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Riemann–Hilbert Problems, Polynomial Lax Pairs, Integrable Equations and Their Soliton Solutions †

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† To the memory of professor V. E. Zakharov and professor A. B. Shabat

Abstract: The standard approach to integrable nonlinear evolution equations (NLEE) usually uses the following steps: 1) Lax representation $[L, M] = 0$; 2) construction of fundamental analytic solutions (FAS); 3) reducing the inverse scattering problem (ISP) to a Riemann-Hilbert problem (RHP) $\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t\lambda)$ on a contour Γ with sewing function $G(x,t,\lambda)$; 4) soliton solutions and possible applications. Step 1) involves several assumptions: the choice of the Lie algebra $\mathfrak{g}$ underlying L, as well as its dependence on the spectral parameter, typically linear or quadratic in λ. In the present paper we propose another approach which substantially extends the classes of integrable NLEE. Its first advantage is that one can effectively use any polynomial dependence in both L and M. We use the following steps: A) Start with canonically normalized RHP with predefined contour Γ; B) Specify the x and t dependence of the sewing function defined on Γ; C) Introduce convenient parametrization for the solutions $\xi^\pm(x,t,\lambda)$ of the RHP and formulate the Lax pair and the nonlinear evolution equations (NLEE); D) use Zakharov-Shabat dressing method to derive their soliton solutions. This needs correctly taking into account the symmetries of the RHP. E) Define the resolvent of the Lax operator and use it analyze its spectral properties.

Keywords: Riemann-Hilbert problem; polynomial Lax pair; integrable equations; soliton solutions

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1. Introduction

In 1968 one of the great discoveries in mathematical physics took place. Its authors: P. Lax, C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura and N. J. Zabusky after several years of analysis proved that the KdV equation can be exactly solved by the inverse scattering method (ISM). This was the first and for some time, the only NLEE that could be solved exactly. Soon after that it was demonstrated that the KdV is completely integrable infinite-dimensional Hamiltonian system; its action-angle variables were found by Zakharov and Faddeev [137]. The whole story is well described by N. J. Zabusky in his review paper [128].

The second big step in this direction followed in 1971 by the seminal paper of Zakharov and Shabat who discovered the second equation integrable by the ISM: the nonlinear Schrödinger (NLS) equation [132]; in 1973 the same authors demonstrated that the NLS equation is integrable also under nonvanishing boundary conditions [133]. Both versions of NLS equations described interesting and important physical applications in nonlinear optics, plasma physics, hydrodynamics and others. This inspired many scientists, mathematicians and physicists alike to join the scientific community interested in the study of soliton equations. As a result new soliton equations started to appear one after another. Here we only mention the modified KdV (mKdV) equation [125], the N-wave equations [131], the Manakov system known also as the vector NLS equation [92] and many others. Many of them have already been included in monographs: see e.g. [1,10,18,64,104] and the numerous references therein.

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The first few NLEE were related to the algebra $sl(2)$, so the corresponding inverse scattering problem could be solved using the famous Gelfand-Levitan-Marchenko (GLM) equation. For the Manakov system it was necessary to use 2×2 block matrix Lax operator, so the GLM equation was naturally generalized also for that case. However, the ISM for the N-wave system came up to be substantially more difficult. Indeed, for the 2×2 (block) matrix Lax operator the Jost solutions possess analyticity properties which are basic for the GLM eq. However the Lax pair for the N-wave system is the generalized $sl(n)$ Zakharov-Shabat system:

$$\begin{aligned} &L\psi \equiv i\frac{\partial \psi}{\partial x} + ([J, Q(x,t)] - \lambda J)\psi(x,t,\lambda) = 0, \qquad M\psi \equiv i\frac{\partial \psi}{\partial t} + ([K, Q(x,t)] - \lambda K)\psi(x,t,\lambda) = 0, \\ &J = \operatorname{diag}(a_1, a_2, \dots, a_n), \qquad K = \operatorname{diag}(b_1, b_2, \dots, b_n), \end{aligned} \tag{1.1}$$

where a_k and b_k are real constants such that $\operatorname{tr} J = 0$, $\operatorname{tr} K = 0$. Without restrictions we can assume that $a_1 > a_2 > \cdots > a_n$. In this case only the first and the last column of the corresponding Jost solutions allow analytic extension in the spectral parameter λ. This, however, was not enough to derive GLM equation. It was Shabat who discovered the way out of this difficulty [106,107]. He was able to modify the integral equations for the Jost solutions into integral equations that provide the fundamental analytic solutions (FAS) $\chi^+(x,t,\lambda)$ and $\chi^+(x,t,\lambda)$ of L which allowed analytic extensions for $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ respectively. As a result the interrelation between the FAS and the sewing function $G_0(x,t,\lambda)$:

$$\begin{aligned} \chi^+(x,t,\lambda) &= \chi^-(x,t,\lambda)G_0(t,\lambda), \qquad \operatorname{Im}\lambda = 0, \\ i\frac{\partial G_0}{\partial t} &= \lambda[K, G_0(x,t,\lambda)]. \end{aligned} \tag{1.2}$$

can be reformulated as Riemann-Hilbert problem (RHP). Now we can solve the ISP for L by using the RHP with canonical normalization:

$$\begin{aligned} &\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \operatorname{Im}\lambda = 0, \\ &i\frac{\partial G_0}{\partial x} = \lambda[J, G_0(x,t,\lambda)], \qquad i\frac{\partial G_0}{\partial t} = \lambda[K, G_0(x,t,\lambda)]. \end{aligned} \tag{1.3}$$

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}, \tag{1.4}$$

The normalization condition (1.4) ensures that the RHP has unique regular solution [104]. It also allowed Zakharov and Shabat to develop their dressing method, which enables one to calculate the N-soliton solutions not only for the N-wave system, but also for the whole hierarchy of NLEE related to L (1.1); see [134,135] and also [32,104]. In short, the dressing Zakharov-Shabat factor came up as one of the most effective methods for a) constructing soliton solutions, and b) understand that the dressed Lax operator has some additional discrete eigenvalues, as well as the explicit form of the dressed FAS.

The GZS system has natural reductions after which the potential $Q(x,t)$ and J belong either to $so(n)$ or to $sp(2r)$ algebra. Other reductions were proposed by Mikhailov [96] which substantially enlarged the classes of integrable NLEE. Some of these reductions require that J has complex-valued eigenvalues. Constructing FAS for such systems poses additional difficulties, which were overcome by Beals and Coifman [7] for the GZS systems related to $sl(n)$ algebra. Later the results of [7] were extended first for the systems related to $so(n)$ or to $sp(2r)$ algebra, (see [64]) as well as to Mikhailov's reductions [57,64].

The FAS play an important role in soliton theory. Indeed, they can be used to introduce

1. Scattering data The minimal sets of scattering data are determined by the asymptotics of

$$\lim_{x\to-\infty} \chi^{\pm}(x,t,\lambda)e^{iJ\lambda x} = S^{\pm}(\lambda,t) \qquad \text{or} \qquad \lim_{x\to\infty} \chi^{\pm}(x,t,\lambda)e^{iJ\lambda x} = T^{\pm}(\lambda,t).$$

Here $S^{\pm}(t,\lambda)$ and $T^{\pm}(\lambda,t)$ are the factors of the Gauss decompositions of the scattering matrix $T(\lambda,t)$.

2. **Resolvent** The FAS determine the kernel of the resolvent $R^{\pm}(x,y,t,\lambda)$ of L. Applying contour integration method on $R^{\pm}(x,y,t,\lambda)$ one can derive the spectral expansions for L, i.e. the completeness relation of FAS.
3. **Dressing method** Zakharov-Shabat dressing method is a very effective and convenient method to construct the class of reflectionless potentials of L and to derive the soliton solutions of the NLEE. The simplest dressing factor $u(x,t,\lambda)$ has pole singularities at $\lambda_1^{\pm}$, which determine the new discrete eigenvalues that are added to the spectrum of the initial Lax operator.
4. **Generalized Fourier transforms** Here we start with GZSh system related to a simple Lie algebra $\mathfrak{g}$ with Cartan-Weyl basis H_j, E_α [67] and construct the so-called 'squared solutions'

$$e_\alpha^{\pm}(x,t,\lambda) = \pi_{0J}(\chi^{\pm}E_\alpha(\chi^{\pm})^{-1}(x,t,\lambda) \qquad \text{and} \qquad h_j^{\pm}(x,t,\lambda) = \pi_{0J}(\chi^{\pm}H_j(\chi^{\pm})^{-1}(x,t,\lambda),$$

where π_{0J} is the projector onto the image of the operator ad_J. It is known that the 'squared solution' are complete set of functions in the space of allowed potentials [30]. In particular, if we expand the potential $Q(x,t)$ over the 'squared solutions' the expansion coefficients will provide the minimal set of scattering data. Similarly, the expansion coefficients of $\mathrm{ad}_J^{-1}\delta Q$ are the variations of the minimal set of scattering data. Therefore the 'squared solutions' can be viewed as FAS in the adjoint representation of $\mathfrak{g}$, see [2,12,30,45,59,62–64,78,108,121] as well as [12,24,39,65].
5. **Hierarchies of Hamiltonian structures** The GFT described above allow one to prove that each of the NLEE related to L allows a hierarchy of Hamiltonian structures. More precisely, each NLEE allows a hierarchy of Hamiltonians $H^{(n)}$ and a hierarchy of symplectic forms $\Omega^{(n)}$ (or a hierarchy of Poisson brackets) such that for any n they produce the relevant NLEE. [30,81,86]
6. **Complete integrability and action-angle variables.** Starting from the famous paper by Zakharov and Faddeev [137] it is known that some of the NLEE allow action-angle variables. The difficulty here is that these NLEE are Hamiltonian system with infinitely many degrees of freedom. Therefore the strict derivation of the proof must be based on the completeness relation for the 'squared solutions'. In fact VG and E. Khristov [45,59] (see also [63]) proposed the so-called 'symplectic basis' of squared solutions, which maps the variation of the potential $\sigma_3\delta Q(x,t)$ of the AKNS system to the variation of the action-variables. Unfortunately for many multi-component systems such bases are not yet known.

The above arguments lead us to the hypothesis that we could use more effective approach to the integrable NLEE which starts from the RHP rather than from a specific Lax operator. In the first part of this paper we will demonstrate that FAS could be constructed and used also for quadratic pencils. We also formulate explicitly the corresponding RHP. For quadratic pencils we have additional natural symmetry which maps $\lambda \to -\lambda$. This symmetry is also inherent in the contour in the complex λ-plane, on which the RHP is defined.

In Section 2 we first demonstrate that the well known methods for analysis of NLEE can be generalized also for Lax operators quadratic in λ, see eq. (2.28) below. On this level we for the first time meet with purely algebraic problem for constructing two commuting quadratic pencils. For polynomial pencils of higher orders those problems will be more and more difficult to solve. In particular we outline the construction of the FAS $\chi^+(x,t,\lambda)$ and $\chi^-(x,t,\lambda)$ which are analytic in $\Omega_0 \cup \Omega_3$ and $\Omega_2 \cup \Omega_4$ respectively, see Figure 4 below. As a result, the continuous spectrum of these Lax operators fill up the union of the real and imaginary axis of the complex λ-plane. As a consequence contours of the corresponding RHP will be $\mathbb{R} \cup i\mathbb{R}$ unless an additional factor complicates the picture. Thus we see that the symmetries of the NLEE or of its Lax pair determine the contour of the RHP.

In Section 3 we remind the notion of Mikhailov's reduction group and briefly outline characteristic contours of the relevant RHP. We also demonstrate that Zakharov–Shabat theorem is valid for a larger class of Lax operators than it was proved before. It is important to request that the RHP is canonically normalized. This ensures that the RHP has unique regular solution, which is important for the application of the Zakharov-Shabat dressing method.

Another important factor in formulating the RHP is Mikhailov's reduction group. In Section 4 we outline some of the obvious effects which the reduction group may have on the contour of the RHP. Therefore it is not only the order of the polynomial in λ, but also the symmetry (the reduction group) which determine the contour of the RHP. For example, if we add an additional Mikhailov's symmetry that maps $\lambda \to 1/\lambda^*$ then the corresponding Lax operator will be polynomial in λ and $1/\lambda$ which in turn will require adjustments in the techniques for deriving the dressing factors and soliton solutions.

In Section 5 we propose a parametrization of the solutions $\xi(x,t,\lambda)$ of RHP for the class of RHP related to homogeneous spaces, see eq. (5.1). Here we require that the coefficients $Q_s(x,t)$ provide local coordinates of the corresponding homogeneous space. Thus we are able to derive a new systems of N-wave equations, see also [36,52,53,56,74]. We also demonstrate that the dressing Zakharov-Shabat method [100,129,134–136] can be naturally extended to derive the soliton solutions of these new N-wave equations. At the same time the structure of the dressing factors depends substantially on the symmetries of the Lax operators. Thus even for the one-soliton solutions we need to solve linear block-matrix equations. The situation when we have two involutions: the Hermitian one $(\chi^+(x,t,\lambda))^\dagger = (\chi^-(x,t,\lambda^*))^{-1}$ and the $\lambda \to -\lambda$ symmetry $C_0\chi^\pm(x,t,\lambda)C_0^{-1} = \chi^\pm(x,t,-\lambda)$ are typical for Lax operators L related to the algebras $sl(n,\mathbb{C})$. But if we request in addition that L is related to symplectic or orthogonal Lie algebra then we have to deal with three involutions, and the corresponding linear equations get more involved. That is why we focus first on the one-soliton solutions. The derivation of the N soliton solutions is discussed later.

Section 6 is devoted to the MNLS equations which require the use of symmetric spaces, see Refs. [21,40,67,92,94,118,119,122,126]. We start again with the parametrization of the RHP which now must be compatible with the structure of the symmetric spaces. To us it was natural to limit ourselves to the four classes Hermitian symmetric spaces related to the non-exceptional Lie algebras, see [67]. Again we parametrize the coefficients $Q_s(x,t)$ as local coordinates of the corresponding symmetric spaces. In fact $\mathcal{Q}(x,t,\lambda)$ must have the same grading as the symmetric space, but we were able to apply additional reductional requesting $Q_{2s} = 0$, see eq. (6.11) below. Thus we formulate the typical MNLS equations related to the four classes of symmetric spaces.

In Section 7 we derive the one soliton solutions of MNLS. Again, like in Section 5, we treat separately the MNLS related to A.III type symmetric spaces, because the corresponding FAS have only two involutions. The MNLS related to C.I and D.III symmetric spaces possess three involutions; the corresponding linear equations are similar to the ones for the class of N-wave equations, but the solutions are different. The symmetric spaces of BD.I class are treated separately, because their typical representation is provided by 3×3 block matrices, so many of the calculations are indeed different. At the end of this Section we derive the soliton interactions for the BD.I class of MNLS [41]. More precisely we use the asymptotic of the dressing factor for $x \to \pm\infty$ applying it to the two-soliton solution in order to calculate the center-of-mass and the phase shifts of the solitons.

In Section 8 we introduce the resolvent of the Lax operators in terms of the FAS. The diagonal of the resolvent after a regularization can be expressed in terms of the solution of the RHP by $\xi^\pm(x,t,\lambda)J\hat{\xi}^\pm(x,t,\lambda)$; here by 'hat' we denote the inverse matrix. It can be viewed as generating functional of the integrals of motion.

We end the paper by discussion and conclusions. Some technical aspects in the calculations such as the structure of the symmetric spaces, and the root systems of the simple Lie algebras as well as the Gauss decompositions of the elements of the simple Lie groups are given in the appendices.

2. From the Lax representation to the RHP

2.1. N-waves according to Manakov and Zakharov

The N-wave equations were discovered by Manakov and Zakharov in 1974 [131]. The Lax operator is the classical Zakharov-Shabat system:

$$L_{\rm ZS}\psi \equiv i\frac{\partial \psi}{\partial x} + ([J, Q(x,t)] - \lambda J)\psi(x,t,\lambda) = 0, \tag{2.1}$$

with real-valued diagonal $J = \operatorname{diag}(a_1, a_2, \dots, a_n)$. The second operator in the Lax representation is also linear in λ

$$M\psi \equiv i\frac{\partial \psi_{\rm ZS}}{\partial x} + ([K, Q(x,t)] - \lambda K)\psi(x,t,\lambda) = 0, \tag{2.2}$$

with real-valued diagonal $K = \operatorname{diag}(b_1, b_2, \dots, b_n)$. Then the N-wave equations, which are the compatibility condition $[L, M] = 0$ take the form:

$$\left[J, \frac{\partial Q}{\partial t}\right] - \left[K, \frac{\partial Q}{\partial x}\right] + [[J, Q(x,t)], [K, Q(x,t)]] = 0. \tag{2.3}$$

For some time solving the ISP for the Lax operator $L_{\rm ZS}$ was an open problem. However in 1974 Shabat [106,107] proved that $L_{\rm ZS}$ has fundamental solutions $\chi^+(x,t,\lambda)$ and $\chi^-(x,t,\lambda)$ which are analytic in the upper- and lower-half of $\mathbb{C}$.

Let us briefly outline the construction of FAS proposed by A. B. Shabat [106,107]; for more details see also [32,104]. We will assume for simplicity that the potential $q(x)$ is defined on the whole axis and satisfies the following conditions:

C.1 By $Q(x) \in \mathcal{M}_S$ we mean that $Q(x)$ possesses smooth derivatives of all orders and falls off to zero for $|x| \to \infty$ faster than any power of x:

$$\lim_{x\to\pm\infty} |x|^k Q(x) = 0, \qquad \forall k = 0, 1, 2, \dots$$

C.2 $Q(x)$ is such that the corresponding operator L has only a finite number of simple discrete eigenvalues.

We will impose also the typical reduction of the Lax operator: the GZSs with $\mathbb{Z}_2$-reduction:

$$U^\dagger(x,t,\epsilon\lambda^*) = U(x,t,\lambda). \tag{2.4}$$

We start by introducing the Jost solutions of L:

$$\lim_{x\to-\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \tag{2.5}$$

and the scattering matrix:

$$T(t,\lambda) = \psi^{-1}(x,t,\lambda)\phi(x,t,\lambda). \tag{2.6}$$

The Jost solutions satisfy the following Volterra type integral equations:

$$\begin{aligned} \tilde{\psi}_{ij}(x,\lambda) &= \delta_{ij} + i\int_\infty^x dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^h (a_i - a_p) Q_{ip}(y) \tilde{\psi}_{pj}(y,\lambda), \\ \tilde{\phi}_{ij}(x,\lambda) &= \delta_{ij} + i\int_{-\infty}^x dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^h (a_i - a_p) Q_{ip}(y) \tilde{\phi}_{pj}(y,\lambda), \end{aligned} \tag{2.7}$$

where $\tilde{\psi}(x,\lambda) = \psi(x,\lambda)e^{iJ\lambda x}$ and $\tilde{\phi}(x,\lambda) = \phi(x,\lambda)e^{iJ\lambda x}$. It is well known that Volterra type equations possess solutions provided the integrals of the equations are convergent. In our case this will be true for real λ. Indeed, in this case the exponential factors in the equations (2.7) will be bounded and the convergence of the integrals is ensured by the fact that $Q(x)$ satisfies Condition 1. However, for complex λ the exponential factors are not growing only for the first and the last columns of the Jost solutions.

Shabat introduced the FAS $\chi^{\pm}(x,\lambda)$ of L by modifying the integral equations for them as follows:

$$\begin{aligned} \xi^+_{ij}(x,\lambda) &= \delta_{ij} + i\int_{-\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} (a_i - a_p) Q_{ip}(y)\xi^+_{pj}(y,\lambda), \qquad i \geq j; \\ \xi^+_{ij}(x,\lambda) &= i\int_{\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} (a_i - a_p) Q_{ip}(y)\xi^+_{pj}(y,\lambda), \qquad i < j; \end{aligned} \tag{2.8}$$

$$\begin{aligned} \xi^-_{ij}(x,\lambda) &= i\int_{\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} (a_i - a_p) Q_{ip}(y)\xi^-_{pj}(y,\lambda), \qquad i > j; \\ \xi^-_{ij}(x,\lambda) &= \delta_{ij} + i\int_{-\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} (a_i - a_p) Q_{ip}(y)\xi^-_{pj}(y,\lambda), \qquad i \leq j; \end{aligned} \tag{2.9}$$

where

$$\xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)e^{i\lambda Jx}. \tag{2.10}$$

Now it is not difficult to check that all exponential factors in the integrands of (2.8) are falling off for $\lambda \in \mathbb{C}_+$, while the exponential factors in the integrands of (2.9) are falling off for $\lambda \in \mathbb{C}_-$. In other words, $\xi^+(x,t,\lambda)$ allows analytic extensions for λ in the upper complex half-plane, while $\xi^-(x,t,\lambda)$ is analytic for λ in the lower complex half-plane. Obviously $\xi^+(x,\lambda)$ and $\xi^-(x,\lambda)$ are FAS of a slightly modified Lax operator:

$$\tilde{L}\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(x,\lambda) - \lambda[J, \xi^{\pm}(x,\lambda)] = 0. \tag{2.11}$$

Theorem 1. *Let $q(x) \in \mathcal{M}_S$ satisfies conditions (C.1), (C.2) and let the matrix elements of J be ordered $a_1 > a_2 > \cdots > a_n$. Then the solution $\xi^+(x,\lambda)$ of the eqs. (2.8) (resp. $\xi^-(x,\lambda)$ of the eqs. (2.9)) exists and allows analytic extension for $\lambda \in \mathbb{C}_+$ (resp. for $\lambda \in \mathbb{C}_-$).*

Remark 1. *Due to the fact that in eq. (2.8) we have both ∞ and $-\infty$ as lower limits the equations are rather of Fredholm than of Volterra type. Therefore we have to consider also the Fredholm alternative, i.e. there may exist finite number of values of $\lambda = \lambda_k^{\pm} \in \mathbb{C}_{\pm}$ for which the solutions $\xi^{\pm}(x,\lambda)$ have zeroes and pole singularities in λ. The points $\lambda_k^{\pm}$ in fact are the discrete eigenvalues of $L(\lambda)$ in $\mathbb{C}_{\pm}$.*

The reduction condition (2.4) with $\epsilon = 1$ means that the FAS and the scattering matrix $T(\lambda)$ satisfy:

$$\left(\chi^+(x,t,\lambda^*)\right)^{\dagger} = (\chi^-(x,t,\lambda))^{-1}, \qquad T^{\dagger}(t,\lambda^*) = T^{-1}(t,\lambda). \tag{2.12}$$

Each fundamental solution of the Lax operator is uniquely determined by its asymptotic for $x \to \infty$ or $x \to -\infty$. Therefore in order to determine the linear relations between the FAS and the Jost solutions for $\lambda \in \mathbb{R}$ we need to calculate the asymptotics of FAS for $x \to \pm\infty$. Taking the limits in the right hand sides of the integral equations (2.8) and (2.9) we get:

$$\lim_{x\to-\infty} \xi^+_{ij}(x,\lambda) = \delta_{ij}, \quad \text{for } i \geq j; \qquad \lim_{x\to\infty} \xi^+_{ij}(x,\lambda) = 0, \quad \text{for } i < j; \tag{2.13a}$$

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$$\lim_{x\to-\infty} \xi^-_{ij}(x,\lambda) = \delta_{ij}, \quad \text{for } i \le j; \qquad \lim_{x\to\infty} \xi^-_{ij}(x,\lambda) = 0, \quad \text{for } i > j; \tag{2.13b}$$

This can be written in compact form using (2.10):

$$\chi^\pm(x,\lambda) = \psi_-(x,\lambda)S^\pm(\lambda) = \psi_+(x,\lambda)T^\mp(\lambda)D^\pm(\lambda), \tag{2.14}$$

where the matrices $S^\pm(\lambda)$, $D^\pm(\lambda)$ and $T^\pm(\lambda)$ are of the form:

$$S^+(\lambda) = \begin{pmatrix} 1 & S^+_{12} & \dots & S^+_{1n} \\ 0 & 1 & \dots & S^+_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \qquad T^+(\lambda) = \begin{pmatrix} 1 & T^+_{12} & \dots & T^+_{1n} \\ 0 & 1 & \dots & T^+_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \tag{2.15a}$$

$$D^+(\lambda) = \mathrm{diag}\,(D^+_1, D^+_2, \dots, D^+_n), \qquad D^-(\lambda) = \mathrm{diag}\,(D^-_1, D^-_2, \dots, D^-_n), \tag{2.15b}$$

$$S^-(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ S^-_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S^-_{n1} & S^-_{n2} & \dots & 1 \end{pmatrix}, \qquad T^-(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ T^-_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T^-_{n1} & T^-_{n2} & \dots & 1 \end{pmatrix}, \tag{2.15c}$$

Let us now relate the factors $T^\pm(\lambda)$, $S^\pm(\lambda)$ and $D^\pm(\lambda)$ to the scattering matrix $T(\lambda)$. Comparing (2.14) with (2.6) we find

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda), \tag{2.16}$$

i.e. $T^\pm(\lambda)$, $S^\pm(\lambda)$ and $D^\pm(\lambda)$ are the factors in the Gauss decomposition of $T(\lambda)$.

It is well known how given $T(\lambda)$ one can construct explicitly its Gauss decomposition, see the Appendix B. Here we need only the expressions for $D^\pm(\lambda)$:

$$D^+_j(\lambda) = \frac{m^+_j(\lambda)}{m^+_{j-1}(\lambda)}, \qquad D^-_j(\lambda) = \frac{m^-_{n-j+1}(\lambda)}{m^-_{n-j}(\lambda)}, \tag{2.17}$$

where $m^\pm_j$ are the principal upper and lower minors of $T(\lambda)$ of order j.

Remark 2. *Gauss decomposition of of $T(t,\lambda) \in SL(n)$ is well known, see Appendix B and [32]. The derivation requires the knowledge of the fundamental representations of the Lie algebra $sl(n)$. If $T(t,\lambda)$ belongs to an orthogonal or symplectic group, see [30].*

Corollary 1. *The upper (resp. lower) principal minors $m^+_j(\lambda)$ (resp. $m^-_j(\lambda)$ of $T(\lambda)$ are analytic functions of λ for $\lambda \in \mathbb{C}_+$ (resp. for $\lambda \in \mathbb{C}_-$).*

Proof. Follows directly from theorem 1, from the limits:

$$\lim_{x\to\infty} \xi^+_{jj}(x,\lambda) = D^+_j(\lambda), \qquad \lim_{x\to\infty} \xi^-_{jj}(x,\lambda) = D^-_j(\lambda), \tag{2.18}$$

and from (2.15b) and (2.17). □

Thus the solution of the ISP for L_{ZS} was reduced to a RHP:

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda)G_0(t,\lambda), \qquad \lambda \in \mathbb{R} \tag{2.19}$$

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on the real axis of $\mathbb{C}$. The solutions $\chi^+(x,t,\lambda)$ defined by (2.10) satisfy an alternative RHP:

$$\begin{aligned} &\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \\ &i\frac{\partial G}{\partial x} - \lambda[J, G(x,t,\lambda) = 0, \qquad i\frac{\partial G}{\partial t} - \lambda[K, G(x,t,\lambda) = 0, \end{aligned} \tag{2.20}$$

which is canonically normalized, i.e.

$$\lim_{\lambda\to\infty} \xi^\pm(x,t,\lambda) = \mathbb{1}. \tag{2.21}$$

2.2. MNLS equations according to Manakov, Fordy and Kulish

The first MNLS:

$$i\frac{\partial \vec{q}}{\partial t} + \frac{\partial^2 \vec{q}}{\partial x^2} + 2(\vec{q}^{\,\dagger}, \vec{q})\vec{q}(x,t) = 0 \tag{2.22}$$

where $\vec{q}$ is a 2-component vector, was proposed by Manakov [92] in 1974. Later in their seminal paper [21] Fordy and Kulish demonstrated the deep relations between the MNLS equations and the symmetric spaces [67]. As a result the Lax representation for the MNLS takes the form:

$$\begin{aligned} L_{\rm MNLS}\psi &\equiv i\frac{\partial \psi}{\partial x} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \\ M_{\rm MNLS}\psi &\equiv i\frac{\partial \psi}{\partial t} + (V_2(x,t) + \lambda Q(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0 \end{aligned} \tag{2.23}$$

For the Manakov model (2.22) $J = \operatorname{diag}(1,-1,-1)$ and $Q(x,t) = \begin{pmatrix} 0 & \vec{q}^{\,T} \\ \vec{q}^{\,*} & 0 \end{pmatrix}$. Special role here is played by the Cartan element J of the corresponding simple Lie algebra. It determines the Cartan involution, which picks up the symmetric space from the corresponding Lie algebra. The Manakov model is related to the symmetric space $SU(3)/S(U(1)\times U(2))$. The symmetric spaces have been classified about a century ago by E. Cartan, see e.g. [67].

In 1976 Kaup and Newell [81] derived the generalizations of NLS corresponding to Lax operator, quadratic in λ. This generalization of NLS now is known as the derivative NLS (DNLS), because its nonlinear terms depend on the x-derivative of q. Another form of the DNLS eq. is known since 1978 as the Gerdjikov-Ivanov (GI) equation [42,58], see also [14,19,20,90] and the references therein. It is gauge equivalent to DNLS and besides the terms with x-derivative, contains also nonlinearities of 5-th order. In Section 6 below we describe the multi-component generalizations to GI equations.

2.3. Generic Lax representation

We start with the idea of constructing more general polynomial Lax pairs:

$$\begin{aligned} L_0\psi_0 &\equiv i\frac{\partial \psi_0}{\partial x} + (U(x,t,\lambda) - \lambda^{N_1} J)\psi_0(x,t,\lambda) = 0, \\ M_0\psi_0 &\equiv i\frac{\partial \psi_0}{\partial t} + (V(x,t,\lambda) - \lambda^{N_2} K)\psi_0(x,t,\lambda) = \psi_0(x,t,\lambda)C(\lambda), \\ U(x,t,\lambda) &= \sum_{s=1}^{N_1} U_s \lambda^{N_1-s}, \qquad V(x,t,\lambda) = \sum_{s=1}^{N_2} V_s \lambda^{N_2-s}. \end{aligned} \tag{2.24}$$

The compatibility condition $[L_0, M_0] = 0$ of the pair (2.24) holds for any $C(\lambda)$ and has the form:

$$i\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial t} + [U(x,t,\lambda), V[x,t,\lambda)] = 0. \tag{2.25}$$

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Here we will assume that both $N_1 \geq 2$ and $N_2 \geq 2$. In addition we will fix up the gauge by requiring that

$$U_{N_1}(x,t) = -J, \qquad V_{N_2}(x,t) = -K, \tag{2.26}$$

where J and K are constant diagonal matrices.

Eq. (2.25) must hold identically with respect to the spectral parameter λ. Thus here there comes the first technical difficulty related with the parametrization of L and M. Indeed, let us consider the example where $N_1 = 3$ and $N_2 = 3$. The the left hand side of (2.25) is a polynomial of order 6 with respect to λ, whose highest 4 coefficients are given by

$$\begin{aligned} &\lambda^6 \quad [K,J] = 0, && \lambda^5 \quad -[V_2,J] - [K,U_2] = 0, \\ &\lambda^4 \quad -[V_1,J] + [V_2,U_2] - [K,U_1] = 0, \\ &\lambda^3 \quad -[V_2,J] + [V_1,U_2] + [V_2,U_1] - [K,U_2] = 0, \end{aligned} \tag{2.27}$$

The coefficient at λ^6 is vanishing because J and K are diagonal. The vanishing of the coefficient at λ^5 is means that V_2 is expressed through U_2. Indeed, if we put $J = \operatorname{diag}(a_1, a_2, \dots, a_n)$ and $K = \operatorname{diag}(b_1, b_2, \dots, b_n)$ then

$$(V_2)_{jk} = \frac{b_j - b_k}{a_j - a_k}(U_2)_{jk}.$$

The next two relations coming from λ^4 and λ^3 are not so easy to satisfy.

Gel'fand and Dickey [15,25–28] provide a very effective solution to the problem. They suggest a general construction for the Lax pairs of the form (2.24) using the fractional powers of the Lax operator. Below we will outline, along with the effective parametrization of the RHP, an equivalently effective method for generic Lax representations.

2.4. *Jost solutions and FAS of L*

Here we start with generic Lax operator, quadratic in λ with vanishing boundary conditions and canonical gauge. In our case this is:

$$\begin{aligned} L\psi &\equiv i\frac{\partial \psi}{\partial x} + (U_2(x,t) + U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\ J &= \operatorname{diag}(a_1, a_2, \dots, a_n), \qquad a_1 > a_2 > \cdots > a_n, \qquad \operatorname{tr} J = 0. \end{aligned} \tag{2.28}$$

Remark 3. *For the potentials $U_2(x,t)$ and $U_1(x,t)$ we assume that they $n \times n$ matrices which are smooth functions of x for all values of t, tending to 0 fast enough for $x \to \pm\infty$; for simplicity we could take them to be Schwartz-type functions*

The Jost solutions to (2.28) are defined as follows:

$$\lim_{x\to\infty} \psi(x,t,\lambda)e^{i\lambda^2 Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,t,\lambda)e^{i\lambda^2 Jx} = \mathbb{1}. \tag{2.29}$$

Both Jost solutions are fundamental solutions: indeed they are non-degenerate $n \times n$ matrix-valued functions. In what follows we will choose the potentials U_2 and U_1 to take values in a given simple Lie algebra $\mathfrak{g}$. Then the fundamental solutions ψ and ϕ will belong to the corresponding simple Lie group $\mathcal{G}$.

It is also well known that any two fundamental solutions are linearly related. In other words:

$$\phi(x,t,\lambda) = \psi(x,t,\lambda)T(t,\lambda). \tag{2.30}$$

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The matrix $T(t,\lambda)$ belongs to the Lie group $\mathcal{G}$.

