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Covariant Integral Quantization of the Semi-Discrete SO(3)-Hypercylinder

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Abstract: Covariant integral quantization with rotational symmetry SO(3) is established for the quantum motion on this group manifold, or, alternatively, for Gabor signal analysis on this group. We revisit the action of the related (non-unitary) Weyl-Gabor operator on the Hilbert space of square integrable functions on SO(3) and disclose a set of various properties. By selecting weight functions on the corresponding discrete-continuous phase space, one derives related coherent states and Wigner transform.

Keywords: Covariant Weyl-Heisenberg integral quantization; semi-discrete hypercylinder; coherent states; Weyl-Gabor operator; quantum models on SO(3); Wigner function; phase space portrait

MSC: 46L65; 81S10; 81S30; 81R30

1. Introduction

The group SO(3) offers particular interest in physics as it is the configuration space for the motion of a rigid body fixed in a point. It is also of interest in signal analysis and processing. The rigid rotor is a classic problem in classical and quantum mechanics, describing the dynamics of a rigid body with its center of mass held fixed [1–4]. On the quantum level, it allows a consistent description of the rotational spectra of molecules [5–17]. Moreover, using SO(3) as a configuration manifold leads to several applications including texture analysis [18–20], protein-protein docking [21,22], air tempertature control [23], structure of interest rates (in economics) [24], attitude of rigid body [25–30], quantum information [31] or spherical image analysis [32,33].

This work can be viewed as the direct continuation of a previous ones devoted to the semi-discrete cylinder [34,35]. We also draw our inspiration from the insightful works by Mukunda et al [36–38]. These authors were concerned by the finding of a consistent Wigner function for compact Lie groups, and they illustrate their approach with the example of SU(2), the double covering of SO(3). For more recent related works, see for instance [39,40] and references therein.

Here, we investigate the so-called covariant integral quantization of functions or distributions on the phase space $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$, where $\widehat{\mathrm{SO}(3)}$ is the discrete set $\{(l,m,n)\,,\, l \in \mathbb{N}\,,\, -l \leq m,n \leq l\}$ labelling the matrix elements of the unitary irreducible representations (UIR) of SO(3) with respect to the spherical harmonic Hilbertian basis of $\mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\Omega)$. In Section 2 we briefly introduce the general concept of covariant integral quantization, and in Section 3 we apply it to the quantum description of motion of a particle on the SO(3) manifold. In Section 4 we derive the (non-unitary) Weyl-Gabor operator U acting on the Hilbert space $\mathcal{K} = L^2(\mathrm{SO}(3), \mathrm{d}\mathbf{x})$ of square integrable functions on the manifold SO(3) equipped with its Haar measure. This operator leads to a decomposition of the identity on space $\mathcal{K}$. In Section 5, we first define our quantization tools, namely a weight function ϖ defined on the phase space and the related integral operator M^{ϖ} acting on the representation space. We then define the quantization map which transforms a function or distribution f on the phase space Γ into an operator A^{ϖ}_f acting on $\mathcal{K}$. We compute the quantization of separable functions in position and momentum, in momentum only, and position only. In Section 6 we compute the so-called semi-classical portrait (or lower symbol) of the operator A^{ϖ}_f and study how much they are closed

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to the original f. In Section 7, we give examples of quantum operators obtained through coherent state quantization. In Section 8 we introduce a Wigner function built from what we define as the *squaring rotation operator*. In the concluding section 9, we present some appealing investigations in the continuation of the present work and we give some insights about the application of our formalism to the analysis of signals defined on the manifold SO(3). Interesting formulae are given in Appendix A.

2. Resolution of the identity as the common guideline

Here we give an outline of integral quantization. Detailed presentations can be found in [41,42], and more recently in [43–45] with references therein.

Given a measure space (X, μ) and a (separable) Hilbert space $\mathcal{K}$, an operator-valued function

$$X \ni x \mapsto \mathsf{M}(x) \text{ acting in } \mathcal{K}\,,$$

resolves the identity operator $\mathbb{1}$ in $\mathcal{K}$ with respect the measure μ if

$$\int_X \mathsf{M}(x)\,\mathrm{d}\mu(x) = \mathbb{1} \tag{1}$$

holds in a weak sense.

In Signal Analysis, *analysis* and *reconstruction* are grounded in the application of (1) on a signal, *i.e.*, a vector in $\mathcal{K}$

$$\mathcal{K} \ni |s\rangle \overset{\text{reconstruction}}{=} \int_X \overbrace{\mathsf{M}(x)|s\rangle}^{\text{analysis}}\,\mathrm{d}\mu(x)\,.$$

In quantum formalism, *integral quantization* is grounded in the linear map of a function on X to an operator in $\mathcal{K}$

$$f(x) \mapsto \int_X f(x)\mathsf{M}(x)\,\mathrm{d}\mu(x) = A_f\,, \quad 1 \mapsto \mathbb{1}\,.$$

2.1. *Probabilistic content of integral quantization: semi-classical portraits*

If the operators $\mathsf{M}(x)$ in

$$\int_X \mathsf{M}(x)\,\mathrm{d}\mu(x) = \mathbb{1}\,, \tag{2}$$

are nonnegative, *i.e.*, $\langle\phi|\mathsf{M}(x)|\phi\rangle \geq 0$ for all $x \in X$, one says that they form a (normalised) positive operator-valued measure (POVM) on X.

If they are further unit trace-class, *i.e.* $\mathrm{tr}(\mathsf{M}(x)) = 1$ for all $x \in X$, *i.e.*, if the $\mathsf{M}(x)$'s are density operators, then the map

$$f(x) \mapsto \check{f}(x) := \mathrm{tr}(\mathsf{M}(x)A_f) = \int_X f(x')\,\mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x'))\,\mathrm{d}\mu(x') \tag{3}$$

is a local averaging of the original $f(x)$ (which can very singular, like the Dirac defined in (8) below) with respect to the probability distribution on X,

$$x' \mapsto \mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x'))\,. \tag{4}$$

This averaging, or semi-classical portrait of the operator A_f, is in general a regularisation, depending of course on the topological nature of the measure space (X, μ) and the functional properties of the $\mathsf{M}(x)$'s.

2.2. Classical limit

Consider a set of parameters $\boldsymbol{\kappa}$ and corresponding families of POVM $\mathsf{M}_{\boldsymbol{\kappa}}(x)$ solving the identity

$$\int_X \mathsf{M}_{\boldsymbol{\kappa}}(x)\, \mathrm{d}\mu(x) = \mathbb{1}\,, \tag{5}$$

One says that the classical limit $f(x)$ holds at $\boldsymbol{\kappa}_0$ if

$$\check{f}_{\boldsymbol{\kappa}}(x) := \int_X f(x')\, \mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x'))\, \mathrm{d}\mu(x') \to f(x) \quad \text{as} \quad \boldsymbol{\kappa} \to \boldsymbol{\kappa}_0\,, \tag{6}$$

where the convergence $\check{f} \to f$ is defined in the sense of a certain topology.

Otherwise said, $\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x'))$ tends to

$$\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x')) \to \delta_x(x') \tag{7}$$

where δ_x is a Dirac measure with respect to μ,

$$\int_X f(x')\, \delta_x(x')\, \mathrm{d}\mu(x') = f(x)\,. \tag{8}$$

Of course, these definitions should be given a rigorous mathematical sense, and nothing guarantees the existence of such a limit.

