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Generalized Minkowski-Type Integral Formulas for Compact Hypersurfaces in Pseudo-Riemannian Manifolds

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Abstract: We obtain some generalised Minkowski-type integral formulas for compact Riemannian (resp. spacelike) hypersurfaces in Riemannian (resp. Lorentzian) manifolds admitting an arbitrary vector field that we assume to be timelike in the case where the ambient space is Lorentzian. Some of these formulas generalize existing formulas in the case of conformal and Killing vector fields. We apply these integral formulas to obtain interesting results concerning the characterization of such hypersurfaces in some particular cases such as when the ambient space is Einstein admitting an arbitrary (in particular, conformal or Killing) vector field, and when the hypersurface has constant mean curvature.

Keywords: Minkowski-type integral formulas; conformal and Killing vector fields; Ricci and scalar curvatures; constant mean curvature (CMC) hypersurfaces; Minimal and maximal hypersurfaces

MSC: 53A10; 53C40; 53C42; 53C65

1. Introduction

In 1903 H. Minkowski published in [16] his two famous integral formulas for compact surfaces in three dimensional Euclidean space. After that, many authors obtained integral formulas that generalized the two Minkowski formulas to hypersurfaces in Euclidean space and then in a general Riemannian manifold that admits a Killing or conformal vector field. For instance, in [11] and [12], C. C. Hsiung obtained generalized integral formulas of Minkowsi type for embedded hypersurfaces in Riemannian manifolds (see also [13]). In [14] and [15], Y. Katsurada generalized the work of Hsiung and derived some integral formulas of Minkowski type that are valid for Einstein manifolds and used them to prove that given a hypersurface (M, g) with constant mean curvature in an Einstein Riemannian manifold $(\overline{M}, \overline{g})$, and given a homothetic vector field ξ of $(\overline{M}, \overline{g})$ such that the inner product of ξ and the normal to M does not change sign and does not vanish on M, then M is necessarily umbilical. In [19], K. Yano obtained three integral formulas of Minkowski type for hypersurfaces with constant mean curvature in a Riemannian manifold admitting a homothetic vector field. Then, over time, several integral formulas of Minkowski type appeared in literature that were used to obtain rigidity results for isometrically immersed hypersurfaces in pseudo-Riemannian manifolds admitting a conformal vector filed. In [6] and [7] (resp. [8]), L. J. Alias, A. Romero, and M. Sanchez obtained the first and second integral formulas of Minkowski type for compact spacelike hypersurfaces in a generalized Robertson–Walker spacetime (resp. conformally stationary spacetime), and applied them to the study of compact spacelike hypersurfaces with constant mean curvature. Two years later, in [17], S. Montiel provided another proof of the first and second Minkowski formulas in the case where the ambient spacetime is equipped with a conformal timelike vector field. In 2003, L. J. Alias, A. Brasil JR, and A. G. Colares generalized in [2] the integral formulas obtained in [6–8] for spacelike hypersurfaces in conformally stationary spacetimes. See also [3] and [4].

The assumption that the ambient space admits a conformal vector field is inspired by the fact that the position vector field in Euclidean space is a closed conformal vector field (which in some references

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is called a concircular vector field). The importance of conforml vector fields comes from the use of conformal mappings as a mathematical tool in general relativity. In fact, although a conformal vector field does not leave the Einstein tensor invariant, its existence in a pseudo-Riemannian manifold (M, g) is a symmetry assumption for g that can be used (for example) to obtain exact solutions of Einstein's equation.

Consider now an $(n+1)$-dimensional either Riemannian or Lorentzian manifold $(\overline{M}, \overline{g})$ admitting a conformal vector field $\overline{\xi}$ that we assume to be timelike in the case where $(\overline{M}, \overline{g})$ is Lorentzian. Let (M, g) be a connected n-dimensional Riemannian manifold that is isometrically immersed as a hypersurface into $(\overline{M}, \overline{g})$, and let ξ denote the restriction of $\overline{\xi}$ to M. Consider the function $\theta = \overline{g}(\xi, N)$, where $\overline{\xi}$ is an arbitrary vector field and N is a globally defined unit vector field normal to M. In the case where is Riemannian, L. J. Alias, M. Dajczer, and J. Ripoll gave in [5] an expression for the Laplacian $\Delta\theta$ in terms of the Ricci curvature of $(\overline{M}, \overline{g})$ and the norm of the shape operator of (M, g). One year later, in 2008, A. Barros, A. Brasil, and A. Caminha obtained in [9] the analogous expression when $(\overline{M}, \overline{g})$ is Lorentzian.

In 2010, A. L. Albujer, J. A. Aledo, and L. J. Alias gave in [1] an expression for $\Delta\theta$ in a slightly different way as given in [5] and [9]. Then, they used this expression to obtain a Minkowski-type integral formula for compact Riemannian and spacelike hypersurfaces, and applied this to deduce some interesting results concerning the characterization of compact Riemannian and spacelike hypersurfaces under certain hypotheses like the constancy of the mean curvature of the assumption that the ambient space is Einstein or a product space.

In this paper, we mainly wish to generalize previous results concerning Minkowski-type integral formulas for Riemannian (resp. spacelike) hypersurfaces in Riemannian (resp. Lorentzian) manifolds admitting an arbitrary vector field that we assume to be timelike in the case where is Lorentzian, and apply these integral forms to compact Riemannian and spacelike hypersurfaces in order to obtain interesting results concerning the characterization of such hypersurfaces in some particular cases, such as the ambient space is Einstein admitting an arbitrary (and in particular, a conformal Killing) vector field, or the hypersurface is minimal (resp. maximal) or has constant mean curvature.