The integral equations for the Jost solutions take the form:

$$\begin{aligned} \boldsymbol{\psi}(x,t,\lambda) &= \mathbb{1} + i\int_{\infty}^{x} dy\, e^{-i\lambda^2 J(x-y)}(U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\psi}(y,t,\lambda)e^{i\lambda^2 J(x-y)}, \\ \boldsymbol{\phi}(x,t,\lambda) &= \mathbb{1} + i\int_{-\infty}^{x} dy\, e^{-i\lambda^2 J(x-y)}(U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\phi}(y,t,\lambda)e^{i\lambda^2 J(x-y)}, \end{aligned} \tag{2.31}$$

where $\boldsymbol{\psi}(x,t,\lambda) = \psi(x,t,\lambda)e^{i\lambda^2 Jx}$ and $\boldsymbol{\phi}(x,t,\lambda) = \phi(x,t,\lambda)e^{i\lambda^2 Jx}$ and $\mathbb{1}$ is the unit $n\times n$ matrix. In components we have:

$$\begin{aligned} \boldsymbol{\psi}_{jk}(x,t,\lambda) &= \delta_{jk} + i\int_{\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)}\left((U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\psi}(y,t,\lambda)\right)_{jk}, \\ \boldsymbol{\phi}_{jk}(x,t,\lambda) &= \delta_{jk} + i\int_{-\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)}\left((U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\phi}(y,t,\lambda)\right)_{jk}. \end{aligned} \tag{2.32}$$

Note that both integral equations are Volterra type equations.

The boundary conditions on the potentials $U_2(x,t)$ and $U_1(x,t)$ in Remark 3 ensure that equations (2.31) always have solutions provided the exponentials $e^{-i\lambda^2(a_j-a_k)(x-y)}$ do not grow for $x\to\infty$ or $x\to-\infty$. Obviously, this holds true for $\operatorname{Im}\lambda^2=0$, i.e. for $\lambda\in\mathbb{R}\oplus i\mathbb{R}$. As we will see below, this is the continuous spectrum of L (2.28).

However the equations (2.31) allow also for important exceptions. These will be easier seen if we use the equations (2.32). Let us, for example consider the equations for the first (resp. for the last) columns of the Jost solutions; So we have to consider the equations (2.32) for $k=1$ (resp. for $k=n$). Let us assume also that $\operatorname{Im}\lambda^2<0$, i.e. λ is in the second or fourth quadrant of the complex λ-plane. Then it is easy to check that the exponential factors in all the equations for $(\boldsymbol{\psi})_{j1}$, $j=1,\dots,n$ decay for $x,y\to\pm\infty$. The same holds true also for the equations for $(\boldsymbol{\phi})_{jn}$, $j=1,\dots,n$. Therefore we find that the first column of $\boldsymbol{\psi}$ and the last column of $\boldsymbol{\phi}$ allow analytic extensions to the second and fourth quadrants of $\mathbb{C}$. Similarly we find, that the first column of $\boldsymbol{\phi}$ and the last column of $\boldsymbol{\psi}$ allow analytic extensions for $\operatorname{Im}\lambda^2>0$, or to the first and third quadrants of $\mathbb{C}$. The other columns of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are defined only on the continuous spectrum of L. Indeed, the corresponding set of equations (2.32) some of the exponential factors will decay, but others will grow up.

Nevertheless our aim will be to demonstrate that one can construct fundamental analytic solutions (FAS) for L. These will be $n\times n$ matrix solutions, one of which $\chi^+(x,t,\lambda)$ allows analytic extension for $\operatorname{Im}\lambda^2>0$ and the other one $\chi^-(x,t,\lambda)$ – for $\operatorname{Im}\lambda^2<0$. This can be done using Shabat's method [106,107] based on proper modification of the integral equations (2.32). So let $\chi^\pm(x,t,\lambda)$ be fundamental solutions of L, i.e. $L\chi^\pm(x,t,\lambda)=0$ and let us introduce $\boldsymbol{\chi}^\pm(x,t,\lambda)=\chi^\pm(x,t,\lambda)e^{i\lambda^2 Jx}$. These solutions will be different from the Jost solutions, because, as we will see, their behavior for $x\to\pm\infty$ is different.

Following Shabat's idea we define $\boldsymbol{\chi}^+(x,t,\lambda)$ as the solution of the following set of integral equations:

$$\begin{aligned} \boldsymbol{\chi}^+_{jk}(x,t,\lambda) &= \delta_{jk} + i\int_{-\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)}\left((U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\chi}^+(y,t,\lambda)\right)_{jk}, \quad j\geq k \\ \boldsymbol{\chi}^+_{jk}(x,t,\lambda) &= i\int_{\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)}\left((U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\chi}^+(y,t,\lambda)\right)_{jk}, \quad j<k. \end{aligned} \tag{2.33}$$

Likewise, $\boldsymbol{\chi}^-(x,t,\lambda)$ are defined as the solution of the following set of integral equations:

$$\begin{aligned} \boldsymbol{\chi}^-_{jk}(x,t,\lambda) &= \delta_{jk} + i\int_{-\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)}\left((U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\chi}^-(y,t,\lambda)\right)_{jk}, \quad j\leq k \\ \boldsymbol{\chi}^-_{jk}(x,t,\lambda) &= i\int_{\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)}\left((U_2(y,t)+\lambda U_1(y,t))\boldsymbol{\chi}^-(y,t,\lambda)\right)_{jk}, \quad j>k. \end{aligned} \tag{2.34}$$

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Note that the only difference with the equations (2.32) is in index inequalities in the right hand sides; this changes the signs of the factors $a_j - a_k$.

There is an alternative possibility to introduce FAS with a minor change of the integral equations (2.33) and (2.34)

$$\begin{aligned} \chi^{+,\prime}_{jk}(x,t,\lambda) &= i\int_{\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)} \left((U_2(y,t)+\lambda U_1(y,t))\chi^{+,\prime}(y,t,\lambda)\right)_{jk}, \quad j>k \\ \chi^{+,\prime}_{jk}(x,t,\lambda) &= \delta_{jk} + i\int_{-\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)} \left((U_2(y,t)+\lambda U_1(y,t))\chi^{+,\prime}(y,t,\lambda)\right)_{jk}, \quad j\leq k. \end{aligned} \tag{2.35}$$

Likewise, $\chi^-(x,t,\lambda)$ are defined as the solution of the following set of integral equations:

$$\begin{aligned} \chi^{-,\prime}_{jk}(x,t,\lambda) &= i\int_{-\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)} \left((U_2(y,t)+\lambda U_1(y,t))\chi^{-,\prime}(y,t,\lambda)\right)_{jk}, \quad j<k \\ \chi^{-,\prime}_{jk}(x,t,\lambda) &= \delta_{jk} + i\int_{\infty}^{x} dy\, e^{-i\lambda^2(a_j-a_k)(x-y)} \left((U_2(y,t)+\lambda U_1(y,t))\chi^{-,\prime}(y,t,\lambda)\right)_{jk}, \quad j\geq k. \end{aligned} \tag{2.36}$$

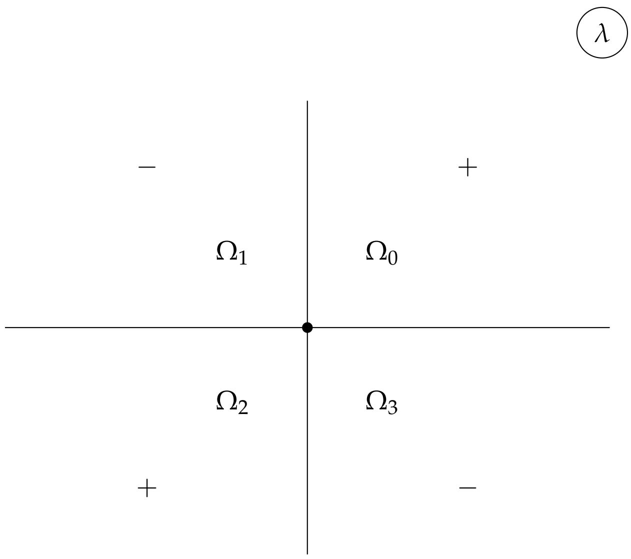

Figure 1. The continuous spectrum of the Lax operator (2.28) and the contour of the related Riemann-Hilbert problem $\mathbb{R} \cup i\mathbb{R}$.

Now we have to establish the linear relations between the FAS and the Jost solutions.

The first consequence of the equations (2.33), (2.34), (2.35) and (2.36) concerns the limits of their diagonal matrix elements for $x \to \infty$, namely:

$$\begin{aligned} \lim_{x\to\infty} \operatorname{diag}\left(\chi^{\pm}(x,t,\lambda)\right) &= D^{\pm}(\lambda) = \operatorname{diag}\left(D_1^{\pm}(\lambda), D_2^{\pm}(\lambda), \dots, D_n^{\pm}(\lambda)\right), \\ \lim_{x\to\infty} \operatorname{diag}\left(\chi^{\pm,\prime}(x,t,\lambda)\right) &= D^{\pm,\prime}(\lambda) = \operatorname{diag}\left(D_1^{\pm,\prime}(\lambda), D_2^{\pm,\prime}(\lambda), \dots, D_n^{\pm,\prime}(\lambda)\right), \end{aligned} \tag{2.37}$$

Obviously the matrices $D^+(\lambda)$ and $D^{+,\prime}(\lambda)$ are x and t independent and in addition they are analytic for $\operatorname{Im}\lambda^2 > 0$; analogously $D^-(\lambda)$ and $D^{-,\prime}(\lambda)$ are analytic for $\operatorname{Im}\lambda^2 < 0$.

Next we find that:

$$\begin{aligned} &\lim_{x\to-\infty} e^{i\lambda^2 Jx}\chi^{\pm}(x,t,\lambda) = S^{\pm}(t,\lambda), &&\lim_{x\to\infty} e^{i\lambda^2 Jx}\chi^{\pm}(x,t,\lambda) = T^{\mp}(t,\lambda)D^{\pm}(\lambda), \\ &\lim_{x\to\infty} e^{i\lambda^2 Jx}\chi^{\pm,\prime}(x,t,\lambda) = T^{\mp}(t,\lambda), &&\lim_{x\to-\infty} e^{i\lambda^2 Jx}\chi^{\pm,\prime}(x,t,\lambda) = S^{\pm}(t,\lambda)\hat{D}^{\pm}(\lambda), \end{aligned} \tag{2.38}$$

where $S^{+}(t,\lambda)$ and $T^{+}(t,\lambda)$ (resp. $S^{-}(t,\lambda)$ and $T^{-}(t,\lambda)$) are upper-triangular matrices (resp. lower triangular) with 1 on the diagonal. Remember, since the Jost solutions and the FAS belong to the Lie group $\mathcal{G}$, then all the limits in (2.38) must also belong to $\mathcal{G}$.

From (2.38) it follows that the FAS $\chi^{\pm}_{jk}(x,t,\lambda)$ are related to the Jost solutions as follows:

$$\begin{aligned} \chi^{\pm}(x,t,\lambda) &= \phi(x,t,\lambda)S^{\pm}(t,\lambda), & \chi^{\pm}(x,t,\lambda) &= \psi(x,t,\lambda)T^{\mp}(t,\lambda)D^{\pm}(\lambda), \\ \chi^{\pm,\prime}(x,t,\lambda) &= \phi(x,t,\lambda)S^{\pm}(t,\lambda)\hat{D}^{\pm}, & \chi^{\pm,\prime}(x,t,\lambda) &= \psi(x,t,\lambda)T^{\mp}(t,\lambda), \end{aligned} \tag{2.39}$$

More detailed analysis shows that these triangular and diagonal matrices are in fact the factors in the Gauss decompositions of the scattering matrix $T(t,\lambda)$:

$$T(t,\lambda) = T^{-}(t,\lambda)D^{+}(\lambda)\hat{S}^{+}(t,\lambda) = T^{+}(t,\lambda)D^{-}(\lambda)\hat{S}^{-}(t,\lambda), \tag{2.40}$$

see Appendix B.

Remark 4. *Let us consider cubic pencils in λ assuming that the leading terms $\lambda^3 J$ and $\lambda^3 K$ of L and M respectively are such that J and K have different real eigenvalues. Their FAS can be constructed quite analogously as we did above for the quadratic pencils. The substantial difference between the cubic and the quadratic cases are in the regions of analyticity. The solutions $\chi^{+}(x,t,\lambda)$ (resp. $\chi^{-}(x,t,\lambda)$) for the cubic pencils are analytic for Im $\lambda^3 > 0$ (resp. Im $\lambda^3 < 0$), see sectors $\Omega_0 \cup \Omega_2 \cup \Omega_4$ (resp. sectors $\Omega_1 \cup \Omega_3 \cup \Omega_5$) on Figure 4.*

2.5. The time-dependence of $T(t,\lambda)$

The Lax representation of a given NLEE requires the commutativity of two ordinary differential operators:

$$[L, M] = 0 \tag{2.41}$$

where L is given by (2.28). We assume that M has the same form as L:

$$M\psi \equiv \left(i\frac{\partial}{\partial t} + V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K \right) \psi(x,t,\lambda) = \psi(x,t,\lambda)C(\lambda). \tag{2.42}$$

Below we will treat also M-operators that are higher order polynomials of λ. The first remark is that the commutativity condition (2.41) must hold identically with respect to λ. Note also that (2.41) holds true for any $C(\lambda)$ in (2.42). We will use this fact to determine the t-dependence of the scattering matrix $T(t,\lambda)$.

Indeed, let us consider $M\boldsymbol{\phi}(x,t,\lambda) = \boldsymbol{\phi}(x,t,\lambda)C(\lambda)$ where $\boldsymbol{\phi}(x,t,\lambda)$ is the Jost solution in (2.29) and let us take the limit $x \to -\infty$ in (2.42). Due to the vanishing boundary conditions we get:

$$\left(i\frac{\partial}{\partial t} - \lambda^2 K \right) e^{-i\lambda^2 Jx} = e^{-i\lambda^2 Jx} C(\lambda), \tag{2.43}$$

i.e. $C(\lambda) = -\lambda^2 K$. Thus we determined the function $C(\lambda)$ for this choice of the M-operator. Let us now take the limit $x \to \infty$ in (2.42). This gives:

$$\left(i\frac{\partial}{\partial t} - \lambda^2 K \right) \left(e^{-i\lambda^2 Jx} T(t,\lambda) \right) = e^{-i\lambda^2 Jx} T(t,\lambda) C(\lambda), \tag{2.44}$$

i.e.

$$i\frac{\partial T}{\partial t} - \lambda^2 [K, T(t,\lambda)] = 0. \tag{2.45}$$

From eq. (2.45) there follow also the time-dependence of the Gauss factors:

$$i\frac{\partial S^{\pm}}{\partial t} - \lambda^2[K, S^{\pm}(t,\lambda)] = 0, \qquad i\frac{\partial T^{\pm}}{\partial t} - \lambda^2[K, T^{\pm}(t,\lambda)] = 0. \tag{2.46}$$

For the diagonal factors we get:

$$i\frac{\partial D^{\pm}(\lambda)}{\partial t} = 0. \tag{2.47}$$

In other words we have two sets of functions of λ that are t-independent. Note also, that from the canonical normalization of FAS there follows that $\lim_{\lambda\to\infty} D_k^{\pm} = 1$. Obviously they will generate integrals of motion. Indeed, let us consider their asymptotic expansions:

$$\ln D_k^{\pm} = \sum_{s=1}^{\infty} \lambda^{-s} I_k. \qquad k = 1, \dots, n; \tag{2.48}$$

Obviously $dI_k/dt = 0$. As we shall see below each of the integrals I_k can be expressed as an integral of the potentials U_2 and U_1. The advantage of the choice (2.47) is that the integrands of I_k are local, i.e. they depend only on U_2 and U_1 and their x-derivatives.

Above we described the time dependence for the N-wave equations related to the Lax operators (2.28). In general, each of these operators generates a hierarchy of NLEE. Indeed the NLEE is determined by fixing up along with L also the second operator M in the Lax pair. For example, if we want to treat DNLS-type equations we need, to specify two things. First, the structure of L must be compatible with a Hermitian symmetric space [21], and second, the potential of M must polynomial of order 4 in λ with leading term $\lambda^4 J$. Then $C(\lambda) = -\lambda^4 J$ and the time dependence of the scattering data will be given by:

$$i\frac{\partial T}{\partial t} - \lambda^4[J, T(t,\lambda)] = 0. \tag{2.49}$$

Then from eq. (2.49) there follow also the time-dependence of the Gauss factors:

$$i\frac{\partial S^{\pm}}{\partial t} - \lambda^4[J, S^{\pm}(t,\lambda)] = 0, \qquad i\frac{\partial T^{\pm}}{\partial t} - \lambda^4[J, T^{\pm}(t,\lambda)] = 0. \tag{2.50}$$

More details about this we will give in the Section devoted to the DNLS type equations.

3. RHP and integrable NLEE

The simplest nontrivial Lax operators, i.e. the ones that are linear in λ have been studied in detail by now, see [18,63,104] and the numerous references therein. That is why in this paper we will concentrate mostly on Lax operators that are quadratic in λ like (2.28). In the previous section we constructed the FAS of this operators for the case of vanishing boundary conditions and the result was as follows. The continuous spectrum of L fills up the union of the real and purely imaginary axis $\mathbb{R} \cup i\mathbb{R}$, see Figure 1; The analyticity regions for the FAS of (2.28) are $\mathbb{Q}_1 \cup \mathbb{Q}_3$ for χ^+ and $\mathbb{Q}_2 \cup \mathbb{Q}_4$ for χ^-, where by $\mathbb{Q}_k$, $k = 1, 2, 3, 4$ we denote the quadrants of the complex λ-plane.

3.1. Uniqueness of the regular solution of RHP

The FAS derived above are also linearly related to each other. From equations (2.39) there easily follows that:

$$\begin{aligned} \chi^{+}(x,t,\lambda) &= \chi^{-}(x,t,\lambda)G_0(t,\lambda), \qquad & G_0 &= \hat{S}^{-}S^{+}(t,\lambda), \\ \chi^{+,\prime}(x,t,\lambda) &= \chi^{-,\prime}(x,t,\lambda)G_0'(t,\lambda), \qquad & G_0' &= \hat{S}^{-,\prime}S^{+,\prime}(t,\lambda), \end{aligned} \qquad \lambda \in \mathbb{R} \oplus i\mathbb{R}. \tag{3.1}$$

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These relations allow us to relate the Lax pair to a RHP. Indeed, our FAS are not very suitable because for large $x \to \pm\infty$ they strongly oscillate (like $\exp(-i\lambda^2 Jx)$). In order to avoid these singularities we introduce $\xi^{\pm}(x,t,\lambda) = \chi^{\pm}(x,t,\lambda)\exp(i\lambda^2 Jx)$. Then the RHP can be formulated as:

$$\begin{aligned} \xi^{+}(x,t,\lambda) &= \xi^{-}(x,t,\lambda)G(x,t,\lambda), & G(x,t,\lambda) &= e^{-i\lambda^2 Jx}G_0(t,\lambda)e^{i\lambda^2 Jx}, \\ \xi^{+,\prime}(x,t,\lambda) &= \xi^{-,\prime}(x,t,\lambda)G'(x,t,\lambda), & G'(x,t,\lambda) &= e^{-i\lambda^2 Jx}G'_0(t,\lambda)e^{i\lambda^2 Jx}, \end{aligned} \quad \lambda \in \mathbb{R} \oplus i\mathbb{R}. \tag{3.2}$$

which allows canonical normalization:

$$\lim_{\lambda\to\infty} \xi^{+}(x,t,\lambda) = \lim_{\lambda\to\infty} \xi^{-}(x,t,\lambda) = \mathbb{1} \tag{3.3}$$

In the previous Subsections we outlined how, starting with the Lax representation one can derive the RHP. Here, following Zakharov and Shabat we will demonstrate how starting from the RHP one can derive the relevant Lax representation. We will use slightly modified RHP with canonical normalization:

$$\begin{aligned} \xi^{+}(x,t,\lambda) &= \xi^{-}(x,t,\lambda)G(x,t,\lambda), & &\lambda \in \mathbb{R} \oplus i\mathbb{R}, \\ i\frac{\partial G}{\partial x} - \lambda^2[J, G(x,t,\lambda)] &= 0, & i\frac{\partial G}{\partial t} - \lambda^2[K, G(x,t,\lambda)] &= 0, \\ \lim_{\lambda\to\pm\infty} \xi^{\pm}(x,t,\lambda) &= \mathbb{1} \end{aligned} \tag{3.4}$$

compare with (3.2).

We will say that the solution $\xi_0^{\pm}(x,t,\lambda)$ is regular if $\xi_0^{+}(x,t,\lambda)$ and $\xi_0^{-}(x,t,\lambda)$ have neither zeroes nor singularities in their regions of analyticity.

Corollary 2. *The RHP (3.4) with canonical normalization has unique regular solution.*

Proof. Let us assume that $\xi_1^{\pm}(x,t,\lambda)$ is another regular solution to the RHP (3.4). Consider:

$$g_0^{\pm}(x,t,\lambda) = \xi_0^{\pm}(x,t,\lambda)\hat{\xi}_1^{\pm}(x,t,\lambda); \tag{3.5}$$

we remind that $\hat{\xi} \equiv \xi^{-1}$. Using (3.4) we easily find that:

$$g_0^{+}(x,t,\lambda) = \xi_0^{+}(x,t,\lambda)\hat{\xi}_1^{+}(x,t,\lambda) = \xi_0^{-}(x,t,\lambda)G(x,t,\lambda)\hat{G}(x,t,\lambda)\hat{\xi}_1^{-}(x,t,\lambda) = g_0^{-}(x,t,\lambda), \tag{3.6}$$

i.e. $g_0^{+}(x,t,\lambda)$ is analytic in the whole complex plane λ. In addition, it tends to $\mathbb{1}$ for $\lambda \to \infty$. Then according to the great Liouville theorem, $g_0^{+}(x,t,\lambda) = g_0^{-}(x,t,\lambda) = \mathbb{1}$. □

Remark 5. *The RHP (3.4) obviously allows the trivial regular solution when $\xi_0^{+} = \xi_0^{-} = \mathbb{1}$. In this case the corresponding FAS of L will take the form:*

$$\chi_0^{+}(x,t,\lambda) = \chi_0^{-}(x,t,\lambda) = e^{-i\lambda^2 Jx - i\lambda^2 Kt}. \tag{3.7}$$

3.2. Zakharov-Shabat theorem

Theorem 2 (Zakharov-Shabat [135]). *Let $\xi^{\pm}(x,t,\lambda)$ satisfy the RHP (3.4). Then $\chi^{\pm}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda)e^{-i\lambda^2 Jx}$ are FAS of L (2.28) and M (2.42).*

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Proof. Let us introduce the functions

$$\begin{aligned} g_1^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^2 \xi^{\pm}(x,t,\lambda) J\hat{\xi}^{\pm}(x,t,\lambda), \\ g_2^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^2 \xi^{\pm}(x,t,\lambda) K\hat{\xi}^{\pm}(x,t,\lambda), \end{aligned} \tag{3.8}$$

Then, from eqs. (3.4) we find:

$$\begin{aligned} g_1^{+}(x,t,\lambda) &= i\frac{\partial(\xi^{-}G)}{\partial x}\hat{G}\hat{\xi}^{-}(x,t,\lambda) + \lambda^2 \xi^{-}(x,t,\lambda) GJ\hat{G}\hat{\xi}^{-}(x,t,\lambda) \\ &= i\frac{\partial \xi^{-}}{\partial x}\hat{\xi}^{-}(x,t,\lambda) + \xi^{-}(x,t,\lambda)\left(i\frac{\partial G}{\partial x}\hat{G}(x,t,\lambda) + \lambda^2 GJ\hat{G}(x,t,\lambda)\right)\hat{\xi}^{-}(x,t,\lambda) \\ &= i\frac{\partial \xi^{-}}{\partial x}\hat{\xi}^{-}(x,t,\lambda) + \lambda^2 \xi^{-}(x,t,\lambda) J\hat{\xi}^{-}(x,t,\lambda) = g_1^{-}(x,t,\lambda). \end{aligned} \tag{3.9}$$

Thus $g_1^{\pm}(x,t,\lambda)$ are analytic in the whole complex plane $\mathbb{C}$. The canonical normalization of the RHP means, however, that $g_1^{\pm}(x,t,\lambda)$ are singular for $\lambda \to \infty$. More specifically, $g_1^{\pm}(x,t,\lambda) - \lambda^2 J$ is linear function of λ. Using again the great Liouville theorem we conclude that there must exist the linear in λ function $-U_2(x,t) - \lambda U_1(x,t)$ such that:

$$g_1^{\pm}(x,t,\lambda) - \lambda^2 J = -U_2(x,t) - \lambda U_1(x,t), \tag{3.10}$$

i.e.

$$g_1^{\pm}(x,t,\lambda) + U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J = 0. \tag{3.11}$$

Multiplying both sides of (3.11) by $\xi^{\pm}(x,t,\lambda)$ we find that the solutions to the RHP (3.4) must satisfy the ODE:

$$i\frac{\partial \xi^{\pm}}{\partial x} + (U_2(x,t) + \lambda U_1(x,t))\xi^{\pm}(x,t,\lambda) - \lambda^2[J, \xi^{\pm}(x,t,\lambda)] = 0. \tag{3.12}$$

It remains to insert $\chi^{\pm}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda)e^{-i\lambda^2 Jx}$ into eq. (3.12) to obtain that $\chi^{\pm}(x,t,\lambda)$ are FAS of eq. (2.28).

Applying the same considerations on the second function $g_2^{\pm}(x,t,\lambda)$ we find that it is analytic on the whole complex λ-plane and has the form:

$$g_2^{\pm}(x,t,\lambda) + V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K = 0. \tag{3.13}$$

This means that $\xi^{\pm}(x,t,\lambda)$ must satisfy also

$$i\frac{\partial \xi^{\pm}}{\partial t} + (V_2(x,t) + \lambda V_1(x,t))\xi^{\pm}(x,t,\lambda) - \lambda^2[K, \xi^{\pm}(x,t,\lambda)] = 0. \tag{3.14}$$

and therefore $\chi^{\pm}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda)e^{-i\lambda^2 Jx}$ are FAS of the M-operator (2.42).
□

Remark 6. *Zakharov-Shabat theorem initially was proven for Lax operators with linear dependence on the spectral parameter λ. The above proof is elementary generalization of the original idea. Obviously it can easily be extended for any polynomial Lax pair, see e.g. (2.24). The details of these generalizations are left to the readers.*

4. Mikhailov's reduction groups and the contours of RHP

4.1. General theory

The reduction groups introduced by [96] are a powerful tool for deriving new integrable equations, admitting Lax representation. It has been substantially developed since its discovery, see [9,22,31,37, 88,89,120,127]. Mikhailov's reductions were used in the seminal paper of Drinfeld and Sokolov [16] as an important tool to analyze the gradings of simple Lie algebras and their consequences for the integrable equations. It also stimulated the development of the infinite dimensional Lie algebras and Kac-Moody algebras [8,16,67].

A reduction group G_R is a finite group acting on the solution set of (2.24) which preserves the Lax representation [96], i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations: i) $G_R \subset \mathrm{Aut}\mathfrak{g}$ and ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [96]:

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda), \tag{4.1}$$

where $C_k \in \mathrm{Aut}\,\mathfrak{g}$ and $\Gamma_k(\lambda) \in \mathrm{Conf}\,\mathbb{C}$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

The finite subgroups of Conf $(\mathbb{C})$ were classified by Klein, see [13]. They consist of two infinite series: i) $\mathbb{Z}_h$ - cyclic group of order h; ii) $\mathbb{D}_h$ - dihedral group of order $2h$; and the groups related to the Platonic solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron. In what follows we will most attention to $\mathbb{Z}_h$ and $\mathbb{D}_h$; although examples of systems with Platonic solids as group of reductions are also known, [82,88,89,95]

It is important to note that the form of the equations depends not only on the chosen reduction group, but also on its realization.

It is well known, that every finite group can be imbedded as a subgroup of some finite symmetric group $\mathcal{S}_m$; this is the group of permutations of the numbers $\{1, 2, \dots, m\}$. The group $\mathbb{Z}_h$ consists of all cyclic permutations of the numbers $1, 2, \dots, h$.

The symmetric group $\mathcal{A}_h$ is the commutator subgroup of the symmetric group $\mathcal{S}_h$ with index 2 and has therefore $h!/2$ elements. It is the kernel of the signature group homomorphism sgn $: \mathcal{S}_h \to \{1, -1\}$ explained under symmetric group. It is isomorphic to $\mathbb{D}_h$.

Generically each of these groups is rigorously defined by their genetic codes. In other words one introduces one or more generating elements of the group and defines the relationes they must satisfy. For the two types of groups we have:

$$\begin{aligned} \mathcal{S}_2 \quad &: \quad s_1^2 = \mathbb{1}, \\ \mathcal{S}_h \quad &: \quad s_1^h = \mathbb{1}, \\ \mathcal{D}_h \quad &: \quad s_1^2 = \mathbb{1}, \qquad s_2^2 = \mathbb{1}, \qquad (s_1 s_2)^h = \mathbb{1}, \end{aligned} \tag{4.2}$$

These are the formal definitions of these groups. Below we will outline their realizations as subgroups of the group of automorphisms Aut $\mathfrak{g}$ of the algebra $\mathfrak{g}$, as well as subgroups of the conformal group. These realizations are specific both for the explicit form of the corresponding NLEE as well as for the spectral properties of their Lax operators.

Other important facts are the orbits and the fundamental domains of the groups. Each element Γ_k of G_R is of finite order, i.e. there exist an integer n_k such that $\Gamma_k^{n_k} = \mathbb{1}$. Acting on a point, say λ_0 it produces an orbit in the complex λ-plane consisting of the points $\{\Gamma_k^s \lambda_0, s = 1, \dots, n_k$. Then by fundamental domain of G_R we mean the manifold A_{G_R} which contains only one point of each orbit. The orbits depend not only on the element Γ_k, but also on the specific realization of G_R. Below we will specify the fundamental domains for each of the realizations of the reduction groups.

We start with $\mathbb{Z}_2$-reductions, (or involutions) for two of the best known classes of NLEE:

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4.2. Involutive reductions

We will start with typical $\mathbb{Z}_2$ reductions (involutions) on the Lax representations (2.24) [96]:

$$\begin{aligned} &1) \quad C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), && C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), \\ &2) \quad C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), && C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \\ &3) \quad C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), && C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), \\ &4) \quad C_4(U(\kappa_2(\lambda))) = U(\lambda), && C_4(V(\kappa_2(\lambda))) = V(\lambda), \end{aligned} \tag{4.3}$$

By C_j above we have denoted involutive automorphisms $C_j^2 = \mathbb{1}$ of the simple Lie algebra $\mathfrak{g}$ in which $U(x,t,\lambda)$ and $v(x,t,\lambda)$ take values.

Working with Lax operators which are quadratic pencils it is natural to impose two basic symmetries: i) the Hermitian symmetry 1) in (4.3) with $C_1 = \mathbb{1}$ and $\kappa_1(\lambda) = \lambda^*$; and the symmetry 2) in (4.3) mapping $\lambda \to -\lambda$. A third involution iii) may appear if we request in addition that U and V belong to a simple Lie algebra of the series B_r, C_r or D_r. This means:

$$U(x,t,\lambda) = -S_a U^T(x,t,\lambda) S_a^{-1}, \qquad V(x,t,\lambda) = -S_a V^T(x,t,\lambda) S_a^{-1},$$

where the matrices S_a are introduced in the Appendix A.

Note that equations (4.3) in fact take into account the all typical external automorphisms of the simple Lie algebras. Therefore we need to consider only those realizations of C_j which are elements of the Weyl group $\mathcal{W}_\mathfrak{g}$ of $\mathfrak{g}$, or belong to the Cartan subgroup of $\mathfrak{g}$. This means that C_j may have the form:

$$C_j \to S_j \equiv S_{\beta_1} S_{\beta_2} \dots S_{\beta_k}, \qquad \text{or} \qquad C_j \to \boldsymbol{h}_j = \exp(\pi i H_{\vec{b}_j}), \tag{4.4}$$

where the roots $\beta_1, \beta_2, \dots, \beta_k$ are all orthogonal to each other. This will ensure that S_j is an involution: $S_j^2 = \mathbb{1}$. Similarly the vector $\vec{b}_j$ must be such that $\vec{b}_j = \sum_{s=1}^r k_{j,s}\omega_s$, where $k_{j,s}$ are nonnegative integers and ω_s are the fundamental weights of $\mathfrak{g}$. Then the similarity transformations with $\boldsymbol{h}_j$ will also have the property $\boldsymbol{h}_j^2 = \mathbb{1}$.