3. Overview: Scalar fields on the rotation group SO(3), Fourier and Gabor transform

3.1. Quantum formalism on SO(3)

An element $\mathbf{x}$ of SO(3) can be parametrised in several ways.

a) In the Euler angles parametrization with ZYZ convention, α and γ are rotation angles about the 3rd axis and β is a rotation angle about the 2nd axis, with $\alpha \in [0, 2\pi]$, $\beta \in [0, \pi]$, and $\gamma \in [0, 2\pi]$. The corresponding rotation matrix reads en terms of these one-dimensional matrices as:

$$\mathrm{R}(\alpha, \beta, \gamma) = \mathrm{R}_3(\alpha)\mathrm{R}_2(\beta)\mathrm{R}_3(\gamma) \equiv \mathbf{x}(\alpha, \beta, \gamma)\,. \tag{9}$$

The related (non normalised) Haar measure is given by

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}(\alpha, \beta, \gamma) = \sin\beta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\,, \tag{10}$$

which yields $\mathrm{Vol}(\mathrm{SO}(3)) = 8\pi^2$.

b) In the axis-angle parametrization, $\omega \in [0, 2\pi)$ is the anticlockwise (or right-hand rule) rotation angle about the oriented axis $\hat{\mathbf{n}} \in \mathbb{S}^2$ determined by the usual angular spherical coordinates (θ, φ), $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$.

$$\mathbf{x}(\omega, \theta, \varphi) \equiv \mathbf{x}(\omega, \hat{\mathbf{n}}(\theta, \varphi))\,, \tag{11}$$
$$\hat{\mathbf{n}}(\theta, \phi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,.$$

The matrix representation of $\mathbf{x}(\omega, \hat{\mathbf{n}})$ is given by

$$\mathbf{x}(\omega, \hat{\mathbf{n}}) = \cos\omega\, \mathbb{1}_3 + (1 - \cos\omega)\, \hat{\mathbf{n}}\, {}^t\hat{\mathbf{n}} + \sin\omega\, \hat{\mathbf{n}}\times = \exp(\omega\hat{\mathbf{n}}\times)\,, \tag{12}$$

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where $\hat{\mathbf{n}}\,{}^t\hat{\mathbf{n}}$ is the orthogonal projector on $\hat{\mathbf{n}}$, $\hat{\mathbf{n}}\,{}^t\hat{\mathbf{n}}\,\mathbf{v} = \hat{\mathbf{n}}\cdot\mathbf{v}$ and $\hat{\mathbf{n}}\times$ is linearly acts on $\mathbb{R}^3$ as

$$\hat{\mathbf{n}}\times\mathbf{v} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_z & n_x & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}. \tag{13}$$

The related unnormalized Haar measure is:

$$\mathrm{d}\mathbf{x}(\omega,\hat{\mathbf{n}}) = (1-\cos\omega)\mathrm{d}\omega\,\mathrm{d}\hat{\mathbf{n}} = (1-\cos\omega)\sin\theta\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}\varphi\,. \tag{14}$$

We now consider the Hilbert space $\mathcal{K} = L^2(\mathrm{SO}(3),\mathrm{d}\mathbf{x})$ of square integrable functions ψ on the rotation group SO(3), that is, functions satisfying the condition,

$$\langle\psi|\psi\rangle \equiv \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,|\psi(\mathbf{x})|^2 < \infty\,. \tag{15}$$

The group multiplication on the left induces the unitary action of the operator L on $\mathcal{K}$,

$$\mathrm{SO}(3)\ni\mathbf{q}\mapsto\mathsf{L}(\mathbf{q})\,,\quad \psi\in\mathcal{K}\,,\quad (\mathsf{L}(\mathbf{q}))\psi)\,(\mathbf{x}) = \psi(\mathbf{q}^{-1}\mathbf{x})\,. \tag{16}$$

The 3 basic generators (angular momentum components) of this action in the Euler angles parametrization (9) are given by [46]:

$$L_x = -\mathrm{i}\left(-\cos\alpha\,\cot\beta\,\frac{\partial}{\partial\alpha} - \sin\alpha\,\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right), \tag{17}$$

$$L_y = -\mathrm{i}\left(-\sin\alpha\,\cot\beta\,\frac{\partial}{\partial\alpha} + \cos\alpha\,\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right), \tag{18}$$

$$L_z = -\mathrm{i}\frac{\partial}{\partial\alpha}\,. \tag{19}$$

One can decompose any function $\psi\in\mathcal{K}$ using the Wigner $\mathcal{D}$-functions $\mathcal{D}^l_{m\,n}$, with $l\in\mathbb{N}$, and $m,n = -l,-l+1,...,0,...,l-1,l$. Hence, these functions form an Hilbertian basis of $\mathcal{K}$, and the set of triplets of integers

$$\{(l,m,n)\,,\,l\in\mathbb{N}\,,\,m,n = -l,-l+1,...,0,...,l-1,l\} \equiv \widehat{SO(3)} \tag{20}$$

form the Fourier dual of SO(3). The Wigner $\mathcal{D}$-functions $\mathcal{D}^l_{m\,n}$ are matrix elements of the irreducible unitary representation of SO(3) with respect to the Hilbertian basis of normalised spherical harmonics $Y_{lm}(\theta,\varphi)$ in $\mathcal{H} = L^2(\mathbb{S}^2,\mathrm{d}\hat{\mathbf{n}})$. Our convention concerning the latter is that one given by Edmonds [46]:

$$\begin{gathered} Y_{lm}(\theta,\varphi) = (-1)^m\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} P^l_m(\cos\theta)\,e^{\mathrm{i}m\varphi}\,, \\ \int_{\mathbb{S}^2}\overline{Y_{lm}(\theta,\varphi)}Y_{l'm'}(\theta,\varphi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{ll'}\,\delta_{mm'}\,, \end{gathered} \tag{21}$$

where the $P^l_m(x)$ are the associated Legendre functions [47].

In the Euler angle parametrization the Wigner $\mathcal{D}$-functions appear in the expansion [5,46]:

$$Y_{ln}(\mathbf{x}^{-1}(\alpha,\beta,\gamma)\cdot(\theta,\varphi)) = \sum_{m=-l}^{l}\mathcal{D}^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))Y_{lm}(\theta,\varphi) \tag{22}$$

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and are given by

$$\mathcal{D}^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma)) = \int_{\mathbb{S}^2} \overline{Y_{lm}(\theta,\varphi)} Y_{l'n}(\mathbf{x}^{-1}(\alpha,\beta,\gamma)\cdot(\theta,\varphi))\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = e^{\mathrm{i}m\alpha} d^l_{m\,n}(\beta) e^{\mathrm{i}n\gamma}\,. \tag{23}$$

In this expression the functions $d^l_{m\,n}$ are expressed in terms of the Jacobi polynomials for $m-n\geq 0$ and $m+n\geq 0$ [47]:

$$d^l_{m\,n}(\beta) = \left[\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!}\right]^{1/2} \left(\cos\frac{\beta}{2}\right)^{m+n} \left(\sin\frac{\beta}{2}\right)^{m-n} P^{(m-n,m+n)}_{l-m}(\cos\beta)\,. \tag{24}$$

