\ &\le f(t)k\left[(km-1)\sum_{i=1}^{m}\frac{\hbar_i^2}{(1-t\hbar_i)^2}\right] \\ &\le 0. \end{aligned}$$

This shows that $f(t)$ is a concave function. It follows that

$$g(t) = 1 - f(t) = 1 - \prod_{i=1}^{m}(1-t\hbar_i)^k, \quad t\in[0,1],$$

is a convex function and we have

$$1 - \prod_{i=1}^{m}(1-t\hbar_i)^k \le t[1 - \prod_{i=1}^{m}(1-\hbar_i)^k]. \tag{2.9}$$

The very definition of $\|\xi\|_p^2$ shows that

$$\begin{aligned} \|t\xi\|_p^2 &= |t\xi_1|^{2p_1} + |t\xi_2|^{2p_2} + \cdots + |t\xi_r|^{2p_r} \\ &\le t^2(|\xi_1|^{2p_1} + |\xi_2|^{2p_2} + \cdots + |\xi_r|^{2p_r}) \\ &= t^2\|\xi\|_p^2. \end{aligned} \tag{2.10}$$

Hence, by inequalities (2.9) and (2.10), we obtain

$$1 - \det(I - t^2 Z\overline{Z}')^k + \|t\xi\|_p^2 \le t^2\left[1 - \det(I - Z\overline{Z}')^k + \|\xi\|_p^2\right].$$

□

Lemma 2.10 *Let us consider* $0 < mk \le 1$*, some positive intergers* p_j $(j = 1,2,\cdots,r)$*,* $t \in [0,1]$*,* $(Z,\xi)\in \mathrm{GHE_I}$*,* $q = \max\{p_1, p_2, \cdots, p_r\}$*. Then, the following inequality holds*

$$|\overline{(Z,\xi)}| \le M\sqrt{1 - \det(I - Z\overline{Z}')^{\frac{k}{q}} + \|\xi\|_p^{\frac{2}{q}}},$$

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where $M = \max\{\sqrt{\frac{q}{k}}, \sqrt{r^{1-\frac{1}{q}}}\}$.

Proof. If $t \in [0,1], \forall (Z,\xi) \in \mathrm{GHE_I}$, then $(tZ, t\xi) \in \mathrm{GHE_I}$, $|Z|^2 = \mathrm{tr}(Z\overline{Z}') = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2$. By Lemma 2.4 and (2.3), we get

$$\begin{aligned}
\det(I - Z\overline{Z}')^{\frac{k}{q}} &= \prod_{i=1}^{m}(1-\lambda_i^2)^{\frac{k}{q}} = \left\{\prod_{i=1}^{m}(1-\lambda_i^2)^{\frac{1}{m}}\right\}^{\frac{mk}{q}} \\
&\leq \left\{\frac{1}{m}(m - \sum_{i=1}^{m}\lambda_i^2)\right\}^{\frac{mk}{q}} \\
&= (1 - \frac{1}{m}\sum_{i=1}^{m}\lambda_i^2)^{\frac{mk}{q}} \\
&\leq 1 - \frac{mk}{q}\cdot\frac{1}{m}|Z|^2 \\
&= 1 - \frac{k}{q}|Z|^2,
\end{aligned}$$

then

$$|Z|^2 \leq \frac{q}{k}[1 - \det(I - Z\overline{Z}')^{\frac{k}{q}}]. \tag{2.11}$$

Using (2.7) one has

$$\begin{aligned}
\|\xi\|_p^{\frac{2}{q}} &= (|\xi_1|^{2p_1} + |\xi_2|^{2p_2} + \cdots + |\xi_r|^{2p_r})^{\frac{1}{q}} \\
&\geq r^{\frac{1}{q}-1}(|\xi_1|^{\frac{2p_1}{q}} + |\xi_2|^{\frac{2p_2}{q}} + \cdots + |\xi_r|^{\frac{2p_r}{q}}) \\
&\geq r^{\frac{1}{q}-1}(|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_r|^2) \\
&= r^{\frac{1}{q}-1}|\xi|^2,
\end{aligned}$$

then

$$|\xi|^2 \leq r^{1-\frac{1}{q}}\|\xi\|_p^{\frac{2}{q}}. \tag{2.12}$$

Therefore, by combining (2.12) and (2.11) we have

$$\begin{aligned}
|\overline{(Z,\xi)}| &= \sqrt{|Z|^2 + |\xi|^2} \\
&\leq \sqrt{\frac{q}{k}[1 - \det(I - Z\overline{Z}')^{\frac{k}{q}}] + r^{1-\frac{1}{q}}\|\xi\|_p^{\frac{2}{q}}} \\
&\leq M\sqrt{1 - \det(I - Z\overline{Z}')^{\frac{k}{q}} + \|\xi\|_p^{\frac{2}{q}}},
\end{aligned} \tag{2.13}$$

where $M = \max\{\sqrt{\frac{q}{k}}, \sqrt{r^{1-\frac{1}{q}}}\}$. □

Lemma 2.11 *Given* $0 < km \leq 1$, p_j *some positive integers* $(j = 1, 2, \cdots, r)$, $\forall (Z,\xi) \in \mathrm{GHE_I}, q = \max\{p_1, p_2, \cdots, p_r\}$ *and* f *a holomorphic function on* $\mathcal{B}^\alpha(\mathrm{GHE_I})$, *then there exists a constant* C *such that*

$$|f(Z,\xi)| \leq \begin{cases} C\|f\|_{\mathcal{B}^\alpha} & 0 < \alpha < 1 \\ C\|f\|_{\mathcal{B}^\alpha} \ln \dfrac{2q}{\det(I - Z\overline{Z}')^k - \|\xi\|_p^2} & \alpha = 1 \\ C\|f\|_{\mathcal{B}^\alpha} \dfrac{1}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^{\alpha-1}} & \alpha > 1 \end{cases} \tag{2.14}$$

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where $(Z,\xi) = (z_{11}, z_{12}, \cdots, z_{mn}, \xi_1, \xi_2, \cdots, \xi_r)$.

Proof. According to Lemmas 2.2, 2.9 and (2.13),

$$\begin{aligned}
|f(Z,\xi)| &= |f(0,0) + \int_0^1 \langle \nabla f(tZ,t\xi), \overline{(Z,\xi)} \rangle \mathrm{d}t| \\
&\leq |f(0,0)| + \int_0^1 |\nabla f(tZ,t\xi)||\overline{(Z,\xi)}|\mathrm{d}t \\
&= |f(0,0)| + |\overline{(Z,\xi)}| \int_0^1 \frac{[\det(I - t^2 Z\overline{Z}')^k - \|t\xi\|_p^2]^\alpha |\nabla f(tZ,t\xi)|}{[\det(I - t^2 Z\overline{Z}')^k - \|t\xi\|_p^2]^\alpha} \mathrm{d}t \\
&\leq \left[1 + \int_0^1 \frac{|\overline{(Z,\xi)}|}{[\det(I - t^2 Z\overline{Z}')^k - \|t\xi\|_p^2]^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&= \left[1 + \int_0^1 \frac{|\overline{(Z,\xi)}|}{[1 - (1 - \det(I - t^2 Z\overline{Z}')^k + \|t\xi\|_p^2)]^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&\leq \left[1 + M \int_0^1 \frac{\sqrt{1 - \det(I - Z\overline{Z}')^{\frac{k}{q}} + \|\xi\|_p^{\frac{2}{q}}}}{[1 - t^2(1 - \det(I - Z\overline{Z}')^k + \|\xi\|_p^2)]^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&\leq \left[1 + M \int_0^1 \frac{\sqrt{1 - \frac{1}{q}(\det(I - Z\overline{Z}')^k - \|\xi\|_p^2)}}{[1 - t^2(1 - \frac{1}{q}(\det(I - Z\overline{Z}')^k - \|\xi\|_p^2))]^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&= \left[1 + M \int_0^1 \frac{\Im}{[1 - t^2 \Im^2]^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&= \left[1 + M \int_0^1 \frac{\Im}{[(1 - t\Im)(1 + t\Im)]^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&\leq \left[1 + M \int_0^1 \frac{\Im}{(1 - t\Im)^\alpha} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha},
\end{aligned}$$

where $\Im = \sqrt{1 - \frac{1}{q}(\det(I - Z\overline{Z}')^k - \|\xi\|_p^2)}$.

Case 1: $0 < \alpha < 1$,

$$\begin{aligned}
|f(Z,\xi)| &\leq \left\{1 + \frac{M}{1-\alpha}[1 - (1 - \Im)^{1-\alpha}]\right\} \|f\|_{\mathcal{B}^\alpha} \\
&\leq (1 + \frac{M}{1-\alpha}) \|f\|_{\mathcal{B}^\alpha} \\
&\leq C\|f\|_{\mathcal{B}^\alpha},
\end{aligned} \tag{2.15}$$

where $C = 1 + \frac{M}{1-\alpha}$.