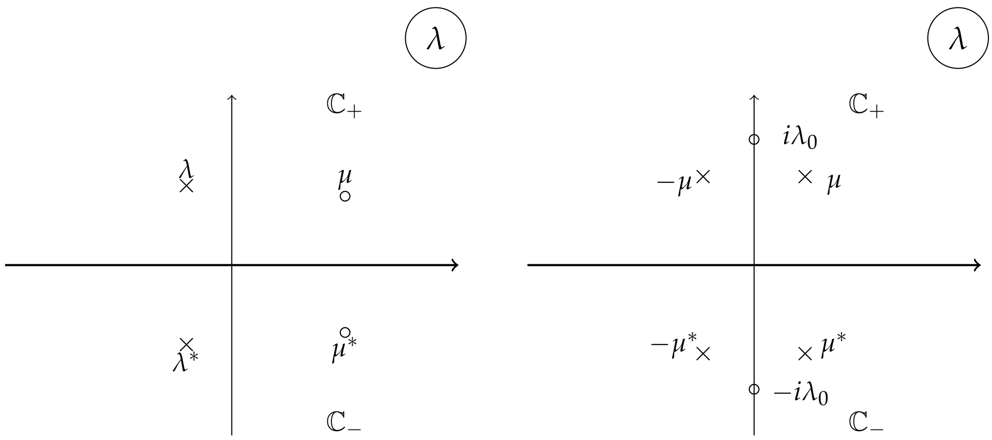

Figure 2. Examples of involutions for Lax operators linear in λ. The continuous spectrum fills up the real axis; the FAS $\chi^{\pm}(x,t,\lambda)$ are analytic for $\lambda \in \mathbb{C}_\pm$ respectively. Left panel: for reductions of types C_1 and C_3 the discrete eigenvalues come in complex-conjugate pairs, shown by $\times$ and $\circ$; Right panel: for reductions of type C_2 and C_3 there are two types of eigenvalues: pairs of purely imaginary ones $\pm i\lambda_0$ shown by $\circ$; quadruplets $\pm\mu$ and $\pm\mu^*$ shown by $\times$

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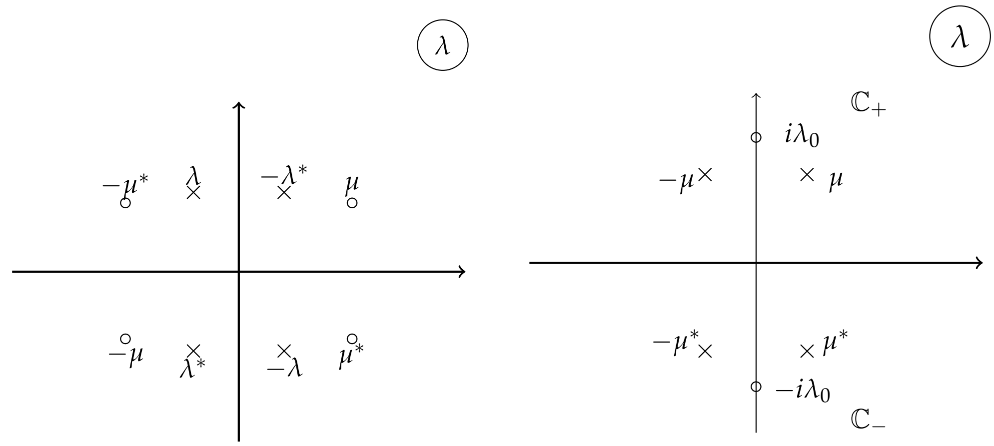

Figure 3. Examples of involutions for Lax operators quadratic in λ; the continuous spectrum fills up $\mathbb{R} \oplus i\mathbb{R}$. Left panel: for reductions of types C_1 and C_3 the discrete eigenvalues come in complex-conjugate pairs, shown by $\times$; for reductions of type C_2 they come in pairs shown by $\circ$. In both cases the contour of the RHP is the real axis. Right panel: the first involution maps $\lambda \to \lambda^{-1}$; or to $\lambda \to -\lambda^{-1}$; the second one maps $\mu \to \mu^{*,-1}$ or to $\mu \to -\mu^{*,-1}$. In both cases the contour of the RHP is given by the unit circle $|\lambda|^2 = 1$, [119]

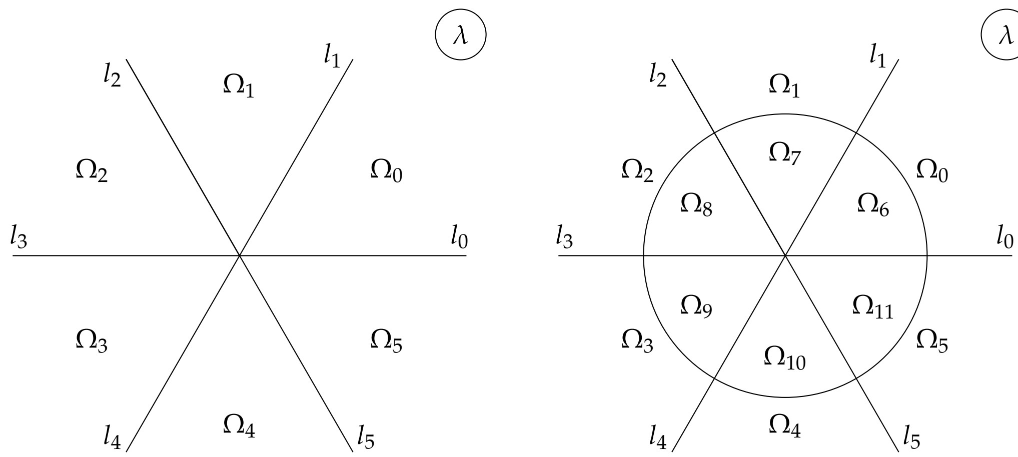

Figure 4. Left panel: contour of a RHP with $\mathbb{Z}_3$ symmetry. The FAS $\chi^+(x,\lambda)$ is analytic in the sectors $\Omega_0 \cup \Omega_2 \cup \Omega_4$, while $\chi^-(x,\lambda)$ is analytic in the sectors $\Omega_1 \cup \Omega_3 \cup \Omega_5$; Right panel: contour of a RHP with $\mathbb{Z}_3$ symmetry with additional involution mapping $\lambda \to \lambda^{-1}$.

Another important factor for the correct realization of the reduction groups is N_1 – the leading power λ in the Lax operator (2.24). In the majority of cases people consider generalized Zakharov-Shabat systems, i.e. $N_1 = 1$. Let us combine this choice with real-valued eigenvalues of J and vanishing boundary conditions on $Q(x,t)$. Then the analysis of the corresponding integral equations for the Jost solutions show that the FAS $\chi_0^+(x,t,\lambda)$ (resp. $\chi_0^-(x,t,\lambda)$) of L_0 have analyticity properties for $\operatorname{Im}\lambda > 0$ (resp. $\operatorname{Im}\lambda < 0$). That means that the contour of the corresponding RHP coincides with the real axis $\mathbb{R}$. Then the spectrum of L_0 consists of continuous part filling up $\mathbb{R}$ and pairs of complex-valued discrete eigenvalues $\lambda_j^\pm \in \mathbb{C}_\pm$, see Figure 3. As fundamental domain of this group we can choose the upper half-plane $\mathbb{C}_+$ of the complex λ-plane.

In what follows we will concentrate on Lax operators with $N_1 > 1$; most often we will have $N_1 = 2$. In Section 2 we proved that $\chi^+(x,t,\lambda)$ (resp. $\chi^-(x,t,\lambda)$) of L (2.28) have analyticity properties for $\operatorname{Im}\lambda^2 > 0$ (resp. $\operatorname{Im}\lambda^2 < 0$). That means that the contour of the corresponding RHP coincides with the union of the real and purely imaginary axis $\mathbb{R} \oplus i\mathbb{R}$. Then the spectrum of L consists of continuous part filling up $\mathbb{R} \oplus i\mathbb{R}$. Typically such Lax operators have additional symmetry $\lambda \leftrightarrow -\lambda$, so the discrete eigenvalues come in quadruplets $\pm\lambda_j^+ \in \mathbb{Q}_1 \cup \mathbb{Q}_3$ and $\pm\lambda_j^- \in \mathbb{Q}_2 \cup \mathbb{Q}_4$, see Figure 3. Effectively in these cases we deal with $\mathbb{Z}_2 \otimes \mathbb{Z}_2$= Then as fundamental domain of this realizations of the group we can choose the first quadrant $\mathbb{Q}_1$ of the complex λ-plane.

4.3. $\mathbb{Z}_h$ reduction groups

We will consider here the cyclic groups $\mathbb{Z}_h$ of order $h > 2$. These groups have only one generating elements:

$$\mathbb{Z}_h \quad : \quad s^h = \mathbb{1}. \tag{4.5}$$

The cyclic group has h elements: $\mathbb{1}, s^k, k = 1, \dots, h-1$; typically its realization on the complex λ-plane is given by $s(\lambda) = \lambda\omega$, where $\omega = \exp(2\pi i/h)$.

4.4. $\mathbb{D}_h$ reduction groups

We will consider here the dihedral groups $\mathbb{D}_h$ of order $h > 2$. These groups have two generating elements:

$$\mathbb{D}_h \quad : \quad r^2 = s^h = \mathbb{1}, \qquad srs^{-1} = s^{-1}. \tag{4.6}$$

The dihedral group $\mathbb{D}_h$ has $2h$ elements: $\{s^k, rs^k, k = 1, \dots, h\}$ and allows several inequivalent realization on the complex λ-plane. Some of them are:

$$\begin{aligned} &\text{(i)} \quad s(\lambda) = \lambda\omega, \quad r(\lambda) = \epsilon\lambda^*, \qquad &&\text{(ii)} \quad s(\lambda) = \lambda\omega, \quad r(\lambda) = \frac{\epsilon}{\lambda^*}, \\ &\text{(iii)} \quad s(\lambda) = \lambda\omega, \quad r(\lambda) = \epsilon\lambda, \qquad &&\text{(iv)} \quad s(\lambda) = \lambda\omega, \quad r(\lambda) = \frac{\epsilon}{\lambda}, \end{aligned}$$

where again $\omega = \exp(2\pi i/h)$ and $\epsilon = \pm 1$. An important realization in the case of a $\mathbb{D}_2$ reduction group is given by

$$\text{(v)} \quad s(\lambda) = \lambda^*, \quad r(\lambda) = \frac{\epsilon}{\lambda}. \tag{4.7}$$

5. Parametrizing the RHP with canonical normalization

An important tool in our investigation is the theory of the simple Lie algebras and the methods of their gradings.

The reason to limit our selves with the simple Lie algebras is due to the fact, that we need to have unique solution of the inverse spectral problem of the Lax operator. The mapping between the potential and the scattering matrix for generic, linear in λ operators have been studied using the Wronskian relations [10,30,63]. They require the existence of a non-degenerate metric. A metric, characteristic for the Lie algebras is the famous Killing form, which is non-degenerate for the semi-simple Lie algebras. In fact we will limit ourselves by considering only the simple Lie algebras.

We will limit ourselves also by considering only two families of NLEE. The first family is known as the N-wave equations, discovered by Zakharov and Manakov [131], see Section 2.1 above. Typically they contain first order derivatives in both x and t and quadratic nonlinearities. In this Section we will describe a new class of N-wave equations whose Lax operators are both quadratic in λ [36,52]. We will see, that they have higher order nonlinearities.

The second family of NLEE we will focus on are the multi-component NLS (MNLS) equations. It is well known that they are related to the symmetric spaces [21]. Their Lax operators will also be quadratic in λ, so they will be multicomponent generalizations of the derivative MLS eq. [81] and GI equations [19,20,42,58].

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5.1. Generic parametrization of the RHP with canonical normalization

We can introduce a parametrization for $\xi^{\pm}(x,t,\lambda)$ using its asymptotic expansion:

$$\xi^{\pm}(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)), \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} Q_s(x,t). \tag{5.1}$$

Obviously, if we want that $\xi^{\pm}(x,t,\lambda)$ be elements of a simple Lie group $\mathcal{G}$, then the coefficients $Q_s(x,t)$ must be elements of the corresponding simple Lie algebra $\mathfrak{g}$. In addition we request that Q_s provides local coordinates of the corresponding homogeneous space. Besides, the solution $\xi^{\pm}(x,t,\lambda)$ is canonically normalized, because

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}. \tag{5.2}$$

The most general parametrization of $\xi^{\pm}(x,t,\lambda)$ requires that $Q_s(x,t)$ are generic elements of the algebra $\mathfrak{g}$. However such approach has a disadvantage: the corresponding NLEE involve too many independent functions. There are two ways to avoid it: first, we can fix up the gauge of the Lax operators; second we can and will impose reductions of Mikhailov type. Typically we will fix the gauge by requesting that the leading terms in the Lax operators are chosen as diagonal constant matrices, i.e. constant elements of the Cartan subalgebra $\mathfrak{h}$. Another important issue is to explain how, using $Q_s(x,t)$ from eq. (5.1) we can parameterize any generic Lax pair related to that RHP.

Let us choose, following the ideas of Gel'fand and Dickey

$$\begin{aligned} L\psi &\equiv \frac{\partial\psi}{\partial x} - U(x,t,\lambda)\psi(x,t,\lambda) = 0, \qquad U(x,t,\lambda) = \left(\lambda^{N_1}\xi J\xi^{-1}(x,t,\lambda)\right)_+, \\ M\psi &\equiv \frac{\partial\psi}{\partial t} - V(x,t,\lambda)\psi(x,t,\lambda) = 0, \qquad V(x,t,\lambda) = \left(\lambda^{N_2}\xi K\xi^{-1}(x,t,\lambda)\right)_+, \end{aligned} \tag{5.3}$$

where the subscript $+$ means that we retain only the non-negative in λ terms in the right hand sides of (5.3) and explain how one can calculate $U(x,t,\lambda)$ and $V(x,t,\lambda)$. First, we remind that since $J,K \in \mathfrak{h}$ and $\xi^{\pm}(x,t,\lambda) \in \mathcal{G}$, then both $U(x,t,\lambda), V(x,t,\lambda) \in \mathfrak{g}$. From the general theory of Lie algebras we know that

$$\xi^{\pm} J\hat{\xi}^{\pm}(x,t,\lambda) = J + \sum_{k=1}^{\infty} \frac{\lambda^{-k}}{k!} \mathrm{ad}_{\mathcal{Q}}^{k} J = J - \sum_{k=1}^{\infty} \lambda^{-k} V_k. \tag{5.4}$$

where $\mathrm{ad}_{\mathcal{Q}} Y \equiv [\mathcal{Q}, Y]$, $\mathrm{ad}_{\mathcal{Q}}^2 \equiv [[\mathcal{Q},[\mathcal{Q},Y]]$ etc. The first few coefficients in these expansions take the form:

$$\begin{aligned} V_1 &= \mathrm{ad}_J Q_1, \qquad V_2 = \mathrm{ad}_J Q_2 - \frac{1}{2}\mathrm{ad}_{Q_1}^2 J, \\ V_3 &= \mathrm{ad}_J Q_3 - \frac{1}{2}\left(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2} + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1}\right) J - \frac{1}{6}\mathrm{ad}_{Q_1}^3 J, \\ V_4 &= \mathrm{ad}_J Q_4 - \frac{1}{2}\left(\mathrm{ad}_{Q_1}\mathrm{ad}_{Q_3} + \mathrm{ad}_{Q_2}^2 + \mathrm{ad}_{Q_3}\mathrm{ad}_{Q_1}\right) J \\ &\quad - \frac{1}{6}\left(\mathrm{ad}_{Q_1}^2\mathrm{ad}_{Q_2} + \mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1} + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1}^2\right) J - \frac{1}{24}\mathrm{ad}_{Q_1}^4 J. \end{aligned} \tag{5.5}$$

Thus we see that $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are parameterized by the first few coefficients Q_s.

Another formula from the general theory of Lie algebras which we will need below is:

$$i\frac{\partial\xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) = i\frac{\partial\mathcal{Q}}{\partial x} + i\sum_{k=1}^{\infty} \frac{1}{(k+1)!}\mathrm{ad}_{\mathcal{Q}}^{k}\frac{\partial\mathcal{Q}}{\partial x} = i\sum_{k=1}^{\infty} \mathcal{X}_k \lambda^{-k}, \tag{5.6}$$

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In general:

$$i\mathcal{X}_k = i\frac{\partial Q_k}{\partial x} + i\sum_{s=1}^{k}\frac{1}{(s+1)!}\sum_{j_1+j_2+\cdots+j_s=k}\mathrm{ad}_{Q_{j_1}}\mathrm{ad}_{Q_{j_2}}\cdots\mathrm{ad}_{Q_{j_{s-1}}}\frac{\partial Q_{j_s}}{\partial x}. \tag{5.7}$$

The first few coefficients are:

$$\begin{aligned}
\mathcal{X}_1 &= \frac{\partial Q_1}{\partial x}, \qquad \mathcal{X}_2 = \frac{\partial Q_2}{\partial x} + \frac{1}{2}\mathrm{ad}_{Q_1}\frac{\partial Q_1}{\partial x},\\
\mathcal{X}_3 &= \frac{\partial Q_3}{\partial x} + \frac{1}{2}\left(\mathrm{ad}_{Q_1}\frac{\partial Q_2}{\partial x} + \mathrm{ad}_{Q_2}\frac{\partial Q_1}{\partial x}\right) + \frac{1}{6}\mathrm{ad}^2_{Q_1}\frac{\partial Q_1}{\partial x},\\
\mathcal{X}_4 &= \frac{\partial Q_4}{\partial x} + \frac{1}{2}\left(\mathrm{ad}_{Q_1}\frac{\partial Q_3}{\partial x} + \mathrm{ad}_{Q_2}\frac{\partial Q_2}{\partial x} + \mathrm{ad}_{Q_3}\frac{\partial Q_1}{\partial x}\right)\\
&\quad + \frac{1}{6}\left(\mathrm{ad}^2_{Q_1}\frac{\partial Q_2}{\partial x} + \mathrm{ad}_{Q_1}\mathrm{ad}_{Q_2}\frac{\partial Q_1}{\partial x} + \mathrm{ad}_{Q_2}\mathrm{ad}_{Q_1}\frac{\partial Q_1}{\partial x}\right) + \frac{1}{24}\mathrm{ad}^3_{Q_1}\frac{\partial Q_1}{\partial x},
\end{aligned} \tag{5.8}$$

The effectiveness of the general form of the Lax pair (5.3) follows from the relation which is easy to check:

$$\left[\lambda^{N_1}\xi J\xi^{-1}(x,t,\lambda), \lambda^{N_2}\xi K\xi^{-1}(x,t,\lambda)\right] = \lambda^{N_1+N_2}\xi[J,K]\xi^{-1}(x,t,\lambda) = 0,$$

because the matrices K and J are diagonal. Therefore, the commutator $[U(x,t,\lambda), V(x,t,\lambda)]$ must contain only negative powers of λ.

In addition we may impose on $\mathcal{Q}$ Mikhailov type reductions. Each of them uses a finite order automorphism, which introduces a grading in the algebra $\mathfrak{g}$. Below we will use several types of $\mathbb{Z}_2$-reductions based on automorphisms of order 2 of the Lie algebra:

$$\begin{aligned}
&1) \quad \mathcal{Q}^\dagger(x,t,\kappa_1(\lambda)) = -\mathcal{Q}(x,t,\lambda), && \kappa_1(\lambda) = \lambda^*,\\
&2) \quad \mathcal{Q}^T(x,t,\lambda) = -S_a\mathcal{Q}(x,t,\lambda)S_a^{-1}, && \kappa_2(\lambda) = \pm\lambda,\\
&3) \quad \mathcal{Q}^*(x,t,\kappa_3(\lambda)) = -\mathcal{Q}(x,t,\lambda), && \kappa_3(\lambda) = \lambda^*,\\
&4) \quad C_0\mathcal{Q}(x,t,\lambda)C_0^{-1} = \mathcal{Q}(x,t,-\lambda), && C_0^2 = \mathbb{1}, \quad C_0 \in \mathfrak{h}.
\end{aligned} \tag{5.9}$$

compare with (4.3). The last reduction C_0 is typical for Lax operators which are quadratic in λ.

Another important $\mathbb{Z}_2$ reduction is provided by the Cartan involutions $\mathcal{J}$, which determines the hermitian symmetric spaces [67] and acts on $\mathcal{Q}(x,t,\lambda)$ as follows:

$$\mathcal{Q}(x,t,\lambda) = \mathcal{J}\mathcal{Q}(x,t,-\lambda)\mathcal{J}. \tag{5.10}$$

5.2. The family of N-wave equations with cubic nonlinearities

If we generalize to Lax pairs quadratic in λ we find:

$$\begin{aligned}
L_{\mathrm{Nw}}\psi &\equiv i\frac{\partial\psi}{\partial x} + (U_2(x,t) + \lambda[J,Q(x,t)] - \lambda^2 J)\psi(x,t,\lambda) = 0,\\
M_{\mathrm{Nw}}\psi &\equiv i\frac{\partial\psi}{\partial x} + (V_2(x,t) + \lambda[K,Q(x,t)] - \lambda^2 K)\psi(x,t,\lambda) = 0,
\end{aligned} \tag{5.11}$$

where U_2, V_2, Q again belong to a simple Lie algebra $\mathfrak{g}$, J and K are constant elements of $\mathfrak{h}$. Examples of N-wave type equations will be given below; here we just note that they contain first order derivatives with respect to x and t and cubic (not quadratic) nonlinearities with respect to Q_{jk}.

Below we will impose two types of Mikhailov reductions:

$$\begin{aligned}
&C_0U(x,t,\lambda)C_0^{-1} = U(x,t,-\lambda), && C_0V(x,t,\lambda)C_0^{-1} = V(x,t,-\lambda),\\
&U(x,t,\lambda) = -U^\dagger(x,t,\lambda^*), && V(x,t,\lambda) = -V^\dagger(x,t,\lambda^*),
\end{aligned} \tag{5.12}$$

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where

$$U(x,t,\lambda) = U_2(x,t) + \lambda[J,Q] - \lambda^2 J, \qquad V(x,t,\lambda) = V_2(x,t) + \lambda[K,Q] - \lambda^2 K. \tag{5.13}$$

and $C_0 \in \operatorname{Aut}\mathfrak{g}$, $C_0^2 = \mathbb{1}$. In particular for the n-wave equations (see eqs. (2.3) and (2.1), (2.2)), we get $Q = Q^\dagger$ and J and K must be real. For the FAS and the scattering matrix these reductions give:

$$\begin{aligned} \chi^{\pm}(x,t,\lambda) &= (\chi^{\mp})^\dagger(x,t,\lambda^*), \qquad T_{\rm Nw}(\lambda,t) = T_{\rm Nw}^\dagger(\lambda^*,t), \\ C_0\chi^{\pm}(x,t,\lambda)C_0^{-1} &= \chi^{\pm}(x,t,-\lambda), \qquad C_0 T_{\rm Nw}(\lambda,t)C_0^{-1} = T_{\rm Nw}(-\lambda,t), \end{aligned} \tag{5.14}$$

see [36,52].

First we will derive the N-wave equations in general form; then we will illustrate them by a couple of examples. Using the generic parametrization (5.1) we obtain:

$$\begin{aligned} U(x,t,\lambda) &= U_2 + \lambda U_1 - \lambda^2 J, \qquad U_2 = [J,Q_2] + \frac{1}{2}[Q_1,U_1], \qquad U_1 = [J,Q_1], \\ V(x,t,\lambda) &= V_2 + \lambda V_1 - \lambda^2 K, \qquad V_2 = [K,Q_2] + \frac{1}{2}[Q_1,V_1], \qquad V_1 = [K,Q_1], \end{aligned} \tag{5.15}$$

The compatibility condition in this case is:

$$i\frac{\partial}{\partial x}(V_2 + \lambda V_1) - i\frac{\partial}{\partial t}(U_2 + \lambda U_1) + [U_2 + \lambda U_1 - \lambda^2 J, V_2 + \lambda V_1 - \lambda^2 K] = 0. \tag{5.16}$$

It must hold identically with respect to λ. It is easy to check that the coefficients at λ^4 and λ^3 vanish. Some more efforts are needed to check that the coefficient at λ^2:

$$-[J,V_2] + [U_1,V_1] - [U_2,K] = 0 \tag{5.17}$$

also vanishes identically due to the proper parametrization of U and V. The compatibility conditions must hold identically with respect to λ. The first three of these relations:

$$\lambda^4: \quad [J,K] = 0, \qquad \lambda^3: \quad [K,U_1] - [J,V_1] = 0, \qquad \lambda^2: \quad [K,U_2] - [J,V_2] + [U_1,V_1] = 0, \tag{5.18}$$

are satisfied identically due to the correct parametrization of $\xi^{\pm}(x,t,\lambda)$. In more details

$$\begin{aligned} U_1 &= [J,Q_1], \quad U_2 = -\frac{1}{2}[U_1,Q_1] + [J,Q_2], \\ V_1 &= [K,Q_1], \quad V_2 = -\frac{1}{2}[V_1,Q_1] + [K,Q_2]. \end{aligned} \tag{5.19}$$

The last two coefficients at λ^1 and λ^0 vanish provided $Q_1(x,t)$ and $Q_2(x,t)$ satisfy the following N-wave type equations:

$$\begin{aligned} \lambda^1 \quad &: -i\frac{\partial U_1}{\partial t} + i\frac{\partial V_1}{\partial x} + [U_2,V_1] + [U_1,V_2] = 0, \\ \lambda^0 \quad &: -i\frac{\partial U_2}{\partial t} + i\frac{\partial V_2}{\partial x} + [U_2,V_2] = 0. \end{aligned} \tag{5.20}$$

Note, that while U_1 and V_1 are linear in Q_1, U_2 and V_2 are quadratic in Q_1. Therefore the nonlinearities in this N-wave equations are cubic in Q_1.

We assume that the root system of $\mathfrak{g}$ is split into $\Delta = \Delta_0 \cup \Delta_1$, such that

$$C_0 E_\alpha C_0^{-1} = E_\alpha, \quad \alpha \in \Delta_0, \qquad C_0 E_\alpha C_0^{-1} = -E_\alpha, \quad \alpha \in \Delta_1. \tag{5.21}$$

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We also denote positive and negative roots by a plus or minus superscript. Then considering (5.12) we must have:

$$\begin{aligned} &C_0 Q_1(x,t) C_0^{-1} = -Q_1(x,t), \quad C_0 Q_2(x,t) C_0^{-1} = Q_2(x,t), \\ &Q_{2s-1}(x,t) = \sum_{\alpha\in\Delta_1^+} (q_{2s-1,\alpha} E_\alpha - q^*_{2s-1,\alpha} E_{-\alpha}), \qquad Q_{2s}(x,t) = \sum_{\alpha\in\Delta_0^+} (q_{2s,\alpha} E_\alpha - q^*_{2s,\alpha} E_{-\alpha}), \end{aligned} \tag{5.22}$$

It is easy to check that this choice of Q_s is compatible with the following two involutions of the RHP

$$\begin{aligned} &\text{a)} \qquad \mathcal{Q}(x,t,\lambda) = \mathcal{Q}^\dagger(x,t,-\lambda^*) = \sum_{s=1}^{\infty} Q_s(x,t)\lambda^{-s}, \qquad Q_s^\dagger(x,t) = -Q_s(x,t), \\ &\text{b)} \qquad \mathcal{Q}(x,t,\lambda) = C_0 \mathcal{Q}(x,t,-\lambda) C_0^{-1}, \qquad C_0 Q_{2s-1} C_0^{-1} = -Q_{2s-1}, \quad C_0 Q_{2s} C_0^{-1} = Q_{2s}, \end{aligned} \tag{5.23}$$

which means that

$$\text{a)} \quad (\xi^\pm)^\dagger(x,t,\lambda) = (\xi^\mp)^{-1}(x,t,\lambda^*), \qquad \text{b)} \quad C_0 \xi^\pm(x,t,-\lambda) C_0^{-1} = \xi^\pm(x,t,-\lambda). \tag{5.24}$$

Example 1 (6-wave type equations: $sl(4)-case$)**.** *The involution is given by*

$$C_0 = \exp(\pi i H_{e_2}) = diag\,(1,-1,-1,1). \tag{5.25}$$

The potentials are:

$$Q_1 = \begin{pmatrix} 0 & q_1 & q_2 & 0 \\ -q_1^* & 0 & 0 & q_3 \\ -q_2^* & 0 & 0 & q_4 \\ 0 & -q_3^* & -q_4^* & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & q_6 \\ 0 & 0 & q_5 & 0 \\ 0 & -q_5^* & 0 & 0 \\ -q_6^* & 0 & 0 & 0 \end{pmatrix}, \tag{5.26}$$
$$J = diag\,(a_1, a_2, -a_2, -a_1), \quad K = diag\,(b_1, b_2, -b_2, -b_1).$$

The corresponding NLEE ensure that the coefficients at λ^1 and λ^0 also vanish. These give:

$$\begin{aligned} &-i(a_1-a_2)\frac{\partial q_1}{\partial t} + i(b_1-b_2)\frac{\partial q_1}{\partial x} + \kappa(q_1(|q_3|^2 - |q_2|^2) + 2(q_2 q_5^* + q_6 q_3^*)) = 0, \\ &-i(a_2+a_1)\frac{\partial q_2}{\partial t} + i(b_2+b_1)\frac{\partial q_2}{\partial x} + \kappa(q_2(|q_1|^2 - |q_4|^2) + 2(q_1^* q_5 - q_4^* q_6)) = 0, \\ &-i(a_2+a_1)\frac{\partial q_3}{\partial t} + i(b_2+b_1)\frac{\partial q_3}{\partial x} - \kappa(q_3(|q_1|^2 - |q_4|^2) - 2(q_1^* q_6 - q_4 q_5)) = 0, \\ &-i(a_1-a_2)\frac{\partial q_4}{\partial t} + i(b_1-b_2)\frac{\partial q_4}{\partial x} - \kappa(q_4(|q_3|^2 - |q_2|^2) + 2(q_2^* q_6 + q_3 q_5^*)) = 0 \end{aligned} \tag{5.27}$$

where $\kappa = a_1 b_2 - a_2 b_1$ and

$$\begin{aligned} &-2ia_2\frac{\partial q_5}{\partial t} + 2ib_2\frac{\partial q_5}{\partial x} - ia_1\frac{\partial(q_3 q_4^* - q_1^* q_2)}{\partial t} + ib_1\frac{\partial(q_3 q_4^* - q_1^* q_2)}{\partial x} \\ &\qquad + \kappa(q_1^* q_2 - q_3 q_4^*)(|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2) - 2\kappa q_5(|q_1|^2 - |q_2|^2 - |q_3|^2 + |q_4|^2) = 0, \\ &-2ia_1\frac{\partial q_6}{\partial t} + 2ib_1\frac{\partial q_6}{\partial x} - ia_2\frac{\partial(q_1 q_3 - q_2 q_4)}{\partial t} + ib_2\frac{\partial(q_1 q_3 - q_2 q_4)}{\partial x} \\ &\qquad + \kappa(q_1 q_3 - q_2 q_4)(|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2) + 2\kappa q_6(|q_1|^2 - |q_2|^2 - |q_3|^2 + |q_4|^2) = 0. \end{aligned} \tag{5.28}$$

Example 2 (4-wave type equations: $so(5)-case$)**.** *The involution is given by*

$$C_0 = \exp(\pi i H_{e_1}) = diag\,(-1,1,1,1,-1). \tag{5.29}$$

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We first choose the potentials Q_1, Q_2, J, K and the involution C_0 as follows:

$$Q_1(x,t) = \begin{pmatrix} 0 & q_1 & q_2 & q_3 & 0 \\ -q_1^* & 0 & 0 & 0 & q_3 \\ -q_2^* & 0 & 0 & 0 & -q_2 \\ -q_3^* & 0 & 0 & 0 & q_1 \\ 0 & -q_3^* & q_2^* & -q_1^* & 0 \end{pmatrix}, \quad Q_2(x,t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_4 & 0 & 0 \\ 0 & -q_4^* & 0 & q_4 & 0 \\ 0 & 0 & -q_4^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (5.30)$$
$$J = diag\,(a_1, a_2, 0, -a_2, -a_1), \quad K = diag\,(b_1, b_2, 0, -b_2, -b_1).$$

It is easy to check that $C_0Q_1C_0^{-1} = -Q_1$ and $C_0Q_2C_0^{-1} = Q_2$, and consequently the FAS of $L_{\rm Nw}$ (5.11) satisfies $C_0\chi^{\pm}(x,t,\lambda)C_0^{-1} = \chi^{\pm}(x,t,-\lambda)$. The corresponding equations (5.20) become:

$$\begin{aligned} &-2i(a_1-a_2)\frac{\partial q_1}{\partial t} + 2i(b_1-b_2)\frac{\partial q_1}{\partial x} + \kappa(q_1|q_2|^2 + q_2^2q_3^* - 2q_2q_4^*) = 0, \\ &-2ia_1\frac{\partial q_2}{\partial t} + 2ib_1\frac{\partial q_2}{\partial x} - \kappa(q_2(|q_2|^2 - |q_3|^2) + 2q_3q_4^* - 2q_1q_4) = 0, \\ &-2i(a_1+a_2)\frac{\partial q_3}{\partial t} + 2i(b_1+b_2)\frac{\partial q_3}{\partial x} - \kappa(q_3|q_2|^2 + q_2^2q_1^* - 2q_2q_4) = 0, \end{aligned} \qquad (5.31)$$

and

$$\begin{aligned} &-2ia_2\frac{\partial q_4}{\partial t} + 2ib_2\frac{\partial q_4}{\partial x} - i(2a_1-a_2)\frac{\partial(q_2q_1^*)}{\partial t} + i(2b_1-b_2)\frac{\partial(q_2q_1^*)}{\partial x} - i(2a_1+a_2)\frac{\partial(q_3q_2^*)}{\partial t} \\ &+i(2b_1+b_2)\frac{\partial(q_3q_2^*)}{\partial x} + \kappa\left(|q_1|^2(3q_3q_2^* + q_2q_1^*) + |q_3|^2(3q_2q_1^* + q_3q_2^*) + 2q_4(|q_1|^2 - |q_3|^2)\right) = 0, \end{aligned} \qquad (5.32)$$

5.3. The main idea of the dressing method

In this section we will generalize Zakharov-Shabat dressing method [134,135] for the quadratic pencils. We will start with the simplest possible form of the dressing factor which generates the one-soliton solutions. We do this for two reasons. The first one is that due to the additional involutions inherent in the quadratic pencils the dressing factors for the one-soliton solutions require solving block-matrix linear equations. The other reason is that we will be able to calculate the asymptotics of the one-soliton dressing factors which will allow us to study the soliton interactions for the corresponding NLEE. The N-soliton solutions can be derived either by repeating N-times the one-soliton dressing or by considering dressing factors whose pole singularities determined by $\lambda_k^{\pm}$, $k = 1, 2, \dots, N$. In this case one has to solve much more complicated block-matrix linear equations.

In order to avoid unnecessary repetitions of formulae we will introduce the notations for the 'naked' and one-soliton solutions FAS of the Lax pairs.

$$L_0\chi_0^{\pm}(x,t,\lambda) = 0, \qquad M_0\chi_0^{\pm}(x,t,\lambda) = 0, \qquad L_{1s}\chi_1^{\pm}(x,t,\lambda) = 0, \qquad M_{1s}\chi_1^{\pm}(x,t,\lambda) = 0, \qquad (5.33)$$

where L_0 and M_0 are the Lax pair whose potentials U_k and V_k are vanishing. By L_{1s} and M_{1s} we denote the Lax pair whose potentials are provided by the one-soliton solutions of the corresponding NLEE. Each time from the context it will be clear which specific La pair we are considering.