The other cases give similar expressions after using symmetries of indexes for these polynomials. More precisely, from the general expression of the Jacobi polynomials

$$P^{(\mu,\nu)}_n(x) = 2^{-n}\sum_r \binom{n+\mu}{r}\binom{n+\nu}{n-r}(x+1)^r(x-1)^{n-r}\,, \tag{25}$$

with

$$P^{(\mu,\nu)}_n(-x) = (-1)^n P^{(\nu,\mu)}_n(x)\,,\quad P^{(-l,\nu)}_n(x) = \frac{\binom{n+\nu}{l}}{\binom{n}{l}}\left(\frac{x-1}{2}\right)^l P^{(l,\nu)}_{n-l}(x)\,, \tag{26}$$

one can derive the Fourier series expansion of the Wigner d-functions (not trivial!):

$$d^l_{m\,n}(\beta) = \mathrm{i}^{n-m}\sum_{m'=-l}^{m'=l} \Delta^l_{m'm}\Delta^l_{m'n} e^{\mathrm{i}\,m'\beta} \tag{27}$$

where $\Delta^l_{m\,n} = d^l_{m\,n}(\pi/2)$ [48,49]. In terms of its matrix elements, the unitarity of the $\mathcal{D}$-matrix at fixed l read

$$\sum_n \mathcal{D}^l_{m\,n}(\mathbf{x})\,\mathcal{D}^l_{n\,m'}(\mathbf{x}^{-1}) = \sum_n \mathcal{D}^l_{m\,n}(\mathbf{x})\,\overline{\mathcal{D}^l_{m'\,n}(\mathbf{x})} = \delta_{mm'}\,, \tag{28}$$

and so

$$\sum_n |\mathcal{D}^l_{m\,n}(\mathbf{x})|^2 = 1\,, \tag{29}$$

while the orthogonality relations obeyed by these $\mathcal{D}$-matrix elements read

$$\left\langle \mathcal{D}^l_{m\,n}|\mathcal{D}^{l'}_{m'\,n'}\right\rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\mathcal{D}^l_{m\,n}(\mathbf{x})}\mathcal{D}^{l'}_{m'\,n'}(\mathbf{x}) = \frac{8\pi^2}{2l+1}\delta_{l\,l'}\delta_{m\,m'}\delta_{n\,n'}\,, \tag{30}$$

Let us introduce the Dirac distribution $\delta_\mathrm{e}(\mathbf{x})$ on SO(3) as having its support at the group identity $\mathrm{e}\equiv \mathbb{1}_3$

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta_\mathrm{e}(\mathbf{x}) = 1 \quad\text{and}\quad \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta_\mathrm{e}(\mathbf{x})\,f(\mathbf{x}) = f(\mathrm{e})\,, \tag{31}$$

for all test functions in some dense subspace of $\mathcal{K}$, e.g., infinitely differentiable. For any $\mathbf{q}\in \mathrm{SO}(3)$ and from the SO(3) invariance of the measure, we have

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta_\mathrm{e}(\mathbf{x})\,f(\mathbf{q}\mathbf{x}) = f(\mathbf{q}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta_\mathrm{e}(\mathbf{q}^{-1}\mathbf{x})\,f(\mathbf{x})\,, \tag{32}$$

which entails the definition of the Dirac distribution $\delta_\mathbf{q}$ with support at any point $\mathbf{q}\in\mathrm{SO}(3)$:

$$\delta_\mathrm{e}(\mathbf{q}^{-1}\mathbf{x}) \equiv \delta_\mathbf{q}(\mathbf{x}) \Rightarrow \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta_\mathbf{q}(\mathbf{x})\,f(\mathbf{x}) = f(\mathbf{q})\,. \tag{33}$$

Using Dirac notations, we introduce kets $|\mathbf{x}\rangle$ and their dual bras $\langle\mathbf{x}|$, both labeled by the points $\mathbf{x} \in \mathrm{SO}(3)$, as obeying the following orthogonality and normalization (in the distributional sense) and resolution of the unity in $\mathcal{K}$

$$\langle\mathbf{x}|\mathbf{x}'\rangle = \delta_{\mathbf{x}'}(\mathbf{x}) = \delta_{\mathbf{x}}(\mathbf{x}') \equiv \delta(\mathbf{x},\mathbf{x}')\,, \tag{34}$$

$$\mathbb{1} = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\mathbf{x}\rangle\langle\mathbf{x}| \tag{35}$$

From its construction, we derive the invariance property of the Dirac distribution on SO(3):

$$\delta(\mathbf{q}\mathbf{x},\mathbf{q}\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}') \ \ \forall \mathbf{x} \in \mathrm{SO}(3)\,. \tag{36}$$

With these notations, one can write for any $\psi \in \mathcal{K}$ (or for suitably defined distributions)

$$\psi(\mathbf{x}) \equiv \langle\mathbf{x}|\psi\rangle\,. \tag{37}$$

With this formalism at hand the completeness of the Hilbertian basis

$$\left\{\mathcal{D}^{l}_{m\,n}\,,\ (l,m,n) \in \widehat{\mathrm{SO}(3)}\right\} \tag{38}$$

in $\mathcal{K}$ reads:

$$\frac{2l+1}{8\pi^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}\overline{\mathcal{D}^{l}_{m\,n}(\mathbf{x})}\mathcal{D}^{l}_{m\,n}(\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')\,, \tag{39}$$

On the other hand, as matrix elements of the unitary operator $\mathsf{L}(\mathbf{q})$ (irreducibly acting in the $(2l+1)$-dimensional subspace $\mathcal{H}_l$ of $\mathcal{H} = \oplus_{l=0}^{\infty}\mathcal{H}_l$) they are uniformly bounded by

$$|\mathcal{D}^{l}_{m\,n}(\mathbf{x})| \leq 1\,. \tag{40}$$

We now define the SO(3) *Fourier transform* of $\psi \in \mathcal{K}$ as the orthogonal projection of ψ on the basis $\left\{\mathcal{D}^{l}_{m\,n}\right\}$, that is its Fourier coefficient:

$$\hat{\psi}_{j\,m\,n} = \sqrt{\frac{2l+1}{8\pi^2}}\langle\mathcal{D}^{l}_{mn}|\psi\rangle = \sqrt{\frac{2l+1}{8\pi^2}}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,\overline{\mathcal{D}^{l}_{mn}(\mathbf{x})}\psi(\mathbf{x})\,, \tag{41}$$

and its inverse is consistently the Fourier series expansion:

$$\psi(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}\mathcal{D}^{l}_{m\,n}(\mathbf{x})\hat{\psi}_{l\,m\,n}\,. \tag{42}$$

3.2. Phase space formalism

Inspired by the Mukunda et al's approach [36,37] we now consider the rotation $\mathbf{x}$ as an element of the configuration space SO(3) and the triple (l,m,n) of its unitary ($\sim$ Fourier) dual $\widehat{\mathrm{SO}(3)}$ as *momentum* or *frequency* variables. Hence, we denote in the following

$$\mathbf{p} \equiv (l,m,n) \in \widehat{\mathrm{SO}(3)}\,, \quad \langle\mathbf{p}|\psi\rangle \equiv \hat{\psi}_{l\,m\,n}\,, \tag{43}$$

with orthogonality relations and resolution of the identity

$$\langle\mathbf{p}|\mathbf{p}'\rangle = \delta_{\mathbf{p}\,\mathbf{p}'}\,, \quad \mathbb{1} = \sum_{\mathbf{p}}|\mathbf{p}\rangle\langle\mathbf{p}|\,. \tag{44}$$

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With these shortened notations, we write the Hilbertian basis as:

$$e_{\mathbf{p}}(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x})\,. \tag{45}$$

The completeness relation (39), Fourier transform (41), and its inverse (42), take the simplest forms:

$$\sum_{\mathbf{p}} \overline{e_{\mathbf{p}}(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')\,, \tag{46}$$

$$\hat{\psi}_{j\,m\,n} \equiv \hat{\psi}(\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})}\psi(\mathbf{x}) \equiv \mathcal{F}\left[\psi\right](\mathbf{p})\,, \tag{47}$$