In particular, we wish to work to generalize the results in [19] and [1] for any arbitrary Riemannian or spacelike hypersurface in any arbitrary ambient space admitting an arbitrary vector field. More precisely, given an $(n+1)$-dimensional either Riemannian or Lorentzian manifold $(\overline{M}, \overline{g})$ admitting an arbitrary vector field $\overline{\xi}$ that we assume to be timelike in the case where $(\overline{M}, \overline{g})$ is Lorentzian, and given a connected n-dimensional Riemannian manifold (M, g) that is isometrically immersed as a hypersurface into $(\overline{M}, \overline{g})$. Let ξ denote the restriction of $\overline{\xi}$ to M, and let N be a globally defined unit vector field normal to M. Of course, N is supposed to be timelike in the case where $(\overline{M}, \overline{g})$ is Lorentzian. Our first main goal in this paper is to give a useful expression for the Laplacian $\Delta\theta$ of the function $\theta = \overline{g}(\xi, N)$ in terms of the Ricci and scalar curvatures of the ambient space, the mean curvature of the hypersurface, and the tangent part of the restriction of the vector field ξ to M. In the particular case where $\overline{\xi}$ is a conformal (resp. Killing) vector field, our expression reduces to that obtained in [1] (resp. [19]). We will deduce from the generalized expression for $\Delta\theta$ different generalized Minkowski-type integral formulas valid for any Riemannian or spacelike hypersurface in any arbitrary Riemannian or Lorentzian manifold admitting an arbitrary vector field. We will, in particular, generalize an integral formula obtained in [1] in the case when $\overline{\xi}$ is conformal to the case of an arbitrary vector field. We will also apply the obtained generalized Minkowski-type formulas to deduce interesting results concerning the characterization of Riemannian and spacelike hypersurfaces in some particular cases, such as the ambient space is Einstein admitting an arbitrary (and in particular, a conformal Killing) vector field, or the hypersurface has constant mean curvature.

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2. Preliminaries

Let $n \geq 2$, and let (M, g) be a connected n-dimensional pseudo-Riemannian manifold. In this paper, we adopt the opposite convention of that in [18] to define the Riemannian tensor. That is, the Riemannian tensor is defined here to be the $(1,3)$ tensor field given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

For every $p \in M$ and every orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, the Ricci curvature tensor Ric and the scalar curvature $Scal$ are, respectively, defined to be

$$Ric(X,Y) = trace(Z \mapsto R(Z,X)Y) = \sum_{i=1}^{n} \epsilon_i g(R(X,e_i)e_i, Y),$$

for all $X, Y \in T_pM$.

$$Scal(p) = trace(Ric) = \sum_{i=1}^{n} \epsilon_i Ric(e_i, e_i),$$

where $\epsilon_i = g(e_i, e_i)$.

Throughout this paper, we will assume that (M, g) is Riemannian (i.e. the metric g has index 0) which is isometrically immersed as a hypersurface into an $(n+1)$-dimensional pseudo-Riemannian manifold $\left(\overline{M}, \overline{g}\right)$ that we assume to be Riemannian or Lorentzian (i.e. the metric $\overline{g}$ has index 0 or 1). Let ∇ and $\overline{\nabla}$ denote the Levi-Civita connections on M and $\overline{M}$, respectively. Let $\mathfrak{X}(M)$ and $\mathfrak{X}(\overline{M})$ denote, respectively, the sets of all tangent vector fields on M and $\overline{M}$, and let $\overline{\mathfrak{X}}(M)$ denote the set of all $\overline{M}$ vector fields on M. We will use the two notations $X \cdot f$ or $X(f)$ to denote the value of a vector field X on a function f.

Let $\overline{\xi} \in \mathfrak{X}(\overline{M})$ that we will assume to be timelike in the case where $\overline{M}$ is Lorentzian, and let $\theta_{\overline{\xi}}$ denote its dual one form, that is, the one form given by $\theta_{\overline{\xi}}(Y) = \overline{g}(\overline{\xi}, Y)$, for every $Y \in \mathfrak{X}(\overline{M})$. Let $A_{\overline{\xi}}$ be the $(1,1)$-tensor (viewed as an endomorphism) defined by

$$A_{\overline{\xi}}(Y) = \overline{\nabla}_Y \overline{\xi}$$

Write as usual

$$L_{\overline{\xi}}\overline{g}(Y,Z) + d\theta_{\overline{\xi}}(Y,Z) = 2\overline{g}\left(A_{\overline{\xi}}(Y), Z\right),$$

for all $Y, Z \in \mathfrak{X}(\overline{M})$, where L is the Lie derivative of the metric $\overline{g}$ with respect to $\overline{\xi}$ a

Let B and ϕ be the symmetric and skew-symmetric parts of $A_{\overline{\xi}}$. In other words, we have

$$L_{\overline{\xi}}\overline{g}(Y,Z) = 2\overline{g}(B(Y), Z) \tag{1}$$

$$d\theta_{\overline{\xi}}(Y,Z) = 2\overline{g}(\phi(Y), Z), \tag{2}$$

Now, in the case where $\overline{M}$ is Riemannian, we will assume that there exists a globally defined unit vector field N normal to M. In this case, M is said to be a two-sided hypersurface. In the case where $\overline{M}$ is Lorentzian, since M is a spacelike hypersurface in $\overline{M}$ and $\overline{\xi}$ is assumed to be timelike, then we can choose a (globally defined) timelike unit vector field N normal to M and in the same time-orientation of $\overline{\xi}$, that is we have $\overline{g}\left(\overline{\xi}, N\right) < 0$ on M. In both cases, if ξ is the restriction of $\overline{\xi}$ to M, then we will denote by θ the smooth function on M, called the support function that is defined by $\theta = \overline{g}(\xi, N)$. It is clear that in the case where $\overline{M}$ is Lorentzian we have $\theta \leq -\sqrt{-\overline{g}(\xi, \xi)} < 0$. If T is the tangential component of ξ to M,then we have

$$\xi = T + \epsilon\theta N, \tag{3}$$

where $\epsilon = \overline{g}(N, N) = \pm 1$, according to whether $\overline{M}$ is Riemannian or Lorentzian, respectively.

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Since $\xi \in \overline{\mathfrak{X}}(M)$, then the operator $A_\xi : \mathfrak{X}(M) \to \overline{\mathfrak{X}}(M)$ given by $A_\xi(X) = \overline{\nabla}_X \xi$ is well-defined (see for instance [18], pp. 97-99). Then, we have

$$A_\xi(X) = \psi(X) + \epsilon\alpha(X)N, \tag{4}$$

where $\psi(X) = \left(\overline{\nabla}_X \xi\right)^\top$ is the tangential component of $\overline{\nabla}_X \xi$ to M and α is a one form on M.