For the N-wave systems the 'naked' FAS are given by:

$$\chi_0^{\pm}(x,t,\lambda) = \exp\left(-i\lambda^2(Jx+Kt)\right), \qquad \chi_1^{\pm}(x,t,\lambda) = u(x,t,\lambda)\chi_0^{\pm}(x,t,\lambda), \qquad (5.34)$$

while for the NLS-type equations

$$\chi_0^{\pm}(x,t,\lambda) = \exp\left(-iJ(\lambda^2x+\lambda^4t)\right), \qquad \chi_1^{\pm}(x,t,\lambda) = u(x,t,\lambda)\chi_0^{\pm}(x,t,\lambda), \qquad (5.35)$$

where the dressing factor $u(x,t,\lambda)$ will be calculated below for each of the relevant cases. The specific form of J and K in (5.34) depends on the specific choice of the corresponding homogeneous space. Likewise the specific form of J in (5.35) is determined by the choice of the relevant symmetric space.

Each dressing factor is a fractional linear function of the spectral parameter λ. As such we will use:

$$c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \qquad c_1(-\lambda) = \frac{\lambda + \lambda_1^+}{\lambda + \lambda_1^-}, \qquad \lambda_1^\pm = \mu_1 \pm \nu_1, \qquad c_1^*(\lambda^*) = \frac{1}{c_1(\lambda)}. \tag{5.36}$$

Indeed, $c_1(-\lambda)$ comes up naturally due to the symmetry $\lambda \to -\lambda$. By $\lambda_1^\pm$ we denote constants such that $\operatorname{Im}(\lambda_1^\pm)^2 \gtrless 0$; i.e $\mu_1\nu_1 \gtrless 0$. As we shall see below $\lambda_1^\pm$, $-\lambda_1^\pm$ and their hermitian conjugate determine the discrete eigenvalues of L_{1s}.

The generic form of the dressing factors is the same for both types of NLEE considered above. If we impose only types of symmetries on L and M, such as:

$$U^\dagger(x,t,\lambda^*) = -U(x,t,\lambda), \qquad C_0 U(x,t,-\lambda)C_0^{-1} = U(x,t,\lambda), \tag{5.37}$$

and similar relations for $V(x,t,\lambda)$. Here C_0 is constant diagonal matrix such that $C_0^2 = \mathbb{1}$. Then $u(x,t,\lambda)$ must satisfy:

$$u^\dagger(x,t,\lambda^*) = u^{-1}(x,t,\lambda), \qquad C_0 u(x,t,-\lambda)C_0^{-1} = u(x,t,\lambda), \tag{5.38}$$

then $u(x,t,\lambda)$ and its inverse have the form

$$\begin{aligned} u(x,t,\lambda) &= \mathbb{1} + (c_1(\lambda) - 1)|N_1\rangle\langle m_1| + (c_1(-\lambda) - 1)C_0|N_1\rangle\langle m_1|C_0^{-1}, \\ u^{-1}(x,t,\lambda) &= \mathbb{1} + \left(\frac{1}{c_1(\lambda)} - 1\right)|n_1\rangle\langle M_1| + \left(\frac{1}{c_1(-\lambda)} - 1\right)C_0|n_1\rangle\langle M_1|C_0^{-1}. \end{aligned} \tag{5.39}$$

Here the 'polarization' vectors $|N_1\rangle$, $\langle m_1|$, $|n_1\rangle$ and $\langle M_1|$ determine the residues of u and u^{-1}. These residues for the one-soliton case will be evaluated explicitly below for each of the NLEE we consider.

5.4. *Dressing of N-wave equations: two involutions*

We start with the N-wave type on homogeneous spaces with two involutions. Using the equations (5.33) we derive the following equation for the dressing factor:

$$i\frac{\partial u}{\partial x} + (U_{2;1s} + \lambda U_{1;1s} - \lambda^2 J)u(x,t,\lambda) - u(x,t,\lambda)(U_{20} + \lambda U_{10} - \lambda^2 J) = 0, \tag{5.40}$$

which also must hold identically with respect to λ. This can be verified by taking the residues of the left hand sides of (5.40) for $\lambda = \lambda_1^-$ and equating them to 0. This gives:

$$\begin{aligned} &\left(i\frac{\partial|N_1\rangle}{\partial x} + (U_{2;1s} + \lambda_1^- U_{1;1s} - (\lambda_1^-)^2 J)|N_1\rangle\right)\langle m_1| \\ &\qquad + |N_1\rangle\left(i\frac{\partial\langle m_1|}{\partial x} - \langle m_1|(U_{20} + \lambda_1^- U_{10} - (\lambda_1^-)^2 J\right) = 0, \end{aligned} \tag{5.41}$$

from which one easily finds, see e q. (5.34):

$$|N_1\rangle = \chi_1^-(x,t,\lambda_1^-)|N_{10}\rangle, \qquad \langle m_1| = \langle m_{10}|\hat{\chi}_0^-(x,t,\lambda_1^-). \tag{5.42}$$

Similarly, we can use the equation satisfied by $\hat{u} \equiv u^{-1}(x,t,\lambda)$ which reads:

$$i\frac{\partial \hat{u}}{\partial x} + (U_{20} + \lambda U_{10} - \lambda^2 J)\hat{u}(x,t,\lambda) - \hat{u}(x,t,\lambda)(U_{2;1s} + \lambda U_{1;1s} - \lambda^2 J) = 0. \tag{5.43}$$

Putting the residue of (5.43) at λ_1^+ to 0 we get:

$$\begin{split} \left(i\frac{\partial |n_1\rangle}{\partial x} + (U_{20} + \lambda_1^+ U_{10} - (\lambda_1^+)^2 J)|n_1\rangle \right) \langle M_1| \\ + |n_1\rangle \left(i\frac{\partial \langle M_1|}{\partial x} - \langle m_1|(U_{2;1s} + \lambda_1^+ U_{1;1s} - (\lambda_1^+)^2 J \right) = 0, \end{split} \tag{5.44}$$

The result is, see eq. (5.34):

$$|n_1\rangle = \chi_0^+(x,t,\lambda_1^+)|n_{10}\rangle, \qquad \langle M_1| = \langle M_{10}|\hat{\chi}_1^+(x,t,\lambda_1^+). \tag{5.45}$$

Remark 7. *We note that the vectors $|n_{10}\rangle$, $|N_{10}\rangle$, $\langle m_{10}|$ and $\langle M_{10}|$ are constants, which must satisfy the (5.38). Due to the same reductions we must also have $\lambda_1^- = (\lambda_1^+)^*$. We have also chosen C_0 to be constant diagonal matrix whose matrix elements equal ± 1.*

Thus, if we know the regular solutions $\chi_0^\pm(x,t,\lambda)$ then we have derived explicitly the x and t dependence of the vectors $|n_1\rangle$ and $\langle m_1|$. In addition we know that $u\hat{u} = \mathbb{1}$ also must hold identically with respect to λ. That means that the residues:

$$\operatorname*{Res}_{\lambda=\lambda_1^-} u(x,t,\lambda)u^{-1}(x,t,\lambda) = 0, \qquad \operatorname*{Res}_{\lambda=\lambda_1^+} u(x,t,\lambda)u^{-1}(x,t,\lambda) = 0. \tag{5.46}$$

must vanish. Inserting u and $\hat{u}$ from eq. (5.39) we obtain the equations:

$$\begin{aligned} \langle M_1| &= \langle m_1| \left(\langle m_1|n_1\rangle \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} \langle m_1|C_0|n_1\rangle C_0 \right)^{-1}, \\ |N_1\rangle &= \left(\langle m_1|n_1\rangle \mathbb{1} - \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} \langle m_1|C_0|n_1\rangle C_0 \right)^{-1} |n_1\rangle. \end{aligned} \tag{5.47}$$

In the specific calculations below we will use more convenient notations, namely:

$$\begin{aligned} &\mathcal{E}_{01}^\pm = \exp\left(-i(\lambda_1^\pm)^2 (Jx + Kt) \right) = \operatorname{diag}\left(e^{\pm z_1 - i\phi_1}, \dots, e^{\pm z_n - i\phi_n} \right), \quad \lambda_1^\pm = \mu_1 \pm i\nu_1, \\ &\quad z_k = 2\mu_1\nu_1(J_k x + K_k t), \quad \phi_k = (\mu_1^2 - \nu_1^2)(J_k x + K_k t), \quad \kappa_1 = \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} = i\frac{\nu_1}{\mu_1}, \\ &|n_1\rangle = \mathcal{E}_{01}^+ |n_{10}\rangle, \qquad \langle m_1| = \langle m_{10}|\hat{\mathcal{E}}_{01}^-. \end{aligned} \tag{5.48}$$

where $\mu_1 > 0$ and $\nu_1 > 0$. The functions $z_k(x,t)$ and $\phi_k(x,t)$ are linear functions of x and t; for each specific example they will be given explicitly.

The last step we need to do is to determine the corresponding singular potentials U_1 and U_2. To this end we come back to the equation (5.40) for the dressing factor and study its limit for $\lambda \to \infty$. Its left hand side is a quadratic polynomial of λ. Skipping the details we obtain:

$$\begin{aligned} U_{1;1s} - U_{10} &= (\lambda_1^+ - \lambda_1^-)[J, \mathcal{W}^+], \qquad \mathcal{W}^\pm = \mp|N_1\rangle\langle m_1| + C_0|N_1\rangle\langle m_1|C_0, \\ U_{2;1s} - U_{20} &= -(\lambda_1^+ - \lambda_1^-)\left(U_1\mathcal{W}^- - \mathcal{W}^- U_{20} + \lambda_1^-[J, \mathcal{W}^-] \right) \end{aligned} \tag{5.49}$$

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We we put $U_{10} = U_{20} = 0$ we get simplified expression for the one-soliton solution:

$$\begin{aligned} U_{1;1s} &= (\lambda_1^+ - \lambda_1^-)[J, \mathcal{W}^-], \\ U_{2;1s} &= -(\lambda_1^+ - \lambda_1^-)\left(U_{1;1s}\mathcal{W}^+ + \lambda_1^-[J, \mathcal{W}^-]\right). \end{aligned} \tag{5.50}$$

More explicit expressions for $U_{1;1s}$ and $U_{2;1s}$ in terms of hyper-trigonometric functions will be given below for each of the examples.

Example 3 (One soliton solutions, $sl(4)$ case)**.** *The Lax representation in that case is provided by the operators (5.11) where J, K, Q_1 and Q_2 are given by (5.26). The 'naked' polarization vectors $\langle m_1|$ (5.42) and $|n_1\rangle$ (5.45) become:*

$$\langle m_1| = \left(m_{10,1}e^{z_1+i\phi_1}, m_{10,2}e^{z_2+i\phi_2}, m_{10,3}e^{-z_2-i\phi_2}, m_{10,2}e^{-z_1-i\phi_1}\right), \qquad |n_1\rangle = \begin{pmatrix} n_{10,1}e^{z_1-i\phi_1} \\ n_{10,2}e^{z_2-i\phi_2} \\ n_{10,3}e^{-z_2+i\phi_2} \\ n_{10,4}e^{-z_1+i\phi_1} \end{pmatrix}. \tag{5.51}$$

Taking into account the typical hermitian reduction of L and M we find $\langle m_1| = (|n_1\rangle)^\dagger$ and $\langle m_{10}| = (|n_{10}\rangle)^\dagger$. The dressed polarization vectors defined by (5.47) are equal to:

$$\begin{aligned} &\langle M_1| = \left(\frac{m_{1;1}}{R_1^+}, \frac{m_{1;2}}{R_1^-}, \frac{m_{1;3}}{R_1^-}, \frac{m_{1;4}}{R_1^+}\right), \qquad |N_1\rangle = \begin{pmatrix} (R_1^-)^{-1} & 0 & 0 & 0 \\ 0 & (R_1^+)^{-1} & 0 & 0 \\ 0 & 0 & (R_1^+)^{-1} & 0 \\ 0 & 0 & 0 & (R_1^-)^{-1} \end{pmatrix} |n_1\rangle, \\ &R_1^\pm = \langle m_1|n_1\rangle \pm \frac{i\nu_1}{\mu_1}\langle m_1|C_0|n_1\rangle, \\ &\langle m_1|n_1\rangle = 2\eta_{01}\cosh(2z_1+\zeta_{01}) + 2\eta_{02}\cosh(2z_2+\zeta_{02}), \quad \eta_{01} = |n_{10,1}n_{10,4}|, \quad \eta_{02} = |n_{10,2}n_{10,3}|, \\ &\langle m_1|C_0|n_1\rangle = 2\eta_{01}\cosh(2z_1+\zeta_{01}) - 2\eta_{02}\cosh(2z_2+\zeta_{02}), \quad \zeta_{01} = \ln\frac{|n_{10,1}|}{|n_{10,4}|}, \quad \zeta_{02} = \ln\frac{|n_{10,2}|}{|n_{10,3}|}, \end{aligned} \tag{5.52}$$

They also satisfy $\langle M_1| = (|N_1\rangle)^\dagger$. Therefore

$$\begin{aligned} W^+ &= -2\begin{pmatrix} 0 & n_{1,1}m_{1,2} & n_{1,1}m_{1,3} & 0 \\ n_{2,1}m_{1,1} & 0 & 0 & n_{1,2}m_{1,4} \\ n_{1,3}m_{1,1} & 0 & 0 & n_{1,3}m_{1,4} \\ 0 & n_{1,4}m_{1,2} & n_{1,4}m_{1,3} & 0 \end{pmatrix}, \\ W^- &= 2\begin{pmatrix} n_{1,1}m_{1,1} & 0 & 0 & n_{1,1}m_{1,4} \\ 0 & n_{1,2}m_{1,2} & n_{1,2}m_{1,3} & 0 \\ 0 & n_{1,3}m_{1,2} & n_{1,3}m_{1,3} & 0 \\ n_{1,4}m_{1,1} & 0 & 0 & n_{1,4}m_{1,4} \end{pmatrix} \end{aligned} \tag{5.53}$$

$$\begin{aligned} q_1(x,t) &= -\frac{4i\nu_1(a_1-a_2)m_{1,2}n_{1,1}}{R_1^+}, \qquad q_2(x,t) = -\frac{4i\nu_1(a_1+a_2)m_{1,3}n_{1,1}}{R_1^+} \\ q_3(x,t) &= -\frac{4i\nu_1(a_1+a_2)m_{1,4}n_{1,2}}{R_1^-}, \qquad q_4(x,t) = -\frac{4i\nu_1(a_1-a_2)m_{1,4}n_{1,3}}{R_1^-} \\ q_5(x,t) &= -\frac{8\nu_1 n_{1,1}m_{1,4}\left((a_1-a_2)n_{1,2}m_{1,2} + (a_1-a_2)n_{1,3}m_{1,3}\right) - i\lambda_1^- a_1 R_1^-}{R_1^+R_1^-}, \\ q_6(x,t) &= \frac{8\nu_1 n_{1,2}m_{1,3}\left((a_1-a_2)n_{1,1}m_{1,1} - (a_1+a_2)n_{1,4}m_{1,4}\right) - i\lambda_1^+ a_2 R_1^+}{R_1^+R_1^-}, \end{aligned} \tag{5.54}$$

$$
\begin{aligned}
q_1(x,t) &= \frac{4i\nu_1\mu_1(a_1-a_2)|n_{10,1}n_{10,2}|e^{z_1+z_2+i(\tilde{\phi}_2-\tilde{\phi}_1)}\left[\lambda_1^+\eta_{01}\cosh(\tilde{z}_1)+\lambda_1^-\eta_{02}\cosh(\tilde{z}_2)\right]}{(\mu_1^2+\nu_1^2)(\eta_{01}^2\cosh^2(\tilde{z}_1)+\eta_{02}^2\cosh^2(\tilde{z}_2))+2\eta_{10}\eta_{20}(\mu_1^2-\nu_1^2)\cosh(\tilde{z}_1)\cosh(\tilde{z}_2)},\\
q_2(x,t) &= \frac{4i\nu_1\mu_1(a_1+a_2)|n_{10,3}n_{10,2}|e^{z_1-z_2-i(\tilde{\phi}_1+\tilde{\phi}_2)}\left[\lambda_1^+\eta_{01}\cosh(\tilde{z}_1)+\lambda_1^-\eta_{02}\cosh(\tilde{z}_2)\right]}{(\mu_1^2+\nu_1^2)(\eta_{01}^2\cosh^2(\tilde{z}_1)+\eta_{02}^2\cosh^2(\tilde{z}_2))+2\eta_{10}\eta_{20}(\mu_1^2-\nu_1^2)\cosh(\tilde{z}_1)\cosh(\tilde{z}_2)},\\
q_3(x,t) &= \frac{4i\nu_1\mu_1(a_1+a_2)|n_{10,4}n_{10,2}|e^{-z_1+z_2-i(\tilde{\phi}_1+\tilde{\phi}_2)}\left[\lambda_1^-\eta_{01}\cosh(\tilde{z}_1)+\lambda_1^+\eta_{02}\cosh(\tilde{z}_2)\right]}{(\mu_1^2+\nu_1^2)(\eta_{01}^2\cosh^2(\tilde{z}_1)+\eta_{02}^2\cosh^2(\tilde{z}_2))+2\eta_{10}\eta_{20}(\mu_1^2-\nu_1^2)\cosh(\tilde{z}_1)\cosh(\tilde{z}_2)},\\
q_4(x,t) &= \frac{4i\nu_1\mu_1(a_1-a_2)|n_{10,1}n_{10,2}|e^{-z_1-z_2+i(\tilde{\phi}_2-\tilde{\phi}_1)}\left[\lambda_1^-\eta_{01}\cosh(\tilde{z}_1)+\lambda_1^+\eta_{02}\cosh(\tilde{z}_2)\right]}{(\mu_1^2+\nu_1^2)(\eta_{01}^2\cosh^2(\tilde{z}_1)+\eta_{02}^2\cosh^2(\tilde{z}_2))+2\eta_{10}\eta_{20}(\mu_1^2-\nu_1^2)\cosh(\tilde{z}_1)\cosh(\tilde{z}_2)},
\end{aligned}
\tag{5.55}
$$

where $\tilde{z}_k = 2z_k + \zeta_{0k}$ *and* $\tilde{\phi}_k = \phi_k + \arg n_{10,k}$. *In addition*

$$
\begin{aligned}
&q_5(x,t) = -16\nu_1|n_{10,1}||n_{10,4}|e^{-2i\tilde{\phi}_1}\times\\
&\times\frac{\left[\nu_1\mu_1\left(a_1\eta_{01}\cosh(\tilde{z}_2)-a_2\eta_{02}\sinh(\tilde{z}_2)\right)-i\lambda_1^-a_1(\lambda_1^-\eta_{01}\cosh(\tilde{z}_1)+\lambda_1^+\eta_{02}\cosh(\tilde{z}_2))\right)]}{\mu_1\left((\mu_1^2+\nu_1^2)(\eta_{01}^2\cosh^2(\tilde{z}_1)+\eta_{02}^2\cosh^2(\tilde{z}_2))+2\eta_{10}\eta_{20}(\mu_1^2-\nu_1^2)\cosh(\tilde{z}_1)\cosh(\tilde{z}_2)\right)},\\
&q_6(x,t) = -16\nu_1|n_{10,2}||n_{10,3}|e^{-2i\tilde{\phi}_2}\times\\
&\times\frac{\left[\nu_1\mu_1\left(a_1\eta_{01}\sinh(\tilde{z}_1)-a_2\eta_{02}\cosh(\tilde{z}_2)\right)-i\lambda_1^+a_2(\lambda_1^-\eta_{01}\cosh(\tilde{z}_1)+\lambda_1^+\eta_{02}\cosh(\tilde{z}_2))\right)]}{\mu_1\left((\mu_1^2+\nu_1^2)(\eta_{01}^2\cosh^2(\tilde{z}_1)+\eta_{02}^2\cosh^2(\tilde{z}_2))+2\eta_{10}\eta_{20}(\mu_1^2-\nu_1^2)\cosh(\tilde{z}_1)\cosh(\tilde{z}_2)\right)}.
\end{aligned}
\tag{5.56}
$$

5.5. Dressing of N-wave equations: three involutions

Here we will consider only the cases, when the two above involutions are combined with condition that $u(x,t,\lambda) \in SO(N)$ or $u(x,t,\lambda) \in SP(2N)$. Then the dressing factors take the form:

$$
\begin{aligned}
u(x,t,\lambda) &= \mathbb{1} + (c_1(\lambda)-1)|N_1\rangle\langle m_1| + (c_1(-\lambda)-1)C_0|N_1\rangle\langle m_1|C_0^{-1}\\
&\quad + \left(\frac{1}{c_1(\lambda)}-1\right)S_a^{-1}|M_1\rangle\langle n_1|S_a + \left(\frac{1}{c_1(-\lambda)}-1\right)S_a^{-1}C_0|M_1\rangle\langle n_1|C_0^{-1}S_a,\\
u^{-1}(x,t,\lambda) &= \mathbb{1} + \left(\frac{1}{c_1(\lambda)}-1\right)|n_1\rangle\langle M_1| + \left(\frac{1}{c_1(-\lambda)}-1\right)C_0^{-1}|n_1\rangle\langle M_1|C_0\\
&\quad + (c_1(\lambda)-1)\,S_a|m_1\rangle\langle N_1|S_a^{-1} + (c_1(-\lambda)-1)\,S_aC_0^{-1}|m_1\rangle\langle N_1|C_0S_a^{-1},
\end{aligned}
\tag{5.57}
$$

where $c_1(\lambda) = (\lambda-\lambda_1^+)/(\lambda-\lambda_1^-)$. Inserting $c_1(\lambda)$ into (5.57) we get:

$$
\begin{aligned}
u(x,t,\lambda) &= \mathbb{1} - \frac{\lambda_1^+-\lambda_1^-}{\lambda-\lambda_1^-}|N_1\rangle\langle m_1| + \frac{\lambda_1^+-\lambda_1^-}{\lambda+\lambda_1^-}C_0|N_1\rangle\langle m_1|C_0^{-1}\\
&\quad + \frac{\lambda_1^+-\lambda_1^-}{\lambda-\lambda_1^+}S_a^{-1}|M_1\rangle\langle n_1|S_a - \frac{\lambda_1^+-\lambda_1^-}{\lambda+\lambda_1^+}S_a^{-1}C_0|M_1\rangle\langle n_1|C_0^{-1}S_a,\\
u^{-1}(x,t,\lambda) &= \mathbb{1} + \frac{\lambda_1^+-\lambda_1^-}{\lambda-\lambda_1^+}|n_1\rangle\langle M_1| - \frac{\lambda_1^+-\lambda_1^-}{\lambda+\lambda_1^+}C_0^{-1}|n_1\rangle\langle M_1|C_0\\
&\quad - \frac{\lambda_1^+-\lambda_1^-}{\lambda-\lambda_1^-}S_a|m_1\rangle\langle N_1|S_a^{-1} + \frac{\lambda_1^+-\lambda_1^-}{\lambda+\lambda_1^-}S_aC_0^{-1}|m_1\rangle\langle N_1|C_0S_a^{-1}.
\end{aligned}
\tag{5.58}
$$

Since the dressing factor is mapping a regular solution to the RHP into a singular solution of the same problem, it must satisfy the reductions (5.24), i.e

$$
u^\dagger(x,t,\lambda^*) = u^{-1}(x,t,\lambda). \tag{5.59}
$$

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In addition one can check that the conditions:

$$C_0 u(x,t,-\lambda) C_0^{-1} = u(x,t,\lambda), \qquad u^{-1}(x,t,\lambda) = S_a u^T(x,t,\lambda) S_a^{-1}, \qquad a = 0,1; \tag{5.60}$$

are identically satisfied. The second condition in (5.60) ensures that $u(x,t,\lambda)$ belongs to the orthogonal group if $S = S_0$; for $S = S_1$ $u(x,t,\lambda)$ belongs to the symplectic group.

Note that the new dressing factors and their inverse satisfy the same differential equations (5.40) and (5.43) respectively. These equations must be satisfied identically with respect to λ. This means that all the residues of these equations at the poles of u and $\hat{u}$ must vanish.

Thus we obtain the generalizations of eqs. (5.45) and (5.42) for the three involutions case:

$$\begin{aligned} |n_1\rangle &= \chi_0^+(x,t,\lambda_1^+)|n_{10}\rangle, & |N_1\rangle &= \chi_1^+(x,t,\lambda_1^+)|N_{10}\rangle, \\ \langle m_1| &= \langle m_{10}|\hat{\chi}_0^-(x,t,\lambda_1^-), & \langle M_1| &= \langle M_{10}|\hat{\chi}_1^-(x,t,\lambda_1^-), \end{aligned} \tag{5.61}$$

where $|n_{10}\rangle$, $|N_{10}\rangle$, $\langle m_{10}|$ and $\langle M_{10}|$ are constant polarization vectors. We assume that we know $\chi_0^\pm(x,t,\lambda)$ which are related to the regular solutions of the RHP. Typically they correspond to vanishing potentials $U_{10} = 0$ and $U_{20} = 0$; thus we must have:

$$\chi_0^\pm(x,t,\lambda) = e^{-i\lambda^2(Jx+Kt)}. \tag{5.62}$$

In addition we need to ensure that the expressions for $uu^{-1} = \mathbb{1}$ and $u^{-1}u = \mathbb{1}$ hold identically with respect to λ. The presence of the third involution makes this problem more difficult, because these expressions have second order poles at the points $\lambda_1^\pm$. It is easy to see that these residues simplify to:

$$\begin{aligned} &\langle m_1|S_a|m_1\rangle = 0, && \langle n_1|S_a|n_1\rangle = 0, && \langle m_1|C_0S_aC_0|m_1\rangle = 0, && \langle n_1|C_0S_aC_0|n_1\rangle = 0, \\ &\langle N_1|S_a|N_1\rangle = 0, && \langle M_1|S_a|M_1\rangle = 0, && \langle N_1|C_0S_aC_0|N_1\rangle = 0, && \langle M_1|C_0S_aC_0|N_1\rangle = 0, \end{aligned} \tag{5.63}$$

Remember that these vectors depend on x and t and the conditions (5.63) must be identities. But we also know that these polarization vectors must satisfy (5.61). Therefore we have:

$$\begin{aligned} \langle n_1|S_a|n_1\rangle &= \langle n_{10}|(\chi_0^+(x,\lambda_1^+))^T S_a \chi_0^+(x,\lambda_1^+)|n_{10}\rangle = \langle n_{10}|S_a|n_{10}\rangle, \\ \langle N_1|S_a|N_1\rangle &= \langle N_{10}|(\chi^-(x,\lambda_1^-))^T S_a \chi^-(x,\lambda_1^-)|N_{10}\rangle = \langle N_{10}|S_a|N_{10}\rangle, \\ \langle N_1|C_0S_aC_0|N_1\rangle &= \langle N_{10}|(\chi^-(x,\lambda_1^-))^T C_0S_aC_0\chi^-(x,\lambda_1^-)|N_{10}\rangle \\ &= \langle N_{10}|C_0(\chi^-(x,-\lambda_1^-))^T S_a \chi^-(x,-\lambda_1^-)C_0|N_{10}\rangle = \langle N_{10}|C_0S_aC_0|N_{10}\rangle. \end{aligned} \tag{5.64}$$

The proof that all other scalar products in (5.63) are x and t independent is done similarly. Thus the conditions (5.63) in fact impose restrictions only on the initial polarization vectors:

$$\begin{aligned} &\langle m_{10}|S_a|m_{10}\rangle = 0, && \langle n_{10}|S_a|n_{10}\rangle = 0, && \langle m_{10}|C_0S_aC_0|m_{10}\rangle = 0, && \langle n_{10}|C_0S_aC_0|n_{10}\rangle = 0, \\ &\langle N_{10}|S_a|N_{10}\rangle = 0, && \langle M_{10}|S_a|M_{10}\rangle = 0, && \langle N_{10}|C_0S_aC_0|N_{10}\rangle = 0, && \langle M_{10}|C_0S_aC_0|N_{10}\rangle = 0, \end{aligned} \tag{5.65}$$

The last condition that is imposed on the polarization vectors comes from eq. (5.59) and reads

$$|n_{10}\rangle = (\langle m_{10}|)^\dagger, \qquad |N_{10}\rangle = (\langle M_{10}|)^\dagger, \qquad C_0S_a = S_aC_0. \tag{5.66}$$

Taking the residues at λ_1^-, λ_1^+ leads to

$$\langle m_1| \left(\mathbb{1} - |n_1\rangle\langle M_1| - \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} C_0 |n_1\rangle\langle M_1| C_0 + \frac{\lambda_1^+ - \lambda_1^-}{2\lambda_1^-} S C_0 |m_1\rangle\langle N_1| C_0 S^{-1} \right) = 0,$$
$$\left(\mathbb{1} - |N_1\rangle\langle m_1| + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} C_0 |N_1\rangle\langle m_1| C_0 - \frac{\lambda_1^+ - \lambda_1^-}{2\lambda_1^+} S^{-1} C_0 |M_1\rangle\langle n_1| C_0 S \right) |n_1\rangle = 0. \tag{5.67}$$

i.e., transposing the first of the above equations we find:

$$|m_1\rangle = \left(\langle m_1|n_1\rangle \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} C_0 \langle m_1|C_0|n_1\rangle \right) |M_1\rangle - \frac{\lambda_1^+ - \lambda_1^-}{2\lambda_1^-} \langle m_1|C_0 S|m_1\rangle C_0 S |N_1\rangle,$$
$$|n_1\rangle = \left(\langle m_1|n_1\rangle \mathbb{1} - \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} \langle m_1|C_0|n_1\rangle C_0 \right) |N_1\rangle + \frac{\lambda_1^+ - \lambda_1^-}{2\lambda_1^+} \langle n_1|C_0 S|n_1\rangle S^{-1} C_0 |M_1\rangle \tag{5.68}$$

By the way, $\lambda_1^{\pm} = \mu_1 \pm i\nu_1$, so

$$\frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} = \frac{i\nu_1}{\mu_1}.$$

Thus eq. (5.68) is rewritten as:

$$\begin{pmatrix} |m_1\rangle \\ |n_1\rangle \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix} \begin{pmatrix} |M_1\rangle \\ |N_1\rangle \end{pmatrix} \tag{5.69}$$

where the block matrices $X_{1,1}$ and $Y_{1,2}$ are given by:

$$X_1 = \langle m_1|n_1\rangle \mathbb{1} + \frac{i\nu_1}{\mu_1} C_0 \langle m_1|C_0|n_1\rangle, \qquad X_2 = \frac{i\nu_1}{\lambda_1^-} \langle m_1|C_0 S|m_1\rangle C_0 S,$$
$$Y_1 = -\frac{i\nu_1}{\lambda_1^+} \langle n_1|C_0 S|n_1\rangle S^{-1} C_0, \qquad Y_2 = \langle m_1|n_1\rangle \mathbb{1} - \frac{i\nu_1}{\mu_1} \langle m_1|C_0|n_1\rangle C_0, \tag{5.70}$$

The inverse of the matrix in the right hand side of eq. (5.69) is given by:

$$\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{-1} = \begin{pmatrix} (X_1 - X_2 Y_2^{-1} Y_1)^{-1} & -X_1^{-1} X_2 (Y_2 - Y_1 X_1^{-1} X_2)^{-1} \\ -Y_2^{-1} Y_1 (X_1 - X_2 Y_2^{-1} Y_1)^{-1} & (Y_2 - Y_1 X_1^{-1} X_2)^{-1} \end{pmatrix} \tag{5.71}$$

and

$$X_1^{-1} = \frac{\mu_1^2 \langle m_1|n_1\rangle - i\mu_1\nu_1 \langle m_1|C_0|n_1\rangle C_0}{\mu_1^2 \langle m_1|n_1\rangle^2 + \nu_1^2 \langle m_1|C_0|n_1\rangle^2}, \quad Y_2^{-1} = \frac{\mu_1^2 \langle m_1|n_1\rangle + i\mu_1\nu_1 \langle m_1|C_0|n_1\rangle C_0}{\mu_1^2 \langle m_1|n_1\rangle^2 + \nu_1^2 \langle m_1|C_0|n_1\rangle^2}, \tag{5.72}$$

i.e., we introduce the relations and the notations

$$X_1^{-1} = g_1 Y_2, \qquad Y_2^{-1} = g_1 X_1, \qquad g_1 = \frac{\mu_1^2}{\mu_1^2 \langle m_1|n_1\rangle + \nu_1^2 \langle m_1|C_0|n_1\rangle},$$
$$\beta_1 = \langle m_1|S C_0|m_1\rangle, \qquad \bar{\beta}_1 = \langle n_1|S C_0|n_1\rangle. \tag{5.73}$$

Using the fact that S and C_0 commute we can simplify the matrix in (5.71) into:

$$\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{-1} = \frac{g_1}{w_1} \begin{pmatrix} Y_2 & \frac{i\nu_1}{\lambda_1^-} \beta_1 C_0 S \\ -\frac{i\nu_1}{\lambda_1^+} \bar{\beta}_1 C_0 S^{-1} & X_1 \end{pmatrix} \tag{5.74}$$

where

$$\frac{g_1}{w_1} = \frac{\mu_1^2(\mu_1^2+\nu_1^2)}{(\mu_1^2+\nu_1^2)(\mu_1^2\langle m_1|n_1\rangle^2 + \nu_1^2\langle m_1|C_0|n_1\rangle^2) - \mu_1^2\nu_1^2\beta_1\bar{\beta}_1}. \tag{5.75}$$

Thus we find the following expressions for $|M_1\rangle$ and $|N_1\rangle$ in terms of $|m_1\rangle$ and $|n_1\rangle$:

$$\begin{aligned} |M_1\rangle &= \frac{g_1}{w_1}\left(\left(\langle m_1|n_1\rangle + \frac{i\nu_1}{\mu_1}\langle m_1|C_0|n_1\rangle C_0\right)|m_1\rangle + \frac{i\nu_1}{\lambda_1^-}\beta_1 C_0 S|n_1\rangle\right), \\ |N_1\rangle &= \frac{g_1}{w_1}\left(\left(\langle m_1|n_1\rangle - \frac{i\nu_1}{\mu_1}\langle m_1|C_0|n_1\rangle C_0\right)|n_1\rangle - \frac{i\nu_1}{\lambda_1^+}\bar{\beta}_1 C_0 S|m_1\rangle\right), \end{aligned} \tag{5.76}$$

It remains to calculate the potentials. To this end we rewrite eq. (5.40) in the form:

$$i\frac{\partial u}{\partial x}\hat{u} + U_{2,1s} + \lambda U_{1,1s} = \lambda^2(J - uJ\hat{u}(x,t,\lambda)), \tag{5.77}$$

where we have assumed $U_{20} = 0$ and $U_{1,0} = 0$. Taking the limit $\lambda \to \infty$ of (5.77) we obtain:

$$U_{1,1s} = -2i\nu_1(J\hat{A} + AJ), \qquad U_{2,1s} = 4\nu_1^2(AJ\hat{B} + BJ\hat{A}) - 2i\nu_1(BJ + J\hat{B}), \tag{5.78}$$

where

$$\begin{aligned} A &= \left(-|N_1\rangle\langle m_1| + C_0|N_1\rangle\langle m_1|C_0 + S_a^{-1}|M_1\rangle\langle n_1|S_a - S_a^{-1}C_0|M_1\rangle\langle n_1|C_0S_a\right), \\ \hat{A} &= \left(|n_1\rangle\langle M_1| - C_0|n_1\rangle\langle M_1|C_0 - S_a|m_1\rangle\langle N_1|S_a^{-1} + S_aC_0|m_1\rangle\langle N_1|C_0S_a^{-1}\right), \\ B &= \left(-\lambda_1^-(|N_1\rangle\langle m_1| + C_0|N_1\rangle\langle m_1|C_0) - \lambda_1^+(S_a^{-1}|M_1\rangle\langle n_1|S_a + S_a^{-1}C_0|M_1\rangle\langle n_1|C_0S_a)\right), \\ \hat{B} &= \left(\lambda_1^+(|n_1\rangle\langle M_1| + C_0|n_1\rangle\langle M_1|C_0) - \lambda_1^-(S_a|m_1\rangle\langle N_1|S_a^{-1} + S_aC_0|m_1\rangle\langle N_1|C_0S_a^{-1})\right). \end{aligned} \tag{5.79}$$

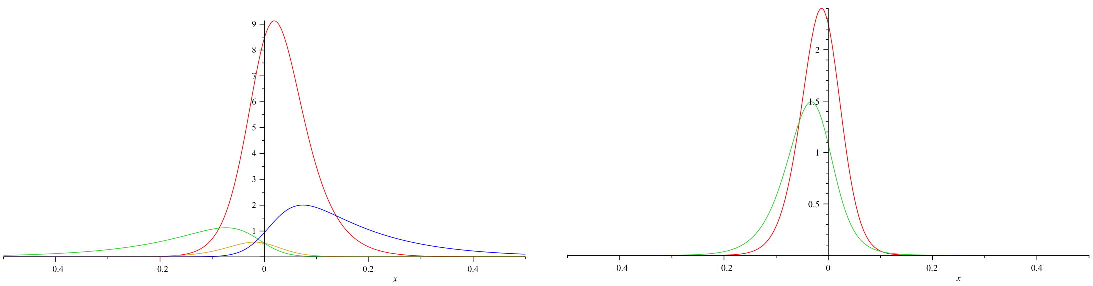

Figure 5. One soliton solution for the 6-wave equations: q_1 (blue), q_2 (red), q_3 (green) and q_4 (yellow) on the left panel; q_5 (green) and q_6 (red) are on the right panel. All functions are evaluated for $t = 0$. The values of the parameters are $|n_{10}\rangle = (2,2,-1,-2)^T$, $\langle m_{10} = (2,2,-1,-2)$, $a_1 = 2$, $a_2 = 1$, $b_1 = 3$, $b_2 = 1$.