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}} e_{\mathbf{p}}(\mathbf{x})\hat{\psi}(\mathbf{p}) \equiv \overline{\mathcal{F}}\left[\hat{\psi}\right](\mathbf{x})\,. \tag{48}$$

3.3. SO(3)-Weyl-Gabor operator, coherent states and Gabor transform

3.3.1. SO(3)-Weyl-Gabor operator

Besides the unitary representation operator L introduced in (16) we define the non-unitary modulation operator by the momentum variable $\mathbf{p} = (l, m, n)$ as the non-Hermitian bounded multiplication operator:

$$(\mathsf{E}_{\mathbf{p}}\psi)(\mathbf{x}) = e_{\mathbf{p}}(\mathbf{x})\psi(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x})\psi(\mathbf{x})\,, \quad \|\mathsf{E}_{\mathbf{p}}\|_{\mathcal{K}} = \sqrt{\frac{2l+1}{8\pi^2}}\,. \tag{49}$$

Note that it is the sum of unitary operators due to (23) and (27):

$$(\mathsf{E}_{\mathbf{p}}\psi)(\mathbf{x}) = \mathrm{i}^{n-m}\sum_{m'=-l}^{m'=l} \Delta^l_{m'm}\Delta^l_{m'n}e^{\mathrm{i}\,(m\alpha+m'\beta+n\gamma)}\,\psi(\mathbf{x}(\alpha,\beta,\gamma))\,. \tag{50}$$

Its adjoint $\mathsf{E}^{\dagger}$ is defined by:

$$(\mathsf{E}^{\dagger}_{\mathbf{p}}\psi)(x) = \overline{e_{\mathbf{p}}(\mathbf{x})}\psi(\mathbf{x}) == \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{n\,m}(\mathbf{x}^{-1})\psi(\mathbf{x}) = e_{\mathbf{p}^{\mathrm{t}}}(\mathbf{x}^{-1})\,, \tag{51}$$

where the transpose $\mathbf{p}^{\mathrm{t}}$ of $\mathbf{p}$ means

$$\mathbf{p}^{\mathrm{t}} = (l,m,n)^{\mathrm{t}} = (l,n,m)\,. \tag{52}$$

Combining these operators leads to the (non-unitary) "SO(3)-Weyl-Gabor" operator

$$\mathsf{U}(\mathbf{q},\mathbf{p}) := \mathsf{E}_{\mathbf{p}}\mathsf{L}(\mathbf{q}) = e_{\mathbf{p}}(\cdot)\mathsf{L}(\mathbf{q}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\cdot)\mathsf{L}(\mathbf{q})\,, \tag{53}$$

acting on $\mathcal{K}$ as

$$(\mathsf{U}(\mathbf{q},\mathbf{p})\psi)(\mathbf{x}) = e_{\mathbf{p}}(\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{x})\,. \tag{54}$$

Its adjoint $\mathsf{U}^{\dagger} = \mathsf{L}^{\dagger}(\mathbf{q})\mathsf{E}^{\dagger}_{\mathbf{p}}$ acts on $\mathcal{K}$ as:

$$(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\psi)(\mathbf{x}) = \overline{e_{\mathbf{p}}(\mathbf{qx})}\psi(\mathbf{qx}) = \sqrt{\frac{2l+1}{8\pi^2}}\overline{\mathcal{D}^l_{m\,n}(\mathbf{qx})}\psi(\mathbf{qx})\,. \tag{55}$$

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We then have the following actions on $\psi \in \mathcal{K}$:

$$\left(\mathsf{U}^{\dagger}(\mathbf{q}',\mathbf{p}')\mathsf{U}(\mathbf{q},\mathbf{p})\psi\right)(\mathbf{x}) = \overline{e_{\mathbf{p}'}(\mathbf{q}'\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{q}'\mathbf{x})\psi(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x})\,, \tag{56}$$

$$\left(\mathsf{U}(\mathbf{q},\mathbf{p})\mathsf{U}(\mathbf{q}',\mathbf{p}')\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\psi\right)(\mathbf{x}) = e_{\mathbf{p}}(\mathbf{x})e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})}\psi(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})\,. \tag{57}$$

In particular, the lack of unitarity of U and $\mathsf{U}^{\dagger}$ is obvious from the fact that $\mathsf{U}\,\mathsf{U}^{\dagger}$ and $\mathsf{U}^{\dagger}\,\mathsf{U}$ are nontrivial bounded multiplication operators:

$$\left(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\,\mathsf{U}(\mathbf{q},\mathbf{p})\,\psi\right)(\mathbf{x}) = |e_{\mathbf{p}}(\mathbf{q}\mathbf{x})|^2\,\psi(\mathbf{x})\,, \quad \left(\mathsf{U}(\mathbf{q},\mathbf{p})\,\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\,\psi\right)(\mathbf{x}) = |e_{\mathbf{p}}(\mathbf{x})|^2\,\psi(\mathbf{x}) \tag{58}$$

3.3.2. Coherent states

Let us pick a normalised vector ϕ in $\mathcal{K}$ and consider the family of family of states labelled by the elements of the phase space $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$:

$$|\mathbf{q},\mathbf{p}\rangle_{\phi} := \mathsf{U}(\mathbf{q},\mathbf{p})\phi\,, \quad \langle\mathbf{x}|\mathbf{q},\mathbf{p}\rangle_{\phi} = e_{\mathbf{p}}(\mathbf{x})\phi(\mathbf{q}^{-1}\mathbf{x})\,. \tag{59}$$

These states will be named Γ-coherent states with *fiducial* vector ϕ for the reason that they solve the identity in $\mathcal{K}$, as asserted in the following.

Proposition 3.1. *Let us equip the phase space Γ with the measure*

$$\int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p} := \sum_{\mathbf{p}=(lmn)} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,. \tag{60}$$

Then the states $|\mathbf{q},\mathbf{p}\rangle_{\phi}$ resolve the identity $\mathbb{1}$ in $\mathcal{K}$ with respect to this measure:

$$\mathbb{1} = \int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,|\mathbf{q},\mathbf{p}\rangle_{\phi\phi}\langle\mathbf{q},\mathbf{p}|\,. \tag{61}$$

Proof. Pick $\psi, \psi' \in \mathcal{K}$ and compute

$$\begin{aligned}\langle\psi|\left[\int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,|\mathbf{q},\mathbf{p}\rangle_{\phi\phi}\langle\mathbf{q},\mathbf{p}|\right]|\psi'\rangle &= \int_{\Gamma} \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}\,\langle\psi|\mathbf{q},\mathbf{p}\rangle_{\phi\phi}\langle\mathbf{q},\mathbf{p}|\psi'\rangle \\ &= \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\,\psi'(\mathbf{x}')\overline{e_{\mathbf{p}}(\mathbf{x})} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\phi\left(\mathbf{q}^{-1}\mathbf{x}\right)\,\overline{\phi\left(\mathbf{q}^{-1}\mathbf{x}'\right)}\,.\end{aligned}$$

First performing the sum on $\mathbf{p} = (l,m,n)$ yields $\delta(\mathbf{x},\mathbf{x}')$ by application of (46). By integrating the latter and using the invariance of the Haar measure $\mathrm{d}\mathbf{q}$ we end with

$$\begin{aligned}&\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}\,\psi'(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q})}\,\phi(\mathbf{q}) \\ &= \langle\psi|\psi'\rangle\,\|\phi\|^2 = \langle\psi|\psi'\rangle\,.\end{aligned}$$