Case 2: $\alpha = 1$,

$$\begin{aligned}
|f(Z,\xi)| &\leq \left[1 + M \int_0^1 \frac{\Im}{1 - t\Im} \mathrm{d}t\right] \|f\|_{\mathcal{B}^\alpha} \\
&= \left[1 + M \ln \frac{1}{1 - \Im}\right] \|f\|_{\mathcal{B}^\alpha} \\
&= \left[1 + M \ln \frac{1 + \Im}{(1 - \Im)(1 + \Im)}\right] \|f\|_{\mathcal{B}^\alpha}
\end{aligned} \tag{2.16}$$

$$
\begin{aligned}
&\le \left[1 + M\ln\frac{2}{1-\Im^2}\right]\|f\|_{\mathcal{B}^\alpha} \\
&\le \left[\frac{1}{\ln 2}\ln\frac{2}{1-\Im^2} + M\ln\frac{2}{1-\Im^2}\right]\|f\|_{\mathcal{B}^\alpha} \\
&\le \left[\frac{1}{\ln 2} + M\right]\ln\frac{2}{1-\Im^2}\|f\|_{\mathcal{B}^\alpha} \\
&= C\|f\|_{\mathcal{B}^\alpha}\ln\frac{2q}{\det(I - Z\overline{Z}')^k - \|\xi\|_p^2},
\end{aligned}
$$

where $C = \frac{1}{\ln 2} + M$.

Case 3: $\alpha > 1$,

$$
\begin{aligned}
|f(Z,\xi)| &\le \left[1 + \frac{M}{\alpha - 1}\left(\frac{1}{(1-\Im)^{\alpha-1}} - 1\right)\right]\|f\|_{\mathcal{B}^\alpha} \\
&\le \left[C' + C'\left(\frac{1}{(1-\Im)^{\alpha-1}} - 1\right)\right]\|f\|_{\mathcal{B}^\alpha} \\
&= C'\|f\|_{\mathcal{B}^\alpha}\frac{1}{(1-\Im)^{\alpha-1}} \\
&= C'\|f\|_{\mathcal{B}^\alpha}\frac{(1+\Im)^{\alpha-1}}{[(1-\Im)(1+\Im)]^{\alpha-1}} \\
&\le 2^{\alpha-1}C'\|f\|_{\mathcal{B}^\alpha}\frac{1}{(1-\Im^2)^{\alpha-1}} \\
&= C\|f\|_{\mathcal{B}^\alpha}\frac{1}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^{\alpha-1}}
\end{aligned}
\tag{2.17}
$$

where $C = (2q)^{\alpha-1}C'$, $C' = \max\{1, \frac{M}{\alpha-1}\}$.

By combining (2.15)(2.16) and (2.17), the proof of the Lemma is complete. □

Lemma 2.12 *Let $\phi = (\phi_{11}, \phi_{12} \dots \phi_{mn+r})$ be a holomorphic self-map of $\mathrm{GHE_I}$ and $\psi \in H(\mathrm{GHE_I})$. The weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact if and only if ψC_ϕ is bounded and for any bounded sequence $\{f_n\}_{n\ge1}$ in $\mathcal{B}^\alpha(\mathrm{GHE_I})$ converging to 0 uniformly on compact subsets of $\mathrm{GHE_I}$, $\|\psi C_\phi f_n\|_{\mathcal{A}_\beta} \to 0$ as $n \to \infty$.*

Proof. Assume that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact. Let $\{f_n\}_{n\ge1}$ be a bounded sequence in $\mathcal{B}^\alpha(\mathrm{GHE_I})$ and $f_n \to 0$ uniformly on compact subsets of $\mathrm{GHE_I}$ as $n \to \infty$.

If $\|\psi C_\phi f_n\|_{\mathcal{A}_\beta} \to 0$ as $n \to \infty$, then there exists a subsequence $\{f_{n_j}\}_{j\ge1}$ of $\{f_n\}_{n\ge1}$ such that

$$\inf_{j\in N}\|\psi C_\phi f_{n_j}\|_{\mathcal{A}_\beta} > 0.$$

Since ψC_ϕ is compact, there exists a subsequence of the bounded sequence $\{f_{n_j}\}_{j\ge1}$(without loss of generality, we still write $\{f_{n_j}\}_{j\ge1}$), such that

$$\lim_{j\to\infty}\|\psi C_\phi f_{n_j} - f\|_{\mathcal{A}_\beta} = 0, \quad f \in \mathcal{A}_\beta(\mathrm{GHE_I}).$$

Let K be a compact subspace of $\mathrm{GHE_I}$. From Lemma 2.1, it follows that

$$|(\psi C_\phi f_{n_j} - f)(Z,\xi)| \le \frac{\|\psi C_\phi f_{n_j} - f\|_{\mathcal{A}_\beta}}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}, \quad j \to \infty.$$

for $\forall(Z,\xi) \in K \subset \mathrm{GHE_I}$. Thus, $\psi C_\phi f_{n_j} - f \to 0$ uniformly on K. This means that for arbitrary $\varepsilon > 0, \exists N_1 > 0$, such that for $j > N_1$, we have

$$|\psi(Z,\xi)f_{n_j}(\phi(Z,\xi)) - f(Z,\xi)| < \varepsilon.$$

for all $(Z,\xi) \in K$. Since $f_{n_j} \to 0$ on compact subsets of $\mathrm{GHE_I}$ as $j \to \infty$, also there exists a positive integer N_2, $|f_{n_j}(\phi(Z,\xi))| < \varepsilon$ for $(Z,\xi) \in K$ whenever $j > N_2$. Let $N = \max\{N_1, N_2\}$ and $M = \max_{(Z,\xi)\in K} |\psi(Z,\xi)|$, whenever $j > N$, we have

$$|f(Z,\xi)| \le |f_{n_j}(\phi(Z,\xi))| \max_{(Z,\xi)\in K} |\psi(Z,\xi)| + \varepsilon \le (M+1)\varepsilon, \ \forall(Z,\xi) \in K.$$

From the arbitrariness of ε, we obtain $f(Z,\xi) \equiv 0, \forall(Z,\xi) \in K$. By the uniqueness theorem of analytic functions, we have $f(Z,\xi) \equiv 0$, $\forall(Z,\xi) \in \mathrm{GHE_I}$. This shows that $\lim_{j\to\infty} \|\psi C_\phi f_{n_j}\|_{\mathcal{A}_\beta} = 0$, which contradicts the assumption $\inf_{j\in N} \|\psi C_\phi f_{n_j}\|_{\mathcal{A}_\beta} > 0$.

Conversely, suppose that $\{f_n\}_{n\ge1}$ is a bounded sequence in $\mathcal{B}^\alpha(\mathrm{GHE_I})$, then $\|f_n\|_{\mathcal{B}^\alpha} \le D_1$, for all n. Clearly $\{f_n\}_{n\ge1}$ is uniformly bounded on compact subsets of $\mathrm{GHE_I}$. By Montel's theorem, there exists a subsequence $\{f_{n_j}\}_{j\ge1}$ of $\{f_n\}_{n\ge1}$ such that $f_{n_j} \to f$ uniformly on every compact subset of $\mathrm{GHE_I}$ and $f \in \mathcal{B}^\alpha(\mathrm{GHE_I})$. For all $(Z_0,\xi_0) \in \mathrm{GHE_I}$, there exists a compact set $K_{(Z_0,\xi_0)}$ such that $(Z_0,\xi_0) \in K_{(Z_0,\xi_0)} \subset \mathrm{GHE_I}$. By Weierstrass's theorem and because $f_{n_j} \to f$ as $j \to \infty$, for $(Z,\xi) \in K_{(Z_0,\xi_0)}$, we obtain $\nabla f_{n_j} \to \nabla f$ as $j \to \infty$. Then, there exists a $J_0 > 0$, such that for $j > J_0$, we have $|\nabla f_{n_j}(Z,\xi) - \nabla f(Z,\xi)| < 1$, for $(Z,\xi) \in K_{(Z_0,\xi_0)}$. In addition, $|\nabla f(Z_0,\xi_0)| \le |\nabla f(Z_0,\xi_0) - \nabla f_{n_j}(Z_0,\xi_0)| + |\nabla f_{n_j}(Z_0,\xi_0)|$, which suffices to obtain

$$\begin{aligned}
&[\det(I - Z_0\overline{Z_0}')^k - \|\xi_0\|_p{}^2]^\alpha |\nabla f(Z_0,\xi_0)| \\
&\le [\det(I - Z_0\overline{Z_0}')^k - \|\xi_0\|_p^2]^\alpha |\nabla f(Z_0,\xi_0) - \nabla f_{n_j}(Z_0,\xi_0)| \\
&+ [\det(I - Z_0\overline{Z_0}')^k - \|\xi_0\|_p^2]^\alpha |\nabla f_{n_j}(Z_0,\xi_0)| \\
&\le 1 + \|f_{n_j}\|_{\mathcal{B}^\alpha} \\
&\le 1 + D_1.
\end{aligned}$$