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Example 4 (One soliton solutions, $so(5)$ case)**.** *We start with the 'naked' polarization vectors:*

$$
\begin{aligned}
\langle m_1| &= \langle m_{10}|e^{i\lambda_1^{-,2}(Jx+Kt)} = \left(n_{10,1}^* e^{z_1+i\phi_1}, n_{10,2}^* e^{z_2+i\phi_2}, n_{10,3}^*, n_{10,4}^* e^{-z_2-i\phi_2}, n_{10,5}^* e^{-z_1-i\phi_1}\right), \\
|n_1\rangle &= e^{-i\lambda_1^{+,2}(Jx+Kt)}|n_{10}\rangle = \begin{pmatrix} e^{z_1-i\phi_1} n_{10,1} \\ e^{z_2-i\phi_2} n_{10,2} \\ n_{10.3} \\ e^{-z_2+i\phi_2} n_{10,4} \\ e^{-z_1+i\phi_1} n_{10,5} \end{pmatrix}, \quad z_k = 2\mu_1\nu_1 s_k, \quad \phi_k = (\mu_1^2-\nu_1^2)s_k, \\
z_k &= 2\mu_1\nu_1(a_k x + b_k t + \zeta_{1,k}), \qquad \phi_k = (\mu_1^2-\nu_1^2)(a_k x + b_k t + \phi_{0,k}), \quad C_0 = \mathrm{diag}\,(1,-1,1,-1,1).
\end{aligned}
\tag{5.80}
$$

where $s_k = (a_k x + b_k t)$. In addition the constant polarization vectors must satisfy eqs. (5.64). For convenience we choose the following parametrization of $|n_{10}\rangle$, $|N_{10}\rangle$, $\langle m_{10}|$ and $\langle M_{10}|$:

$$
\begin{aligned}
n_{10,k} &= r_{10,k} e^{i\phi_{0k}}, & n_{10,6-k} &= s_{10,6-k} e^{-i\phi_{0k}}, & k &= 1,2; \\
m_{10,k} &= r_{10,k} e^{-i\phi_{0k}}, & m_{10,6-k} &= s_{10,6-k} e^{i\phi_{0k}}, & k &= 1,2; \\
N_{10,k} &= R_{10,k} e^{i\Phi_{0k}}, & N_{10,6-k} &= S_{10,6-k} e^{-i\Phi_{0k}}, & k &= 1,2; \\
M_{10,k} &= R_{10,k} e^{-i\Phi_{0k}}, & M_{10,6-k} &= S_{10,6-k} e^{i\Phi_{0k}}, & k &= 1,2;
\end{aligned}
\tag{5.81}
$$

where the real constants $r_{10,k}$, $s_{10,k}$, $R_{10,k}$ and $S_{10,k}$ satisfy the relations:

$$
\begin{aligned}
n_{10,3}^2 &= -2r_{10,1}s_{10,5} + 2r_{10,2}s_{10,4}, & N_{10,3}^2 &= -2R_{10,1}S_{10,5} + 2R_{10,2}S_{10,4}, \\
m_{10,3}^2 &= -2r_{10,1}s_{10,5} + 2r_{10,2}s_{10,4}, & M_{10,3}^2 &= -2R_{10,1}S_{10,5} + 2R_{10,2}S_{10,4}.
\end{aligned}
\tag{5.82}
$$

With polarization constants the relations (5.64) are automatically satisfied; in addition the relations (5.66) are also satisfied. With these notations for the scalar products of the 'naked' polarization vectors we obtain:

$$
\begin{aligned}
\langle m_1|n_1\rangle &= |n_{10,1}|^2 e^{2z_1} + |n_{10,2}|^2 e^{2z_2} + |n_{10,3}|^2 + |n_{10,4}|^2 e^{-2z_2} + |n_{10,5}|^2 e^{-2z_1} \\
&= 4\eta_{10}\sinh^2(\tilde{z}_1) + 4\eta_{20}\cosh^2(\tilde{z}_2), \qquad \tilde{z}_k = z_k + \zeta_{1,k}, \quad \eta_{k0} = r_{10,k}s_{10,6-k}, \\
&= 2|n_{10,1}n_{10,5}|\cosh(2z_1) + 2|n_{10,2}n_{10,4}|\cosh(2z_2) + |n_{10,3}|^2, \\
\langle m_1|C_0|n_1\rangle &= -4\eta_{10}\cosh^2(\tilde{z}_1) + 4\eta_{20}\cosh^2(\tilde{z}_2), \qquad \zeta_{1,k} = \frac{1}{2}\ln\frac{r_{10,k}}{s_{10,6-k}} \\
&= 2|n_{10,1}n_{10,5}|\cosh(2z_1) - 2|n_{10,2}n_{10,4}|\cosh(2z_2) + |n_{10,3}|^2.
\end{aligned}
\tag{5.83}
$$

The 'dressed' polarization vectors:

$$
\begin{pmatrix} |M_1\rangle \\ |N_1\rangle \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{-1} \begin{pmatrix} |m_1\rangle \\ |n_1\rangle \end{pmatrix},
\tag{5.84}
$$

where

$$
\begin{aligned}
X_1 &= \langle m_1|n_1\rangle + \frac{i\nu_1}{\mu_1}\langle m_1|C_0|n_1\rangle C_0, & X_2 &= -\frac{i\nu_1}{\lambda_1^-}\beta_1 S C_0, \\
Y_1 &= \frac{i\nu_1}{\lambda_1^+}\bar{\beta}_1 S^{-1} C_0, & Y_2 &= \langle m_1|n_1\rangle - \frac{i\nu_1}{\mu_1}\langle m_1|C_0|n_1\rangle C_0,
\end{aligned}
\tag{5.85}
$$

Note that $\beta_1 = \bar{\beta}_1 = -4\eta_{10}$ are real constants. Skipping the details we find:

$$
\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{-1} = \frac{g_1}{w_1}\begin{pmatrix} Y_2 & -X_2 \\ -Y_1 & X_1 \end{pmatrix},
\tag{5.86}
$$

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where $\frac{g_1}{w_1}$, *provided in general by eq. (5.75) now takes the form:*

$$\frac{g_1}{w_1} = \frac{\mu_1^2(\mu_1^2+\nu_1^2)}{(\mu_1^2+\nu_1^2)\left((\mu_1^2-\nu_1^2)4\eta_{10}\cosh^2(\tilde{z}_1)+(\mu_1^2+\nu_1^2)4\eta_{20}\cosh^2(\tilde{z}_2)-4\mu_1^2\eta_{10}\right)-16\mu_1^2\nu_1^2\eta_{10}^2} \tag{5.87}$$

Thus we obtain the dressed polarization vectors as follows:

$$\begin{aligned} |M_1\rangle &= \frac{g_1}{w_1}\left\{\begin{pmatrix} S_1^+ & 0 & 0 & 0 & 0 \\ 0 & S_1^- & 0 & 0 & 0 \\ 0 & 0 & S_1^- & 0 & 0 \\ 0 & 0 & 0 & S_1^- & 0 \\ 0 & 0 & 0 & 0 & S_1^+ \end{pmatrix}|m_1\rangle + \frac{i\lambda_1^+\nu_1\beta_1}{\mu_1^2\nu_1^2}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}|n_1\rangle\right\}, \\ |N_1\rangle &= \frac{g_1}{w_1}\left\{\begin{pmatrix} S_1^- & 0 & 0 & 0 & 0 \\ 0 & S_1^+ & 0 & 0 & 0 \\ 0 & 0 & S_1^+ & 0 & 0 \\ 0 & 0 & 0 & S_1^+ & 0 \\ 0 & 0 & 0 & 0 & S_1^- \end{pmatrix}|n_1\rangle - \frac{i\lambda_1^-\nu_1\bar{\beta}_1}{\mu_1^2\nu_1^2}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}|m_1\rangle\right\}, \end{aligned} \tag{5.88}$$

where

$$\begin{aligned} S_1^+ &= \frac{4}{\mu_1}\left(\lambda_1^+\eta_{10}\cosh^2(\tilde{z}_1)+\lambda_1^-\eta_{20}\cosh^2(\tilde{z}_2)-\mu_1\eta_{10}\right), \\ S_1^- &= \frac{4}{\mu_1}\left(\lambda_1^-\eta_{10}\cosh^2(\tilde{z}_1)+\lambda_1^+\eta_{20}\cosh^2(\tilde{z}_2)-\mu_1\eta_{10}\right). \end{aligned} \tag{5.89}$$

It remains to use eq. (5.78) and calculate the corresponding one-soliton solutions of the $so(5)$ *4-wave NLEE (5.31) and (5.32). The result is:*

$$\begin{aligned} q_{1,1s}(x,t) &= 2i\nu_1\left(a_1(N_{1,4}m_{1,5}+M_{1,2}n_{1,1}-a_2(N_{1,1}m_{1,2}+M_{1,5}n_{1,4})\right), \\ q_{2,1s}(x,t) &= 4i\nu_1 a_1(N_{1,3}m_{1,5}-M_{1,3}n_{1,1}), \\ q_{3,1s}(x,t) &= 4i\nu_1\left(a_1(N_{1,2}m_{1,5}+M_{1,4}n_{1,1})+a_2(N_{1,1}m_{1,4}+M_{1,5}n_{1,2})\right), \end{aligned} \tag{5.90}$$

and

$$q_{4,1s} = 4i\nu_1 a_2(\lambda_1^- N_{1,3}m_{1,4}+\lambda_1^+ M_{1,3}n_{1,2})+16\nu_1^2 a_1(N_{1,2}M_{1,3}+M_{1,4}N_{1,3})(n_{1,1}m_{1,1}-m_{1,5}n_{1,5}). \tag{5.91}$$

6. MNLS family and symmetric spaces

We already pointed out the importance of the seminal paper of Fordy and Kulish [21] for the NNLS equations. Each symmetric space is generated by a Cartan involution [67], which determines the principal symmetry for the Lax pair and for the solution of the RHP. For this family of NLEE it will be more convenient to use another parametrization of $\xi(x,t,\lambda)$:

$$\xi(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)), \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty}\frac{Q_s}{\lambda^s}, \tag{6.1}$$

This is similar to the parametrization in (5.1), however now each of the coefficients $Q_s(x,t)$ provides local coordinates for the relevant symmetric space. The typical reduction here is to assume that $\mathcal{Q}(x,t,\lambda)$ is anti-hermitian:

$$\mathcal{Q}(x,t,\lambda) = -\mathcal{Q}^\dagger(x,t,\lambda^*), \quad \text{i.e.} \quad Q_s(x,t) = -Q_s^\dagger(x,t), \quad \text{and} \quad (\xi^\pm(x,t,\lambda))^{-1} = \xi^\mp(x,t,\lambda^*). \tag{6.2}$$

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In the appendix we briefly describe the root systems of the simple Lie algebras, as well as the Cartan involutions that generate the corresponding Hermitian spaces. We remind that the Cartan involution is determined by a special element of the Cartan subalgebra $J \in \mathfrak{h}$. By $\vec{h}$ we will denote the vector in the space $\mathbb{E}^r$ which is dual to J. Using $\vec{h}$ we can split the positive roots Δ_+ of $\mathfrak{g}$ into two subsets:

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+, \qquad \Delta_0^+ \equiv \{\alpha \in \Delta^+, \quad (\vec{h}, \alpha) = 0\}, \qquad \Delta_1^+ \equiv \{\beta \in \Delta_1^+, \quad (\vec{h}, \beta) > 0\}, \tag{6.3}$$

Below we specify the Cartan involutions for the four classes of Hermitian symmetric spaces and their realizations as factor groups:

$$\begin{aligned}
&\text{A.III} \quad J = \sum_{j=1}^{m} E_{jj} - \frac{m}{r+1} \sum_{j=1}^{r+1} E_{jj}, && SU(m+n)/S(U(m) \otimes U(n)),\\
&\text{BD.I} \quad J = H_{e_1} = E_{11} - E_{2r+1,2r+1}, && SO(2r+1)/S(O(2r-1) \otimes O(2)),\\
&\text{C.I} \quad J = \sum_{j=1}^{r} H_{e_j} = \sum_{j=1}^{r} (E_{jj} - E_{2r+1-j,2r+1-j}), && SP(2r)/U(r),\\
&\text{D.I} \quad J = \sum_{j=1}^{r} H_{e_j} = \sum_{j=1}^{r} (E_{jj} - E_{2r+1-j,2r+1-j}), && SO(2r)/U(r),
\end{aligned} \tag{6.4}$$

in A.III case $m + n = r + 1$. The corresponding vectors $\vec{h}$ and the subsets of roots Δ_1^+ are given by:

$$\begin{aligned}
&\text{A.III} \quad \vec{h} = \sum_{j=1}^{m} \vec{e}_j - \frac{m}{r+1} \vec{\epsilon}, && \Delta_1^+ = \{\vec{e}_j - \vec{e}_k,\ m \leq j < k \leq r+1\},\\
&\text{BD.I} \quad \vec{h} = \vec{e}_1, && \Delta_1^+ = \{\vec{e}_1 - \vec{e}_j,\ \vec{e}_1,\ \vec{e}_1 + \vec{e}_j,\ 2 \leq j \leq r,\},\\
&\text{C.I} \quad \vec{h} = \sum_{j=1}^{r} \vec{e}_j, && \Delta_1^+ = \{\vec{e}_1 + \vec{e}_j,\ 2\vec{e}_j,\ 1 \leq i < j \leq r,\},\\
&\text{D.III} \quad \vec{h} = \sum_{j=1}^{r} \vec{e}_j, && \Delta_1^+ = \{\vec{e}_1 + \vec{e}_j,\ 1 \leq i < j \leq r,\},
\end{aligned} \tag{6.5}$$

where $\vec{\epsilon} = \sum_{j=1}^{r+1} \vec{e}_j$.

$$Q_s(x,t) = \sum_{\alpha \in \Delta_+} (q_{\alpha,s}(x,t) E_\alpha - q_{\alpha,s}^*(x,t) E_{-\alpha}). \tag{6.6}$$

The Cartan involution splits the root system Δ of the relevant simple Lie algebra into two subsets:

$$\Delta = \Delta_0 \cup \Delta_1, \qquad \Delta_0 \equiv \{\alpha \in \Delta, \quad \alpha(J) = 0\}, \qquad \Delta_1 \equiv \{\alpha \in \Delta, \quad \alpha(J) \neq 0\}. \tag{6.7}$$

In the appendix we will briefly describe the roots systems and the Cartan involutions corresponding to these symmetric spaces. Here we just note that for the A.III, C.I and D.III symmetric spaces we can introduce local coordinates, which in the typical representation is given by block 2×2 matrices:

$$Q_s(x,t) = \sum_{\alpha \in \Delta_1^+} (q_{s,\alpha} E_\alpha - q_{s,\alpha}^* E_{-\alpha}) = \begin{pmatrix} 0 & \boldsymbol{q}_s \\ -\boldsymbol{q}_s^\dagger & 0 \end{pmatrix}. \tag{6.8}$$

Indeed, for the A.III type symmetric spaces the blocks $\boldsymbol{q}$ and $\boldsymbol{p}$ may be arbitrary, apart from the fact that they must be hermitian conjugate to each other. For the symmetric spaces of type C.I and D.III

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these blocks must be constrained in such a way that the corresponding matrix $Q_1(x,t)$ must be an element of the $sp(2r)$ or $so(2r)$ algebra.

The fourth hermitian symmetric space is $SO(m+2)/S(O(m)\times O(2))$ has different block-matrix structure. In this case we a dealing with the root system of B_r algebras and the Cartan involution is provided by J which is dual to e_1. As a result we have:

$$\begin{aligned} J = H_{e_1}, \qquad &\Delta_0 \equiv \{\pm(e_k \pm e_j), \quad 1<k<j\leq r\} \cup \{\pm e_k, \quad 1<k\leq r\}, \\ &\Delta_1 \equiv \{\pm(e_1 \pm e_j), \quad 1<j\leq r\} \cup \{\pm e_1\}. \end{aligned} \tag{6.9}$$

As a result, in the typical representation the local coordinates are provided by a 3×3 block matrices $Q_s(x,t)$ with the following structure:

$$Q_s(x,t) = \sum_{\alpha\in\Delta_1^+} (q_{s,\alpha}E_\alpha - q_{s,\alpha}^* E_{-\alpha}) = \begin{pmatrix} 0 & \vec{q}_s^{\,T} & 0 \\ -\vec{q}_s^{\,*} & 0 & s_0\vec{q}_1 \\ 0 & -\vec{q}_s^{\,\dagger} s_0 & 0 \end{pmatrix}. \tag{6.10}$$

For these reasons we will consider separately these two types of symmetric space.

Here we assume that the exponent $\mathcal{Q}(x,t,\lambda)$ provides the local coordinates of the corresponding hermitian symmetric space; we also assume that it is an odd function of λ. In short:

$$\mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-2s+1} Q_{2s-1}(x,t), \qquad \xi^{\pm}(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)). \tag{6.11}$$

Note that if we evaluate the exponential in (6.11) we will have two types of terms. The even powers of $\mathcal{Q}(x,t,\lambda)$ will be block diagonal matrices which are even functions of λ; the odd powers of $\mathcal{Q}(x,t,\lambda)$ will be block-off-diagonal matrices which will be odd functions of λ. As we shall see below, this will give us more 'economic' way for the MNLS equations. Indeed, the equations that we will derive will be parametrized solely by the coefficients of $Q_1(x,t)$. In addition we have additional symmetry due to the fact that $JQ_1J = -Q_1$ and $J\mathcal{Q}(x,t,\lambda)J = -\mathcal{Q}(x,t,\lambda)$. This means that

$$C_0\xi^{\pm}(x,t,\lambda)C_0^{-1} = \xi^{\pm}(x,t,-\lambda). \qquad C_0 = \exp(\pi i J). \tag{6.12}$$

Let us now consider Lax operators which are quadratic in λ, e.g. (2.28). The construction of their FAS is outlined in Subsection 2.4 above. Obviously the solutions $\chi^+(x,t,\lambda)$ and $\chi^-(x,t,\lambda)$ of the equations (2.33) are analytic for $\operatorname{Im}\lambda^2 > 0$ and $\operatorname{Im}\lambda^2 < 0$ respectively. This means that now $\chi^+(x,t,\lambda)$ is analytic in the first and third quadrant of the complex λ-plane, while $\chi^-(x,t,\lambda)$ is analytic in the second and the third quadrant, see Figure 1.

6.1. Lax pairs on symmetric spaces. Generic case

We start with the realization of Cartan involution which are convenient for Lax pairs. Here we assume that the reader is familiar with the theory of simple Lie algebras $\mathfrak{g}$ and their root systems [67,87]. We already mentioned above that the Cartan involution is determined by the Cartan subalgebra element J which provides the leading terms of the operators in the Lax pair. Below we will assume that L is quadratic in λ; then M must be a quartic polynomial of λ:

$$\begin{aligned} L_{\rm MNLS}\psi &\equiv i\frac{\partial\psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\ M_{\rm MNLS}\psi &\equiv i\frac{\partial\psi}{\partial t} + (V_4(x,t) + \lambda V_3(x,t) + \lambda^2 V_2(x,t) + \lambda^3 V_1(x,t) - \lambda^4 J)\psi(x,t,\lambda) = 0 \end{aligned} \tag{6.13}$$

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This Lax pair has the form of (5.3), only now $J = K$. Therefore $U_k(x,t) = V_k(x,t)$ for $k = 1,2$. Then the compatibility condition takes the form:

$$i\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial t} + [U(x,t,\lambda), V(x,t,\lambda)] = 0, \tag{6.14}$$

and equate to zero the terms with the different powers of λ. It is easy to check that the terms at λ^6, λ^5 and λ^4 vanish identically. The rest of the terms are given by:

$$\begin{aligned} \lambda^3 &: i\frac{\partial Q}{\partial x} = [J, V_3], & \lambda^2 &: i\frac{\partial V_2}{\partial x} + [Q, V_3] = [J, V_4], \\ \lambda^1 &: i\frac{\partial V_3}{\partial x} - i\frac{\partial Q}{\partial t} + [V_2, V_3] + [Q, V_4] = 0, & \lambda^0 &: i\frac{\partial V_4}{\partial x} - \frac{\partial V_2}{\partial t} + [V_2, V_4] = 0. \end{aligned} \tag{6.15}$$

where we used the traditional $Q(x,t) \equiv V_1(x,t)$. The first two of the above equations allow us to express V_3 and V_4 in terms of Q and Q_x.

Here we assume that functions $U_k(x,t)$, $k = 1,2$, $V_j(x,t)$ belong to the simple Lie algebra $\mathfrak{g}$; J is a constant elements in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The classification of all symmetric spaces, and therefore, of all possible Cartan involutions has been done by Cartan more than a century ago and can be found in many monographs, see e.g. [67]. The Cartan involution introduces a $\mathbb{Z}_2$-grading in $\mathfrak{g}$, i.e.

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}. \tag{6.16}$$

Let us assume that $X_1, X_2 \in \mathfrak{g}^{(0)}$ and $Y_1, Y_2 \in \mathfrak{g}^{(1)}$. Then:

$$[X_1, X_2] \in \mathfrak{g}^{(0)}, \qquad [Y_1, Y_2] \in \mathfrak{g}^{(0)}, \qquad [X_1, Y_2] \in \mathfrak{g}^{(1)}. \tag{6.17}$$

In particular this means that the Cartan involution splits the roots system Δ into two subsets, which are determined by the choice of J:

$$\Delta \equiv \Delta_0 \cup \Delta_1, \qquad \alpha \in \Delta_0 \quad \text{iff} \quad \alpha(J) = 0; \qquad \beta \in \Delta_1 \quad \text{iff} \quad \alpha(J) = 1 \mod (2); \tag{6.18}$$

In particular this means, that the generic elements $X \in \mathfrak{g}^{(0)}$ and $Y \in \mathfrak{g}^{(1)}$ take the form:

$$X = \sum_{j=1}^{r} x_j H_j + \sum_{\alpha \in \Delta_0} x_\alpha E_\alpha, \qquad Y = \sum_{\beta \in \Delta_1} y_\beta E_\beta, \tag{6.19}$$

The Cartan involution can be realized as a similarity transformation by the Cartan subgroup element $\mathfrak{J} = \exp(\pi i J)$ which acts on X and Y above by:

$$\mathfrak{J}^{-1} X \mathfrak{J} = X^\dagger, \qquad \mathfrak{J}^{-1} Y \mathfrak{J} = -Y^\dagger. \tag{6.20}$$

Therefore the imposed reduction can be written as:

$$\mathcal{Q}^\dagger(x,t,\lambda^*) = -\mathcal{Q}(x,t,\lambda), \qquad \text{i.e.} \qquad (\xi^\pm)^\dagger(x,t,\lambda^*) = (\xi^\mp)^{-1}(x,t,\lambda). \tag{6.21}$$

Note that this is in agreement with the chosen parametrization (5.1).

Remark 8. *Note that the mapping $\lambda \to -\lambda$ maps $Q_1 \to Q_3$ and $Q_2 \to Q_4$; as a result its preserves the analyticity regions of both ξ^+ and ξ^-. The mapping $\lambda \to \lambda^*$ maps $Q_1 \to Q_2$ and $Q_3 \to Q_4$, i.e. it maps the analyticity region of ξ^+ into the analyticity region of ξ^-.*

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6.2. NLEE on symmetric spaces: A.III

We will limit ourselves with the Hermitian symmetric spaces, see [67] and [118].

Here we will consider multicomponent derivative NLS-type (DMNLS) equations [81], [123] and multicomponent GI (MGI) equations [42,58]. Note that the RHP for the DMNLS equations is not canonically normalized which requires slight modifications in applying the dressing method. However DMNLS and MGI equations are gauge equivalent.

For L linear in λ we get the well known multicomponent NLS equations, see [21]. For the symmetric spaces of the types A.III, C.I and D.III the Cartan involution is fixed up by the choice of the matrix J which takes the form [67]:

$$J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{6.22}$$

The coefficients of the Lax operators are:

$$\begin{aligned} &Q(x,t) = \begin{pmatrix} 0 & q_1 \\ q_1^\dagger & 0 \end{pmatrix}, \quad V_2(x,t) = \frac{1}{2}\begin{pmatrix} q_1q_1^\dagger & 0 \\ 0 & -q_1^\dagger q_1 \end{pmatrix}, \quad V_3(x,t) = \begin{pmatrix} 0 & q_3 - \frac{1}{6}q_1q_1^\dagger q_1 \\ q_3^\dagger - \frac{1}{6}q_1^\dagger q_1 q_1^\dagger & 0 \end{pmatrix}, \\ &V_4(x,t) = \frac{1}{2}\begin{pmatrix} q_1q_3^\dagger + q_3q_1^\dagger - \frac{1}{12}q_1q_1^\dagger q_1q_1^\dagger & 0 \\ 0 & -q_1^\dagger q_3 - q_3^\dagger q_1 + \frac{1}{12}q_1^\dagger q_1 q_1^\dagger q_1 \end{pmatrix}, \\ &V_5(x,t) = \begin{pmatrix} 0 & V_{5;12} \\ V_{5;21} & 0 \end{pmatrix}, \qquad \begin{array}{l} V_{5;12} = q_5 - \frac{1}{6}(q_1q_1^\dagger q_3 + q_1q_3^\dagger q_1 + q_3q_1^\dagger q_1) + \frac{1}{120}q_1q_1^\dagger q_1q_1^\dagger q_1 \\ V_{5;21} = q_5^\dagger - \frac{1}{6}(q_1^\dagger q_1 q_3^\dagger + q_1^\dagger q_3 q_1^\dagger + q_3^\dagger q_1 q_1^\dagger) + \frac{1}{120}q_1^\dagger q_1 q_1^\dagger q_1 q_1^\dagger \end{array}. \end{aligned} \tag{6.23}$$

The first of the equations in (6.15) is satisfied if

$$V_3(x,t) = -i\,\mathrm{ad}_J^{-1}\frac{\partial Q}{\partial x} = -\frac{i}{2}\begin{pmatrix} 0 & q_{1,x} \\ -q_{1,x}^\dagger & 0 \end{pmatrix}. \tag{6.24}$$

Comparing this expression for V_3 with the one from (6.23) we find that

$$Q_3 \equiv \frac{1}{2}\begin{pmatrix} 0 & q_3 \\ -q_3^\dagger & 0 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 0 & iq_{1,x} - \frac{1}{3}q_1q_1^\dagger q_1 \\ iq_{1,x}^\dagger + \frac{1}{3}q_1^\dagger q_1 q_1^\dagger & \end{pmatrix}. \tag{6.25}$$

Inserting Q_3 into the expression for V_4 in (6.23) we find:

$$V_4 = \frac{1}{4}\begin{pmatrix} i(q_1q_{1,x}^\dagger - q_{1,x}q_1^\dagger) + \frac{1}{2}q_1q_1^\dagger q_1q_1^\dagger & 0 \\ 0 & i(q_1^\dagger q_{1,x} - q_{1,x}^\dagger q_1) - \frac{1}{2}q_1^\dagger q_1 q_1^\dagger q_1 \end{pmatrix} \tag{6.26}$$

The corresponding MNLS is obtained from the third of the equations (6.15). Taking into account that

$$[V_3, V_2] = \frac{i}{4}\begin{pmatrix} 0 & q_{1,x}q_1^\dagger q_1 + q_1q_1^\dagger q_{1,x} \\ q_{1,x}^\dagger q_1 q_1^\dagger + q_1^\dagger q_1 q_{1,x}^\dagger & 0 \end{pmatrix},$$

and

$$[V_4, Q] = \frac{1}{4}\begin{pmatrix} 0 & Y_{4;12} \\ Y_{4;21} & 0 \end{pmatrix}, \qquad \begin{array}{l} Y_{4;12} = 2iq_1q_{1,x}^\dagger q_1 - iq_{1,x}q_1^\dagger q_1 - iq_1q_1^\dagger q_{1,x} + q_1q_1^\dagger q_1q_1^\dagger q_1 \\ Y_{4,21} = 2iq_1^\dagger q_{1,x}q_1^\dagger - iq_{1,x}^\dagger q_1 q_1^\dagger - iq_1^\dagger q_1 q_{1,x}^\dagger - q_1^\dagger q_1 q_1^\dagger q_1 q_1^\dagger \end{array},$$

we obtain the MNLS in the form:

$$-iq_{1,t} + \frac{1}{2}q_{1,xx} + \frac{i}{2}q_1q_{1,x}^\dagger q_1 + \frac{1}{4}q_1q_1^\dagger q_1q_1^\dagger q_1 = 0, \tag{6.27}$$

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If we put $\tau = -2t$ the equations obtain more familiar form:

$$iq_{1,\tau} + q_{1,xx} + iq_1 q_{1,x}^\dagger q_1 + \frac{1}{2} q_1 q_1^\dagger q_1 q_1^\dagger q_1 = 0, \tag{6.28}$$

It remains to analyze the last of the equations (6.15). To this end we need:

$$[V_4, V_2] = \frac{i}{8} \begin{pmatrix} W_{4;11} & \\ 0 & W_{4;22} \end{pmatrix}. \qquad \begin{array}{l} W_{4,11} = q_1 (q_1^\dagger q_1)_x q_1^\dagger - q_{1,x} q_1^\dagger q_1 q_1^\dagger - q_1 q_1^\dagger q_1 q_{1,x}^\dagger, \\ W_{4;22} = -q_1^\dagger (q_1 q_1)_x^\dagger q_1 + q_{1,x}^\dagger q_1 q_1^\dagger q_1 + q_1^\dagger q_1 q_1^\dagger q_{1,x} \end{array} .$$

Then the last of the equations (6.15) become:

$$i(q_1 q_1^\dagger)_t + \frac{1}{2}(q_1 q_{1,xx}^\dagger + q_{1,xx} q_1^\dagger) - \frac{i}{2} q_1 (q_1^\dagger q_1)_x q_1^\dagger = 0. \tag{6.29}$$

It is easy to check that eq. (6.29) directly follows from (6.27).