□

3.3.3. Gabor transform

The Gabor transform, denoted by $\mathcal{L}_{\phi}$, maps $\psi \in \mathcal{K}$ to a function $\Psi(\mathbf{q},\mathbf{p})$ in the Hilbert space $\mathfrak{K} = L^2(\Gamma, \mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p})$ of square integrable functions on the phase space Γ equipped with the measure $\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{p}$:

$$\mathcal{L}_{\phi} : \psi \mapsto \Psi\,, \quad \Psi(\mathbf{q},\mathbf{p}) = (\mathcal{L}_{\phi}\psi)(\mathbf{q},\mathbf{p}) = {}_{\phi}\langle\mathbf{p},\mathbf{q}|\psi\rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}}(\mathbf{x})}\,\overline{\phi(\mathbf{q}^{-1}\mathbf{x})}\psi(\mathbf{x}) \tag{62}$$

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Proposition 3.2. *The map $\mathcal{L}_\phi$ satisfies the following properties:*

(i) it is an isometry:

$$||\psi||^2 = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\psi(\mathbf{x})|^2 = \int_\Gamma \mathrm{d}\mathbf{q}\, \mathrm{d}\mathbf{p}\, |\Psi(\mathbf{q},\mathbf{p})|^2 = ||\Psi||^2 , \tag{63}$$

(ii) it can be inverted on its range:

$$\psi(\mathbf{x}) = \int_\Gamma \mathrm{d}\mathbf{q}\, \mathrm{d}\mathbf{p}\, \Psi(\mathbf{q},\mathbf{p}) \langle \mathbf{x}|\mathbf{q},\mathbf{p}\rangle_\phi , \tag{64}$$

(iii) the closure of the range of $\mathcal{L}_\phi$ is a reproducing kernel Hilbert space:

$$\begin{aligned} (\mathcal{L}_\phi\psi)(\mathbf{q},\mathbf{p}) = \Psi(\mathbf{q},\mathbf{p}) &= \int_\Gamma \mathrm{d}\mathbf{q}'\, \mathrm{d}\mathbf{p}'\, {}_\phi\langle \mathbf{q},\mathbf{p}|\mathbf{q}',\mathbf{p}'\rangle_\phi \Psi(\mathbf{q}',\mathbf{p}') \\ &\equiv \int_\Gamma \mathrm{d}\mathbf{q}'\, \mathrm{d}\mathbf{p}'\, K_\phi(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\, \Psi(\mathbf{q}',\mathbf{p}') \end{aligned} \tag{65}$$

Proof. All statements are straighforward consequence of the resolution of the identity (61). □

Proposition 3.3. *We have the following trace formulas for the SO(3)-Weyl-Gabor operator:*

$$\mathrm{Tr}[\mathsf{U}(\mathbf{q},\mathbf{p})] = \delta_{\mathbf{p}\,\mathbf{0}}\, \delta_{\mathrm{e}}(\mathbf{q}) , \quad \mathbf{p} = (l,m,n) , \quad \mathbf{0} = (0,0,0) , \tag{66}$$

$$\mathrm{Tr}\left[\mathsf{U}^\dagger(\mathbf{q}',\mathbf{p}')\mathsf{U}(\mathbf{q},\mathbf{p})\right] = \delta_{\mathbf{p}\mathbf{p}'}\delta(\mathbf{q},\mathbf{q}') . \tag{67}$$

Proof. For (66), using (46) and the orthonormality of the $e_\mathbf{p}$'s,

$$\mathrm{Tr}[\mathsf{U}(\mathbf{q},\mathbf{p})] = \sum_{\mathbf{p}'} \int_{SO(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}'}(\mathbf{x})}\, e_\mathbf{p}(\mathbf{x}) e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x}) = \delta_\mathrm{e}(\mathbf{q}^{-1})\delta_{\mathbf{p}\mathbf{0}} = \delta_\mathrm{e}(\mathbf{q})\delta_{\mathbf{p}\mathbf{0}} .$$

For (67):

$$\begin{aligned} \mathrm{Tr}[\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})\mathsf{U}(\mathbf{q}',\mathbf{p}')] &= \sum_\mathbf{b} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_\mathbf{b}(\mathbf{x})} \left(\mathsf{E}^\dagger_\mathbf{p} \mathsf{L}(\mathbf{q}^{-1}\mathbf{q}')\mathsf{E}_{\mathbf{p}'} e_\mathbf{b}\right)(\mathbf{x}) \\ &= \sum_\mathbf{b} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_b(\mathbf{x})}\, \overline{e_\mathbf{p}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x}) e_\mathbf{b}(\mathbf{q}'^{-1}\mathbf{q}\mathbf{x}) \\ &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_\mathbf{p}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x})\delta(\mathbf{x}^{-1}\mathbf{q}'^{-1}\mathbf{q}\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_\mathbf{p}(\mathbf{x})}\, e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{q}'\mathbf{x})\delta(\mathbf{q}'^{-1}\mathbf{q}) \\ &= \delta_{\mathbf{p}\mathbf{p}'}\, \delta(\mathbf{q},\mathbf{q}') . \end{aligned}$$

□

3.4. Example of fiducial vectors and coherent states

As seen above, for any square integrable function on SO(3), including the completely non localized function $\phi = 1$ on the manifold SO(3), is a fiducial vector, our coherent states form the family $\{\langle \mathbf{x}|\mathbf{q},\mathbf{p}\rangle_\phi\}$ of transported ϕ through the SO(3)-Weyl-Gabor operator. It is, of course, interesting to consider fiducial vectors that are well "localized" in position and momentum. Although it is not the main purpose of this paper, we present a few fiducial vectors that can be of interest. Some of these examples are extracted from signal processing on SO(3) as related to probability densities.

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1. Eigenfunctions of certain operators [50]. The first example is the free rotor fiducial vector which is the eigenfunction of $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$.

$$e_{\mathbf{p}}(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,. \tag{68}$$

The second example is the highest fiducial vector for SO(3) which is cancelled by $L_+ = L_x + \mathrm{i}\,L_y$, that is:

$$e_{l\,l\,l}(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathcal{D}^l_{l\,l}(\mathbf{x}(\alpha,\beta,\gamma))\,. \tag{69}$$

2. Some radial fiducial vectors. Below, we give examples of fiducial vectors depending only on

$$|\mathbf{x}(\alpha,\beta,\gamma)| := \arccos\left(\frac{\mathrm{Tr}[\mathbf{x}(\alpha,\beta,\gamma)]-1}{2}\right), \tag{70}$$

which defines a metric on SO(3). Details about this metric can be found in [22],

(a) The κ-dependent von Mises-Fisher Kernel fiducial vectors ϕ [22], their derivatives with respect to β and α, and their difference at two different κ:

$$\phi(\mathbf{x}(\alpha,\beta,\gamma)) = \frac{e^{\kappa\cos(|\,\mathbf{x}(\alpha,\beta,\gamma)\,|)}}{I_0(\kappa)-I_1(\kappa)} = \frac{e^{\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa)-I_1(\kappa)}, \quad \kappa > 0\,. \tag{71}$$

$$\begin{aligned}\phi^{(1)}_{\beta}(\mathbf{x}(\alpha,\beta,\gamma)) &= -\frac{K}{2}\,\sin\frac{\beta}{2}\,\cos\frac{\alpha+\gamma}{2}\,\frac{e^{\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa)-I_1(\kappa)}, \quad \kappa > 0\,.\\ \phi^{(1)}_{\alpha}(\mathbf{x}(\alpha,\beta,\gamma)) &= -\frac{K}{2}\,\cos\frac{\beta}{2}\,\sin\frac{\alpha+\gamma}{2}\,\frac{e^{\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa)-I_1(\kappa)}, \quad \kappa > 0\,.\end{aligned} \tag{72}$$

$$\phi_{dov}(\mathbf{x}(\alpha,\beta,\gamma)) = \frac{e^{\kappa_1\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa_1)-I_1(\kappa_1)} - \frac{e^{\kappa_2\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})}}{I_0(\kappa_2)-I_1(\kappa_2)}, \quad \kappa_1,\kappa_2 > 0\,. \tag{73}$$

where I_n, $n \in \mathbb{N}$ denotes the modified Bessel functions of first kind.