For all $(Z,\xi) \in \mathrm{GHE_I}$, $[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha |\nabla f(Z,\xi)| \le 1 + D_1$. We thus have $\|f\|_{\mathcal{B}^\alpha} \le 1 + D_1$ and $\|f_{n_j} - f\|_{\mathcal{B}^\alpha} \le \|f_{n_j}\|_{\mathcal{B}^\alpha} + \|f\|_{\mathcal{B}^\alpha} \le 2D_1 + 1$ and $f_{n_j} - f \to 0$ on every compact subset of $\mathrm{GHE_I}$ as $j \to \infty$. Consequently, we have

$$\lim_{j\to\infty} \|\psi C_\phi (f_{n_j} - f)\|_{\mathcal{A}_\beta} = \lim_{j\to\infty} \|\psi C_\phi f_{n_j} - \psi C_\phi f\|_{\mathcal{A}_\beta} = 0,$$

which shows that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact. □

Lemma 2.13 *Let $(Z,\xi), (S,t) \in \mathrm{GHE_I}$, if $0 < km \le 1$, then*

$$\det(I_m - Z\overline{Z}')^k + \det(I_m - S\overline{S}')^k \le 2|\det(I_m - Z\overline{S}')^k|, \tag{2.18}$$

and $'' = ''$ holds if and only if $(Z,\xi) = (S,t)$. If $km > 1$, then

$$\det(I_m - Z\overline{Z}')^k + \det(I_m - S\overline{S}')^k \le 2^{mk}|\det(I_m - Z\overline{S}')^k|. \tag{2.19}$$

Proof. For $m = n$, since $(Z,\xi),(S,t) \in \mathrm{GHE_I}$, there exist two $m \times m$ unitary matrices U_1, U_2 and two $n \times n$ unitary matrices V_1, V_2 (by Lemma 2.6) such that

$$Z = U_1 \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix} V_1 = U_1\Lambda_1V_1 \ \ (1 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0)$$

$$S = U_2 \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_m \end{pmatrix} V_2 = U_2\Lambda_2V_2 \ \ (1 > \mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0).$$

Then, one has

$$\begin{aligned} \det(I - Z\overline{S}') &= \det(I - U_1\Lambda_1V_1\overline{V_2}'\overline{\Lambda_2}'\overline{U_2}') \\ &= \det(U_1\overline{U_1}' - U_1\Lambda_1V_1\overline{V_2}'\overline{\Lambda_2}'\overline{U_2}') \\ &= \det U_1 \det(\overline{U_1}' - \Lambda_1V_1\overline{V_2}'\overline{\Lambda_2}'\overline{U_2}') \\ &= \det(I - \Lambda_1V_1\overline{V_2}'\overline{\Lambda_2}'V_2\overline{V_1}'V_1\overline{V_2}'\overline{U_2}'U_1). \end{aligned}$$

By Lemma 2.7, there exists a square matrix P such that

$$|\det(I - Z\overline{S}')| \geq |\det(I - \Lambda_1P\Lambda_2P')| = \prod_{i=1}^{m}(1 - \lambda_i\mu_{k_i}),$$

where $k_1, k_2, \cdots, k_m$ is a permutation of $1, 2, \cdots, m$.
If $0 < km \leq 1$,and using(2.7) and Lemma 2.8, we get

$$\begin{aligned} 2|\det(I - Z\overline{S}')^k| &= 2^{1-mk} \cdot 2^{mk}|\det(I - Z\overline{S}')^k| \\ &= 2^{1-mk}[2^m|\det(I - Z\overline{S}')|]^k \\ &\geq 2^{1-mk}\left[2^m\prod_{i=1}^{m}(1 - \lambda_i\mu_{k_i})\right]^k \\ &\geq 2^{1-mk}\left\{\{\prod_{i=1}^{m}[(1 - \lambda_i{}^2) + (1 - \mu_{k_i}{}^2)]\}^{\frac{1}{m}}\right\}^{mk} \\ &\geq 2^{1-mk}\left\{\left[\prod_{i=1}^{m}(1 - \lambda_i{}^2)\right]^{\frac{1}{m}} + \left[\prod_{i=1}^{m}(1 - \mu_{k_i}{}^2)\right]^{\frac{1}{m}}\right\}^{mk} \\ &\geq 2^{1-mk} \cdot 2^{mk-1}\left\{\left[\prod_{i=1}^{m}(1 - \lambda_i{}^2)\right]^{\frac{1}{m}\times mk} + \left[\prod_{i=1}^{m}(1 - \mu_{k_i}{}^2)\right]^{\frac{1}{m}\times mk}\right\} \\ &= \left[\prod_{i=1}^{m}(1 - \lambda_i{}^2)\right]^k + \left[\prod_{i=1}^{m}(1 - \mu_{k_i}{}^2)\right]^k \\ &= \det(I - Z\overline{Z}')^k + \det(I - S\overline{S}')^k. \end{aligned} \tag{2.20}$$

If $km > 1$, by using (2.6) and Lemma 2.8, we get

$$\begin{aligned}2^{mk}|\det(I-Z\overline{S}')^k| &= [2^m|\det(I-Z\overline{S}')|]^k \\ &\geq \left[2^m\prod_{i=1}^{m}(1-\lambda_i\mu_{k_i})\right]^k \\ &\geq \left\{\{\prod_{i=1}^{m}[(1-\lambda_i{}^2)+(1-\mu_{k_i}{}^2)]\}^{\frac{1}{m}}\right\}^{mk}\end{aligned} \tag{2.21}$$

$$\begin{aligned}&\geq \left\{\left[\prod_{i=1}^{m}(1-\lambda_i{}^2)\right]^{\frac{1}{m}}+\left[\prod_{i=1}^{m}(1-\mu_{k_i}{}^2)\right]^{\frac{1}{m}}\right\}^{mk} \\ &\geq \left[\prod_{i=1}^{m}(1-\lambda_i{}^2)\right]^{\frac{1}{m}\times mk}+\left[\prod_{i=1}^{m}(1-\mu_{k_i}{}^2)\right]^{\frac{1}{m}\times mk} \\ &= \left[\prod_{i=1}^{m}(1-\lambda_i{}^2)\right]^{k}+\left[\prod_{i=1}^{m}(1-\mu_{k_i}{}^2)\right]^{k} \\ &= \det(I-Z\overline{Z}')^k+\det(I-S\overline{S}')^k.\end{aligned}$$

For $m < n$, there exists a unitary matrix $U^{(n)}$ such that

$$Z=(Z_1{}^{(m)},0)U,\ \ S=(S_1{}^{(m)},S_2)U.$$

According to (2.20), we have

$$\begin{aligned}2|\det(I-Z\overline{S}')^k| &= 2|\det(I-Z_1\overline{S_1}')^k| \\ &\geq \det(I-Z_1\overline{Z_1}')^k+\det(I-S_1\overline{S_1}')^k \\ &\geq \det(I-Z_1\overline{Z_1}')^k+\det(I-S_1\overline{S_1}'-S_2\overline{S_2}')^k \\ &= \det(I-Z\overline{Z}')^k+\det(I-S\overline{S}')^k.\end{aligned}$$

Thus, the inequality

$$2|\det(I-Z\overline{S}')^k| \geq \det(I-Z\overline{Z}')^k+\det(I-S\overline{S}')^k, \tag{2.22}$$

holds when $m \leq n$, whereas the equal sign holds if and only if $Z = S$.

According to (2.21), we see that

$$\begin{aligned}2^{mk}|\det(I-Z\overline{S}')^k| &= 2^{mk}|\det(I-Z_1\overline{S_1}')^k| \\ &\geq \det(I-Z_1\overline{Z_1}')^k+\det(I-S_1\overline{S_1}')^k \\ &\geq \det(I-Z_1\overline{Z_1}')^k+\det(I-S_1\overline{S_1}'-S_2\overline{S_2}')^k \\ &= \det(I-Z\overline{Z}')^k+\det(I-S\overline{S}')^k.\end{aligned}$$

Thus, the inequality

$$2^{mk}|\det(I-Z\overline{S}')^k| \geq \det(I-Z\overline{Z}')^k+\det(I-S\overline{S}')^k, \tag{2.23}$$

holds when $m \leq n$, with the equality holding if and only if $Z = S$ and $mk = 1$.

□

Lemma 2.14 *Assume* $(Z,\xi),(S,t) \in \mathrm{GHE_I}$ *and* $0 < km \leq 1$, *then*

$$[\det(I_m-Z\overline{Z}')^k-\|\xi\|_p^2]+[\det(I_m-S\overline{S}')^k-\|t\|_p^2] \leq 2|\,|\det(I_m-Z\overline{S}')^k|-\|\xi\|_p\|t\|_p|, \tag{2.24}$$

with equality that holds if and only if $(Z,\xi)=(S,t)$.