Let $\eta \in \mathfrak{X}(M)$ be the vector field associated to α. Therefore, for all $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} g(\eta, X) &= \alpha(X) \\ &= \overline{g}\left(A_\xi(X), N\right) \\ &= \overline{g}\left(\phi(X) + B(X), N\right) \\ &= -\overline{g}\left(X, \phi(N)\right) + \overline{g}\left(X, B(N)\right) \\ &= -g\left(X, \phi(N)\right) + g\left(X, B(N)^\top\right) \end{aligned} \tag{5}$$

Since ϕ is skew-symmetric, we have $\overline{g}(\phi(N), N) = 0$, that is $\phi(N) \in \mathfrak{X}(M)$. Therefore, (5) implies that

$$\eta = B(N)^\top - \phi(N) \tag{6}$$

On the other hand, the Gauss and Weingarten formulae for M as a hypersurface of $\overline{M}$ are given by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \epsilon g(A(X), Y)N \\ A(X) &= -\overline{\nabla}_X N, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, where A is the shape operator of M with respect to N.

Therefore, for all $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \overline{\nabla}_X \xi &= \overline{\nabla}_X (T + \epsilon\theta N) \\ &= \overline{\nabla}_X . T + \epsilon(X \cdot \theta)N + \epsilon\theta\overline{\nabla}_X N \\ &= \nabla_X T + \epsilon g(A(X), T)N + \epsilon g(\nabla\theta, X)N - \epsilon\theta A(X) \\ &= \nabla_X T - \epsilon\theta A(X) + \epsilon g(A(T) + \nabla\theta, X)N \end{aligned} \tag{7}$$

From (4) and (7) we deduce that

$$\nabla_X T = \psi(X) + \epsilon\theta A(X) \tag{8}$$

$$\nabla\theta = \eta - A(T) \tag{9}$$

For that we need to recall some definitions. In general, recall that for a $(1,1)$-tensor S, the covariant derivative ∇S of S is defined as follows

$$\nabla S(X, Y) = (\nabla_X S)(Y) = \nabla_X (S(Y)) - S(\nabla_X Y)$$

The divergence of a vector field $X \in \mathfrak{X}(M)$ is defined as the function

$$div(X) = \sum_{i=1}^{n} g\left(\nabla_{e_i} X, e_i\right), \tag{10}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of vector fields in $\mathfrak{X}(M)$.

The divergence of a $(1,1)$-tensor S on M is defined as the vector field

$$div(S) = trace(\nabla S) = \sum_{i=1}^{n} (\nabla S)(e_i, e_i),$$

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where, as above, $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of vector fields in $\mathfrak{X}(M)$.

We observe that, without loss of generality, we may assume $\{e_1, \ldots, e_n\}$ to be parallel. In this case, we see that

$$div(S) = \sum_{i=1}^{n} \nabla_{e_i}(S(e_i))$$

We also recall that the curvature tensor R of M is given in terms of the curvature tensor $\overline{R}$ of $\overline{M}$ and the shape operator by the so-called Gauss equation

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^{\top} + \epsilon(g(A(Y),Z)A(X) - g(A(X),Z)A(Y)), \tag{11}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Recalling that the mean curvature of M is defined to be

$$H = \frac{\epsilon}{n} trace(A), \tag{12}$$

it follows from (11) that the Ricci curvatures Ric and $\overline{Ric}$ of M and $\overline{M}$ are related as follows

$$Ric(X,Y) = \overline{Ric}(X,Y) - \epsilon\overline{g}(\overline{R}(N,X)Y,N) + g(A(X), nHY - \epsilon A(Y)), \tag{13}$$

for all $X, Y \in \mathfrak{X}(M)$.

Also, by tracing (13), we see that the scalar curvatures $Scal$ and $\overline{Scal}$ of M and $\overline{M}$ are related as follows

$$Scal = \overline{Scal} - 2\epsilon\overline{Ric}(N,N) + \epsilon\left(n^2H^2 - \|A\|^2\right) \tag{14}$$

3. Some useful tensor formulas

With the notations above, let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional either Riemannian or Lorentzian manifold, and let $\overline{\xi} \in \mathfrak{X}(\overline{M})$ be an arbitrary vector field that we assume to be timelike in the case where $(\overline{M}, \overline{g})$ is Lorentzian. Let (M, g) be a connected n-dimensional Riemannian manifold that is isometrically immersed as a hypersurface into $(\overline{M}, \overline{g})$, and let ξ denote the restriction of $\overline{\xi}$ to M.

Our main goal in this section is to give a useful expression for the Laplacian $\Delta\theta$ of the function $\theta = \overline{g}(\xi, N)$, where $\overline{\xi}$ is an arbitrary vector field and N is a globally defined unit vector field normal to M. In the case where $(\overline{M}, \overline{g})$ is Riemannian and $\overline{\xi}$ is a Killing (resp. conformal) vector field an expression for $\Delta\theta$ has been given in [10] (resp. [5]) in terms of the Ricci curvature of $(\overline{M}, \overline{g})$ and the norm of the shape operator. An analogous formula has been obtained in [9] in the case where $(\overline{M}, \overline{g})$ is Lorentzian and $\overline{\xi}$ is a timelike conformal vector field. As we have mentioned in the introduction, in [1], a formula for $\Delta\theta$ was obtained in a slightly different way as given in [5] and [9].

Let us denote by $B^{\top}$ the restriction of B to TM, and let $f = \overline{g}\left(\overline{\nabla}_N\xi, N\right)$. It is clear that f is a smooth function on M. In fact, from (1), we see that

$$f = \frac{1}{2}L_{\xi}\overline{g}(N,N) = \overline{g}(B(N), N) \tag{15}$$

To calculate $\Delta\theta$, we will use (9). So, we start by computing the divergences of T and $A(T)$.