In Appendix B, we give plots of these fiducial vectors in α and β variables at a fixed γ and for a few values of κ (Figures B.1 and B.2).

(b) The Abel-Poisson fiducial vector ϕ [22]:

$$\begin{aligned}\phi(\mathbf{x}(\alpha,\beta,\gamma)) &= \frac{1}{2}\left[\frac{1-\kappa^2}{1+2\kappa\cos(|\,\mathbf{x}(\alpha,\beta,\gamma)\,|)+\kappa^2} - \frac{1-\kappa^2}{1-2\kappa\cos(|\,\mathbf{x}(\alpha,\beta,\gamma)\,|)+\kappa^2}\right]\\ &= \frac{1}{2}\left[\frac{1-\kappa^2}{1+2\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})\,|)+\kappa^2} - \frac{1-\kappa^2}{1-2\kappa\cos(\frac{\beta}{2})\cos(\frac{\alpha+\gamma}{2})+\kappa^2}\right], \quad \kappa > 0\,.\end{aligned} \tag{74}$$

4. Quantization operators and the quantization map

Following previous works [43–45], we pick a function ϖ, called weight (but not necessarily positive), on the phase space Γ. We then define the operator M^{ϖ} by

$$\mathsf{M}^{\varpi} = \sum_{\mathbf{p}}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}\,\varpi(\mathbf{q},\mathbf{p})\,\mathsf{U}(\mathbf{q},\mathbf{p})\,, \tag{75}$$

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and we choose the weight such that the operator M^{ϖ} is bounded and symmetric, *i.e.*, is self-adjoint on the Hilbert $\mathcal{K} = L^2(\mathrm{SO}(3), \mathrm{d}\mathbf{x})$ of "physical states".

In what follows, we compute the kernel of this operator and the related trace.

Proposition 4.1. *With the assumption that the weight ϖ has been chosen such that the operator M^{ϖ}, defined by (75), is bounded:*

(i) The operator M^{ϖ} is the integral operator:

$$(\mathsf{M}^{\varpi}\psi)(\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}' \, \mathcal{M}^{\varpi}(\mathbf{x}, \mathbf{x}')\psi(\mathbf{x}')\,, \tag{76}$$

where the kernel $\mathcal{M}^{\varpi}(\mathbf{x}, \mathbf{x}')$ is given by:

$$\mathcal{M}^{\varpi}(\mathbf{x}, \mathbf{x}') = \sum_{\mathbf{p}} \varpi(\mathbf{x}\mathbf{x}'^{-1}, \mathbf{p}) e_{\mathbf{p}}(\mathbf{x}) = \widetilde{\varpi}_p(\mathbf{x}\mathbf{x}'^{-1}, \mathbf{x}) \tag{77}$$

Here, $\widetilde{\varpi}_p$ is the partial inverse discrete Fourier transform (48) of ϖ with respect to the discrete variables.

(ii) The operator M^{ϖ} is symmetric if and only the weight satisfies;

$$\mathsf{M}^{\varpi} = \mathsf{M}^{\varpi\dagger} \Leftrightarrow \widetilde{\varpi}_p(\mathbf{x}\mathbf{x}'^{-1}, \mathbf{x}) = \overline{\widetilde{\varpi}}_p(\mathbf{x}'\mathbf{x}^{-1}, \mathbf{x}')\,. \tag{78}$$

(iii) The trace of M^{ϖ} is given by

$$\mathrm{Tr}(\mathsf{M}^{\varpi}) = \varpi(\mathrm{e}, \mathbf{0})\,. \tag{79}$$

Proof. (i) The action of M^{ϖ} on ψ is given by:

$$(\mathsf{M}^{\varpi}\psi)(\mathbf{x}) = \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q} \, \varpi(\mathbf{q}, \mathbf{p}) e_{\mathbf{p}}(\mathbf{x}) \psi(\mathbf{q}^{-1}\mathbf{x})\,.$$

Using the change of variable $\mathbf{q} \mapsto \mathbf{x}' = \mathbf{q}^{-1}\mathbf{x}$, the invariance of the Haar measure $\mathrm{d}\mathbf{q}$, and the partial inverse discrete Fourier transform (48), we get the expected kernel (77).

(ii) The action of $\mathsf{M}^{\varpi\dagger}$ on ψ is given by:

$$(\mathsf{M}^{\varpi\dagger}\psi)(\mathbf{x}) = \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q} \, \overline{\varpi(\mathbf{q}, \mathbf{p})}\, \overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})} \psi(\mathbf{q}\mathbf{x})\,.$$

Using the change of variable $\mathbf{q} \mapsto \mathbf{x}' = \mathbf{q}\mathbf{x}$, the invariance of the Haar measure $\mathrm{d}\mathbf{q}$, and the partial discrete Fourier transform (47) we formally get (78) by comparison with (77).

(iii) The relation (79) trivially results from (66).

□

In turn,we show in the following proposition that one retrieves the weight ϖ from the quantization operator M^{ϖ} through a tracing operation.

Proposition 4.2. *The trace of the operator $\mathsf{U}^{\dagger}(\mathbf{q}, \mathbf{p})\mathsf{M}^{\varpi}$ is given by:*

$$\mathrm{Tr}[\mathsf{U}^{\dagger}(\mathbf{q}, \mathbf{p})\mathsf{M}^{\varpi}] = \varpi(\mathbf{q}, \mathbf{p})\,. \tag{80}$$

Proof. This relation trivially results from (67). □

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As a first example, let us examine the case $\varpi(\mathbf{q},\mathbf{p}) = e_{\mathbf{p}}(\mathbf{q})$. Then the operator $\mathsf{M}^{e_{\mathbf{p}}}$ is determined through its action on basis elements $e_{\mathbf{p}'}(x)$:

$$\begin{aligned}
\left(\mathsf{M}^{e_{\mathbf{p}}} e_{\mathbf{p}'}\right)(\mathbf{x}) &= \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, e_{\mathbf{p}}(\mathbf{q})\, e_{\mathbf{p}}(\mathbf{x}) e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x}) \\
&= \sum_{l,m,n} \sum_{m''} \frac{2l+1}{8\pi^2} \sqrt{\frac{(2l'+1}{8\pi^2}} \left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathcal{D}^{l}_{mn}(\mathbf{q})\, \mathcal{D}^{l'}_{m'm''}(\mathbf{q}^{-1})\right] \mathcal{D}^{l}_{mn}(\mathbf{x})\, \mathcal{D}^{l'}_{m''n'}(\mathbf{x}) \\
&= \sum_{l,m,n} \sum_{m''} \frac{2l+1}{8\pi^2} \sqrt{\frac{(2l'+1}{8\pi^2}} \left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathcal{D}^{l}_{mn}(\mathbf{q})\, \overline{\mathcal{D}^{l'}_{m''m'}(\mathbf{q})}\right] \mathcal{D}^{l}_{mn}(\mathbf{x})\, \mathcal{D}^{l'}_{m''n'}(\mathbf{x}) \\
&= \sum_{l,m,n} \sum_{m''} \sqrt{\frac{(2l'+1}{8\pi^2}}\, \delta_{ll'}\, \delta_{mm''}\, \delta_{nm'}\, \mathcal{D}^{l}_{mn}(\mathbf{x})\, \mathcal{D}^{l'}_{m''n'}(\mathbf{x}) \\
&= \sum_{m''=-l'}^{l'} \mathcal{D}^{l'}_{m''m'}(\mathbf{x})\, e_{l'm''n'}(\mathbf{x}) = \sum_{m''=-l'}^{l'} \overline{\mathcal{D}^{l'}_{m'm''}(\mathbf{x}^{-1})}\, e_{l'm''n'}(\mathbf{x})\,.
\end{aligned} \tag{81}$$