Up to now in this subsection we treated the matrix q_1 as generic $k \times p$ rectangular matrix. However below we would like to outline the special case: the vector NLS known also as the Manakov model. Then $k = 1$ and the number of vector components[1] $p \geq 2$ can be any:

$$i\frac{\partial \vec{q}}{\partial \tau} + \frac{\partial^2 \vec{q}}{\partial x^2} + i\left(\frac{\partial \vec{q}^{\,\dagger}}{\partial x}, \vec{q}\right) \vec{q} + \frac{1}{2}\left(\vec{q}^{\,\dagger}, \vec{q}\right)^2 \vec{q}(x,t) = 0. \tag{6.30}$$

6.3. MNLS equations related to D.III and C.I symmetric spaces

Eq. (6.28) provides the NLEE related to the A.III type symmetric spaces. Similar considerations may be applied to other two classes of symmetric spaces: D.III and C.I. Indeed the block structure of the Lax pair for these two classes of symmetric spaces is the same as the one of (6.23) and J has the form of (6.22) where the unit matrices $\mathbb{1}$ have equal dimensions N. The substantial difference between these symmetric spaces is in the fact that they are subject of additional reduction. Indeed, for D.III we must require that $Q_s \in so(2N)$ which means that they must satisfy in addition (see [16]):

$$Q_s + S_0 Q_s^T S_0^{-1} = 0, \qquad S_0 = \begin{pmatrix} 0 & s_0 \\ s_0' & 0 \end{pmatrix}, \quad s_0 = \sum_{p=1}^{N} (-1)^{p+1} E_{p,N-p}, \quad s_0' s_0 = \mathbb{1}_N. \tag{6.31}$$

Therefore $S_0^2 = \mathbb{1}_{2N}$ and

$$Q_s = \begin{pmatrix} 0 & \boldsymbol{q} \\ -\boldsymbol{p} & 0 \end{pmatrix}, \qquad \boldsymbol{q}_s = s_0 \boldsymbol{q}_s^T s_0, \qquad \boldsymbol{p}_s = s_0' \boldsymbol{p}_s^T s_0'. \tag{6.32}$$

For C.I type symmetric spaces we must have $Q_s \in sp(2N)$ which means that

$$Q_s + S_1 Q_s^T S_1^{-1} = 0, \qquad S_1 = \begin{pmatrix} 0 & s_1 \\ s_1' & 0 \end{pmatrix}, \quad s_1 = \sum_{p=1}^{N} (-1)^{p+1} E_{p,N-p+1}, \quad s_1' s_1 = -\mathbb{1}_N. \tag{6.33}$$

---

1 Manakov proposed the first vector NLS equation with only 2 components [92]. The reason was that he wanted to treat a special case of VNLS which described the propagation of birefringent optical pulses in optical fibers. Since our Lax operator above is a quadratic pencil, the equation (6.30) we derived is a vector generalization of GI equation. Of course the method of solution of the VNLS is easily generalized to p-component vectors.

Therefore $S_1^2 = -\mathbb{1}_{2N}$ and

$$Q_s = \begin{pmatrix} 0 & \boldsymbol{q} \\ -\boldsymbol{p} & 0 \end{pmatrix}, \qquad \boldsymbol{q}_s = -s_1 \boldsymbol{q}_s^T s_1, \qquad \boldsymbol{p}_s = -s_1' \boldsymbol{p}_s^T s_1'. \tag{6.34}$$

The reasons for such choice of the definitions of the algebras $so(2N)$ and $sp(2N)$ is that the Cartan subalgebras are represented by diagonal matrices. The explicit form of S_0 and S_1 are given in [16], see also the Appendix.

For example, the NLEE related to the symmetric spaces C.I and D.III related to $sp(6)$ and $so(8)$ respectively are obtained by inserting into the equation below:

$$i\boldsymbol{q}_{1,\tau} + \boldsymbol{q}_{1,xx} + i\boldsymbol{q}_1 \boldsymbol{q}_{1,x}^\dagger \boldsymbol{q}_1 + \frac{1}{2} \boldsymbol{q}_1 \boldsymbol{q}_1^\dagger \boldsymbol{q}_1 \boldsymbol{q}_1^\dagger \boldsymbol{q}_1 = 0, \tag{6.35}$$

the following matrices for $\boldsymbol{q}(x,t)$:

$$\boldsymbol{q}_{\rm C.I} = \begin{pmatrix} q_1 & q_3 & q_4 \\ q_2 & q_5 & -q_3 \\ q_6 & -q_2 & q_1 \end{pmatrix}, \qquad \boldsymbol{q}_{\rm D.III} = \begin{pmatrix} q_1 & q_2 & q_3 & 0 \\ q_5 & q_4 & 0 & q_3 \\ q_6 & 0 & q_4 & -q_2 \\ 0 & q_6 & -q_5 & q_1 \end{pmatrix}. \tag{6.36}$$

In both cases such substitutions into (6.28) can be viewed as special reductions to 6-component MNLS. Of course these MNLS can not be equivalent since they are related to non-isomorphic symmetric spaces. The parametrizations in eq. (6.36) have been obtained using the Cartan-Weyl basis given in Appendix A.

6.4. MNLS related to BD.I-type symmetric spaces

The basic parametrization for BD.I which is isomorphic to $SO(2r+1)/SO(2r-1) \otimes SO(2)$ has different block matrix form from the one for A.III, namely:

$$\begin{gathered} \xi(x,t,\lambda) = \exp(\mathcal{Q}(x,t,\lambda)), \qquad \mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-2s+1} Q_{2s-1}(x,t) = -\mathcal{Q}^\dagger(x,t,\lambda^*), \\ (\xi^{\pm})^\dagger(x,t,\lambda) = (\xi^{\mp})^{-1}(x,t,\lambda), \quad Q_{2s-1}(x,t) = \begin{pmatrix} 0 & \vec{q}_{2s-1}^{\,T} & 0 \\ -\vec{q}_{2s-1}^{\,*} & 0 & s_0 \vec{q}_{2s-1} \\ 0 & -\vec{q}_{2s-1}^{\,\dagger} s_0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{gathered} \tag{6.37}$$

Here the matrices are $(2r+1) \times (2r+1)$, while $\vec{q}_{2s-1}$ are $2r-1$-component vectors; this fixes up the block-structure of the matrices in this subsection. We also introduce

$$\xi J \xi^{-1}(x,t,\lambda) = J - \sum_{s=1}^{\infty} \lambda^{-s} V_s(x,t), \tag{6.38}$$

where

$$V_1 \equiv Q(x,t) = [J, Q_1(x,t)], \qquad V_2 = \frac{1}{2}[Q_1, Q], \qquad V_3 = [J, Q_3] - \frac{1}{6} \mathrm{ad}_{Q_1}^{\,3} J, \tag{6.39}$$

$$\begin{gathered} V_1 = \begin{pmatrix} 0 & \vec{q}_1^{\,T} & 0 \\ \vec{q}_1^{\,*} & 0 & s_0 \vec{q}_1 \\ 0 & \vec{q}_1^{\,\dagger} s_0 & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} (\vec{q}_1, \vec{q}_1^{\,*}) & 0 & 0 \\ 0 & -\vec{q}_1^{\,*} \vec{q}_1^{\,T} + s_0 \vec{q}_1 \vec{q}_1^{\,\dagger} s_0 & 0 \\ 0 & 0 & -(\vec{q}_1, \vec{q}_1^{\,*}) \end{pmatrix}, \\ V_3 = \begin{pmatrix} 0 & \vec{q}_3^{\,T} + \vec{w}_3^{\,T} & 0 \\ \vec{q}_{red}^{\,*} 3 + \vec{u}_3 & 0 & s_0(\vec{q}_3 + \vec{w}_3) \\ 0 & (\vec{q}_3^{\,\dagger} + \vec{u}_3^{\,T}) s_0 & 0 \end{pmatrix}, \qquad V_4 = \begin{pmatrix} V_{4;11} & 0 & 0 \\ 0 & V_{4;22} & 0 \\ 0 & 0 & -V_{4,11} \end{pmatrix}. \end{gathered} \tag{6.40}$$

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We have used the following notations above:

$$\begin{aligned} \vec{w}_3 &= \frac{1}{3}\left(-2(\vec{q}_1, \vec{q}_1^{\,*})\vec{q}_1 + (\vec{q}_1 s_0 \vec{q}_1) s_0 \vec{q}_1^{\,*}\right), \qquad \vec{u}_3 = \frac{1}{3}\left(-2(\vec{q}_1, \vec{q}_1^{\,*})\vec{q}_1^{\,*} + (\vec{q}_1^{\,*} s_0 \vec{q}_1^{\,*}) s_0 \vec{q}_1\right), \\ V_{4;11} &= (\vec{q}_1, \vec{q}_3^{\,*}) + (\vec{q}_3, \vec{q}_1^{\,*}) - \frac{1}{3}(\vec{q}_1, \vec{q}_1^{\,*})^2 + \frac{1}{6}(\vec{q}_1 s_0 \vec{q}_1)(\vec{q}_1^{\,*} s_0 \vec{q}_1^{\,*}), \\ V_{4;22} &= \frac{1}{3}(\vec{q}_1^{\,*}\vec{q}_1^T - s_0 \vec{q}_1 \vec{q}_1^{\,\dagger} s_0)(\vec{q}_1, \vec{q}_1^{\,*}) - \vec{q}_1^{\,*}\vec{q}_3^T + s_0 \vec{q}_3 \vec{q}_1^{\,\dagger} s_0 - \vec{q}_3^{\,*}\vec{q}_1^T + s_0 \vec{q}_1 \vec{q}_3^{\,\dagger} s_0. \end{aligned} \tag{6.41}$$

The potentials are:

$$V_3(x,t) = \begin{pmatrix} 0 & V_{3;2}^T & 0 \\ V_{3;1} & 0 & s_0 V_{3;2} \\ 0 & V_{3;1}^T s_0 & 0 \end{pmatrix}, \qquad \begin{aligned} V_{3;1} &= \vec{w} + \frac{1}{3}(\vec{p}^T s_0 \vec{p}) s_0 \vec{q} - \frac{2}{3}(\vec{q}, \vec{p})\vec{p}, \\ V_{3;2} &= s_0 \vec{v} + \frac{1}{3}(\vec{q}^T s_0 \vec{q})\vec{p} - \frac{2}{3}(\vec{q}, \vec{p}) s_0 \vec{q}. \end{aligned} \tag{6.42}$$

$$\begin{aligned} &V_4 = \begin{pmatrix} V_{4;11} & 0 & 0 \\ 0 & V_{4;22} & 0 \\ 0 & 0 & V_{4;33} \end{pmatrix}, \qquad Q_3(x,t) = \begin{pmatrix} 0 & \vec{v}^T & 0 \\ -\vec{w} & 0 & s_0 \vec{v} \\ 0 & -\vec{w}^T s_0 & 0 \end{pmatrix}, \\ &V_{4;11} = -V_{4;33} = i\left((\vec{q}_x, \vec{p}) - (\vec{q}, \vec{p}_x)\right) + (\vec{q}, \vec{p})^2 - \frac{1}{2}(\vec{q}^T s_0 \vec{q})(\vec{p}^T s_0 \vec{p}), \\ &V_{4;22} = i\left(\vec{p}_x \vec{q}^T - \vec{p}\vec{q}_x^T + s_0(\vec{q}_x \vec{p}^T - \vec{q}\vec{p}_x^T) s_0\right) - (\vec{q}.\vec{p})\left(\vec{p}\vec{q}^T - s_0 \vec{q}\vec{p}^T s_0\right). \end{aligned} \tag{6.43}$$

Since $U_2 = V_2$ the first of the equations (6.15) gives:

$$V_3(x,t) = \mathrm{ad}_J^{-1} i \frac{\partial Q}{\partial x} = i \begin{pmatrix} 0 & \vec{q}_x^T & 0 \\ -\vec{p}_x & 0 & s_0 \vec{q}_x \\ 0 & -\vec{p}_x^T s_0 & 0 \end{pmatrix}, \tag{6.44}$$

which in components give:

$$\vec{v} = i\frac{\partial \vec{q}}{\partial x} + \frac{2}{3}(\vec{p}, \vec{q})\vec{q} - \frac{1}{3}(\vec{q}^T s_0 \vec{q}) s_0 \vec{p}, \qquad \vec{w} = -i\frac{\partial \vec{p}}{\partial x} + \frac{2}{3}(\vec{p}, \vec{q})\vec{p} - \frac{1}{3}(\vec{p}^T s_0 \vec{p}) s_0 \vec{q}. \tag{6.45}$$

The second equation in (6.15) is identically satisfied as a consequence of (6.45).

Finally the equations of motion:

$$\lambda: \quad i\frac{\partial V_3}{\partial x} - i\frac{\partial Q}{\partial t} + [Q, V_4] + [U_2, V_3] = 0, \qquad \lambda^0: \quad i\frac{\partial V_4}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2, V_4] = 0. \tag{6.46}$$

Since in addition we put $\vec{p} = \epsilon \vec{q}^{\,*}$ we get [38]:

$$i\frac{\partial \vec{q}}{\partial t} + \frac{\partial^2 \vec{q}}{\partial x^2} + i\epsilon s_0(\vec{q} s_0 \vec{q})\frac{\partial \vec{q}^{\,*}}{\partial x} + s_0(\vec{q}, \vec{q}^{\,*})(\vec{q} s_0 \vec{q}) s_0 \vec{q}^{\,*} - \left(\frac{1}{2}|(\vec{q} s_0 \vec{q})|^2 + 2i\epsilon(\vec{q}, \vec{q}^{\,*})\right)\vec{q} = 0. \tag{6.47}$$

One can check that the second equation in (6.46) holds identically as a consequence of (6.47).

7. Soliton solutions of the MNLS equations

7.1. Dressing for NLEE on symmetric spaces: A.III case

In this case the dressing factor satisfies two involutions:

$$u^\dagger(x,t,\lambda^*) = u^{-1}(x,t,\lambda), \qquad J u(x,t,-\lambda) J^{-1} = u(x,t,\lambda), \tag{7.1}$$

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then $u(x,t,\lambda)$ and its inverse have the form

$$
\begin{aligned}
u(x,t,\lambda) &= \mathbb{1} + (c_1(\lambda)-1)|N_1\rangle\langle m_1| + (c_1(-\lambda)-1)J|N_1\rangle\langle m_1|J^{-1}, \\
u^{-1}(x,t,\lambda) &= \mathbb{1} + \left(\frac{1}{c_1(\lambda)}-1\right)|n_1\rangle\langle M_1| + \left(\frac{1}{c_1(-\lambda)}-1\right)J|n_1\rangle\langle M_1|J^{-1}.
\end{aligned} \tag{7.2}
$$

Here the 'polarization' vectors $|N_1\rangle$, $\langle m_1|$, $|n_1\rangle$ and $\langle M_1|$ determine the residues of u and u^{-1}. They must satisfy eq. (5.40) which must be valid identically with respect to λ. Repeating the arguments as in Subsection 5.4 we find the x and t dependence of the polarization vectors:

$$
\begin{aligned}
|n_1\rangle &= \chi_0^+(x,t,\lambda_1^+)|n_{10}\rangle, \qquad & |N_1\rangle &= \chi_1^+(x,t,\lambda_1^+)|N_{10}\rangle, \\
\langle m_1| &= \langle m_{10}|\hat{\chi}_0^-(x,t,\lambda_1^-), \qquad & \langle M_1| &= \langle M_{10}|\hat{\chi}_1^-(x,t,\lambda_1^-),
\end{aligned} \tag{7.3}
$$

where $|n_{10}\rangle$, $|N_{10}\rangle$, $\langle m_{10}|$ and $\langle M_{10}|$ are constant polarization vectors. We assume that we know $\chi_0^\pm(x,t,\lambda)$ which are related to the regular solutions of the RHP. Typically they correspond to vanishing potentials $U_{10}=0$ and $U_{20}=0$; thus we must have:

$$
\chi_0^\pm(x,t,\lambda) = e^{-i(\lambda^2 x + \lambda^4 t)J)}. \tag{7.4}
$$

We will need also to impose the proper normalization conditions on the polarization vectors so that

$$
\begin{aligned}
&\langle m_1|\hat{\chi}_0^-(x,t,\lambda_1^-) = \langle \tilde{m}_{10}|e^{(z_1+i\phi_1)J}. \qquad |n_1\rangle = e^{(z_1-i\phi_1)J}|\tilde{n}_{10}\rangle, \qquad \langle \tilde{m}_{10;k}|\tilde{n}_{10;k}\rangle = 1, \\
&\sum_{s=1}^{m} \arg \tilde{n}_{10;k} = 0, \quad \sum_{s=1}^{m} \arg \tilde{m}_{10;k} = 0, \quad \sum_{s=m+1}^{n} \arg \tilde{n}_{10;k} = 0, \quad \sum_{s=m+1}^{n} \arg \tilde{m}_{10;k} = 0, \\
&z_1 = 2\mu_1\nu_1 x + 4\mu_1\nu_1(\mu_1^2-\nu_1^2)t + x_{01}, \qquad \phi_1 = (\mu_1^2-\nu_1^2)x + (\mu_1^4 - 6\mu_1^2\nu_1^2 + \nu_1^4)t + \phi_{10},
\end{aligned} \tag{7.5}
$$

Thus we find the following expressions for $|M_1\rangle$ and $|N_1\rangle$ in terms of $|m_1\rangle$ and $|n_1\rangle$:

$$
\begin{aligned}
|M_1\rangle &= \left(\langle m_1|n_1\rangle + \frac{i\nu_1}{\mu_1}\langle m_1|J|n_1\rangle J\right)^{-1}|m_1\rangle, \\
|N_1\rangle &= \left(\langle m_1|n_1\rangle + \frac{i\nu_1}{\mu_1}\langle m_1|J|n_1\rangle J\right)^{-1}|n_1\rangle,
\end{aligned} \tag{7.6}
$$

It remains to calculate the potentials. To this end we must consider the limit of eq. (5.40) for $\lambda\to\infty$, which takes the form:

$$
U_{2,1s} + \lambda U_{1,1s} = \lambda^2(J - uJ\hat{u}(x,t,\lambda)), \tag{7.7}
$$

where we have assumed $U_{20}=0$ and $U_{1,0}=0$. Taking the limit $\lambda\to\infty$ of (7.7) we obtain:

$$
U_{1,1s} = -2i\nu_1[J, -|N_1\rangle\langle m_1| + J|N_1\rangle\langle m_1|J], \qquad U_{2,1s} = -2i\nu_1 U_{1,1s}(-|N_1\rangle\langle m_1| + J|N_1\rangle\langle m_1|J). \tag{7.8}
$$

Skipping the details we find:

$$
\begin{aligned}
U_1(x,t) &= -\begin{pmatrix} 0 & \boldsymbol{q}_{,1s} \\ \boldsymbol{q}_{,1s}^\dagger & 0 \end{pmatrix}, \qquad \boldsymbol{q}_{,1s} = \frac{4\nu_1}{r_{01}}\frac{|\tilde{n}_{10,2}\rangle\langle\tilde{m}_{10,1}|e^{2i\phi_1}}{\cosh(z_1) - i\frac{\nu_1}{\mu_1}\sinh(z_1)}, \\
U_2(x,t) &= \frac{8\nu_1^2}{r_{01}^2}\frac{1}{\cosh^2(z_1) + \frac{\nu_1^2}{\mu_1^2}\sinh^2(z_1)}\begin{pmatrix} |\tilde{n}_{10,1}\rangle\langle\tilde{m}_{10,1}| & 0 \\ 0 & -|\tilde{n}_{10,2}\rangle\langle\tilde{m}_{10,2}| \end{pmatrix}
\end{aligned} \tag{7.9}
$$

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Example 5 (A.III type symmetric spaces, generic case). *Let us specify the form of the naked polarization vectors:*

$$\begin{aligned} |n_{1;0}\rangle &= \begin{pmatrix} |n_{1;01}\rangle \\ |n_{1;02}\rangle \end{pmatrix}, & \langle m_{1;0}| &= \left(\langle n^*_{1;01}|, \langle n^*_{1;02}|\right), \\ |n_1\rangle &= \begin{pmatrix} e^{\tilde{z}_1 - i\tilde{\phi}_1}|n_{1;01}\rangle \\ e^{-\tilde{z}_1 + i\tilde{\phi}_1}|n_{1;02}\rangle \end{pmatrix}, & \langle m_1| &= \left(\langle n^*_{1;01}|e^{\tilde{z}_1 + i\tilde{\phi}_1}, \langle n^*_{1;02}|e^{-\tilde{z}_1 - i\tilde{\phi}_1}\right), \end{aligned} \tag{7.10}$$

Therefore the scalar products are given by:

$$\begin{aligned} \langle m_1|n_1\rangle &= 2r_{10}\cosh(2z_1), & \langle m_1|J|n_1\rangle &= 2r_{10}\sinh(2z_1), \qquad z_1 = \tilde{z}_1 + x_{01}, \\ r_{01} &= \sqrt{\langle m_{1;01}|n_{1;01}\rangle\langle m_{1;02}|n_{1;02}\rangle}, & x_{01} &= \frac{1}{2}\ln\frac{\langle m_{1;01}|n_{1;01}\rangle}{\langle m_{1;02}|n_{1;02}\rangle}. \end{aligned} \tag{7.11}$$

As a result:

$$|N_1\rangle = \frac{\mu_1^2}{\mu_1^2\cosh^2(2z_1) + \nu_1^2\sinh^2(2z_1)} \begin{pmatrix} (\cosh(2z_1) - \frac{i\nu_1}{\mu_1}\sinh(2z_1))e^{\tilde{z}_1 - i\tilde{\phi}_1}|n_{1;01}\rangle \\ (\cosh(2z_1) + \frac{i\nu_1}{\mu_1}\sinh(2z_1))e^{\tilde{z}_1 + i\tilde{\phi}_1}|n_{1;02}\rangle \end{pmatrix} \tag{7.12}$$

The corresponding $U_1(x,t)$ will be given by:

$$\begin{aligned} U_1(x,t) &= 2i\nu_1\,[J, -|N_1\rangle\langle m_1| + J|N_1\rangle\langle m_1|J] = \begin{pmatrix} 0 & \boldsymbol{q}_{1s} \\ \boldsymbol{p}_{1s} & 0 \end{pmatrix}, \\ \boldsymbol{q}_{1s} &= \frac{4i\nu_1\mu_1^2|n_{1;01}\rangle\langle m_{1;02}|}{\mu_1^2\cosh^2(2z_1) + \nu_1^2\sinh^2(2z_1)}\left(\cosh(2z_1) - \frac{i\nu_1}{\mu_1}\sinh(2z_1)\right)e^{-2i\tilde{\phi}_1}, \qquad \boldsymbol{p}_{1s} = \boldsymbol{q}_{1s}^\dagger. \end{aligned} \tag{7.13}$$

The coefficient $U_2(x,t)$ is expressed through the matrix elements of $U_1(x,t)$ by:

$$U_2 = 2\begin{pmatrix} \boldsymbol{q}_{1s}\boldsymbol{q}_{1s}^\dagger & 0 \\ 0 & -\boldsymbol{q}_{1s}^\dagger\boldsymbol{q}_{1s} \end{pmatrix}. \tag{7.14}$$

Example 6 (A.III type symmetric spaces, VNLS case). *Now the naked polarization vectors:*

$$\begin{aligned} |n_{1;0}\rangle &= \begin{pmatrix} n_{10;1} \\ \vec{n}_{10;2}\rangle \end{pmatrix}, & \langle m_{1;0}| &= \left(n^*_{10;1}, \vec{n}^{\,\dagger}_{10;2}|\right), \\ |n_1\rangle &= \begin{pmatrix} e^{\tilde{z}_1 - i\tilde{\phi}_1}n_{10;1} \\ e^{-\tilde{z}_1 + i\tilde{\phi}_1}\vec{n}_{10;2} \end{pmatrix}, & \langle m_1| &= \left(n^*_{10;1}e^{\tilde{z}_1 + i\tilde{\phi}_1}, \vec{n}^{\,\dagger}_{10;2}|e^{-\tilde{z}_1 - i\tilde{\phi}_1}\right), \end{aligned} \tag{7.15}$$

Therefore the scalar products are given by:

$$\begin{aligned} \langle m_1|n_1\rangle &= 2r_{10}\cosh(2z_1), & \langle m_1|J|n_1\rangle &= 2r_{10}\sinh(2z_1), \qquad z_1 = \tilde{z}_1 + x_{01}, \\ r_{01} &= |n_{10;1}|\sqrt{(\vec{n}_{10;2}{}^\dagger, \vec{n}_{10;2})}, & x_{01} &= \frac{1}{2}\ln\frac{|n_{10;1}|^2}{(\vec{n}_{10;2}{}^\dagger, \vec{n}_{10;2})}. \end{aligned} \tag{7.16}$$

Therefore

$$|\vec{N}_1\rangle = \frac{\mu_1^2}{\mu_1^2\cosh^2(2z_1) + \nu_1^2\sinh^2(2z_1)} \begin{pmatrix} (\cosh(2z_1) - \frac{i\nu_1}{\mu_1}\sinh(2z_1))e^{\tilde{z}_1 - i\tilde{\phi}_1}n_{10;1} \\ (\cosh(2z_1) + \frac{i\nu_1}{\mu_1}\sinh(2z_1))e^{\tilde{z}_1 + i\tilde{\phi}_1}\vec{n}_{10;2} \end{pmatrix} \tag{7.17}$$

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The corresponding $U_1(x,t)$ will be given by:

$$
\begin{aligned}
U_1(x,t) &= 2i\nu_1 \left[J, -\vec{N}_1 \vec{m}_1^T + J\vec{N}_1 \vec{m}_1^T J \right] = \begin{pmatrix} 0 & \vec{q}_{1s} \\ \vec{p}_{1s} & 0 \end{pmatrix}, \\
\vec{q}_{1s} &= \frac{4i\nu_1 \mu_1^2 n_{10;1} \vec{m}_{10;2}|}{\mu_1^2 \cosh^2(2z_1) + \nu_1^2 \sinh^2(2z_1)} \left(\cosh(2z_1) - \frac{i\nu_1}{\mu_1} \sinh(2z_1) \right) e^{-2i\tilde{\phi}_1}, \qquad \vec{p}_{1s} = \vec{q}_{1s}^{\dagger}.
\end{aligned}
\tag{7.18}
$$

The coefficient $U_2(x,t)$ is expressed through the matrix elements of $U_1(x,t)$ by:

$$
U_2 = 2 \begin{pmatrix} (\vec{q}^{\,\dagger}_{1s}, \vec{q}_{1s}) & 0 \\ 0 & -\vec{q}_{1s} \vec{q}^{\,\dagger}_{1s} \end{pmatrix}.
\tag{7.19}
$$

The Figure 6 shows that the **one soliton** solution of (6.28) has **two maxima**. Another important difference as compare with the scalar case is that the phase of the A.III type soliton **is not linear** in x.

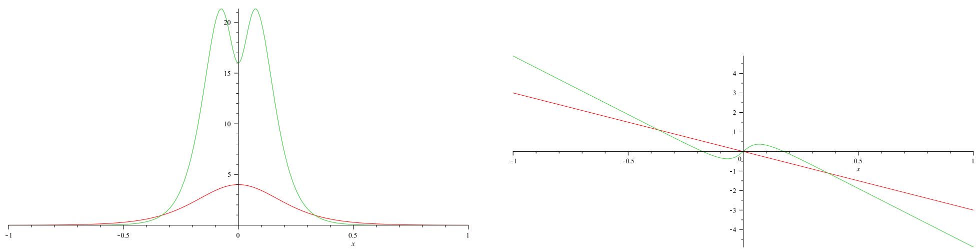

Figure 6. The one-soliton solution for the A.III type vector NLS (6.28). From eq. (7.18) one finds that all components of $\vec{q}_{1s}$ are proportional to the same function, whose modulus squared is plotted on the left panel in green. For comparison we have plotted also the modulus squared of the scalar NLS (red). On the right panel we have plotted the phase for both equations: in green is the phase of (6.28), while in red is phase of the scalar NLS. The plots are done for $\mu_1 = 2$ and $\nu_1 = 1$.

7.2. Dressing for NLEE on symmetric spaces: C.I and D.III cases

In this case the dressing factors are subject to three involutions. Therefore the dressing factor must also satisfy these involutions. They determine the structure of the dressing factors. The first two involutions are the same as in eq. (6.23). The third involution comes from the requirement that the FAS and the dressing factors belong to the corresponding Lie group ($SP(2r)$ or $SO(2r)$), see Appendix A.

The dressing factors that satisfy the above constraints take the form

$$
\begin{aligned}
u(x,t,\lambda) &= \mathbb{1} - \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^-} |N_1\rangle\langle m_1| + \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^-} J|N_1\rangle\langle m_1|J^{-1} \\
&\quad + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} S_a^{-1}|M_1\rangle\langle n_1|S_a - \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^+} S_a^{-1} J|M_1\rangle\langle n_1|J^{-1} S_a, \\
u^{-1}(x,t,\lambda) &= \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} |n_1\rangle\langle M_1| - \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^+} J^{-1}|n_1\rangle\langle M_1|J \\
&\quad - \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^-} S_a|m_1\rangle\langle N_1|S_a^{-1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^-} S_a J^{-1}|m_1\rangle\langle N_1|J S_a^{-1}.
\end{aligned}
\tag{7.20}
$$

First we need to ensure that $u(x,t,\lambda)$ and $u^{-1}(x,t,\lambda)$ in (7.20) map a FAS of L_0 into a FAS of L_1 and vice versa. This will be true if they satisfy the equations:

$$
\begin{aligned}
&i\frac{\partial u}{\partial x} + (U_2 + \lambda U_1 - \lambda^2) u(x,t,\lambda) - u(x,t,\lambda)(U_{20} + \lambda U_{10} - \lambda^2) = 0, \\
&i\frac{\partial \hat{u}}{\partial x} + (U_{20} + \lambda U_{10} - \lambda^2) \hat{u}(x,t,\lambda) - \hat{u}(x,t,\lambda)(U_2 + \lambda U_1 - \lambda^2) = 0,
\end{aligned}
\tag{7.21}
$$

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identically with respect to λ. But we have chosen fraction linear dependence on λ, it is easy to guess that eqs. (7.21) will hold true if all the residues in the left hand sides of (7.21) vanish. For example, taking the residue at $\lambda = \lambda_1^\pm$ we get:

$$\begin{aligned} &i\frac{\partial |N_1\rangle\langle m_1|}{\partial x} + (U_2 + \lambda_1^- U_1 - \lambda_1^{-,2})|N_1\rangle\langle m_1| - |N_1\rangle\langle m_1|(U_{20} + \lambda_1^- U_{10} - \lambda_1^{-,2}) = 0, \\ &i\frac{\partial |n_1\rangle\langle M_1|}{\partial x} + (U_{20} + \lambda_1^+ U_{10} - \lambda_1^{+,2})|n_1\rangle\langle M_1| - |n_1\rangle\langle M_1|(U_2 + \lambda_1^+ U_1 - \lambda_1^{+,2}) = 0, \end{aligned} \tag{7.22}$$

Thus it is easy to check, that the left hand sides of eq. (7.22) vanish if:

$$\begin{aligned} |N_1\rangle &= \chi_1^-(x,t,\lambda_1^-)|N_{10}\rangle, \qquad \langle m_1| = \langle m_{10}|\hat{\chi}_0^-(x,t,\lambda_1^-), \\ \langle M_1| &= \langle M_{10}|\hat{\chi}_1^+(x,t,\lambda_1^+), \qquad |n_1\rangle = \chi_0^+(x,t,\lambda_1^+)|n_{10}\rangle, \end{aligned} \tag{7.23}$$

where $|N_{10}\rangle$, $|N_{10}\rangle$, $\langle m_{10}|$ and $\langle n_{10}|$ are constant polarization vectors. Analogously, using the symmetries we find that all the other residues of eq. (7.21) vanish.