Proposition 1. *Let the notation and assumptions be as above. Then, we have*

$$div(T) = (div(\xi) - \epsilon f) + nH\theta \tag{16}$$

$$div(A(T)) = g(T, div(A)) + trace\left(A \circ B^{T}\right) + \epsilon\theta\|A\|^2 \tag{17}$$

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Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame of vector fields in $\mathfrak{X}(M)$. When $e_1, \ldots, e_n$ and N are extended arbitrarily to vector fields on $\overline{M}$, then according to (10) and making use of (3) we have

$$\begin{aligned} div(T) &= \sum_{i=1}^{n} g(\nabla_{e_i} T, e_i) \\ &= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{e_i} T, e_i) \\ &= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{e_i} \xi, e_i) - \sum_{i=1}^{n} \epsilon \overline{g}(\overline{\nabla}_{e_i}(\theta N), e_i) \\ &= (div\xi - \epsilon f) - \sum_{i=1}^{n} \epsilon \overline{g}(\theta \overline{\nabla}_{e_i} N + (e_i \cdot \theta) N, e_i) \\ &= (div\xi - \epsilon f) + nH\theta. \end{aligned}$$

Using the same formula (10) and making use of (8) and of the fact that if S is self-adjoint operator then $\nabla_X S$ is so, we obtain

$$\begin{aligned} div(A(T)) &= \sum_{i=1}^{n} g(\nabla_{e_i} A(T), e_i) \\ &= \sum_{i=1}^{n} g((\nabla_{e_i} A)(T), e_i) + \sum_{i=1}^{n} g(A(\nabla_{e_i} T), e_i) \\ &= g(T, \sum_{i=1}^{n} (\nabla_{e_i} A)(e_i)) + \sum_{i=1}^{n} g(\psi(e_i) + \epsilon\theta A(e_i), A(e_i)) \\ &= g(T, \sum_{i=1}^{n} (\nabla A)(e_i, e_i)) + \sum_{i=1}^{n} g(\psi(e_i), A(e_i)) + \epsilon\theta \|A\|^2 \\ &= g(T, div(A)) + \sum_{i=1}^{n} \overline{g}(\psi(e_i) + \epsilon\alpha(e_i) N, A(e_i)) + \epsilon\theta \|A\|^2 \\ &= g(T, div(A)) + \sum_{i=1}^{n} \overline{g}(A_\xi(e_i), A(e_i)) + \epsilon\theta \|A\|^2 \\ &= g(T, div(A)) + \sum_{i=1}^{n} \overline{g}(B(e_i) + \phi(e_i), A(e_i)) + \epsilon\theta \|A\|^2 \\ &= g(T, div(A)) + \sum_{i=1}^{n} \overline{g}(B(e_i), A(e_i)) + \epsilon\theta \|A\|^2, \end{aligned}$$

where we have also used here (at the last step) the fact that since A is self-adjoint and ϕ is skew-symmetric, then $\overline{g}(\phi(e_i), A(e_i)) = 0$ for all i. □

In the following proposition, we give an explicit useful formula for $div(A(T))$ in terms of the Ricci curvature (compare to formula (14) in [1]).

Proposition 2. *Let the notation and assumptions be as above. Then, we have*

$$div(A(T)) = \epsilon n T(H) + trace(A \circ B^T) + \epsilon\theta \|A\|^2 - \overline{Ric}(N, T) \tag{18}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame of vector fields in $\mathfrak{X}(M)$ that we assume to be parallel. As we have noticed above, when $e_1, \ldots, e_n$ and N are extended arbitrarily to vector fields on $\overline{M}$ then, using symmetric properties of the curvature tensor $\overline{R}$ of $\overline{M}$, we have

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$$
\begin{aligned}
\overline{Ric}(N,T) &= \sum_{i=1}^{n} \overline{g}(\overline{R}(e_i,N)T,e_i) + \epsilon\overline{g}(\overline{R}(N,N)T,N) \\
&= \sum_{i=1}^{n} \overline{g}(\overline{R}(e_i,T)N,e_i) \\
&= \sum_{i=1}^{n} g((\nabla A)(T,e_i) - (\nabla A)(e_i,T),e_i) \\
&= \sum_{i=1}^{n} g(\nabla_T (A(e_i)), e_i) - \sum_{i=1}^{n} g((\nabla_{e_i} A)(T), e_i) \\
&= \sum_{i=1}^{n} T \cdot g(A(e_i), e_i) - g(T, \sum_{i=1}^{n} (\nabla_{e_i} A)(e_i)) \\
&= T \cdot trace\,(A) - g(T, \sum_{i=1}^{n} (\nabla A)(e_i,e_i)) \\
&= \epsilon n T\,(H) - g(T, div\,(A))
\end{aligned}
$$

□

Now, from this last expression and (17) we obtain (18).

We now give an expression for $div(\overline{\nabla}_N\xi)^\top$. For this purpose, we note that

$$\overline{\nabla}_N\xi = (\overline{\nabla}_N\xi)^\top + \epsilon f N,$$

from which we have

$$
\begin{aligned}
\overline{\nabla}_X\overline{\nabla}_N\xi &= \nabla_X(\overline{\nabla}_N\xi)^\top + \overline{g}\left(A\,(X), (\overline{\nabla}_N\xi)^\top\right) N + \epsilon X\,(f)\,N + \epsilon f \overline{\nabla}_X N \\
&= \nabla_X(\overline{\nabla}_N\xi)^\top + \overline{g}\left(A\,(X), (\overline{\nabla}_N\xi)^\top\right) N + \epsilon X\,(f)\,N - \epsilon f A\,(X)
\end{aligned} \tag{19}
$$

Since $\overline{\nabla}_N\xi = B\,(N) + \phi\,(N)$ and $\phi\,(N) \in \mathfrak{X}\,(M)$, it follows that

$$(\overline{\nabla}_N\xi)^\top = B\,(N)^\top + \phi\,(N) \tag{20}$$
$$(\overline{\nabla}_N\xi)^\perp = B\,(N)^\perp \tag{21}$$

Proposition 3. *Let the notation and assumptions be as above. Then, we have*

$$div(\overline{\nabla}_N\xi)^\top = \overline{Ric}(N,\xi) + N \cdot (div\,(\xi) - \epsilon f) - trace\left(A \circ B^\top\right) + nfH \tag{22}$$