Let us introduce the *squaring rotation operator* $\hat{\mathsf{I}}_{\mathrm{sq}}$ defined by

$$\hat{\mathsf{I}}_{\mathrm{sq}} : \psi \in \mathcal{K} \mapsto \hat{\mathsf{I}}_{\mathrm{sq}}\psi\,, \quad (\hat{\mathsf{I}}_{\mathrm{sq}}\psi)(\mathbf{x}) = \psi(\mathbf{x}^2)\,. \tag{82}$$

With this definition we precisely get from (81):

$$\mathsf{M}^{e_{\mathbf{p}}} = \hat{\mathsf{I}}^{\mathrm{t}}_{\mathrm{sq}} \Leftrightarrow \mathsf{M}^{e_{\mathbf{p}^{\mathrm{t}}}} = \hat{\mathsf{I}}_{\mathrm{sq}}\,, \tag{83}$$

where $\hat{\mathsf{I}}^{\mathrm{t}}_{\mathrm{sq}}$ is the transpose of $\hat{\mathsf{I}}_{\mathrm{sq}}$ and we remind that $\mathbf{p}^{\mathrm{t}} = (l,m,n)^{\mathrm{t}} = (l,n,m)$. This operator plays the central role in our definition of the Wigner-like function within the present context (see Section 8). Other examples of weights will be considered in the rest of the paper.

5. SO(3)-covariant integral quantization from weight function

5.1. General results

We now establish general formulas for the integral quantization issued from a weight function $\varpi(\mathbf{q},\mathbf{p})$ on $\Gamma = \mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$ yielding the bounded self-adjoint operator M^{ϖ} defined in (75). This allows us to build a family of operators obtained from SO(3) Weyl-Gabor operator transport of M^{ϖ}:

$$\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p}) = \mathsf{U}(\mathbf{q},\mathbf{p})\mathsf{M}^{\varpi}\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\,. \tag{84}$$

Then, the corresponding integral quantization is given by the linear map:

$$f \mapsto A^{\varpi}_f = \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f(\mathbf{q},\mathbf{p})\, \mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\,. \tag{85}$$

We have the following result.

Proposition 5.1. *A^{ϖ}_f is the integral operator on $L^2(\mathrm{SO}(3),\mathrm{d}\mathbf{x})$*

$$(A^{\varpi}_f\psi)(\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\, \mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}')\, \psi(\mathbf{x}')\,, \tag{86}$$

and its kernel is given by

$$\mathcal{A}^{\varpi}_f(\mathbf{x},\mathbf{x}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \delta^f(\mathbf{x}\mathbf{q}^{-1};(\mathbf{x},\,\mathbf{x}'))\, \widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,, \tag{87}$$

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with a weighted version of the completeness relation:

$$\delta^f(\mathbf{q};(\mathbf{x},\mathbf{x}')) := \sum_{\mathbf{p}} f(\mathbf{q},\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\,, \quad \delta^1(\mathbf{q};(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')\,. \tag{88}$$

The condition that $f = 1$ be mapped to the unit operator imposes that the follow normalization for ϖ holds:

$$\varpi(\mathrm{e},\mathbf{0}) = 1. \tag{89}$$

Proof. The calculation of the kernel of the integral operator A^{ϖ}_f goes through the following steps which follow from the expressions (84) and (57).

$$\begin{aligned}
&(A^{\varpi}_f)(\psi)(\mathbf{x}) \\
&= \sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f(\mathbf{q},\mathbf{p})\, \left(\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\psi\right)(\mathbf{x}) \\
&= \sum_{\mathbf{p}}\sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}' f(\mathbf{q},\mathbf{p})\varpi(\mathbf{q}',\mathbf{p}') \left(U(\mathbf{q},\mathbf{p})U(\mathbf{q}',\mathbf{p}')U^{\dagger}(\mathbf{q},\mathbf{p})\right)\psi(\mathbf{x}) \\
&= \sum_{\mathbf{p}}\sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}' f(\mathbf{q},\mathbf{p})\varpi(\mathbf{q}',\mathbf{p}') e_{\mathbf{p}}(\mathbf{x}) e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})}\psi(\mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x})\,.
\end{aligned}$$

We then proceed with the change of variable

$$\mathbf{q}' \mapsto \mathbf{x}' = \mathbf{q}\mathbf{q}'^{-1}\mathbf{q}^{-1}\mathbf{x}\,,$$

and use the SO(3) invariance of the measures $\mathrm{d}\mathbf{q}'$ to obtain the form

$$(A^{\varpi}_f)(\psi)(\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\, \mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}')\, \psi(\mathbf{x}')\,,$$

with

$$\mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}}\sum_{\mathbf{p}'} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f(\mathbf{q},\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\varpi(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{p}') e_{\mathbf{p}'}(\mathbf{q}^{-1}\mathbf{x})\,.$$

We then proceed with the change of variables $\mathbf{q} \mapsto \mathbf{z} = \mathbf{q}^{-1}\mathbf{x}$ to obtain

$$\begin{aligned}
\mathcal{A}^{\varpi}(\mathbf{x},\mathbf{x}') &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z} \left[\sum_{\mathbf{p}} f(\mathbf{x}\mathbf{z}^{-1},\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}\right] \left[\sum_{\mathbf{p}'} \varpi(\mathbf{z}\mathbf{x}'^{-1}\mathbf{x}\mathbf{z}^{-1},\mathbf{p}') e_{\mathbf{p}'}(\mathbf{z})\right] \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z}\, \delta^f(\mathbf{x}\mathbf{z}^{-1};(\mathbf{x},\mathbf{x}'))\, \widetilde{\varpi}_p\left(\mathbf{z}\mathbf{x}'^{-1}\mathbf{x}\mathbf{z}^{-1},\mathbf{z}\right)\,,
\end{aligned}$$

which is (87) with the notation (88).

Putting $f = 1$ in the above expression and using the completeness relation (46) give $\delta^1(\mathbf{x}\mathbf{z}^{-1};(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')$ and yield (89).

□

5.2. Particular quantizations

In what follows, we compute the quantized operators of the various simplifications of $f(\mathbf{q},\mathbf{p})$. Let us first introduce the function:

$$\Omega(\mathbf{x},\mathbf{x}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,. \tag{90}$$

It obeys

$$\Omega(\mathbf{x},\mathbf{x}) = \varpi(\mathrm{e},\mathbf{0}) = 1\,. \tag{91}$$

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We also introduce the notations:

$$\frac{\partial^j}{\partial\alpha'^j}(\Omega(\mathbf{x},\mathbf{x}')\Big|_{\mathbf{x}'=\mathbf{x}} = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\; \frac{\partial^j}{\partial\alpha'^j}\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\Big|_{\mathbf{x}'=\mathbf{x}} \equiv \Omega^{(j)}_\alpha\,. \tag{92}$$

$$\frac{\partial^j}{\partial\beta'^j}(\Omega(\mathbf{x},\mathbf{x}')\Big|_{\mathbf{x}'=\mathbf{x}} \equiv \Omega^{(j)}_\beta\,, \tag{93}$$

$$\frac{\partial^j}{\partial\gamma'^j}(\Omega(\mathbf{x},\mathbf{x}')\Big|_{\mathbf{x}'=\mathbf{x}} \equiv \Omega^{(j)}_\gamma\,. \tag{94}$$

In Appendix we give examples of such calculations in the case of coherent states.