Next we check that $u(x,t,\lambda)$ and $u^{-1}(x,t,\lambda)$ obviously satisfy conditions b) and c). However it is far from obvious that they belong to the corresponding group. Since the expression for $u^{-1}(x,t,\lambda)$ is obtained using the group properties of $u(x,t,\lambda)$ it is enough to check that $uu^{-1} \equiv \mathbb{1}$. Note that the product uu^{-1} contains residues of second order. It is easy to see that these residues simplify to:

$$\begin{aligned} &\langle m_1|S_a|m_1\rangle = 0, \quad \langle n_1|S_a|n_1\rangle = 0, \quad \langle m_1|JS_aJ|m_1\rangle = 0, \quad \langle n_1|JS_aJ|n_1\rangle = 0, \\ &\langle N_1|S_a|N_1\rangle = 0, \quad \langle M_1|S_a|M_1\rangle = 0, \quad \langle N_1|JS_aJ|N_1\rangle = 0, \quad \langle M_1|JS_aJ|N_1\rangle = 0, \end{aligned} \tag{7.24}$$

Remember that these vectors depend on x and t and the conditions (7.24) must be identities. But we also know that these polarization vectors must satisfy (5.42) and (5.45). Therefore we have:

$$\begin{aligned} \langle n_1|S_a|n_1\rangle &= \langle n_{10}|(\chi_0^+(x,\lambda_1^+))^T S_a \chi_0^+(x,\lambda_1^+)|n_{10}\rangle = \langle n_{10}|S_a|n_{10}\rangle, \\ \langle N_1|S_a|N_1\rangle &= \langle N_{10}|(\chi^-(x,\lambda_1^-))^T S_a \chi^-(x,\lambda_1^-)|N_{10}\rangle = \langle N_{10}|S_a|N_{10}\rangle, \\ \langle N_1|JS_aJ|N_1\rangle &= \langle N_{10}|(\chi^-(x,\lambda_1^-))^T JS_aJ \chi^-(x,\lambda_1^-)|N_{10}\rangle \\ &= \langle N_{10}|J(\chi^-(x,-\lambda_1^-))^T S_a \chi^-(x,-\lambda_1^-)J|N_{10}\rangle = \langle N_{10}|JS_aJ|N_{10}\rangle. \end{aligned} \tag{7.25}$$

The proof that all other scalar products in (7.24) are x and t independent is done similarly. Thus the conditions (7.24) in fact impose restrictions only on the initial polarization vectors:

$$\begin{aligned} &\langle m_{10}|S_a|m_{10}\rangle = 0, \quad \langle n_{10}|S_a|n_{10}\rangle = 0, \quad \langle m_{10}|JS_aJ|m_{10}\rangle = 0, \quad \langle n_{10}|JS_aJ|n_{10}\rangle = 0, \\ &\langle N_{10}|S_a|N_{10}\rangle = 0, \quad \langle M_{10}|S_a|M_{10}\rangle = 0, \quad \langle N_{10}|JS_aJ|N_{10}\rangle = 0, \quad \langle M_{10}|JS_aJ|N_{10}\rangle = 0, \end{aligned} \tag{7.26}$$

The last condition that is imposed on the polarization vectors comes from eq. (5.59) and reads

$$|n_{10}\rangle = (\langle m_{10}|)^\dagger, \qquad |N_{10}\rangle = (\langle M_{10}|)^\dagger. \tag{7.27}$$

Assuming that the regular solution of the Lax operator corresponds to vanishing potentials $U_{20} = U_{10} = 0$ which means that

$$\chi^\pm(x,t,\lambda) = \exp(-i\lambda^2(x + \lambda^2 t)), \tag{7.28}$$

and using for the eigenvalues $\lambda_1^\pm = \mu_1 \pm i\nu_1$ and the polarization vectors the parametrization above we obtain

$$\begin{aligned} &\langle m_1| = \langle \tilde{m}_{10}|e^{(z_1 + i\phi_1)J}, \qquad |n_1\rangle = e^{(z_1 = i\phi_1)J}|\tilde{n}_{10}\rangle, \\ &z_1 = 2\mu_1\nu_1(x + 2(\mu_1^2 + \nu_1^2)t + r_{10}, \qquad \phi_1 = (\mu_1^2 - \nu_1^2)x + (\mu_1^4 - 6\nu_1^2\mu_1^2 + \nu_1^4)t + \phi_{10}, \end{aligned} \tag{7.29}$$

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and the notations and normalization conditions on the polarization vectors are the same as in (7.5)

Taking the residues at λ_1^-, λ_1^+ leads to

$$\begin{aligned} |m_1\rangle &= \left(\langle m_1|n_1\rangle \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} J\langle m_1|J|n_1\rangle \right) |M_1\rangle + \frac{\lambda_1^+ - \lambda_1^-}{2\lambda_1^-} \langle m_1|JS|m_1\rangle JS|N_1\rangle, \\ |n_1\rangle &= \left(\langle m_1|n_1\rangle \mathbb{1} - \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_1^-} \langle m_1|J|n_1\rangle J \right) |N_1\rangle - \frac{\lambda_1^+ - \lambda_1^-}{2\lambda_1^+} \langle n_1|JS|n_1\rangle S^{-1} J|M_1\rangle \end{aligned} \tag{7.30}$$

which can be written down as the following sets of linear equations:

$$\begin{pmatrix} |m_1\rangle \\ |n_1\rangle \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix} \begin{pmatrix} |M_1\rangle \\ |N_1\rangle \end{pmatrix} \tag{7.31}$$

Here the block matrices $X_{1,1}$ and $Y_{1,2}$ are given by:

$$\begin{aligned} X_1 &= \langle m_1|n_1\rangle \mathbb{1} + \frac{i\nu_1}{\mu_1} J\langle m_1|J|n_1\rangle, & X_2 &= -\frac{i\nu_1}{\lambda_1^-} \eta_a \beta_1 JS_a, \\ Y_1 &= -\frac{i\nu_1}{\lambda_1^+} \eta_a \bar{\beta}_1 S_a J, & Y_2 &= \langle m_1|n_1\rangle \mathbb{1} - \frac{i\nu_1}{\mu_1} \langle m_1|J|n_1\rangle J, \\ \beta_1 &= \langle m_1|S_a J|m_1\rangle = \langle m_{10}|S_a J|m_{10}\rangle, & \bar{\beta}_1 &= \langle n_1|JS_a|n_1\rangle = \langle n_{10}|JS_a|n_{10}\rangle. \end{aligned} \tag{7.32}$$

where $S_a^2 = \eta_a \mathbb{1}$, $\eta_a = \pm 1$. This system is solved similarly as eqs. (5.86) and (5.73). Using the fact that $SJ = -JS$ we can simplify the result into:

$$\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{-1} = \frac{g_2}{w_2} \begin{pmatrix} Y_2 & -X_2 \\ -Y_1 & X_1 \end{pmatrix} \tag{7.33}$$

where

$$\begin{aligned} \frac{g_2}{w_2} &= \frac{\mu_1^2(\mu_1^2 + \nu_1^2)}{(\mu_1^2 + \nu_1^2)(\mu_1^2 \langle m_1|n_1\rangle + \nu_1^2 \langle m_1|J|n_1\rangle) + \mu_1^2 \nu_1^2 \eta_a \beta_1 \bar{\beta}_1} \\ &= \frac{\mu_1^2(\mu_1^2 + \nu_1^2)}{4r_{01}^2(\mu_1^2 + \nu_1^2)(\mu_1^2 \cosh^2(2z_1) + \nu_1^2 \sinh^2(2z_1)) + \mu_1^2 \nu_1^2 \eta_a \beta_1 \bar{\beta}_1}, \end{aligned} \tag{7.34}$$

Thus we find the following expressions for $|M_1\rangle$ and $|N_1\rangle$ in terms of $|m_1\rangle$ and $|n_1\rangle$:

$$\begin{aligned} |M_1\rangle &= \frac{g_2}{w_2} \left(\left(\langle m_1|n_1\rangle - \frac{i\nu_1}{\mu_1} \langle m_1|J|n_1\rangle J \right) |m_1\rangle - \frac{i\nu_1 \eta_a}{\lambda_1^-} \beta_1 S_a J|n_1\rangle \right), \\ |N_1\rangle &= \frac{g_2}{w_2} \left(\left(\langle m_1|n_1\rangle - \frac{i\nu_1}{\mu_1} \langle m_1|J|n_1\rangle J \right) |n_1\rangle + \frac{i\nu_1 \eta_a}{\lambda_1^+} \bar{\beta}_1 S_a J|m_1\rangle \right), \end{aligned} \tag{7.35}$$

It remains to calculate the potentials. To this end we use the first equation in (7.21) for $\lambda \to \infty$, assuming $U_{2,0} = 0$ and $U_{1,0} = 0$. In this limit the term $i\frac{\partial u}{\partial x}$ vanishes and the rest of the equation is linear in λ. Thus we obtain:

$$U_{1,1s} = 2i\nu_1 [J, R_1(x,t)], \qquad U_{2,1s} = 2i\nu_1 U_{1,1s} R_1(x,t) = 2i\nu_1 [J, R_2(x,t)], \tag{7.36}$$

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where

$$
\begin{aligned}
R_1(x,t) &= -|N_1\rangle\langle m_1| + J|N_1\rangle\langle m_1|J + S_a^{-1}|M_1\rangle\langle n_1|S_a - S_a^{-1}J|M_1\rangle\langle n_1|JS_a, \\
R_2(x,t) &= -\lambda_1^-(|N_1\rangle\langle m_1| + J|N_1\rangle\langle m_1|J) + \lambda_1^+(S_a^{-1}|M_1\rangle\langle n_1|S_a + S_a^{-1}J|M_1\rangle\langle n_1|JS_a.
\end{aligned} \tag{7.37}
$$

After some calculations we get:

$$
\begin{aligned}
U_{1,1s} &= \begin{pmatrix} 0 & \boldsymbol{q}_{1,1s} \\ \boldsymbol{q}_{1,1s}^\dagger & 0 \end{pmatrix}, \qquad U_{2,1s} = \begin{pmatrix} \boldsymbol{q}_{2,1s} & 0 \\ 0 & -\boldsymbol{q}_{2,1s}^\dagger \end{pmatrix}, \\
\boldsymbol{q}_{1,1s} &= 8i\nu_1 \left(|N_{1,1}\rangle\langle m_{1,2}| - \sigma_1 |M_{1,2}\rangle\langle n_{1,1}|s_1 \right), \\
\boldsymbol{q}_{2,1s} &= 32\nu_1^2 \left(|N_{1,1}\rangle\langle m_{1,2}| - \sigma_1 |M_{1,2}\rangle\langle n_{1,1}|s_1 \right) \left(|N_{1,2}\rangle\langle m_{1,1}| - \sigma_2 |M_{1,1}\rangle\langle n_{1,2}|s_2 \right) \\
&= \frac{1}{2}\boldsymbol{q}_{1,1s}\boldsymbol{q}_{1,1s}^\dagger .
\end{aligned} \tag{7.38}
$$

The expression for $U_{2,1s}$ coincides with V_2 from eq. (6.23).

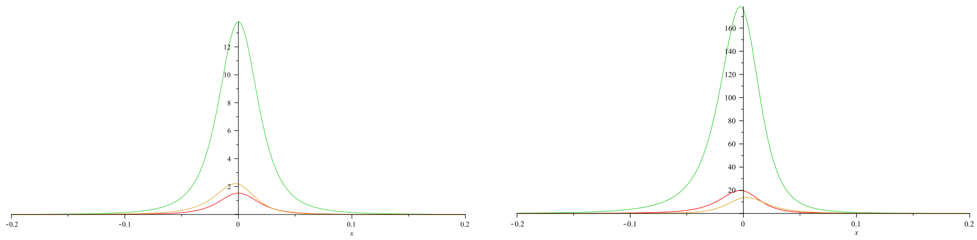

Figure 7. The one-soliton solution for the C.I type NLS (6.28). The plots are done for $\mu_1 = 3$ and $\nu_1 = 1$. Left panel: $q_{2,1s}$ – red, $q_{3,1s}$ – green, $q_{4,1s}$ – orange; Right panel: $q_{1,1s}$ – red, $q_{5,1s}$ – green, $q_{6,1s}$ – orange. In both cases $\mu_1 = 3$, $\nu_1 = 1$ and $n_{10,1} = 1 + i$, $n_{10,2} = 2 - i$, $n_{10,3} = 3(1 + i)$, $n_{10,4} = 3(1 - i)$, $n_{10,5} = 2 + i$,$n_{10,6} = 1 - i$.

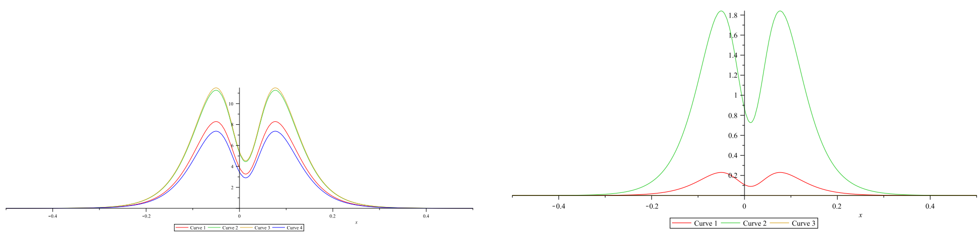

Figure 8. The one-soliton solution for the D.III type NLS (6.28). The plots are done for $\mu_1 = 3$ and $\nu_1 = 1$. Left panel: $q_{1,1s}$ – red, $q_{3,1s}$ – green, $q_{4,1s}$ – orange, $q_{6,1s}$ – blue; Right panel: $q_{2,1s}$ – red, $q_{5,1s}$ – green. In both cases $\mu_1 = 3$, $\nu_1 = 1$ and $n_{10,1} = 2$, $n_{10,2} = 3(1 + i)$, $n_{10,3} = 1 + i$, $n_{10,4} = 2(1 + i)$, $n_{10,5} = 2(1 - i)$,$n_{10,6} = 3(1 - i)$, $n_{10,7} = 1 - i$, $n_{10,8} = 4$.

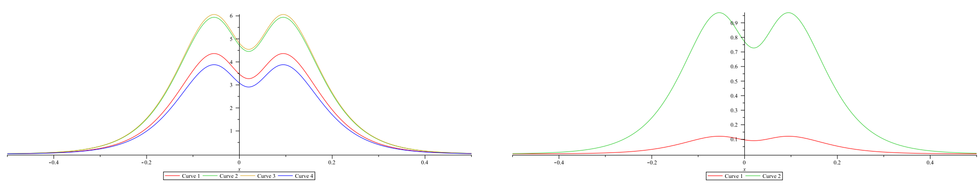

Figure 9. The same as in Figure 8 but now $\mu_1 = 2$ and $\nu_1 = 1$.

7.3. Dressing for NLEE on symmetric spaces: BD.I cases

The Cartan involution is given by a matrix C_0, with

$$C_0 = e^{i\pi J} = \operatorname{diag}(-1, \mathbb{1}, -1), \qquad J = \operatorname{diag}(1, 0, \dots, 0, -1), \qquad S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.39}$$

The dressing factors are like in (7.20)

$$\begin{aligned} u(x,t,\lambda) &= \mathbb{1} - \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^-} |N_1\rangle\langle m_1| + \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^-} J|N_1\rangle\langle m_1|J^{-1} \\ &\quad + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} S_a^{-1}|M_1\rangle\langle n_1|S_a - \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^+} S_a^{-1}J|M_1\rangle\langle n_1|J^{-1}S_a, \\ u^{-1}(x,t,\lambda) &= \mathbb{1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} |n_1\rangle\langle M_1| - \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^+} J^{-1}|n_1\rangle\langle M_1|J \\ &\quad - \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^-} S_a|m_1\rangle\langle N_1|S_a^{-1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda + \lambda_1^-} S_a J^{-1}|m_1\rangle\langle N_1|JS_a^{-1}. \end{aligned} \tag{7.40}$$

where the polarization vectors $|n_{10}\rangle$, $\langle m_{10}|$, $|N_{10}\rangle$ and $\langle M_{10}|$ have $n+1$ components. The explicit form of $\chi_0^\pm(x,t,\lambda)$ is:

$$\begin{aligned} \chi_0^\pm(x,t,\lambda_1^\pm) &= \exp\left(-i(\lambda_1^{\pm,2}x + \lambda_1^{\pm,4}t\right) = \exp(\pm z_1(x,t))\exp(i\phi_1(x,t)), \\ z_1(x,t) &= 2\mu_1\nu_1(x + 2(\mu_1^2 - \nu_1^2)t), \qquad \phi_1(x,t) = (\mu_1^2 - \nu_1^2)x + (\mu_1^4 + \nu_1^4)t. \end{aligned} \tag{7.41}$$

The involutions can then be written as

$$\begin{aligned} Su^T(x,t,\lambda)S^{-1} &= u^{-1}(x,t,\lambda), \quad u^\dagger(x,t,\lambda^*) = u^{-1}(x,t,\lambda), \\ C_0 u(x,t,\lambda)C_0 &= u(x,t,-\lambda), \quad C_0^2 = \mathbb{1}. \end{aligned} \tag{7.42}$$

Note also, that $[C_0, S] = 0$.

The dressing factors subject to three involutions are presented in (7.20). Applying the same technique as above we first derive the constraints on the polarization vectors that annihilate the second order poles in $u(x,t,\lambda)\hat{u}(x,t,\lambda)$:

7.4. N-soliton solutions and soliton interactions of MNLS equations

As you could see from the above even the derivation of the one-soliton solutions is not very easy. The N-soliton solutions however can be derived using one the following suggestions.

The first to try and repeat the dressing method starting however from the one soliton solution, rather than from the naked one $\chi_0(x,t,\lambda)$. Thus:

$$\chi_2(x,t,\lambda) = u_2(x,t,\lambda)\chi_1(x,t,\lambda) \tag{7.43}$$

where $u_2(x,t,\lambda)$ has the same form as (7.20) but now:

$$\begin{aligned} |n_1\rangle &= \chi_1^+(x,t,\lambda_1^+)|n_{10}\rangle, \qquad |N_1\rangle = \chi_2^+(x,t,\lambda_1^+)|N_{10}\rangle, \\ \langle m_1| &= \langle m_{10}|\hat{\chi}_1^-(x,t,\lambda_1^-), \langle M_1| = \langle M_{10}|\hat{\chi}_1^-(x,t,\lambda_1^-), \end{aligned} \tag{7.44}$$

Solving the algebraic equations you can use the same formulae from above, only now the expressions for $\langle m_1|n_1\rangle$, $\langle m_1|J|n_1\rangle$, $\langle m_1|C_0|n_1\rangle$ will be more complicated. Thus you will be able to derive the 2-soliton solution for the corresponding MNLS eq.

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Another way to derive the N-soliton solutions will be to use a more complicated dressing factor:

$$\begin{aligned} u(x,t,\lambda) &= \mathbb{1} + \sum_{j=1}^{N} \left(-\frac{\lambda_j^+ - \lambda_1^-}{\lambda - \lambda_j^-} |N_j\rangle\langle m_j| + \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^-} C_0 |N_j\rangle\langle m_j| C_0^{-1} \right. \\ &\left. + \frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^+} S_a^{-1} |M_j\rangle\langle n_j| S_a - \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^+} S_a^{-1} C_0 |M_j\rangle\langle n_j| C_0^{-1} S_a \right), \\ u^{-1}(x,t,\lambda) &= \mathbb{1} + \sum_{j=1}^{N} \left(\frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^+} |n_j\rangle\langle M_j| - \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^+} C_0^{-1} |n_j\rangle\langle M_j| C_0 \right. \\ &\left. - \frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^-} S_a |m_j\rangle\langle N_j| S_a^{-1} + \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^-} S_a C_0^{-1} |m_j\rangle\langle N_j| C_0 S_a^{-1} \right). \end{aligned} \tag{7.45}$$

Then you will obtain a more complicated linear algebraic system of equations for $\langle N_j|$ and $\langle M_j|$ in terms of $\langle m_j|$ and $\langle n_j|$.

Besides we need the scalar products and the definition of C_0. We assume here that we are considering $so(2N_0)$ or $sp(2N_0)$ algebras and that

$$C_0 = \sum_{k=0}^{N_0} \epsilon_k H_{e_k}, \qquad \epsilon_k = 1 \quad \text{or} \quad -1. \tag{7.46}$$

$$\langle m_1|n_1\rangle = \sum_{s=1}^{2N_0} 2|n_{10,k}|^2 e^{z_1(J_k x + K_k t)} = \sum_{s=1}^{N_0} 2|n_{10,k}||n_{10,\bar{k}}| \cosh(z_{1,k} + x_{0,k}), \quad x_{0,k} = \ln \frac{|n_{10,k}|}{|n_{10,\bar{k}}|} \tag{7.47}$$

8. Multi-soliton solutions

We will outline the dressing method for the N-soliton solutions of MNLS in the case of three involutions.

The dressing factors for N solitons in that case take the form:

$$u_{\rm Ns}(x,t,\lambda) = \mathbb{1} + \sum_{j=1}^{N} u_j(x,t,\lambda), \quad u_{\rm Ns}^{-1}(x,t,\lambda) = \mathbb{1} + \sum_{j=1}^{N} \hat{u}_j(x,t,\lambda), \qquad \hat{u}_j(x,t,\lambda) = S_a u_j^T(x,t,\lambda) S_a^{-1}, \tag{8.1}$$

where

$$\begin{aligned} u_j(x,t,\lambda) &= -\frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^-} |N_j\rangle\langle m_j| + \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^-} J |N_j\rangle\langle m_j| J \\ &+ \frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^+} S_a |M_j\rangle\langle n_j| S_a^{-1} - \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^+} S_a J |M_j\rangle\langle n_j| J S_a^{-1}, \\ \hat{u}_j(x,t,\lambda) &= -\frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^-} S_a |m_j\rangle\langle N_j| S_a^{-1} + \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^-} S_a J |m_j\rangle\langle N_j| J S_a^{-1} \\ &+ \frac{\lambda_j^+ - \lambda_j^-}{\lambda - \lambda_j^+} |n_j\rangle\langle M_j| - \frac{\lambda_j^+ - \lambda_j^-}{\lambda + \lambda_j^+} J |n_j\rangle\langle M_j| J. \end{aligned} \tag{8.2}$$

The N-soliton dressing factor must satisfy eq. (7.21) which must hold identically with respect to λ. That means that all residues for $\lambda = \lambda_j^\pm$ must vanish. Repeating the analysis in Subsection 7.2 we obtain the analogues of eqs. (7.23):

$$\begin{aligned} |N_j\rangle &= \chi_{\rm Ns}^-(x,\lambda_j^-)|N_{j,0}\rangle, \qquad \langle m_j| = \langle m_{j0}|\hat{\chi}_0^-(x,\lambda_j^-), \\ |n_j\rangle &= \chi_0^+(x,\lambda_j^+)|n_{j,0}\rangle, \qquad \langle M_j| = \langle M_{j0}|\hat{\chi}_{\rm Ns}^+(x,\lambda_j^+. \end{aligned} \tag{8.3}$$

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Next we must ensure that $u_{\rm Ns}\hat{u}_{\rm Ns} = \mathbb{1}$. It is obvious that here we encounter residues of second order. Vanishing or these residues requires set of conditions generalizing eq. (7.24):

$$\begin{aligned} -\langle m_j|S_a|m_j\rangle = 0, \qquad \langle n_j|S_a|n_j\rangle = 0, \qquad \langle m_j|JS_aJ|m_j\rangle = 0, \qquad \langle n_j|JS_aJ|n_j\rangle = 0, \\ \langle M_j|S_a|M_j\rangle = 0, \qquad \langle N_j|S_a|N_j\rangle = 0, \qquad \langle M_j|JS_aJ|M_j\rangle = 0, \qquad \langle N_j|JS_aJ|N_j\rangle = 0, \end{aligned} \tag{8.4}$$

for all $j = 1, \dots, N$. Using the fact that the solutions $\chi_0^{\pm}(x, \lambda_j^{\pm})$ and $\chi_{\rm Ns}^{\pm}(x, \lambda_j^{\pm})$ are elements of the orthogonal or symplectic groups one can check that the constraints (8.4) in fact affect only the 'naked' polarization vectors:

$$\langle m_{j0}|S_a|m_{j0}\rangle = 0, \quad \langle n_{j0}|S_a|n_{j0}\rangle = 0, \quad \langle M_{j0}|S_a|M_{j0}\rangle = 0, \quad \langle N_{j0}|S_a|N_{j0}\rangle = 0, \tag{8.5}$$

Analyzing the residues of first order and requiring them to vanish we derive a set of linear equations which allow us to find explicit expressions for the 'dressed' polarization vectors $|N_j\rangle$ and $|M_j\rangle$ in terms of $|m_j\rangle$ and $|n_j\rangle$. Below we will derive this equations for the simplest nontrivial case: $N = 2$:

$$\begin{aligned} |m_1\rangle &= \left(\langle m_1|n_1\rangle + \frac{i\nu_1}{\mu_1}\langle m_1|J|n_1\rangle J\right)|M_1\rangle + \left(\frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^+ - \lambda_1^-}\langle m_1|n_2\rangle + \frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^+ + \lambda_1^-}\langle m_1|J|n_2\rangle J\right)|M_2\rangle \\ &+ \frac{i\nu_1}{\lambda_1^-}\langle m_1|S_aJ|m_1\rangle S_aJ|N_1\rangle + \left(-\frac{\lambda_2^+ - \lambda_2^-}{\lambda_1^- - \lambda_2^-}\langle m_1|S_a|m_2\rangle S_a + \frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^- + \lambda_1^-}\langle m_1|S_aJ|m_2\rangle S_aJ\right)|N_2\rangle, \\ |m_2\rangle &= \left(\frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ - \lambda_2^-}\langle m_2|n_1\rangle + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ + \lambda_2^-}\langle m_2|J|n_1\rangle J\right)|M_1\rangle + \left(\langle m_2|n_2\rangle + \frac{i\nu_2}{\mu_2}\langle m_2|J|n_2\rangle J\right)|M_2\rangle \\ &+ \left(\frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^- - \lambda_1^-}\langle m_2|S_a|m_1\rangle S_a + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^- + \lambda_1^-}\langle m_2|S_aJ|m_1\rangle S_aJ\right)|N_1\rangle + \frac{i\nu_2}{\lambda_2^-}\langle m_2|S_aJ|m_2\rangle S_aJ|N_2\rangle. \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} |n_1\rangle &= \frac{i\nu_1}{\lambda_1^+}\langle n_1|S_a^{-1}J|n_1\rangle S_a^{-1}J|M_1\rangle - \left(\frac{\lambda_2^+ - \lambda_2^-}{\lambda_1^+ - \lambda_2^+}\langle n_1|S_a^{-1}|n_2\rangle S_a^{-1} - \frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^+ + \lambda_1^+}\langle n_1|S_a^{-1}J|n_2\rangle S_a^{-1}J\right)|M_2\rangle \\ &+ \left(\langle n_1|m_1\rangle - \frac{i\nu_1}{\mu_1}\langle n_1|J|m_1\rangle J\right)|N_1\rangle + \left(\frac{\lambda_2^+ - \lambda_2^-}{\lambda_1^+ - \lambda_2^-}\langle n_1|m_2\rangle - \frac{\lambda_2^+ - \lambda_2^-}{\lambda_1^+ + \lambda_2^-}\langle n_1|J|m_2\rangle J\right)|N_2\rangle, \\ |n_2\rangle &= \left(-\frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^+ - \lambda_1^+}\langle n_2|S_a^{-1}|n_1\rangle S_a^{-1} + \frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^+ + \lambda_1^+}\langle n_2|S_a^{-1}J|n_1\rangle S_a^{-1}J\right)|M_1\rangle + \frac{i\nu_2}{\lambda_2^+}\langle n_2|S_a^{-1}J|n_2\rangle S_a^{-1}J|M_2\rangle \\ &\left(\frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^+ - \lambda_1^-}\langle n_2|m_1\rangle - \frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^+ + \lambda_1^-}\langle n_2|J|m_1\rangle J\right)|N_1\rangle + \left(\langle n_2|m_2\rangle - \frac{i\nu_2}{\mu_2}\langle n_2|J|m_2\rangle J\right)|N_2\rangle. \end{aligned} \tag{8.7}$$

$$\begin{pmatrix} |m_1\rangle \\ |m_2\rangle \\ |n_1\rangle \\ |n_2\rangle \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} |M_1\rangle \\ |M_2\rangle \\ |N_1\rangle \\ |N_2\rangle \end{pmatrix}, \tag{8.8}$$

where

$$A_{11} = \langle m_1|n_1\rangle + \frac{i\nu_1}{\mu_1}\langle m_1|J|n_1\rangle J, \quad A_{12} = \frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^+ - \lambda_1^-}\langle m_1|n_2\rangle + \frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^+ + \lambda_1^-}\langle m_1|J|n_2\rangle J,$$

$$
\begin{aligned}
B_{11} &= \frac{i\nu_1}{\lambda_1^-}\langle m_1|S_aJ|m_1\rangle S_aJ, \quad B_{12} = -\frac{\lambda_2^+-\lambda_2^-}{\lambda_1^- - \lambda_2^-}\langle m_1|S_a|m_2\rangle S_a + \frac{\lambda_2^+-\lambda_2^-}{\lambda_2^- + \lambda_1^-}\langle m_1|S_aJ|m_2\rangle S_aJ, \\
A_{21} &= \frac{\lambda_1^+-\lambda_1^-}{\lambda_1^+ - \lambda_2^-}\langle m_2|n_1\rangle + \frac{\lambda_1^+-\lambda_1^-}{\lambda_1^+ + \lambda_2^-}\langle m_2|J|n_1\rangle J, \quad A_{22} = \langle m_2|n_2\rangle + \frac{i\nu_2}{\mu_2}\langle m_2|J|n_2\rangle J, \\
B_{21} &= \frac{\lambda_1^+-\lambda_1^-}{\lambda_2^- - \lambda_1^-}\langle m_2|S_a|m_1\rangle S_a + \frac{\lambda_1^+-\lambda_1^-}{\lambda_2^- + \lambda_1^-}\langle m_2|S_aJ|m_1\rangle S_aJ, \quad B_{22} = \frac{i\nu_2}{\lambda_2^-}\langle m_2|S_aJ|m_2\rangle S_aJ, \\
C_{11} &= -\frac{\lambda_1^+-\lambda_1^-}{\lambda_2^+ - \lambda_1^+}\langle n_2|S_a^{-1}|n_1\rangle S_a^{-1} + \frac{\lambda_1^+-\lambda_1^-}{\lambda_2^+ + \lambda_1^+}\langle n_2|S_a^{-1}J|n_1\rangle S_a^{-1}J, \quad C_{12} = \frac{i\nu_2}{\lambda_2^+}\langle n_2|S_a^{-1}J|n_2\rangle S_a^{-1}J, \\
D_{11} &= \langle n_1|m_1\rangle - \frac{i\nu_1}{\mu_1}\langle n_1|J|m_1\rangle J, \quad D_{12} = \frac{\lambda_2^+-\lambda_2^-}{\lambda_1^+ - \lambda_2^-}\langle n_1|m_2\rangle - \frac{\lambda_2^+-\lambda_2^-}{\lambda_1^+ + \lambda_2^-}\langle n_1|J|m_2\rangle J, \\
C_{21} &= -\frac{\lambda_1^+-\lambda_1^-}{\lambda_2^+ - \lambda_1^+}\langle n_2|S_a^{-1}|n_1\rangle S_a^{-1} + \frac{\lambda_1^+-\lambda_1^-}{\lambda_2^+ + \lambda_1^+}\langle n_2|S_a^{-1}J|n_1\rangle S_a^{-1}J, \quad C_{22} = \frac{i\nu_2}{\lambda_2^+}\langle n_2|S_a^{-1}J|n_2\rangle S_a^{-1}J, \\
D_{21} &= \frac{\lambda_1^+-\lambda_1^-}{\lambda_2^+ - \lambda_1^-}\langle n_2|m_1\rangle - \frac{\lambda_1^+-\lambda_1^-}{\lambda_2^+ + \lambda_1^-}\langle n_2|J|m_1\rangle J, \quad D_{22} = \langle n_2|m_2\rangle - \frac{i\nu_2}{\mu_2}\langle n_2|J|m_2\rangle J \qquad (8.9)
\end{aligned}
$$

In order to find the 2-soliton solution we need to solve eq. (8.8) and invert the 4×4 block matrix; then

$$
\begin{pmatrix} |M_1\rangle \\ |M_2\rangle \\ |N_1\rangle \\ |N_2\rangle \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{pmatrix}^{-1} \begin{pmatrix} |m_1\rangle \\ |m_2\rangle \\ |n_1\rangle \\ |n_2\rangle \end{pmatrix}. \qquad (8.10)
$$

The explicit calculation of the right hand side of eq. (8.10) will be published elsewhere.

9. The resolvent and spectral properties of Lax operators

In this Section following [32] we briefly address the spectral properties of the Lax operators. We intend to demonstrate that the notion of the resolvent introduced in [32] for the generalized Zakharov-Shabat systems can be naturally extended to the class of quadratic pencils that we have studied above.

Below for simplicity we assume that the function $\mathcal{Q}(x,t,\lambda)$ (5.1) which introduces the parametrization of the RHP is a smooth matrix-valued function of x satisfying the following conditions:

C.1 $\mathcal{Q}(x,t,\lambda)$ possesses smooth derivatives of all orders with respect to x and falls off to zero for $|x| \to \infty$ faster than any power of x:

$$
\lim_{x\to\pm\infty} |x|^k \mathcal{Q}(x,t,\lambda) = 0, \qquad \forall\lambda \quad \text{and} \quad k = 0,1,2,\ldots;
$$

C.2 $\mathcal{Q}(x,t,\lambda)$ is such that the corresponding RHP has finite index. For the class of RHP that we have been dealing with this means that the solution of the RHP must have finite number of simple zeroes and pole singularities.

Remark 9. *Let us assume that the zeroes and the pole singularities of $\xi^+(x,t,\lambda)$ (resp. $\xi^-(x,t,\lambda)$) are located at the points $\lambda_j^+ \in \mathbb{Q}_1 \cup \mathbb{Q}_3$ (resp. at $\lambda_j^- \in \mathbb{Q}_2 \cup \mathbb{Q}_4$). Taking into account the symmetries of the FAS (5.14) we conclude that:*

1. *if λ_j^+ is a zero or pole of $\xi^+(x,t,\lambda)$, then there must exist $\lambda_p^+ = -\lambda_j^+$ which is also a zero or pole of $\xi^+(x,t,\lambda)$;*
2. *if λ_j^- is a zero or pole of $\xi^-(x,t,\lambda)$, then there must exist $\lambda_p^- = -\lambda_j^-$ which is also zero or pole of $\xi^-(x,t,\lambda)$.*

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3. *if λ_j^+ is a zero or pole of $\xi^+(x,t,\lambda)$, then there must exist $\lambda_p^- = (\lambda_j^+)^*$ which is a zero or pole of $\xi^-(x,t,\lambda)$.*

Choosing proper enumeration of the zeroes and poles we can arrange them so that $\lambda_j^+ \in \mathbb{Q}_1$, $j = 1,\dots,N$, $\lambda_{j+N}^+ = -\lambda_j^+ \in \mathbb{Q}_3$, $j = 1,\dots,N$, $\lambda_j^- = (\lambda_j^+)^* \in \mathbb{Q}_4$, $j = 1,\dots,N$ and $\lambda_{j+N}^- = (\lambda_{j+N}^+)^* \in \mathbb{Q}_2$, $j = 1,\dots,N$.

The FAS of $L(\lambda)$ which are related to the solution of the RHP by $\chi^\pm(x,\lambda) = \xi^\pm(x,\lambda)e^{I\lambda^2 Jx}$ allow one to construct the resolvent of the operator L and then to investigate its spectral properties. By resolvent of $L(\lambda)$ we understand the integral operator $R(\lambda)$ with kernel $R(x,y,\lambda)$ which satisfies

$$L(\lambda)(R(\lambda)f)(x) = f(x), \tag{9.1}$$

where $f(x)$ is an n-component vector function in $\mathbb{C}^n$ with bounded norm, i.e. $\int_{-\infty}^{\infty} dy(f^\dagger(y)f(y)) < \infty$.

From the general theory of linear operators [5,17] we know that the point λ in the complex λ-plane is a regular point if $R(\lambda)$ is the kernel of a bounded integral operator. In each connected subset of regular points $R(\lambda)$ must be analytic in λ.

The points λ which are not regular constitute the spectrum of $L(\lambda)$. Roughly speaking the spectrum of $L(\lambda)$ consist of two types of points:

- i) the continuous spectrum of $L(\lambda)$ consists of all points λ for which $R(\lambda)$ is an unbounded integral operator;
- ii) the discrete spectrum of $L(\lambda)$ consists of all points λ for which $R(\lambda)$ develops pole singularities.