Proof. Let $\{e_1,\ldots,e_n\}$ be a parallel local orthonormal frame of vector fields in $\mathfrak{X}\,(M)$. Then, by using (19), we deduce that

$$
\begin{aligned}
div\left((\overline{\nabla}_N\xi)^\top\right) &= \sum_{i=1}^{n} g(\nabla_{e_i}(\overline{\nabla}_N\xi)^\top, e_i) \\
&= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_N\xi, e_i) + \sum_{i=1}^{n} g(\epsilon f A\,(e_i), e_i) \\
&= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_N\xi, e_i) + \epsilon f trace\,(A) \\
&= \sum_{i=1}^{n} \overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_N\xi, e_i) + nfH
\end{aligned} \tag{23}
$$

On the other hand, by extending $\{e_1,\ldots,e_n\}$ so that $\overline{\nabla}_N e_i = 0$ for all $i = 1,\ldots,n$, we have

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$$\overline{R}(e_i, N)\xi = \overline{\nabla}_{e_i}\overline{\nabla}_N\xi - \overline{\nabla}_N\overline{\nabla}_{e_i}\xi + \overline{\nabla}_{A(e_i)}\xi \tag{24}$$

By using (24), we see that (23) becomes

$$\begin{aligned} div\left((\overline{\nabla}_N\xi)^\top\right) &= \overline{Ric}(N,\xi) + \sum_{i=1}^{n}\overline{g}(\overline{\nabla}_N\overline{\nabla}_{e_i}\xi, e_i) - \sum_{i=1}^{n}\overline{g}(\overline{\nabla}_{A(e_i)}\xi, e_i) + nfH \\ &= \overline{Ric}(N,\xi) + \sum_{i=1}^{n} N\cdot\overline{g}(\overline{\nabla}_{e_i}\xi, e_i) - \sum_{i=1}^{n}\overline{g}(\overline{\nabla}_{A(e_i)}\xi, e_i) + nfH \\ &= \overline{Ric}(N,\xi) + N\cdot(div\,(\xi) - \epsilon f) - trace\left(A\circ B^\top\right) + nfH, \end{aligned}$$

where we have used here the fact that

$$\overline{g}(\overline{\nabla}_{A(e_i)}\xi, e_i) = \overline{g}\left(B\left(A\left(e_i\right)\right) + \phi\left(A\left(e_i\right)\right), e_i\right) = trace\left(A\circ B^\top\right).$$

_ □

Remark 4. *Note that, since*

$$N\cdot(div\,(\xi) - \epsilon f) = N\cdot trace\left(B^\top\right) = trace\left(\nabla_N B^\top\right)$$

and

$$\overline{g}(\phi\left(A\left(e_i\right)\right), e_i) = 0,\ \ 1\le i\le n,$$

then we can express (22) as follows

$$div\left((\overline{\nabla}_N\xi)^\top\right) = \overline{Ric}(N,\xi) + trace\left(\nabla_N B^\top - A\circ B^\top\right) + nfH \tag{25}$$

We are now ready to give the desired expression for $\Delta\theta$.

Theorem 5. *Let (M, g) be a connected n-dimensional Riemannian manifold that is isometrically immersed as a hypersurface into an $(n+1)$-dimensional either Riemannian or Lorentzian manifold $\left(\overline{M}, \overline{g}\right)$. Let $\overline{\xi}\in\mathfrak{X}(\overline{M})$ be an arbitrary vector field that we assume to be timelike in the case where $\overline{M}$ is Lorentzian. $(\overline{M}, \overline{g})$, and let ξ denote the restriction of $\overline{\xi}$ to M. Let N be a globally defined unit vector field normal to M,and let the notation used here be as above. Then, the Laplacian $\Delta\theta$ of the function $\theta = \overline{g}\,(\xi, N)$ is given by the following expression*

$$\Delta\theta = 2div\left(B\,(N)^\top\right) - \epsilon\theta(\overline{Ric}(N,N) + \|A\|^2) - N\cdot(div\,(\xi) - \epsilon f) - \epsilon n(T\,(H) + \epsilon f H) \tag{26}$$

Proof. By(6) and (9), we have

$$\nabla\theta = B\,(N)^\top - \phi\,(N) - A\,(T) \tag{27}$$

By using (18) and (23), it follows that

$$\begin{aligned} \Delta\theta &= div(B\,(N)^\top - \phi\,(N) - A\,(T)) \\ &= -div(B\,(N)^\top + \phi\,(N)) + 2div\left(B\,(N)^\top\right) - div\,(A\,(T)) \\ &= -div\left((\overline{\nabla}_N\xi)^\top\right) + 2div\left(B\,(N)^\top\right) - div\,(A\,(T)) \\ &= 2div\left(B\,(N)^\top\right) - \overline{Ric}(N,\xi) - N\cdot(div\,(\xi) - \epsilon f) + trace(A\circ B^\top) - nfH \\ &- \epsilon nT\,(H) - trace(A\circ B^T) - \epsilon\theta\,\|A\|^2 + \overline{Ric}(N,T) \\ &= 2div\left(B\,(N)^\top\right) - \epsilon\theta(\overline{Ric}(N,N) + \|A\|^2) - N\cdot(div\,(\xi) - \epsilon f) - \epsilon n(T\,(H) + fH), \end{aligned}$$

as desired. □

As a straightforward consequence of Theorem 5, we obtain the interesting expression for $\Delta\theta$ in the particular case where $\overline{\xi}$ is a conformal Killing vector field on $\overline{M}$, that is a vector field satisfying

$$L_{\overline{\xi}}\overline{g} = 2\psi\overline{g}, \tag{28}$$

for some ψ smooth function on $\overline{M}$, called the conformal factor (or potential function) of $\overline{\xi}$.

Corollary 6. *Let the notation and assumptions be as in Theorem 5, and assume in addition that the vector field $\overline{\xi}$ is conformal. Then, we have*

$$\Delta\theta = -\epsilon\theta(\overline{Ric}(N,N) + \|A\|^2) - \epsilon n\left(\psi H + \epsilon N\left(\psi\right) + T\left(H\right)\right) \tag{29}$$

Proof. From (1) and (28), we deduce that $B = \psi I$, where I is the identity. It follows that $B\left(N\right)^{\top} = 0$ and $B\left(N\right) = \psi N$. Consequently, by using (15), we get

$$f = \frac{1}{2}L_{\xi}\overline{g}(N,N) = \overline{g}(B\left(N\right),N) = \epsilon\psi, \tag{30}$$

where we have identified here the function ψ with its restriction to M.