Proof. Starting from the inequality $a^2+b^2\geq 2ab$, we obtain

$$\|\xi\|_p^2+\|t\|_p^2\geq 2\|\xi\|_p\|t\|_p.$$

Then, by (2.18), we get

$$\begin{aligned}2||\det(I-Z\overline{S}')^k|-\|\xi\|_p\|t\|_p| &= 2|\det(I-Z\overline{S}')^k|-2\|\xi\|_p\|t\|_p\\ &\geq \det(I-Z\overline{Z}')^k+\det(I-S\overline{S}')^k-\|\xi\|_p^2-\|t\|_p^2\\ &= [\det(I-Z\overline{Z}')^k-\|\xi\|_p^2]+[\det(I-S\overline{S}')^k-\|t\|_p^2].\end{aligned}$$

This completes the proof. □

Lemma 2.15 *Assume* $(Z,\xi),(S,t)\in \mathrm{GHE_I}$ *and* $0<km\leq 1$, *then*

$$\left[\det(I_m-Z\overline{Z}')^k-\|\xi\|_p^2\right]\left[\det(I_m-S\overline{S}')^k-\|t\|_p^2\right]\leq ||\det(I_m-Z\overline{S}')^k|-\|\xi\|_p\|t\|_p|^2, \tag{2.25}$$

Proof. By the elementary inequality $\frac{a+b}{2}\geq\sqrt{ab}$ and Lemma 2.14, we have

$$\begin{aligned}&\left[\det(I_m-Z\overline{Z}')^k-\|\xi\|_p^2\right]\left[\det(I_m-S\overline{S}')^k-\|t\|_p^2\right]\\ &\leq \left\{\frac{[\det(I_m-Z\overline{Z}')^k-\|\xi\|_p^2]+[\det(I_m-S\overline{S}')^k-\|t\|_p^2]}{2}\right\}^2\\ &\leq ||\det(I_m-Z\overline{S}')^k|-\|\xi\|_p\|t\|_p|^2\end{aligned}$$

□

Lemma 2.16 (see[18]) *Assume* $Z,S\in\Re_I(m,n)$, *then there exists a constant C such that*

$$|\det(I_m-Z\overline{S}')|\left\{\sum_{\substack{1\leq g\leq m\\ 1\leq l\leq n}}|\mathrm{tr}[(I_m-Z\overline{S}')^{-1}I_{gl}\overline{S}']|^2\right\}^{\frac{1}{2}}\leq C. \tag{2.26}$$

where I_{gl} *is an* $m\times n$ *matrix where the elements of the gth row and lth column are one and the other elements are zero.*

3. Boundedness of $\psi C_\phi:\mathcal{B}^\alpha\to\mathcal{A}_\beta$

Theorem 3.1 *Assume that* $\alpha=1,\beta>0,0<km\leq 1$, *and that* p_j $(j=1,2,\cdots,r)$ *are positive integers. Let* $\phi=(\phi_{11},\phi_{12}\cdots\phi_{mn+r})$ *be a holomorphic self-map of* $\mathrm{GHE_I}$, *with* $\psi\in H(\mathrm{GHE_I})$ *and* $(Z_\phi,\xi_\phi)=\phi(Z,\xi)$. *If*

$$K_1:=\sup_{(Z,\xi)\in\mathrm{GHE_I}}|\psi(Z,\xi)|[\det(I-Z\overline{Z}')^k-\|\xi\|_p^2]^\beta\ln\frac{2q}{\det(I-Z_\phi\overline{Z_\phi}')^k-\|\xi_\phi\|_p^2}<\infty, \tag{3.1}$$

then the weighted composition operator $\psi C_\phi:\mathcal{B}^\alpha(\mathrm{GHE_I})\to\mathcal{A}_\beta(\mathrm{GHE_I})$ *is bounded.*

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Conversely, if the weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded, then

$$K_2 := \sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I - Z_\phi\overline{Z_\phi}')^{1-k} \times \ln \frac{2}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} < \infty. \tag{3.2}$$

Proof. Assume that (3.1) holds. By Lemma 2.11 and for $f \in \mathcal{B}^\alpha(\mathrm{GHE_I})$, we know that

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)||f(\phi(Z,\xi))| \\
&\le C|\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \times \ln \frac{2q}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2}\|f\|_{\mathcal{B}^\alpha} \\
&\le CK_1\|f\|_{\mathcal{B}^\alpha}.
\end{aligned}$$

For all $(Z,\xi) \in \mathrm{GHE_I}$, we have

$$\|\psi C_\phi f\|_{\mathcal{A}_\beta} = \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f)(Z,\xi)| \le CK_1\|f\|_{\mathcal{B}^\alpha},$$

which implies that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded.

Conversely, assume that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded. For any $(S,t) \in \mathrm{GHE_I}$, let us introduce a test function $f_{(S,t)} \in H(\mathrm{GHE_I})$ such that

$$f_{(S,t)}(Z,\xi) := \det(I - S\overline{S}')^{1-k} \ln \frac{2}{\det(I - Z\overline{S}')^k - \langle\xi,t\rangle_p}.$$

This means that

$$\begin{aligned}
\frac{\partial f_{(S,t)}}{\partial z_{gl}} &= \frac{k\cdot\det(I - Z\overline{S}')^{k-1}\cdot\det(I - S\overline{S}')^{1-k}}{\det(I - Z\overline{S}')^k - \langle\xi,t\rangle_p} \times \det(I - Z\overline{S}')\mathrm{tr}[(I - Z\overline{S}')^{-1}I_{gl}\overline{S}'], \\
&\quad 1 \le g \le m, 1 \le l \le n, \\
\frac{\partial f_{(S,t)}}{\partial \xi_j} &= \frac{\det(I - S\overline{S}')^{1-k}\cdot p_j\xi_j^{p_j-1}\overline{t_j}^{p_j}}{\det(I - Z\overline{S}')^k - \langle\xi,t\rangle_p}, \quad j = 1,\cdots,r.
\end{aligned}$$

In view of (2.18), it follows that

$$2|\det(I - Z\overline{S}')|^{\frac{1}{m}} \ge \det(I - Z\overline{Z}')^{\frac{1}{m}} + \det(I - S\overline{S}')^{\frac{1}{m}},$$

then

$$\begin{aligned}
2^{m(1-k)}[|\det(I - Z\overline{S}')|^{\frac{1}{m}}]^{m(1-k)} &\ge [\det(I - Z\overline{Z}')^{\frac{1}{m}} + \det(I - S\overline{S}')^{\frac{1}{m}}]^{m(1-k)} \\
&\ge [\det(I - S\overline{S}')^{\frac{1}{m}}]^{m(1-k)},
\end{aligned}$$

which means that

$$2^{m(1-k)}|\det(I - Z\overline{S}')|^{1-k} \ge \det(I - S\overline{S}')^{1-k}. \tag{3.3}$$

According to (3.3) and Lemmas 2.14, 2.16 , there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]|\nabla f_{(S,t)}(Z,\xi)| \\
&= \frac{\det(I - Z\overline{Z}')^k - \|\xi\|_p^2}{|\det(I - Z\overline{S})^k - \langle \xi, t\rangle_p|} \times \det(I - S\overline{S}')^{1-k} \times \Bigg\{ k^2 |\det(I - Z\overline{S}')^{k-1}|^2 \\
&\quad \times \sum_{\substack{1\le g\le m \\ 1\le l\le n}} |\det(I - Z\overline{S}')\mathrm{tr}[(I - Z\overline{S}')^{-1} I_{gl}\overline{S}']|^2 + \sum_{j=1}^{r} |p_j \xi_j^{p_j-1} \overline{t_j}^{p_j}|^2 \Bigg\}^{\frac{1}{2}} \\
&\le \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2] \times \det(I - S\overline{S}')^{1-k}}{||\det(I - Z\overline{S}')^k| - |\langle \xi, t\rangle_p||} \times \Bigg\{ k|\det(I - Z\overline{S}')|^{k-1} \\
&\quad \times \Bigg[\sum_{\substack{1\le g\le m \\ 1\le l\le n}} |\det(I - Z\overline{S}')|^2 |\mathrm{tr}[(I - Z\overline{S}')^{-1} I_{gl}\overline{S}']|^2 \Bigg]^{\frac{1}{2}} + \Bigg[\sum_{j=1}^{r} |p_j \xi_j^{p_j-1} \overline{t_j}^{p_j}|^2 \Bigg]^{\frac{1}{2}} \Bigg\} \\
&\le \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{||\det(I - Z\overline{S}')^k| - \|\xi\|_p \|t\|_p|} \\
&\quad \times \Bigg\{ kC_1 |\det(I - Z\overline{S}')|^{k-1} \times \det(I - S\overline{S}')^{1-k} + \Bigg[\sum_{j=1}^{r} |p_j \xi_j^{p_j-1} \overline{t_j}^{p_j}|^2 \Bigg]^{\frac{1}{2}} \times \det(I - S\overline{S}')^{1-k} \Bigg\} \\
&\le \frac{2[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2] + [\det(I - S\overline{S}')^k - \|t\|_p^2]} \times \Bigg\{ 2^{m(1-k)} kC_1 + \Bigg[\sum_{j=1}^{r} |p_j|^2 \Bigg]^{\frac{1}{2}} \Bigg\} \\
&\le \frac{2[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{\det(I - Z\overline{Z}')^k - \|\xi\|_p^2} \times \Bigg\{ 2^{m(1-k)} kC_1 + \Bigg[\sum_{j=1}^{r} |p_j|^2 \Bigg]^{\frac{1}{2}} \Bigg\} \\
&\le 2 \times \Bigg\{ 2^{m(1-k)} kC_1 + \Bigg[\sum_{j=1}^{r} |p_j|^2 \Bigg]^{\frac{1}{2}} \Bigg\} \\
&\le C.
\end{aligned}$$