Let us now show how the resolvent $R(\lambda)$ can be expressed through the FAS of $L(\lambda)$. Here and below we will limit ourselves with Lax operators L which are quadratic pencils of λ and have the form (5.3) (i.e. $N_1 = 2$). Assuming that the eigenvalues of J are different and real we can always order them so that

$$J = \operatorname{diag}(a_1, a_2, \dots, a_n), \qquad a_1 > a_2 > \cdots a_k > 0 > a_{k+1} > \cdots > a_n \tag{9.2}$$

In Section 2 above we constructed the FAS of such operators. Using them we can write down $R(\lambda)$ in the form:

$$R(\lambda)f(x) = \int_{-\infty}^{\infty} R(x,y,\lambda)f(y), \tag{9.3}$$

the kernel $R(x,y,\lambda)$ of the resolvent is given by:

$$R(x,y,\lambda) = \begin{cases} R^+(x,y,\lambda) & \text{for } \lambda \in \mathbb{Q}_1 \cup \mathbb{Q}_3, \\ R^-(x,y,\lambda) & \text{for } \lambda \in \mathbb{Q}_2 \cup \mathbb{Q}_4, \end{cases} \tag{9.4}$$

where

$$\begin{aligned} R^\pm(x,y,\lambda) &= \pm i\chi^\pm(x,\lambda)\Theta^\pm(x-y)\hat{\chi}^\pm(y,\lambda), \\ \Theta^\pm(z) &= \theta(\mp z)\Pi_0 - \theta(\pm z)(\mathbb{1} - \Pi_0), \qquad \Pi_0 = \sum_{s=1}^{k_0} E_{ss}, \end{aligned} \tag{9.5}$$

where k_0 is the number of positive eigenvalues of J, see (9.2). Due to the condition $\operatorname{tr} J = \sum_{s=1}^n a_s = 0$, k_0 is fixed up uniquely.

The next theorem establishes that $R(x,y,\lambda)$ is indeed the kernel of the resolvent of $L(\lambda)$.

Theorem 3. *Let $Q(x,t,\lambda)$ satisfy conditions (C.1) and (C.2) and let $\lambda_j^\pm$ be the simple zeroes of the solutions $\xi^\pm(x,t,\lambda)$ of the RHP. Let $\chi^\pm(x,\lambda) = \xi^\pm(x,\lambda)\exp(i\lambda^2 Jx)$ and let $R^\pm(x,y,\lambda)$ be defined as in eq. (9.5). Then:*

1. *$\chi^{\pm}(x,\lambda)$ will be FAS of a quadratic pencil of the form (5.11) whose coefficients will be expressed through $Q_s(x,t)$ as in (5.15).*
2. *$R^{\pm}(x,y,\lambda)$ is a kernel of a bounded integral operator for Im $\lambda^2 \neq 0$;*
3. *$R(x,y,\lambda)$ is uniformly bounded function for $\lambda^2 \in \mathbb{R}$ and provides a kernel of an unbounded integral operator;*
4. *$R^{\pm}(x,y,\lambda)$ satisfy the equation:*

$$L(\lambda)R^{\pm}(x,y,\lambda) = \mathbb{1}\delta(x-y). \tag{9.6}$$

Idea of the proof. 1. is obvious from the fact that $\chi^{\pm}(x,\lambda)$ are the FAS of $L(\lambda)$ (5.11). It is also easy to see that if $\mathcal{Q}(x,t,\lambda)$ satisfies conditions (C.1) and (C.2) then $U_2(x,t)$ and $U_1(x,t)$ will also satisfy condition C1. In addition obviously $\chi^{\pm}(x,\lambda)$ will satisfy condition C2 and will have poles and zeroes at the points $\lambda_j^{\pm}$, see Remark 9.

2. Assume that Im $\lambda^2 > 0$ and consider the asymptotic behavior of $R^{+}(x,y,\lambda)$ for $x,y \to \infty$. From equations (5.11) we find that

$$R^{+}_{ij}(x,y,\lambda) = \sum_{p=1}^{n} \xi^{+}_{ip}(x,\lambda)e^{-i\lambda^2 a_p(x-y)}\Theta^{+}_{pp}(x-y)\hat{\xi}^{+}_{pj}(y,\lambda) \tag{9.7}$$

We use the fact that $\chi^{+}(x,\lambda)$ has triangular asymptotics for $x \to \infty$ and $\lambda^2 \in \mathbb{C}_+$ (see eq. (2.38)). With the choice of $\Theta^{+}(x-y)$ (9.5) we check that the right hand side of (9.7) falls off exponentially for $x \to \infty$ and arbitrary choice of y. All other possibilities are treated analogously.

3. For $\lambda \in \mathbb{R} \cup i\mathbb{R}$ the arguments of 2) can not be applied because the exponentials in the right hand side of (9.7) Im $\lambda = 0$ only oscillate. Thus we conclude that $R^{\pm}(x,y,\lambda)$ for $\lambda \in \mathbb{R}$ is only a bounded function of x and thus the corresponding operator $R(\lambda)$ is an unbounded integral operator.

4. The proof of eq. (9.6) follows from the fact that $L(\lambda)\chi^{+}(x,\lambda) = 0$ and

$$\frac{d\Theta^{\pm}(x-y)}{dx} = \mp\mathbb{1}\delta(x-y). \tag{9.8}$$

□

If the algebra $\mathfrak{g} \simeq sl(n)$ then $\chi^{+}(x,\lambda)$ and its inverse $\hat{\chi}^{+}(x,\lambda)$ do not have common poles. In this case condition C2 is valid also for $R^{+}(x,y,\lambda)$. However this is not true for the orthogonal or symplectic algebras; in this case $R^{+}(x,y,\lambda)$ may have poles of second order, which require additional care.

Now we can derive the completeness relation for the eigenfunctions of the Lax operator L by applying the contour integration method (see e.g. [2,46,47]) to the integral:

$$\mathcal{J}(x,y) = \frac{1}{2\pi i}\oint_{\gamma_1\cup\gamma_3} d\lambda\, \lambda R^{+}(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_2\cup\gamma_4} d\lambda\, \lambda R^{-}(x,y,\lambda), \tag{9.9}$$

where the contours γ_j are shown on the Figure 10. The idea is to calculate $\mathcal{J}(x,y)$ first using the Cauchy residue theorem. Taking into account that the contours γ_1 and γ_3 are positively oriented, while γ_2 and γ_4 are negatively oriented we obtain:

$$\begin{aligned}&\mathcal{J}(x,y)\\ &= \sum_{j=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_j^{+}} \lambda R^{+}(x,y,\lambda) + \operatorname*{Res}_{\lambda=-\lambda_j^{+}} \lambda R^{+}(x,y,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^{-}} \lambda R^{-}(x,y,\lambda) - \operatorname*{Res}_{\lambda=-\lambda_j^{-}} \lambda R^{-}(x,y,\lambda)\right).\end{aligned} \tag{9.10}$$

We can also evaluate $\mathcal{J}(x,y)$ integrating along the contours. The integration along the real and purely imaginary axis provides the contribution coming from the continuous spectrum of L. The

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integration along the infinite arcs of the contours can be evaluated explicitly. To this end we use the fact that $\xi^{\pm}(x,\lambda)$ are canonically normalized we find:

$$\begin{aligned} \mathcal{J}^{+}_{\infty}(x,y) &= \frac{1}{2\pi i}\oint_{\gamma_{1,\infty}\cup\gamma_{3,\infty}} d\lambda\, \lambda R^{+}(x,y,\lambda) = \frac{1}{4\pi i}\oint_{\gamma_{1,\infty}\cup\gamma_{3,\infty}} d\lambda^2\, e^{-i\lambda^2 J(x-y)}\Theta^{+}(x-y) \\ &= \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\lambda^2\, e^{-i\lambda^2 J(x-y)}\Theta^{+}(x-y) \end{aligned} \tag{9.11}$$

$$\begin{aligned} \mathcal{J}^{-}_{\infty}(x,y) &= \frac{1}{2\pi i}\oint_{\gamma_{2,\infty}\cup\gamma_{4,\infty}} d\lambda\, \lambda R^{-}(x,y,\lambda) = \frac{1}{4\pi i}\oint_{\gamma_{2,\infty}\cup\gamma_{4,\infty}} d\lambda^2\, e^{-i\lambda^2 J(x-y)}\Theta^{-}(x-y) \\ &= -\frac{1}{4\pi i}\int_{-\infty}^{\infty} d\lambda^2\, e^{-i\lambda^2 J(x-y)}\Theta^{-}(x-y). \end{aligned} \tag{9.12}$$

Adding these two answers and using the definitions of $\Theta^{\pm}(x-y)$ we find

$$\mathcal{J}^{+}_{\infty}(x,y) + \mathcal{J}^{-}_{\infty}(x,y) = \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\lambda^2\, e^{-i\lambda^2(x-y)}\mathbb{1} = \frac{1}{2}\delta(x-y)\sum_{s=1}^{n}\frac{1}{a_s}E_{ss}. \tag{9.13}$$

Equating the two answers for $\mathcal{J}(x,y)$ we find the following completeness relations for the FAS of L:

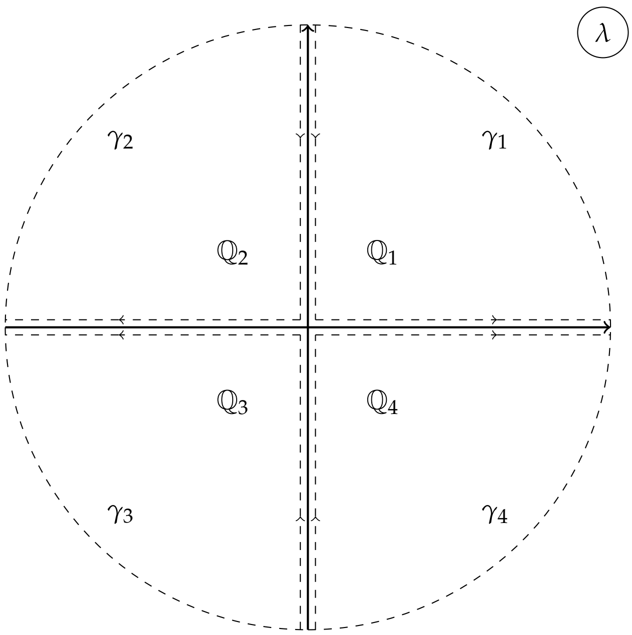

Figure 10. The contours γ_j of the related Riemann-Hilbert problem $\mathbb{R}\cup i\mathbb{R}$.

$$\begin{aligned} \delta(x-y)\frac{1}{2}\sum_{s=1}^{n}\frac{1}{a_s}E_{ss} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda^2 \left\{ \sum_{s=1}^{k_0} |\chi^{[s]+}(x,\lambda)\rangle\langle\hat{\chi}^{[s]+}(y,\lambda)| - \sum_{s=k_0+1}^{n} |\chi^{[s]-}(x,\lambda)\rangle\langle\hat{\chi}^{[s]-}(y,\lambda)| \right\} \\ &+ \sum_{j=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\lambda R^{+}(x,y,\lambda) + \operatorname*{Res}_{\lambda=-\lambda_j^+}\lambda R^{+}(x,y,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-}\lambda R^{-}(x,y,\lambda) - \operatorname*{Res}_{\lambda=-\lambda_j^-}\lambda R^{-}(x,y,\lambda) \right). \end{aligned} \tag{9.14}$$

This relation (9.14) allows one to expand any vector-function $|z(x)\rangle \in \mathbb{C}^n$ over the eigenfunctions of the system (5.11). If we multiply (9.14) on the right by $J \mid z(y)\rangle$ and integrate over y we get:

$$|\, z(x)\rangle = \tfrac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda \left\{ \sum_{s=1}^{k_0} |\chi^{[s]+}(x,\lambda)\rangle \cdot \zeta_s^{+}(\lambda) - \sum_{s=k_0+1}^{n} \chi^{[s]-}(x,\lambda)\rangle \cdot \zeta_s^{-}(\lambda) \right\}$$

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$$+\sum_{j=1}^{N} \nu_j \left(\left| \mathbf{n}_j^+(x) \right\rangle \zeta_j^+ - \left| \mathbf{n}_j^-(x) \right\rangle \zeta_j^- \right), \tag{9.15}$$

where the expansion coefficients are of the form:

$$\zeta_s^{\pm}(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{\chi}^{[s]\pm}(x,\lambda) | J | z(x) \rangle, \qquad \zeta_j^{\pm}(\lambda) = -i \int_{-\infty}^{\infty} dx \left\langle \mathbf{m}_j^{\pm} \right| J \left| z(x) \right\rangle. \tag{9.16}$$

Corollary 3. *The discrete spectrum of the Lax operator (5.11) consists of the poles of the resolvent $R^{\pm}(x,y,\lambda)$ which are described in Remark 9.*

Remark 10. *Here we used also the fact that all eigenvalues of J are non-vanishing. In the case when one (or several) of them vanishes we can prove completeness of the eigenfunctions only on the image of ad_J which is a subspace of $\mathbb{C}^n$.*

Remark 11. *The authors are aware that these type of derivations need additional arguments to be made rigorous. One of the real difficulties is to find explicit conditions on the potentials $U_2(x)$ and $U_1(x)$ that are equivalent to the condition (C.2). Nevertheless there are situations (e.g., the reflectionless potentials) when all these conditions are fulfilled and all eigenfunctions of $L(\lambda)$ can be explicitly calculated. Another advantage of this approach is the possibility to apply it to Lax operators with more general dependence on λ, e.g. polynomial and rational.*

10. Discussion and conclusions

Our idea is to remind some of the old papers which were milestones in the past, but for various reasons tend to be forgotten. Another motivation to do this is our intention to generalize these results for quadratic pencils and pencils of higher degrees.

Of course. many of the results we mention below were first introduced for the GZSh system. One of the important results is the resolvent for the Lax operator L [29,32,33] expressed in terms of the FAS. Combined with the contour integration method it allows one to to construct the spectral decompositions of L. They also allow one to prove that the poles and zeroes of FAS provide the discrete eigenvalues of L. Rigorous proofs would hardly be possible with these methods, but their importance is in the fact that can be applied to wide class of non-self-adjoint Lax operators.

In the Introduction we already listed a number of results about the interpretation of the ISM as GFT and ISM for the GZaSha systems. Our comment on it is that this idea can be generalized also for the quadratic pencils considered as Lax operators above. We intend to continue the idea in our next publications. Of course it will have its versions for the homogeneous as well as for the symmetric spaces.

The initial ideas that we have extended above for quadratic pencils have been developed before by other influential scientists, see some of theirs reviews and monographs [18,32,33,63,70,71,91,101, 104,130]. It is important to note that these ideas are applicable also to discrete evolution equations as Ablowitz-Ladik system [3,43].

Another important approach to integrable NLEE has been developed by Shabat, Mikhailov, Yamilov, Svinolupov et al, see [98,99,101,138]. It allows one to derive systems of equations that possess higher symmetries, or higher integrals of motion. Lately this method has been used by Zhao [138], who covered all integrable systems in two independent functions. Some of these equations were shown to possess Lax representations related to the twisted Kac-Moody algebras [54].

Special attention must be paid to the NLEE with deep Mikhailov reduction groups, like $\mathbb{Z}_h$ and $\mathbb{D}_h$. Typically by h we denote the Coxeter number of the corresponding simple Lie algebra. Here we include the families of KdV and MKdV eqs., as well as 2-dimensional Toda field theories [4,49–51,55,79,80,97,102,103] as well as some Camassa-Holm type eqs. [11,44,66,68,69,75,76].

Kulish-Sklyanin system, or in other words the 3-component NLS related to the BD.I symmetric space has important applications to the spin-1 Bose-Einstein condensate, see [23,24,34,38,38,41,60, 61,72,73,84,85,105,113–117,124]. Using the dressing method in [35,41] we calculated the limit of the dressing factor for $t \to \pm\infty$. This allowed us to describe the soliton interactions, i.e. to calculate the center-of-mass and the phase shifts in the soliton interactions.

In our final remarks we mention the generalizations to $2+1$ dimensions [48,62,83,93], the MNLS related to spectral curves [6,11,77,109–112] and the hierarchies of Hamiltonian structures. It is known that one can relate Poisson brackets on the quadratic pencils which, along with the integrals of motion allow one to interprete the NLEE in the hierarchy as Hamiltonian equations of motion. Such Poisson brackets have been introduced for the $sl(2)$ polynomial pencils by Kulish and Reyman [86]. Of course they allow generalization to any semi-simple Lie algebra. We already mentioned that for the generalized Zakharov-Shabat systems this has been already done. In fact the spectral theory of the recursion operators has been constructed which generate all NLEE along with its Hamilton formulation. For the quadratic pencils the recursion operators have not yet been constructed; we hope that this problem will be solved soon.

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Appendix A. Root systems of simple Lie algebras

This is assumed to be well known material which we display without proofs and in a manner suitable for pedestrians in Lie algebras; for proofs and more details see [8,67]. The fundamental ideas of Cartan and Weil are based on:

- The Cartan subalgebras $\mathfrak{h}$ of all simple Lie algebras can be represented by diagonal matrices;
- There is an one-to-one mapping between the elements of $\mathfrak{h}$ and the vectors in r-dimensional Euclidean space $\mathbb{E}^r$;
- The Weil generators E_α are defined as eigenvectors of all the elements of $\mathfrak{h}$, i.e.

$$[H, E_\alpha] = \alpha(H)E_\alpha = \frac{2(\alpha, \vec{h})}{(\alpha, \alpha)} E_\alpha, \qquad \text{(A.1)}$$

  where $\alpha(H) = (\alpha, \vec{h})$. Here $\vec{h} \in \mathbb{E}^r$ is the vector corresponding to H, and $\alpha \in \Delta$ is a root of the algebra $\mathfrak{g}$ belonging to its system of roots Δ.

Skipping the details we will describe the roots systems Δ for all classical series A_r, B_r, C_r and D_r of simple Lie algebras.

- Root systems
- Dynkin diagrams
- Cartan-Weyl basis
- Automorphisms of finite order

Basic properties of the root systems.

- If $\alpha \in \Delta$ then $-\alpha$ is also a root, $-\alpha \in \Delta_-$.
- Each roots system is split into positive and negative roots:

$$\Delta = \Delta_+ \cup \Delta_-, \qquad \text{if} \qquad \alpha \in \Delta_+ \qquad \text{then} \qquad -\alpha \in \Delta_-.$$

- In each root systems one can introduce a basis, known as **system of simple roots.** By definition α_j, $j = 1, \dots, r$ are simple roots if: i) they are linearly independent and form a basis in $\mathbb{E}^r$; ii) they are positive roots $\alpha_j \in \Delta_+$ such that $\alpha_j - \alpha_k \notin \Delta$;
- Each positive root $\beta \in \Delta_+$ can be expressed as sum of simple roots $\beta = \sum_{j=1}^r n_j \alpha_j$ where all $n_j \geq 0$ are integers;
- There is a maximal (resp. minimal) root $\alpha_{\max}$ (resp $\alpha_{\min}$) such that $\alpha_{\max} + \beta$ (resp. $\alpha_{\min} - \beta$) is not a root;

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- Symmetry properties of Δ and Weyl group. Introduce the Weyl reflections S_β by:

$$S_\beta \vec{x} = \vec{x} - \frac{2(\beta, \vec{x})}{(\beta, \beta)} \beta.$$

  The Weyl reflections form a finite group, which preserves Δ, i.e. $S_\beta \Delta \equiv \Delta$.

Appendix A.1. The root system of $A_r \simeq sl(r+1)$ algebras

The algebras A_r can be represented by $(r+1) \times (r+1)$ matrices whose trace is equal to 0. Their Cartan subalgebras are provided by diagonal matrices $H = \mathrm{diag}\,(h_1, h_2, \dots, h_{r+1})$ whose trace vanishes $\mathrm{tr}\, H = \sum_{j=1}^{r+1} h_j = 0$. Therefore the vector $\vec{h} = (h_1, h_2, \dots, h_{r+1})^T$ dual to H belongs to the $r+1$ dimensional Euclidean space $\mathbb{E}^{r+1}$ which is orthogonal to the vector $\vec{\epsilon} = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_{r+1}$, where $\vec{e}_j$ form the orthonormal basis in $\mathbb{E}^{r+1}$. The Weil generators are provided by the matrices E_{jk} which have only one non-vanishing entry at position j, k equal to 1, i.e.:

$$(E_{jk})_{mn} = \delta_{jk} \delta_{mn}. \tag{A.2}$$

It is easy to check that the commutation relations between the Cartan elements H and the Weyl generators E_{jk} can be written down as:

$$[H, E_{jk}] = (h_j - h_k) E_{jk} = (\vec{h}, \vec{e}_j - \vec{e}_k) E_{jk}, \tag{A.3}$$

It coincides with (A.1) where the root $\alpha = \vec{e}_j - \vec{e}_k$. Thus we find that the root system of A_r consists of the vectors:

$$\Delta_{A_r} \equiv \{\vec{e}_j - \vec{e}_k, \quad j \neq k\} \tag{A.4}$$

Obviously each root is a vector $\vec{e}_j - \vec{e}_k \in \mathbb{E}^{r+1}$ orthogonal to $\vec{\epsilon}$. We will need also to split the root system $\Delta = \Delta_+ \cup \Delta_-$ into positive and negative roots:

$$\Delta_+ = \{\vec{e}_j - \vec{e}_k, \quad j < k\}, \qquad \Delta_- = \{\vec{e}_j - \vec{e}_k, \quad j > k\}, \tag{A.5}$$

i.e. to the positive (resp. negative) roots there correspond upper triangular (resp., lower) triangular matrices. Note that all roots $\beta = \vec{e}_j - \vec{e}_k$ are orthogonal to the vector $\vec{\epsilon}$, so they span r-dimensional subspace of $\mathbb{E}^{r+1}$.

The simple roots of A_r are provided by $\alpha_j = \vec{e}_j - \vec{e}_{j+1}$, $j = 1, \dots, r$. The Dynkin diagram of A_r is:

The basis of the Cartan subalgebra consists of $H_j = E_{jj} - E_{j+1,j+1}$, $j = 1, \dots, r$. The Weyl generators E_α were given above in (A.2).

Remark 12. *For the orthogonal and symplectic Lie algebras we will use slightly different but equivalent definition. Typically the orthogonal algebras $so(P)$ are represented as the algebras of $P \times P$ skew-symmetric matrices, i.e.*

$$X \in so(P) \qquad \text{iff} \qquad X + X^T = 0.$$

In what follows we will use slightly different but equivalent definition, namely:

$$X \in so(N) \qquad \text{iff} \qquad X + S_0 X^T S_0 = 0.$$

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where the matrix S_0 is given by:

$$
\begin{aligned}
P &= 2r+1 \qquad S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k}. \\
P &= 2r \qquad S_0 = \sum_{k=1}^{r} \left((-1)^{k+1} E_{k,2r+1-k} + (-1)^{k+1} E_{2r+1-k,k} \right)
\end{aligned}
\tag{A.6}
$$

Note that $S_0^2 = 1\!\!1$. For the symplectic algebras we have:

$$X \in sp(2r) \qquad \textit{iff} \qquad X + S_1 X^T S_1^{-1} = 0.$$

where

$$S_1 = \sum_{k=1}^{r} \left((-1)^{k+1} E_{k,2r+1-k} - (-1)^{k+1} E_{2r+1-k,k} \right). \tag{A.7}$$

Now $S_1^2 = -1\!\!1$.The advantage of these definitions is, that now the Cartan subalgebras of both $so(P)$ and $sp(2r)$ are represented by diagonal matrices.

Appendix A.2. The root system of $B_r \simeq so(2r+1)$ algebras

Using Remark 12 we find that with the new definition of $so(P)$ the antisymmetry is with respect to the second diagonal. The Cartan subalgebra elements are given by the following $(2r+1) \times (2r+1)$ matrices:

$$H_j = E_{jj} - H_{2r+2-j,2r+2-j}, \qquad H = \sum_{j=1}^{r} h_j H_j. \tag{A.8}$$

The Cartan-Weyl basis in the typical $(2r+1) \times (2r+1)$ representation basis is given by:

$$
\begin{aligned}
E_{\vec{e}_j - \vec{e}_k} &= E_{jk} - (-1)^{k+j} E_{\bar{k},\bar{j}}, \qquad & E_{\vec{e}_j + \vec{e}_k} &= E_{j\bar{k}} - (-1)^{k+j} E_{k,\bar{j}}, \\
E_{\vec{e}_k} &= E_{k,r+1} - (-1)^k E_{r+1,k}, \qquad & H_{\vec{e}_k} &= E_{k,k} - E_{\bar{k},\bar{k}}.
\end{aligned}
\tag{A.9}
$$

where $\bar{k} = 2r + 2 - k$.

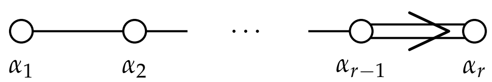
The commutation relations (A.1) show that the root system of B_r is provided by:

$$\Delta_+ = \{\alpha = \vec{e}_k - \vec{e}_j, \quad \beta = \vec{e}_k + \vec{e}_j, \quad \gamma_k = \vec{e}_k\}, \qquad 1 \le k < j \le r. \tag{A.10}$$

The simple roots are given by:

$$\alpha_k = \vec{e}_k - \vec{e}_{k+1}, \quad k = 1, \ldots, r-1, \qquad \alpha_r = \vec{e}_r.$$

The Dynkin diagram is:



Appendix A.3. The root system of $C_r \simeq sp(2r)$ algebras

The Cartan subalgebra elements of $sp(2r)$ are given by the following $(2r) \times (2r)$ matrices:

$$H_j = E_{jj} - H_{2r+1-j,2r+1-j}, \qquad H = \sum_{j=1}^{r} h_j H_j. \tag{A.11}$$

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The Cartan-Weyl basis in the typical $2r \times 2r$ representation basis is given by:

$$\begin{aligned} E_{\vec{e}_j-\vec{e}_k} &= E_{jk} - (-1)^{k+j} E_{\bar{k},\bar{j}}, \qquad & E_{\vec{e}_j+\vec{e}_k} &= E_{j\bar{k}} + (-1)^{k+j} E_{k,\bar{j}}, \\ E_{2\vec{e}_k} &= 2E_{k,\bar{k}}, & H_{\vec{e}_k} &= E_{k,k} - E_{\bar{k},\bar{k}}. \end{aligned} \tag{A.12}$$

where $\bar{k} = 2r + 1 - k$.

The commutation relations (A.1) show that the root system of B_r is provided by:

$$\Delta_+ = \{\alpha = \vec{e}_k - \vec{e}_j, \quad \beta = \vec{e}_k + \vec{e}_j, \quad \gamma_k = 2\vec{e}_k\}, \qquad 1 \leq k < j \leq r. \tag{A.13}$$

The simple roots are given by:

$$\alpha_k = \vec{e}_k - \vec{e}_{k+1}, \quad k = 1, \ldots, r-1, \qquad \alpha_r = 2\vec{e}_r.$$

The Dynkin diagram is:

Appendix A.4. The root system of $D_r \simeq so(2r)$ algebras

Using Remark 12 we find that with the new definition of $so(P)$ the antisymmetry is with respect to the second diagonal. The Cartan subalgebra elements are given by the following $(2r) \times (2r)$ matrices:

$$H_j = E_{jj} - H_{2r+1-j,2r+1-j}, \qquad H = \sum_{j=1}^{r} h_j H_j. \tag{A.14}$$

The Cartan-Weyl basis in the typical $2r \times 2r$ representation basis is given by:

$$\begin{aligned} E_{\vec{e}_j-\vec{e}_k} &= E_{jk} - (-1)^{k+j} E_{\bar{k},\bar{j}}, \qquad & E_{\vec{e}_j+\vec{e}_k} &= E_{j\bar{k}} - (-1)^{k+j} E_{k,\bar{j}}, \\ H_{\vec{e}_k} &= E_{k,k} - E_{\bar{k},\bar{k}}, \end{aligned} \tag{A.15}$$

where $\bar{k} = 2r + 1 - k$.

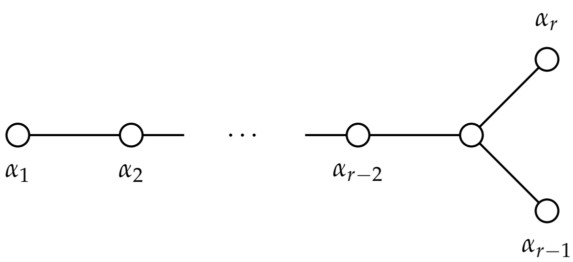
The commutation relations (A.1) show that the root system of B_r is provided by:

$$\Delta_+ = \{\alpha = \vec{e}_k - \vec{e}_j, \quad \beta = \vec{e}_k + \vec{e}_j, \}, \qquad 1 \leq k < j \leq r. \tag{A.16}$$

The simple roots are given by:

$$\alpha_k = \vec{e}_k - \vec{e}_{k+1}, \quad k = 1, \ldots, r-1, \qquad \alpha_r = \vec{e}_{r-1} + \vec{e}_r.$$

The Dynkin diagram is:



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Appendix B. Gauss decompositions

The Gauss decompositions have natural group-theoretical interpretation and can be generalized to any semi-simple Lie algebra:

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda), \tag{B.1}$$

i.e. $T^\pm(\lambda)$, $S^\pm(\lambda)$ and $D^\pm(\lambda)$ are the factors in the Gauss decomposition of $T(\lambda)$. It is well known that if given group element allows Gauss decompositions then its factors are uniquely determined. Below we write down the explicit expressions for the matrix elements of $T^\pm(\lambda)$, $S^\pm(\lambda)$, $D^\pm(\lambda)$ through the matrix elements of $T(\lambda)$:

$$T^-_{pj}(\lambda) = \frac{1}{m^+_j(\lambda)} \left\{ \begin{array}{ccccc} 1, & 2, & \dots, & j-1, & p \\ 1, & 2, & \dots, & j-1, & j \end{array} \right\}^{(j)}_{T(\lambda)}, \tag{B.2}$$

$$\hat{T}^-_{jp}(\lambda) = \frac{(-1)^{j+p}}{m^+_{j-1}(\lambda)} \left\{ \begin{array}{cccccc} 1, & 2, & \dots, & \check{p}, & \dots, & j \\ 1, & 2, & \dots, & p, & \dots, & j-1 \end{array} \right\}^{(j-1)}_{T(\lambda)}, \tag{B.3}$$

$$S^+_{pj}(\lambda) = \frac{(-1)^{p+j}}{m^+_{j-1}(\lambda)} \left\{ \begin{array}{cccccc} 1, & 2, & \dots, & p, & \dots, & j-1 \\ 1, & 2, & \dots, & \check{p}, & \dots, & j \end{array} \right\}^{(j)}_{T(\lambda)}, \tag{B.4}$$

$$\hat{S}^+_{jp}(\lambda) = \frac{1}{m^+_j(\lambda)} \left\{ \begin{array}{ccccc} 1, & 2, & \dots, & j-1, & j \\ 1, & 2, & \dots, & j-1, & p \end{array} \right\}^{(j-1)}_{T(\lambda)}, \tag{B.5}$$

$$T^+_{pj}(\lambda) = \frac{1}{m^-_{n-j+1}(\lambda)} \left\{ \begin{array}{ccccc} p, & j+1, & \dots, & n-1, & n \\ j, & j+1, & \dots, & n-1, & n \end{array} \right\}^{(n-j+1)}_{T(\lambda)}, \tag{B.6}$$

$$\hat{T}^+_{jp}(\lambda) = \frac{(-1)^{p+j}}{m^-_{n-j}(\lambda)} \left\{ \begin{array}{cccccc} j, & j+1, & \dots, & \check{p}, & \dots, & n \\ j+1, & j+2, & \dots, & p, & \dots, & n \end{array} \right\}^{(n-j)}_{T(\lambda)}, \tag{B.7}$$

$$S^-_{pj}(\lambda) = \frac{(-1)^{p+j}}{m^-_{n-j}(\lambda)} \left\{ \begin{array}{cccccc} j+1, & j+2, & \dots, & p, & \dots, & n \\ j, & j+1, & \dots, & \check{p}, & \dots, & n \end{array} \right\}^{(n-j)}_{T(\lambda)}, \tag{B.8}$$

$$\hat{S}^-_{jp}(\lambda) = \frac{1}{m^-_{n-j+1}(\lambda)} \left\{ \begin{array}{ccccc} j, & j+1, & \dots, & n-1, & n \\ p, & j+1, & \dots, & n-1, & n \end{array} \right\}^{(n-j+1)}_{T(\lambda)}, \tag{B.9}$$

where

$$\left\{ \begin{array}{cccc} i_1, & i_2, & \dots, & i_k, \\ j_1, & j_2, & \dots, & j_k \end{array} \right\}^{(k)}_{T(\lambda)} = \det \left| \begin{array}{cccc} T_{i_1 j_1} & T_{i_1 j_2} & \dots T_{i_1 j_k} & \\ T_{i_2 j_1} & T_{i_2 j_2} & \dots T_{i_2 j_k} & \\ \vdots & \vdots & \ddots & \vdots \\ T_{i_k j_1} & T_{i_k j_2} & \dots T_{i_k j_k} & \end{array} \right| \tag{B.10}$$

is the minor of order k of $T(\lambda)$ formed by the rows $i_1, i_2, \dots, i_k$ and the columns $j_1, j_2, \dots, j_k$; by $\check{p}$ we mean that p is missing.

From the formulae above we arrive to the following

Corollary 4. *In order that the group element $T(\lambda) \in SL(n, \mathbb{C})$ allows the first (resp. the second) Gauss decomposition (B.1) is necessary and sufficient that all upper- (resp. lower-) principle minors $m^+_k(\lambda)$ (resp. $m^-_k(\lambda)$) are not vanishing.*

These formulae hold true also if we need to construct the Gauss decomposition of an element of the orthogonal $SO(n)$ group. Here we just note that if $T(\lambda) \in SO(n)$ then

$$S_0(T(\lambda))^T S_0^{-1} = T^{-1}(\lambda), \tag{B.11}$$

where

$$S_0 = \sum_{k=1}^{n_0} (-1)^{k+1}(E_{k,n+1-k} + E_{n+1-k,k}), \quad \text{if} \quad n = 2n_0, \tag{B.12}$$

$$S_0 = \sum_{k=1}^{n_0} (-1)^{k+1}(E_{k,n+1-k} + E_{n+1-k,k}) + (-1)^{n_0} E_{n_0+1,n_0+1}, \quad \text{if} \quad n = 2n_0 + 1.$$

One can check that if $T(\lambda)$ satisfies (B.11) then each of the factors $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ also satisfy (B.11) and thus belong to the same group $\mathfrak{G}$. In addition we have the following interrelations between the principal minors of $T(\lambda)$:

$$\begin{aligned} m_j^{\pm}(\lambda) &= m_{n-j}^{\pm}(\lambda), \qquad \text{for} \quad SO(n), \\ m_j^{\pm}(\lambda) &= m_{n-j}^{\pm}(\lambda), \qquad \text{for} \quad SP(n), \end{aligned} \tag{B.13}$$

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