We also have $div\left(\overline{\xi}\right) = trace\left(A_{\overline{\xi}}\right) = trace\left(B\right) = \left(n+1\right)\psi$, from which we deduce that $div\left(\xi\right) - \epsilon f = n\psi$. Now, by substituting these into (26), we obtain (29). □

It would be of some use to express $\Delta\theta$ in terms of the scalar curvatures of M and $\overline{M}$. This can be done by combining the two formulas (14) and (26), so that we get formula (31) in the following theorem.

Theorem 7. *Let the notation and assumptions be as in Theorem 5. Then, we have*

$$\Delta\theta = 2div\left(B\left(N\right)^{\top}\right) - \theta(\overline{Scal} - Scal - \epsilon\overline{Ric}(N,N) + \epsilon n^2H^2) - N\cdot\left(div\left(\xi\right) - \epsilon f\right) - \epsilon n(T\left(H\right) + fH) \tag{31}$$

It should be noticed that formula (31) is nothing but a generalization to the case of an arbitrary vector field on $\overline{M}$ of formula (9) in [1] which was given in the case where $\overline{\xi}$ is a conformal Killing vector field.

Theorem 8 ([1])**.** *Let the notation and assumptions be as in Theorem 5, and assume in addition that the vector field $\overline{\xi}$ is conformal. Then, we have*

$$\Delta\theta = -\theta(\overline{Scal} - Scal - \epsilon\overline{Ric}(N,N) + \epsilon n^2H^2) - \epsilon n\left(\psi H + \epsilon N\left(\psi\right) + T\left(H\right)\right) \tag{32}$$

On the other hand, we give an expression for $div\left((\overline{\nabla}_N\xi)^{\perp}\right)$.

Proposition 9. *Let the notation and assumptions be as above. Then, we have*

$$div(\overline{\nabla}_N\xi)^{\perp} = N\cdot f - \epsilon nfH \tag{33}$$

Proof. Let $\{e_1,\ldots,e_n\}$ be a parallel local orthonormal frame of vector fields in $\mathfrak{X}\left(M\right)$. Note first that

$$\overline{\nabla}_{e_i}e_i = g(A\left(e_i\right),e_i)N \tag{34}$$

We also note that since $N\cdot\overline{g}(N,N) = 0$, then we have $\overline{\nabla}_N N \in \mathfrak{X}\left(M\right)$. With these in hand, we can calculate

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$$div\left((\overline{\nabla}_N\xi)^{\perp}\right)=\sum_{i=1}^{n}\overline{g}(\overline{\nabla}_{e_i}(\overline{\nabla}_N\xi)^{\perp},e_i)+\overline{g}(\overline{\nabla}_N(\overline{\nabla}_N\xi)^{\perp},N) \tag{35}$$

$$=\sum_{i=1}^{n}(e_i\cdot\overline{g}((\overline{\nabla}_N\xi)^{\perp},e_i)-\overline{g}((\overline{\nabla}_N\xi)^{\perp},g(A\,(e_i)\,,e_i)N) \tag{36}$$

$$+N\cdot\overline{g}((\overline{\nabla}_N\xi)^{\perp},N)-\overline{g}((\overline{\nabla}_N\xi)^{\perp},\overline{\nabla}_N N) \tag{37}$$

$$=-trace(A)\overline{g}(\overline{\nabla}_N\xi)^{\perp},N)+N\cdot\overline{g}(\overline{\nabla}_N\xi)^{\perp},N) \tag{38}$$

$$=-\epsilon nfH+N\cdot f, \tag{39}$$

as desired. □

Remark 10. *By combining the two formulas (22) and (33), we deduce that*

$$div\left(\overline{\nabla}_N\xi\right)=\overline{Ric}(N,\xi)+N\cdot div\left(\xi\right)-trace\left(A\circ B^{\top}\right) \tag{40}$$

We would like to note here that formula (40) for a hypersurface M and which is valid for any arbitrary vector field $\xi\in\overline{\mathfrak{X}}\,(M)$ looks like a formula that we can easily prove its validity for projective vector fields and which says that if ξ is a projective on a pseudo-Riemannian manifod, then

$$div\left(A_\xi Y\right)=Ric\left(Y,\xi\right)+Y\cdot div\left(\xi\right)+trace\left(A_\xi\circ A_Y\right) \tag{41}$$

To obtain (40), it suffices to take $Y=N$ in (41) and remember that $div\left(\overline{\nabla}_N\xi\right)=div\left(A_\xi N\right)$ and $A_N=-A$.

4. Integral formulas for compact Riemannian hypersurfaces in pseudo-Riemannian manifolds

In this section, we assume that (M,g) is an n-dimensional compact Riemannian manifold that is isometrically immersed as a hypersurface in an $(n+1)$-dimensional either Riemannian or Lorentzian manifold $\left(\overline{M},\overline{g}\right)$ with all the assumptions stated at the beginning of the above section. The first integral formula that we can display here results directly from the integration of the simple formula (16).

Proposition 11. *Let (M,g) be as above. Then, we have*

$$\int_M\left(div\left(\xi\right)-\epsilon f+nH\theta\right)dV=0 \tag{42}$$

In particular, if $\overline{\xi}$ is a conformal Killing vector field with conformal factor ψ, then

$$\int_M\left(\psi+H\theta\right)dV=0 \tag{43}$$

By using formula (42) of the previous proposition, the following results can be easily deduced.