Since $f_{(S,t)}(0,0) \le \ln 2$, one has

$$\begin{aligned}
\|f_{(S,t)}\|_{\mathcal{B}^\alpha} &= |f_{(S,t)}(0,0)| + \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha |\nabla f_{(S,t)}(Z,\xi)| \\
&\le C + \ln 2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\infty &> (C + \ln 2)\|\psi C_\phi\|_{\mathcal{B}^\alpha \to \mathcal{A}_\beta} \\
&\ge \|\psi C_\phi f_{(S,t)}\|_{\mathcal{A}_\beta} \\
&= \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi) f_{(S,t)}(\phi(Z,\xi))| \\
&\ge |\psi(Z,\xi)| [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I - S\overline{S}')^{1-k} \times \left| \ln \frac{2}{\det(I - Z_\phi \overline{S}')^k - \langle \xi_\phi, t\rangle_p} \right|.
\end{aligned}$$

Let us now consider

$$(S,t) = (Z_\phi, \xi_\phi) = \phi(Z,\xi),$$

so that

$$\sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)|[\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I-Z_\phi\overline{Z_\phi}')^{1-k} \ln \frac{2}{\det(I-Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} < \infty.$$

The proof is thus completed. □

Theorem 3.2 *Assume that $\alpha > 1, \beta > 0, 0 < km \le 1$, and that p_j are positive integers $(j = 1, 2, \cdots, r)$. Let $\phi = (\phi_{11}, \phi_{12} \cdots \phi_{mn+r})$ be a holomorphic self-map of $\mathrm{GHE_I}$, with $\psi \in H(\mathrm{GHE_I})$ and $(Z_\phi, \xi_\phi) = \phi(Z,\xi)$. If*

$$K_3 := \sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}{[\det(I-Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} < \infty, \tag{3.4}$$

then the weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded.

Conversely, if the weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded, then

$$K_4 := \sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I-Z_\phi\overline{Z_\phi}')^{1-k}}{[\det(I-Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} < \infty. \tag{3.5}$$

Proof. Assume that (3.4) holds. By Lemma 2.11 and for $f \in \mathcal{B}^\alpha(\mathrm{GHE_I})$, we have

$$\begin{aligned}
[\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f)(Z,\xi)| &= [\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)\cdot (C_\phi f)(Z,\xi)| \\
&= [\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)||f(\phi(Z,\xi))| \\
&\le C|\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}{[\det(I-Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha} \\
&\le CK_3\|f\|_{\mathcal{B}^\alpha}.
\end{aligned}$$

For all $(Z,\xi) \in \mathrm{GHE_I}$, we obtain

$$\|\psi C_\phi f\|_{\mathcal{A}_\beta} = \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I-Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f)(Z,\xi)| \le CK_3\|f\|_{\mathcal{B}^\alpha}.$$

which implies that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded.

Conversely, assume that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded. For $(S,t) \in \mathrm{GHE_I}$, define a test function $f_{(S,t)} \in H(\mathrm{GHE_I})$ such that

$$f_{(S,t)}(Z,\xi) := \frac{\det(I-S\overline{S}')^{1-k}}{[\det(I-Z\overline{S}')^k - \langle \xi, t\rangle_p]^{\alpha-1}}.$$

For the test function f, we have

$$\begin{aligned}
\frac{\partial f_{(S,t)}}{\partial z_{gl}} &= \frac{k(\alpha-1)\cdot \det(I-Z\overline{S}')^{k-1}\cdot \det(I-S\overline{S}')^{1-k}}{[\det(I-Z\overline{S}')^k - \langle \xi, t\rangle_p]^\alpha} \\
&\quad \times \det(I-Z\overline{S}')\mathrm{tr}[(I-Z\overline{S}')^{-1} I_{gl} \overline{S}'], \qquad 1 \le g \le m, 1 \le l \le n, \\
\frac{\partial f_{(S,t)}}{\partial \xi_j} &= \frac{(\alpha-1)p_j \xi_j^{p_j-1} \overline{t_j}^{p_j} \cdot \det(I-S\overline{S}')^{1-k}}{[\det(I-Z\overline{S}')^k - \langle \xi, t\rangle_p]^\alpha}, \qquad j = 1, \cdots, r.
\end{aligned}$$

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From (3.3) and Lemmas 2.14, 2.16, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha |\nabla f_{(S,t)}(Z,\xi)| \\
&= \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha}{|\det(I - Z\overline{S}')^k - \langle \xi, t\rangle_p|^\alpha} \times \det(I - S\overline{S}')^{1-k} \times (\alpha-1) \times \Big\{ k^2 |\det(I - Z\overline{S}')^{k-1}|^2 \\
&\times \sum_{\substack{1\le g\le m\\ 1\le l\le n}} |\det(I - Z\overline{S}')\mathrm{tr}[(I - Z\overline{S}')^{-1} I_{gl}\overline{S}']|^2 + \sum_{j=1}^r |p_j \xi_j^{p_j-1} \overline{t_j}^{p_j}|^2 \Big\}^{\frac{1}{2}} \\
&\le \frac{[\det(I - Z\overline{Z}')|^k - \|\xi\|_p^2]^\alpha \times \det(I - S\overline{S}')^{1-k}}{||\det(I - Z\overline{S}')|^k - |\langle \xi, t\rangle_p||^\alpha} \times (\alpha-1) \times \Big\{ k|\det(I - Z\overline{S}')|^{k-1} \\
&\times \Big[\sum_{\substack{1\le g\le m\\ 1\le l\le n}} |\det(I - Z\overline{S}')|^2 |\mathrm{tr}[(I - Z\overline{S}')^{-1} I_{gl}\overline{S}']|^2 \Big]^{\frac{1}{2}} + \Big[\sum_{j=1}^r |p_j \xi_j^{p_j-1} \overline{t_j}^{p_j}|^2 \Big]^{\frac{1}{2}} \Big\} \\
&\le \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha}{||\det(I - Z\overline{S}')|^k - \|\xi\|_p \|t\|_p|^\alpha} \times (\alpha-1) \times \Big\{ kC_1 \det(I - S\overline{S}')^{1-k} |\det(I - Z\overline{S}')|^{k-1} \\
&+ \Big[\sum_{j=1}^r |p_j \xi_j^{p_j-1} \overline{t_j}^{p_j}|^2 \Big]^{\frac{1}{2}} \det(I - S\overline{S}')^{1-k} \Big\} \\
&\le \left\{ \frac{2[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2] + [\det(I - S\overline{S}')^k - \|\xi\|_p^2]} \right\}^\alpha (\alpha-1) \times (kC_1 2^{m(1-k)} + C_2) \\
&\le \left\{ \frac{2[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{\det(I - Z\overline{Z}')^p - \|\xi\|_p^2} \right\}^\alpha (\alpha-1) \times (kC_1 2^{m(1-k)} + C_2) \\
&\le 2^\alpha (\alpha-1) C_3 \\
&= C_4.
\end{aligned}$$

Since $f_{(S,t)}(0,0) \le 1$, we obtain

$$\begin{aligned}
\|f_{(S,t)}\|_{\mathcal{B}^\alpha} &= |f_{(S,t)}(0,0)| + \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha |\nabla f_{(S,t)}(Z,\xi)| \\
&\le C_4 + 1.
\end{aligned}$$

It follows that

$$\begin{aligned}
\infty > (C_4+1)\|\psi C_\phi\|_{\mathcal{B}^\alpha \to \mathcal{A}_\beta} \ge \|\psi C_\phi f_{(S,t)}\|_{\mathcal{A}_\beta} \\
&= \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi) f_{(S,t)}(\phi(Z,\xi))| \\
&\ge |\psi(Z,\xi)| \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I - S\overline{S}')^{1-k}}{|\det(I - Z_\phi \overline{S}')^k - \langle \xi_\phi, t\rangle_p|^{\alpha-1}}.
\end{aligned}$$

We write $(S,t) = (Z_\phi, \xi_\phi) = \phi(Z,\xi)$, then

$$\sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)| \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I - Z_\phi \overline{Z_\phi}')^{1-k}}{[\det(I - Z_\phi \overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} < \infty.$$

□

This completes the proof of the theorem.

Corollary 1. *For $\alpha > 1$, $k = m = 1, p_1 = \cdots = p_r = 1$, we have the case of the unit ball $\mathbf{B} = \{z \in \mathbb{C}^{n+r} : |z|^2 < 1\}$ and $\psi C_\phi : \mathcal{B}^\alpha(\mathbf{B}) \to \mathcal{A}_\beta(\mathbf{B})$ is bounded if and only if*

$$\sup_{z\in B} \frac{|\psi(z)|(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^{\alpha-1}} < \infty$$

when $\beta = 0$. This result is equivalent to that obtained by Li and Stević in [9].

4. Compactness of $\psi C_\phi : \mathcal{B}^\alpha \to \mathcal{A}_\beta$

Theorem 4.1 *Assume that $\alpha = 1, \beta > 0, 0 < km \le 1$, and that p_j $(j = 1, 2, \cdots, r)$ are positive integers. Let $\phi = (\phi_{11}, \phi_{12} \cdots \phi_{mn+r})$ be a holomorphic self-map of $\mathrm{GHE_I}$, with $\psi \in H(\mathrm{GHE_I})$ and $(Z_\phi, \xi_\phi) = \phi(Z, \xi)$. If $\psi \in \mathcal{A}_\beta$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{GHE_I}} |\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \ln \frac{2q}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} = 0, \tag{4.1}$$

then the weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact.

Conversely, if the weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact, then $\psi \in \mathcal{A}_\beta$ and

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{GHE_I}} |\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \det(I - Z_\phi\overline{Z_\phi}')^{1-k} \times \ln \frac{2}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} = 0. \tag{4.2}$$

Proof. Assume that (4.1) holds. We have

$$\sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \ln \frac{2q}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} < \infty.$$

If ψC_ϕ is bounded, consider the bounded sequence $\{f_k\}_{k\ge1}$ in $\mathcal{B}^\alpha(\mathrm{GHE_I})$, which converges to 0 uniformly on compact subsets of $\mathrm{GHE_I}$. Hence, there exists $M_1 > 0$ such that $\|f_k\|_{\mathcal{B}^\alpha} \le M_1, k = 1, 2, \cdots$. By (4.1), this means that $\forall\, \varepsilon > 0, \exists\, \delta \in (0,1)$, such that for $\mathrm{dist}(\phi(Z,\xi), \partial\mathrm{GHE_I}) < \delta$, we have

$$|\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \ln \frac{2q}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} < \varepsilon. \tag{4.3}$$

According to Lemma 2.11, we obtain

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi) \cdot (C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)||f_k(\phi(Z,\xi))| \\
&\le C|\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta \|f_k\|_{\mathcal{B}^\alpha} \\
&\times \ln \frac{2q}{\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2} \\
&\le CM_1\varepsilon.
\end{aligned} \tag{4.4}$$

On the other hand, let us introduce the set

$$E_\delta := \{(Z,\xi) \in \mathrm{GHE_I} : \mathrm{dist}(\phi(Z,\xi), \partial\mathrm{GHE_I}) \ge \delta\},$$

which is a compact subset of $\mathrm{GHE_I}$. By the assumptions, f_k converges to 0 uniformly on any compact subset of $\mathrm{GHE_I}$. From this, and since $\psi \in \mathcal{A}_\beta$, for such ε, we have

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi) \cdot (C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)||f_k(\phi(Z,\xi))| \\
&\leq \|\psi\|_{\mathcal{A}_\beta} \varepsilon.
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\|\psi C_\phi f_k\|_{\mathcal{A}_\beta} = \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f_k)(Z,\xi)| \to 0, \quad k \to \infty.$$

Consequently, making use of Lemma 2.12, we finally have that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact.

Conversely, suppose $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact. Let $f \equiv 1$, we have

$$[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)| = [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |(\psi C_\phi f)(Z,\xi)| < \infty.$$

This shows that $\psi \in \mathcal{A}_\beta$. Consider now a sequence $(S^i, t^i) = \phi(Z^i, \xi^i)$ in $\mathrm{GHE_I}$ such that $\phi(Z^i, \xi^i) \to \partial\mathrm{GHE_I}$ as $i \to \infty$. If such a sequence does not exist, then condition (4.2) obviously holds. Moreover, let us introduce the following sequence of test functions $\{f_i\}_{i\geq 1}$:

$$\begin{aligned}
f_i(Z,\xi) = &\left\{ \ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2} \right\}^{-1} \times \left\{ \ln \frac{2}{\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p} \right\}^2 \\
&\times \det(I - S^i\overline{S^i}')^{1-k}.
\end{aligned}$$

Differentiation gives

$$\begin{aligned}
\frac{\partial f_i}{\partial z_{gl}} &= \frac{2k \times \det(I - Z\overline{S^i}')^{k-1} \det(I - S^i\overline{S^i}')^{1-k}}{\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p} \times \frac{\ln \frac{2}{\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p}}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\quad \times \det(I - Z\overline{S^i}') \times \mathrm{tr}[(I - Z\overline{S^i}')^{-1} I_{gl} \overline{S^i}'], \qquad 1 \leq g \leq m, 1 \leq l \leq n, \\
\frac{\partial f_i}{\partial \xi_j} &= \frac{2p_j \xi_j^{p_j - 1} \overline{t_j}^{p_j} \times \det(I - S^i\overline{S^i}')^{1-k}}{\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p} \times \frac{\ln \frac{2}{\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p}}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}}, \quad j = 1, \cdots, r. \; i = 1, 2, \cdots
\end{aligned}$$

From (3.3) and Lemmas 2.14, 2.16, there exists a constant $C_5 > 0$ such that

$$
\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi|_p^2]|\nabla f_i(Z,\xi)| \\
&= \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]\det(I - S^i\overline{S^i}')^{1-k}}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \times \left| \frac{\ln \frac{2}{\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p}}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \right| \\
&\times \Bigg\{ 4k^2|\det(I - Z\overline{S^i}')^{k-1}|^2 \\
&\times \sum_{\substack{1\le g\le m\\ 1\le l\le n}} |\det(I - Z\overline{S^i}')\mathrm{tr}[(I - Z\overline{S^i}')^{-1} I_{gl}\overline{S^i}']|^2 + 4\sum_{j=1}^r |p_j\xi_j^{p_j-1}\overline{t_j}^{p_j}|^2 \Bigg\}^{\frac{1}{2}} \\
&\le \frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2][\det(I - S^i\overline{S^i}')^{1-k}}{||\det(I - Z\overline{S^i}')^k| - |\langle \xi, t^i\rangle_p||} \times \frac{\left| \ln \frac{2}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\times \Bigg\{ 2k|\det(I - Z\overline{S^i}')|^{k-1} \\
&\times \left[\sum_{\substack{1\le g\le m\\ 1\le l\le n}} |\det(I - Z\overline{S^i}')\mathrm{tr}[(I - Z\overline{S^i}')^{-1} I_{gl}\overline{S^i}']|^2 \right]^{\frac{1}{2}} + 2\left[\sum_{j=1}^r |p_j\xi_j^{p_j-1}\overline{t_j}^{p_j}|^2 \right]^{\frac{1}{2}} \Bigg\} \\
&\le \frac{\det(I - Z\overline{Z}')^k - \|\xi\|_p^2}{||\det(I - Z\overline{S^i}')^k| - \|\xi\|_p\|t^i\|_p|} \times \frac{\left| \ln \frac{2}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\times \left\{ 2kC_5|\det(I - Z\overline{S^i}')|^{k-1}\det(I - S^i\overline{S^i}')^{1-k} + 2\left[\sum_{j=1}^r |p_j\xi_j^{p_j-1}\overline{t_j}^{p_j}|^2 \right]^{\frac{1}{2}} \det(I - S^i\overline{S^i}')^{1-k} \right\} \\
&\le \frac{2[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2] + [\det(I - S^i\overline{S^i}')^k - \|t\|_p^2]} \times \frac{\left| \ln \frac{2}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\times \left\{ 2kC_5|\det(I - Z\overline{S^i}')|^{k-1}\det(I - S^i\overline{S^i}')^{1-k} + C_6 \right\} \\
&\le \frac{2[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]}{\det(I - Z\overline{Z}')^k - \|\xi\|_p^2} \times \frac{\left| \ln \frac{2}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \times (2kC_5 \cdot 2^{m(1-k)} + C_6) \\
&\le 2 \times (2kC_5 \cdot 2^{m(1-k)} + C_6) \times \frac{\left| \ln \frac{2}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le C_7 \times \frac{\left| \ln \frac{2}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2}}.
\end{aligned}
$$

We have now two cases.