Proposition 12. *In a Lorentzian manifold admitting an arbitrary (resp. conformal with potential function ψ) timelike vector field, there is no compact spacelike hypersurface whose mean curvature function H satisfies $\left(div\left(\xi\right)+f\right)H<0$ (resp. $H\psi<0$).*

Proof. It is clear that if $\left(div\left(\xi\right)+f\right)H<0$, then either $div\left(\xi\right)+f<0$ and $H>0$ or $div\left(\xi\right)+f>0$ and $H<0$. It follows that either $div\left(\xi\right)+f+nH\theta<0$ or $div\left(\xi\right)+f+nH\theta>0$. If M is compact, then formula (42) implies in both cases that $div\left(\xi\right)+f=0$ and $H=0$, which is absurd. □

Proposition 13. *With all the notations and assumptions stated at the beginning of the above section, assume further that (M,g) is either a compact Riemannian manifold that is either minimal or maximal according to whether $\left(\overline{M},\overline{g}\right)$ is Riemannian or Lorentzian, respectively. Then, there exists a point $p\in M$ such that $trace\left(B_p^{\top}\right)=0$, that is $div\left(\xi\right)(p)=\overline{g}\left(\overline{\nabla}_N\xi,N\right)_p$ or equivalently $(\overline{\nabla}_N\xi)_p^{\perp}=div\left(\xi\right)(p)\cdot N_p$. In particular, if ξ is affine, then $(\overline{\nabla}_N\xi)_p^{\perp}=0$. If ξ is conformal with conformal factor ψ, then $\psi\left(p\right)=0$.*

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As an immediate consequence of Proposition 13, we have the following

Corollary 14. *With all the notations and assumptions stated at the beginning of the above section, assume that $\overline{\xi}$ is a homothetic vector field. Then, when it is Riemannian (resp. Lorentzian), $(\overline{M}, \overline{g})$ contains no compact minimal (resp. maximal) Riemannian hypersurface.*

A more general result than Proposition 13 is the following

Proposition 15. *With all the notations and assumptions stated at the beginning of the above section, assume that $\overline{\xi}$ is an arbitrary vector field. Assume further that M is compact with constant mean curvature, and assume (in the case where $(\overline{M}, \overline{g})$ is Riemannian) that the function θ is not constant and does not change sign.*

(a) *If $H\theta > 0$, then there exists a point $p \in M$ such that $div\,(\xi)\,(p) < \overline{g}\left(\overline{\nabla}_N \xi, N\right)_p$.*
(b) *If $H\theta < 0$, then there exists a point $p \in M$ such that $div\,(\xi)\,(p) > \overline{g}\left(\overline{\nabla}_N \xi, N\right)_p$.*
(c) *If $H = 0$, then there exists a point $p \in M$ such that $div\,(\xi)\,(p) = \overline{g}\left(\overline{\nabla}_N \xi, N\right)_p$.*

Remark 16. *On the other hand, we easily deduce from (43) that if $\overline{\xi}$ is a Killing vector field (i.e. $\psi = 0$) and M has constant mean curvature, then either θ vanishes somewhere or $H = 0$ (i.e. M is minimal in the case when $\overline{M}$ is Riemannian and maximal in the case when $\overline{M}$ is Lorentzian). Conversely, if $\overline{\xi}$ is a homothetic vector field (i.e. ψ is constant) and $H = 0$, then $\overline{\xi}$ is necessarily a Killing vector field. We also deduce from (43) that if $\overline{\xi}$ is a homothetic vector field and $\theta = 0$, then $\overline{\xi}$ is necessarily a Killing vector field. This is exactly what states Theorem 5.3 in [19].*

Our second integral formula is the following

Theorem 17. *Let (M, g) be an n-dimensional compact Riemannian manifold that is isometrically immersed as a hypersurface in an $(n+1)$-dimensional either Riemannian or Lorentzian manifold $(\overline{M}, \overline{g})$. Then, with the assumptions stated in Theorem 5, we have*

$$\int_M \left(\epsilon\,(n-1)\,T\,(H) + trace\left((A - \epsilon H I)\circ B^{\top}\right) + \epsilon\theta\left(\|A\|^2 - nH^2\right) - \overline{Ric}\,(N,T)\right) dV = 0 \tag{44}$$

In particular, when $\overline{\xi}$ is a conformal Killing vector field with conformal factor ψ, then

$$\int_M \left(\epsilon\,(n-1)\,T\,(H) + \epsilon\theta\left(\|A\|^2 - nH^2\right) - \overline{Ric}\,(N,T)\right) dV = 0 \tag{45}$$

Proof. Using (16), we get

$$\begin{aligned} div\,(HT) &= Hdiv\,(T) + T\,(H) \\ &= H(div\,(\xi) - \epsilon f + nH\theta) + T\,(H) \\ &= H\,(div\,(\xi) - \epsilon f) + nH^2\theta + T\,(H) \end{aligned}$$

Using (18) and recalling that $div\,(\xi) - \epsilon f = trace\left(B^{\top}\right)$, we obtain

$$\begin{aligned} div\,(A\,(T)) - \epsilon div\,(HT) &= \epsilon\,(n-1)\,T\,(H) + trace\left(A \circ B^{\top}\right) - \epsilon H\,(div\,(\xi) - \epsilon f) \\ &\quad + \epsilon\theta\left(\|A\|^2 - nH^2\right) - \overline{Ric}\,(N,T) \\ &= \epsilon\,(n-1)\,T\,(H) + trace\left((A - \epsilon H I)\circ B^{\top}\right) \\ &\quad + \epsilon\theta\left(\|A\|^2 - nH^2\right) - \overline{Ric}\,(N,T) \end{aligned}$$

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Now, by integrating both sides of the above equation, we to obtain formula (44). In the case when $\overline{\xi}$ is a conformal Killing vector field with conformal factor ψ, we have $B^{\top} = \psi I$. Substituting this into (44), we obtain (45). □

Our third integral formula is the following

Theorem 18. *Let (M, g) be an n-dimensional compact Riemannian manifold that is isometrically immersed as a hypersurface in an $(n+1)$-dimensional either Riemannian or Lorentzian manifold $(\overline{M}, \overline{g})$ with the assumptions stated in Theorem 5. Then, we have*

$$\int_M \theta(Scal - \overline{Scal} + \epsilon\overline{Ric}(N,N))dV = \int_M N \cdot (div\,(\xi) - \epsilon f)\, dV - \epsilon n \int_M H\,(div\,(\xi) - 2\epsilon f)\, dV \tag{46}$$

In particular, when $\overline{\xi}$ is a conformal Killing vector field with conformal factor ψ, we meet formula (18) in[1], that is

$$\int_M \theta(Scal - \overline{Scal} + \epsilon\overline{Ric}(N,N))dV = n\int_M N\,(\psi)\, dV - \epsilon n\,(n-1)\int_M H\psi\, dV \tag{47}$$

Proof. By using (16), we have

$$\begin{aligned} T\,(H) &= div\,(HT) - Hdiv\,(T) \\ &= div\,(HT) - H\,(div\,(\xi) - \epsilon f + nH\theta)\,, \end{aligned}$$

and by substituting this into (31), we obtain (46). To obtain (47), it suffices to substitute the values $f = \epsilon\psi$ and $div\,(\xi) = (n+1)\,\psi$ into (46). □

Remark 19. *We notice that, in the above result, formula (47) is nothing but formula (18) in [1], so that (46) can be considered as a generalization of that formula to the case of a general vector field $\overline{\xi}$.*

5. Integral formulas for CMC compact Riemannian hypersurfaces in pseudo-Riemannian manifolds

In this section, we will focus on the case when M has constant mean curvature H.The first result gives an integral formula for a hypersurface with constant mean curvature without any assumption on the ambient space $(\overline{M}, \overline{g})$ or on the vector field $\overline{\xi}$.

Theorem 20. *Under the notations and assumptions stated in Theorem 5, let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional either Riemannian or Lorentzian manifold, and (M, g) an n-dimensional compact Riemannian manifold that is isometrically immersed as a hypersurface with constant mean curvature H in $(\overline{M}, \overline{g})$. Then, we have*

$$\int_M \theta(\overline{Ric}(N,N) + \|A\|^2 - nH^2)dV + \epsilon\int_M N \cdot (div\,(\xi) - \epsilon f)\, dV - H\int_M (div\,(\xi) - \epsilon\,(n+1)\,f)\, dV = 0 \tag{48}$$

In particular, when $\overline{\xi}$ is a conformal Killing vector field with conformal factor ψ, we have

$$\int_M \theta(\overline{Ric}(N,N) + \|A\|^2 - nH^2)dV + \epsilon n\int_M N \cdot (\psi)\, dV = 0, \tag{49}$$

and when $\overline{\xi}$ is homothetic, we have

$$\int_M \theta(\overline{Ric}(N,N) + \|A\|^2 - nH^2)dV = 0 \tag{50}$$

Proof. First, since we know that

$$Scal - \overline{Scal} + \epsilon\overline{Ric}(N,N) = -\epsilon(\overline{Ric}(N,N) + \|A\|^2 - n^2H^2), \tag{51}$$

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then (46) yields

$$\int_M \theta(\overline{Ric}(N,N) + \|A\|^2 - n^2H^2)dV + \epsilon \int_M N\cdot(div\,(\xi) - \epsilon f)\,dV - n\int_M H\,(div\,(\xi) - 2\epsilon f)\,dV = 0, \tag{52}$$

or equivalently

$$\int_M \theta(\overline{Ric}(N,N) + \|A\|^2 - nH^2)dV - n\,(n-1)\int_M H^2\theta dV + \epsilon\int_M N\cdot(div\,(\xi) - \epsilon f)\,dV \\ -n\int_M H\,(div\,(\xi) - 2\epsilon f)\,dV = 0 \tag{53}$$

Since H is constant, (42) yields

$$n\,(n-1)\int_M H^2\theta dV = (n-1)H\int_M nH\theta dV \\ = -\,(n-1)\,H\int_M (div\,(\xi) - \epsilon f)\,dV \tag{54}$$

Now, if we substitute (54) into (53), we obtain (48). Formulas (49) and (50) follow easily from (48) using the facts that $f = \epsilon\psi$ and $div\,(\xi) = (n+1)\,\psi$ when $\overline{\xi}$ is a conformal Killing vector field with conformal factor ψ, and the fact that ψ is constant when $\overline{\xi}$ is homothetic, respectively. □

Since $\|A\|^2 - nH^2 \geq 0$, (50) can be used to deduce the following result which generalizes Theorem 5.1 in [19] to the case of a spacelike hypersurface.

Corollary 21. *Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional either Riemannian or Lorentzian manifold which admits a homothetic vector field $\overline{\xi}$, and let (M, g) be an n-dimensional compact Riemannian manifold that is isometrically immersed in $(\overline{M}, \overline{g})$ as a hypersurface with constant mean curvature. Let N and ξ denote, respectively, the normal to M and the restriction of $\overline{\xi}$ to M. Assume that $\overline{Ric}\,(N,N) \geq 0$ on M and assume (in the case where $(\overline{M}, \overline{g})$ is Riemannian) that the function $\theta = \overline{g}\,(N,\xi)$ does not change sign and is not identically zero. Then, (M, g) is totally umbilical and $\overline{Ric}\,(N,N) = 0$ on M.*

The second result is a direct consequence of the Theorem 17 under the assumptions that $(\overline{M}, \overline{g})$ is Einstein and M has constant mean curvature H. This has been proved in [15] in the case when $(\overline{M}, \overline{g})$ is Riemannian.

Theorem 22. *Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional either Riemannian or Lorentzian Einstein manifold with a conformal Killing vector field $\overline{\xi}$, and let (M, g) be an n-dimensional compact Riemannian manifold that is isometrically immersed as a hypersurface in $(\overline{M}, \overline{g})$, with constant mean curvature H. With all the notations and assumptions stated at the beginning of the above section, assume in addition (in the case where $(\overline{M}, \overline{g})$ is Riemannian) that the function θ does not change sign and is not identically zero. Then, (M, g) is necessarily totally umbilical.*

Proof. Under the assumptions of the proposition, formula (45) becomes

$$\int_M \theta\left(\|A\|^2 - nH^2\right) = 0 \tag{55}$$

Since θ does not change sign and is not identically zero, and since $\|A\|^2 - nH^2 \geq 0$, we should get from the integral above that $\|A\|^2 = nH^2$. We deduce that $A = \epsilon HI$, that is (M, g) is totally umbilical. □

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