Case A_1. If $|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p| \le 2$, then

$$\begin{aligned}
\frac{\left| \ln \frac{2}{|\det(I-Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} &\le \frac{\ln \frac{2}{|\det(I-Z\overline{S^i}')^k| - |\langle \xi, t^i\rangle_p|} + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le \frac{\ln \frac{2}{|\det(I-Z\overline{S^i}')^k| - \|\xi\|_p \|t^i\|_p} + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le \frac{\ln \frac{4}{[\det(I-Z\overline{Z}')^k| - \|\xi\|_p^2] + [\det(I-S^i\overline{S^i}')^k - \|t^i\|^2]_p} + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le \frac{\ln \frac{4}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2} + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le 2 + \frac{\pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le C_8,
\end{aligned} \tag{4.6}$$

where $C_8 = 2 + \frac{\pi}{\ln 2}$.

Case A_2. If $|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i \rangle_p| > 2$, then

$$\frac{\left| \ln \frac{2}{|\det(I-Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|} \right| + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} = \frac{|\ln 2 - \ln|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|| + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \tag{4.7}$$

$$\begin{aligned}
&\le \frac{\ln|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p| + \pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k - \|t^i\|_p^2}} \\
&\le \frac{\ln(|\det(I - Z\overline{S^i}')^k| + |\langle \xi, t^i\rangle_p|) + \pi}{\ln 2}.
\end{aligned}$$

Since $Z\overline{S^i}' = C = (c_{ij})_{m\times m}$, $c_{ij} = \sum_{g=1}^n z_{ig}s_{jg}$ $(i, j = 1, \cdots, m)$, we may write $I - C = D = (d_{ij})_{m\times m}$ with

$$d_{ij} = \begin{cases} 1 - (\sum_{g=1}^n z_{ig}s_{jg}) & i = j \\ -\sum_{g=1}^n z_{ig}s_{jg} & i \ne j \end{cases} \tag{4.8}$$

Using (4.8), we have $\det(I - C) = \sum_{j_1 j_2 \cdots j_m} (-1)^{\tau(j_1 j_2 \cdots j_m)} d_{1j_1} d_{2j_2} \cdots d_{mj_m}$ and

$$|\det(I - C)| \le m!(n+1)^m = G$$

Hence,

$$\frac{\left|\ln \frac{2}{|\det(I-Z\overline{S^i}')^k-\langle\xi,t^i\rangle_p|}\right|+\pi}{\ln \frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}} \le \frac{\ln(G^k+\|\xi\|_p\|t^i\|_p)+\pi}{\ln 2} \le \frac{\ln(G^k+1)+\pi}{\ln 2} \le C_9.$$

By using both cases A_1 and A_2, we have $[\det(I-Z\overline{Z}')-\|\xi\|_p^2]|\nabla f_i(Z,\xi)| \le QC_7$ and then $\|f_i\|_{\mathcal{B}^\alpha} \le QC_7$, which means that $\|f_i\|_{\mathcal{B}^\alpha}$ is bounded, where $Q=\max\{C_8,C_9\}$. It follows that $\{f_i\}_{i\ge 1}\in\mathcal{B}^\alpha(\mathrm{GHE_I})$ and

$$\begin{aligned}|f_i(Z,\xi)| &= \left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p{}^2}\right\}^{-1}\times\left|\ln\frac{2}{\det(I-Z\overline{S^i}')^k-\langle\xi,t^i\rangle_p}\right|^2\\ &\quad\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\le\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\quad\times\left\{\left|\ln\frac{2}{|\det(I-Z\overline{S^i}')^k-\langle\xi,t^i\rangle_p|}\right|+\pi\right\}^2.\end{aligned}$$

If $|\det(I-Z\overline{S^i}')^k-\langle\xi,t^i\rangle_p|\le 2$, then

$$\begin{aligned}|f_i(Z,\xi)| &\le\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\quad\times\left\{\ln\frac{2}{|\det(I-Z\overline{S^i}')^k|-|\langle\xi,t^i\rangle_p|}+\pi\right\}^2\\ &\le\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\quad\times\left\{\ln\frac{2}{|\det(I-Z\overline{S^i}')^k|-\|\xi\|_p\|t^i\|_p}+\pi\right\}^2\\ &\le\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\quad\times\left\{\ln\frac{4}{[\det(I-Z\overline{Z}')^k-\|\xi\|_p^2]+[\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2]}+\pi\right\}^2\\ &\le\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\quad\times\left\{\ln\frac{4}{\det(I-Z\overline{Z}')^k-\|\xi\|_p^2}+\pi\right\}^2\end{aligned}$$

Since $0<\det(I-S^i\overline{S^i}')^{1-k}\le 1$, we take $i\to\infty$ and obtain $(S^i,t^i)\to\partial\mathrm{GHE_I}$. This implies $\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2\to 0$, then $\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\to 0$. Let us now consider a compact subset

E of $\mathrm{GHE_I}$. For $(Z,\xi)\in E$,it is easy to see that $\det(I-Z\overline{Z}')^k-\|\xi\|_p^2$ has a positive lower bound. Thus, we have $f_i(Z,\xi)\to 0,\ i\to\infty$ on all compact subsets of $\mathrm{GHE_I}$. If $|\det(I-Z\overline{S^i}')^k-\langle\xi,t^i\rangle_p|>2$, then

$$\begin{aligned}|f_i(Z,\xi)| \le &\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\times\left\{|\ln 2-\ln(|\det(I-Z\overline{S^i}')^k-\langle\xi,t^i\rangle_p|)|+\pi\right\}^2\\ \le&\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\times\left\{\ln(|\det(I-Z\overline{S^i}')^k|+|\langle\xi,t^i\rangle_p|)+\pi\right\}^2\\ \le&\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\times\det(I-S^i\overline{S^i}')^{1-k}\\ &\times\{\ln(G^k+1)+\pi\}^2\end{aligned}$$

From $0<\det(I-S^i\overline{S^i}')^{1-k}\le 1$ and $\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\to 0$ as $i\to\infty$, one concludes that $f_i(Z,\xi)\to 0,\ i\to\infty$.

The above proof shows that $f_i(Z,\xi)\to 0,\ i\to\infty$ on all compact subsets of $\mathrm{GHE_I}$. By Lemma 2.12, this implies that $\|\psi C_\phi f_i\|_{\mathcal{A}_\beta}\to 0$. Therefore, we conclude that

$$\begin{aligned}0\leftarrow&\|\psi C_\phi f_i\|_{\mathcal{A}_\beta}\\ =&\sup_{\phi(Z,\xi)\in\mathrm{GHE_I}}[\det(I-Z\overline{Z}')^k-\|\xi\|_p^2]^\beta|\psi(Z,\xi)|\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\\ &\times\left|\ln\frac{2}{\det(I-Z_\phi\overline{S^i}')^k-\langle\xi_\phi,t^i\rangle_p}\right|^2\times\det(I-S^i\overline{S^i}')^{1-k}\\ \ge&[\det(I-Z^i\overline{Z^i}')^k-\|\xi^i\|_p^2]^\beta|\psi(Z^i,\xi^i)|\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^{-1}\\ &\times\left\{\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}\right\}^2\times\det(I-S^i\overline{S^i}')^{1-k}\\ =&|\psi(Z^i,\xi^i)|[\det(I-Z^i\overline{Z^i}')^k-\|\xi^i\|_p^2]^\beta\det(I-S^i\overline{S^i}')^{1-k}\times\ln\frac{2}{\det(I-S^i\overline{S^i}')^k-\|t^i\|_p^2}.\end{aligned}$$

□

Theorem 4.2 *Assume that $\alpha>1,\beta>0$, $0<km\le 1$, and that p_j are some positive integers $(j=1,2,\cdots,r)$. Let $\phi=(\phi_{11},\phi_{12}\cdots\phi_{mn+r})$ be a holomorphic self-map of $\mathrm{GHE_I}$, with $\psi\in H(\mathrm{GHE_I})$ and $(Z_\phi,\xi_\phi)=\phi(Z,\xi)$. If $\psi\in\mathcal{A}_\beta$ and*

$$\lim_{\phi(Z,\xi)\to\partial\mathrm{GHE_I}}\frac{|\psi(Z,\xi)|[\det(I-Z\overline{Z}')^k-\|\xi\|_p^2]^\beta}{[\det(I-Z_\phi\overline{Z_\phi}')^k-\|\xi_\phi\|_p^2]^{\alpha-1}}=0, \tag{4.9}$$

then the weighted composition operator $\psi C_\phi:\mathcal{B}^\alpha(\mathrm{GHE_I})\to\mathcal{A}_\beta(\mathrm{GHE_I})$ is compact.

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Conversely, if the weighted composition operator $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact, then $\psi \in \mathcal{A}_\beta$ and

$$\lim_{\phi(Z,\xi)\to\partial\mathrm{GHE_I}} \frac{|\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}{[\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-\frac{1}{k}}} = 0. \tag{4.10}$$

Proof. Assume that (4.9) holds. We have

$$\sup_{(Z,\xi)\in\mathrm{GHE_I}} \frac{|\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}{[\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} < \infty.$$

From Theorem 3.2, it follows that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded. Let $\{f_k\}_{k\geq 1}$ be a bounded sequence in $\mathcal{B}^\alpha(\mathrm{GHE_I})$ with f_k that converges to 0 uniformly on compact subsets of $\mathrm{GHE_I}$. There exists $M_2 > 0$ such that $\|f_k\|_{\mathcal{B}^\alpha} \leq M_2$, $k = 1, 2, \cdots$. By (4.9), for any $\varepsilon > 0$, there is a constant $\delta \in (0,1)$ such that

$$\frac{|\psi(Z,\xi)|[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}{[\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} < \varepsilon, \tag{4.11}$$

for $\mathrm{dist}(\phi(Z,\xi), \partial\mathrm{GHE_I}) < \delta$. Using Lemma 2.11 we have

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|(\psi C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|\psi(Z,\xi)\cdot(C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|\psi(Z,\xi)||f_k(\phi(Z,\xi))| \\
&\leq C|\psi(Z,\xi)|\|f_k\|_{\mathcal{B}^\alpha}\frac{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta}{[\det(I - Z_\phi\overline{Z_\phi}')^k - \|\xi_\phi\|_p^2]^{\alpha-1}} \\
&\leq CM_2\varepsilon.
\end{aligned} \tag{4.12}$$

On the other hand, if we set

$$E_\delta := \{(Z,\xi) \in \mathrm{GHE_I} : \mathrm{dist}(\phi(Z,\xi), \partial\mathrm{GHE_I}) \geq \delta\},$$

we have that E_δ is a compact subset of $\mathrm{GHE_I}$. For ε defined in (4.11), f_k converges to 0 uniformly on any compact subset of $\mathrm{GHE_I}$. For $\psi \in \mathcal{A}_\beta$, we have

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|(\psi C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|\psi(Z,\xi)\cdot(C_\phi f_k)(Z,\xi)| \\
&= [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|\psi(Z,\xi)||f_k(\phi(Z,\xi))| \\
&\leq \|\psi\|_{\mathcal{A}_\beta}\varepsilon.
\end{aligned} \tag{4.13}$$

According to inequalities (4.12) and (4.13), we see that

$$\|\psi C_\phi f_k\|_{\mathcal{A}_\beta} = \sup_{(Z,\xi)\in\mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|(\psi C_\phi f_k)(Z,\xi)| \to 0, \quad k \to \infty.$$

Consequently, making use of Lemma 2.12, we have that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact.

Conversely, suppose that $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact. Then, $\psi C_\phi : \mathcal{B}^\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded. Let $f \equiv 1$, we get

$$[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|\psi(Z,\xi)| = [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta|(\psi C_\phi f)(Z,\xi)| < \infty.$$

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This shows that $\psi \in \mathcal{A}_\beta$. Consider now a sequence $(S^i, t^i) = \phi(Z^i, \xi^i)$ in $\mathrm{GHE_I}$ such that $\phi(Z^i, \xi^i) \to \partial\mathrm{GHE_I}$ as $i \to \infty$. If such a sequence does not exist, then condition (4.10) obviously holds.

Moreover, let us introduce a sequence of test functions $\{f_i\}_{i\geq 1}$:

$$f_i(Z, \xi) := \frac{[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{[\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p]^{2\alpha-1}}.$$

Differentiation gives

$$\begin{aligned}
\frac{\partial f_i}{\partial z_{gl}} &= \frac{(2\alpha-1)k\cdot\det(I - Z\overline{S^i}')^{k-1}[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{[\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p]^{2\alpha}} \\
&\quad \times \det(I - Z\overline{S^i}')\mathrm{tr}[(I - Z\overline{S^i}')^{-1} I_{gl}\overline{S^i}'], \\
\frac{\partial f_i}{\partial \xi_j} &= \frac{(2\alpha-1)p_j\xi_j^{p_j-1}\overline{t_j}^{p_j}[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{[\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p]^{2\alpha}}.
\end{aligned}$$

From (3.3) and Lemma 2.15, it follows that there exists a constant $C_{10} > 0$ such that

$$\begin{aligned}
&[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha |\nabla f_i(Z, \xi)| \\
&= \frac{(2\alpha-1)[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|^{2\alpha}} \times \Big\{k^2|\det(I - Z\overline{S^i}')^{k-1}|^2 \\
&\times \sum_{\substack{1\leq g\leq m\\ 1\leq l\leq n}} |\det(I - Z\overline{S^i}')\mathrm{tr}[(I - Z\overline{S^i}')^{-1} I_{gl}\overline{S^i}']|^2 + \sum_{j=1}^r |p_j\xi_j^{p_j-1}\overline{t_j}^{p_j}|^2\Big\}^{\frac{1}{2}} \\
&\leq \frac{(2\alpha-1)[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\alpha[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^\alpha}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|^{2\alpha}} \\
&\times \Big\{k|\det(I - Z\overline{S^i}')|^{k-1} \times [\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1} \\
&\quad \times \Big[\sum_{\substack{1\leq g\leq m\\ 1\leq l\leq n}} |\det(I - Z\overline{S^i}')\mathrm{tr}[(I - Z\overline{S^i}')^{-1} I_{gl}\overline{S^i}']|^2\Big]^{\frac{1}{2}} \\
&\quad + \Big[\sum_{j=1}^r |p_j\xi_j^{p_j-1}\overline{t_j}'^{p_j}|^2\Big]^{\frac{1}{2}} \times [\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1}\Big\} \\
&\leq (2\alpha-1) \times \Big\{k\cdot C_1|\det(I - Z\overline{S^i}')|^{k-1}[\det(I - S^i\overline{S^i}')^k]^{\frac{1}{k}-1} + C_{10}\Big\} \\
&\leq (2\alpha-1) \times \Big\{k\cdot C_1|\det(I - Z\overline{S^i}')|^{k-1}\det(I - S^i\overline{S^i}')^{1-k} + C_{10}\Big\} \\
&\leq (2\alpha-1) \times \Big\{C_1\cdot k\cdot 2^{m(1-k)} + C_{10}\Big\} \\
&\leq C''.
\end{aligned}$$

This shows that $f_i \in \mathcal{B}^\alpha(\mathrm{GHE_I})$,$i = 1, 2, \cdots$ and

$$\begin{aligned}|f_i(Z,\xi)| &= \frac{[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{|\det(I - Z\overline{S^i}')^k - \langle \xi, t^i\rangle_p|^{2\alpha-1}} \\ &\le \frac{[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{||\det(I - Z\overline{S^i}')^k| - |\langle \xi, t^i\rangle_p||^{2\alpha-1}} \\ &\le \frac{[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{|\det(I - Z\overline{S^i}')^k| - \|\xi\|_p\|t^i\|_p|^{2\alpha-1}} \\ &\le \frac{2^{2\alpha-1}[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2 + \det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{2\alpha-1}} \\ &\le \frac{2^{2\alpha-1}[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{[\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^{2\alpha-1}}.\end{aligned}$$

Taking $i \to \infty$, we have $(S^i, t^i) \to \partial\mathrm{GHE_I}$. This implies that $\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2 \to 0$. If E is a compact subset of $\mathrm{GHE_I}$, for $(Z, \xi) \in E$, we have that $\det(I - Z\overline{Z}')^k - \|\xi\|_p^2$ has a positive lower bound. Thus, we have $f_i(Z, \xi) \to 0$, $i \to \infty$ on all compact subsets of $\mathrm{GHE_I}$. According to Lemma 2.12, we have that $\|\psi C_\phi f_i\|_{\mathcal{A}_\beta} \to 0$. Hence,

$$\begin{aligned}0 \leftarrow \|\psi C_\phi f_i\|_{\mathcal{A}_\beta} &= \sup_{\phi(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \|\xi\|_p^2]^\beta |\psi(Z,\xi)| \frac{[\det(I - S^i\overline{S^i}')^k - \|t^i\|_p^2]^{\frac{1}{k}-1+\alpha}}{|\det(I - Z_\phi\overline{S^i}')^k - \langle \xi_\phi, t^i\rangle_p|^{2\alpha-1}} \\ &\ge [\det(I - Z^i\overline{Z^i}')^k - \|\xi^i\|_p^2]^\beta \frac{|\psi(Z^i, \xi^i)|}{[\det(I - Z^i_\phi\overline{Z^i_\phi}')^k - \|\xi^i_\phi\|_p^2]^{\alpha-\frac{1}{k}}}.\end{aligned}$$

□

Corollary 2. *For $\alpha > 1$, $k = m = 1, p_1 = \cdots = p_r = 1$, we are back to the case of the unit ball $\mathbf{B} = \{z \in \mathbb{C}^{n+r} : |z|^2 < 1\}$, and $\psi C_\phi : \mathcal{B}^\alpha(\mathbf{B}) \to \mathcal{A}_\beta(\mathbf{B})$ is compact if and only if $\psi \in \mathcal{A}_\beta$ and*

$$\lim_{\phi(z)\to\partial B} \frac{|\psi(z)|(1 - |z|^2)^\beta}{(1 - |\phi(z)|^2)^{\alpha-1}} = 0$$

when $\beta = 0$. Also in this case, the result is analogue to that obtained by Li and Stević in [9].

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