Of course, $\frac{\partial}{\partial\alpha'}\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})$ or, equivalently, $\frac{\partial}{\partial\alpha'}\varpi(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{p})$ should be understood as

$$\frac{\partial}{\partial\alpha'}\varpi(\underbrace{\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1}}_{\mathbf{z}},\mathbf{p}) = \nabla_{\mathbf{z}}\varpi\cdot\begin{pmatrix}\frac{\partial\alpha(\mathbf{z})}{\partial\alpha'}\\ \frac{\partial\beta(\mathbf{z})}{\partial\alpha'}\\ \frac{\partial\gamma(\mathbf{z})}{\partial\alpha'}\end{pmatrix}\,,\quad \nabla_{\mathbf{z}}\varpi := \begin{pmatrix}\frac{\partial\varpi}{\partial\alpha(\mathbf{z})}\\ \frac{\partial\varpi}{\partial\beta(\mathbf{z})}\\ \frac{\partial\varpi}{\partial\gamma(bz)}\end{pmatrix}\,, \tag{95}$$

and $\alpha(\mathbf{z})$ means the Euler angle α of the rotation $\mathbf{z}$, etc. We will also need the integral formulae (with suitable conditions on functions appearing in the integrand)

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\alpha}f_1(\mathbf{x})\right)f_2(\mathbf{x}) = -\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, f_1(\mathbf{x})\left(\frac{\partial}{\partial\alpha}f_2(\mathbf{x})\right)\,, \tag{96}$$

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\gamma}f_1(\mathbf{x})\right)f_2(\mathbf{x}) = -\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, f_1(\mathbf{x})\left(\frac{\partial}{\partial\gamma}f_2(\mathbf{x})\right)\,, \tag{97}$$

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\left(\frac{\partial}{\partial\beta}f_1(\mathbf{x})\right)f_2(\mathbf{x}) = -\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, f_1(\mathbf{x})\left(\frac{\partial}{\partial\beta}f_2(\mathbf{x})\right) - \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\cot\beta\, f_1(\mathbf{x})\, f_2(\mathbf{x})\,. \tag{98}$$

5.2.1. Separable functions $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})v(\mathbf{p})$

In this case δ^{uv} in the integral factorises as

$$\delta^{uv}(\mathbf{x}\mathbf{q}^{-1};(\mathbf{x},\mathbf{x}')) = u(\mathbf{x}\mathbf{q}^{-1})\,\delta^{v}(\mathbf{x},\mathbf{x}')\,, \tag{99}$$

with the notation

$$\delta^{v}(\mathbf{x},\mathbf{x}')) = \sum_{\mathbf{p}} v(\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}\,. \tag{100}$$

Hence,

$$A^{\varpi}_f(\mathbf{x},\mathbf{x}') = \delta^{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,. \tag{101}$$

5.2.2. Univariate function $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$

In this case the above (101) simplifies to:

$$\begin{aligned} A^{\varpi}_u(\mathbf{x},\mathbf{x}') &= \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) \\ &= \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{\varpi}_p(\mathrm{e},\mathbf{q}) \\ &= \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{q})\,\widetilde{\varpi}_p(\mathrm{e},\mathbf{q}^{-1}\mathbf{x})\,. \end{aligned} \tag{102}$$

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Hence, the quantization of $u(\mathbf{q})$ is the multiplication operator.

$$\left(A^{\varpi}_{u(\mathbf{q})}\psi\right)(\mathbf{x}) = \left(u *_{\mathrm{SO}(3)} \widetilde{\varpi}_p(\mathrm{e},\cdot)\right)(\mathbf{x})\psi(\mathbf{x})\,, \tag{103}$$

where the (noncommutative) convolution $*_{\mathrm{SO}(3)}$ on the group SO(3) is defined by

$$\left(f_1 *_{\mathrm{SO}(3)} f_2\right)(\mathbf{x}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f_1(\mathbf{x}\mathbf{q}^{-1})\, f_2(\mathbf{q}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, f_1(\mathbf{q})\, f_2(\mathbf{q}^{-1}\mathbf{x})\,. \tag{104}$$

Let us give the quantizations of some basic Fourier or trigonometric functions $v(\mathbf{q})$. In the sequel we put $\left(A^{\varpi}_{u(\mathbf{q})}\psi\right)(\mathbf{x}) \equiv B^{\varpi}_{u(\mathbf{q})}(\mathbf{x})\,\psi(\mathbf{x})$.

- For $u(\mathbf{q}) = e^{\mathrm{i}\alpha}$ we get

$$B^{\varpi}_{e^{\mathrm{i}\alpha}}(\mathbf{x}) = e^{\mathrm{i}\alpha}\left[\sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,0,n))\mathrm{D}^{l}_{1\,n}(0,\beta,\gamma)\mathrm{a}_l\right]. \tag{105}$$

  where:

$$\mathrm{a}_l = \int_0^{\pi} \mathrm{d}\beta_{\mathbf{q}}\,\sin(\beta_{\mathbf{q}})\mathrm{d}^{l}_{10} \tag{106}$$

- For $u(\mathbf{q}) = e^{\mathrm{i}\gamma}$,

$$B^{\varpi}_{e^{\mathrm{i}\gamma}}(\mathbf{x}) == \sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,1,n))\mathrm{D}^{l}_{0\,n}(\alpha,\beta,\gamma)\mathrm{b}_l\,. \tag{107}$$

  where:

$$\mathrm{b}_l = \int_0^{\pi} \mathrm{d}\beta\,\sin(\beta)\,\mathrm{d}^{l}_{01}(\beta) \tag{108}$$

- For $u(\mathbf{q}) = 1/\sin\beta$,

$$B^{\varpi}_{1/\sin\beta}(\mathbf{x}) = \sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,0,n))\mathrm{D}^{l}_{0\,n}(0,\beta,\gamma)\mathrm{c}_l\,. \tag{109}$$

  where:

$$\mathrm{c}_l = \int_0^{\pi} \mathrm{d}\beta\mathrm{d}^{l}_{00}(\beta) \tag{110}$$

- For $u(\mathbf{q}) = \cot\beta$,

$$B^{\varpi}_{\cot\beta}(\mathbf{x}) = \sum_{(l,n)}\sqrt{\frac{2l+1}{4\pi}}\varpi(\mathrm{e},(l,0,n))\mathrm{D}^{l}_{0\,n}(0,\beta,\gamma)\mathrm{d}_l\,. \tag{111}$$

  where:

$$\mathrm{d}_l = \int_0^{\pi} \mathrm{d}\beta\cos(\beta)\mathrm{d}^{l}_{00}(\beta) \tag{112}$$

5.2.3. Univariate function $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$

The integral kernel reads in this case:

$$\mathcal{A}^{\varpi}_{f}(\mathbf{x},\mathbf{x}') = \delta^{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) \tag{113}$$

We know that the values of $\mathbf{p} = (l,m,n)$ are constrained in a forward rectangular discrete pyramid, which is the momentum space. We here work with Euler angle parameters: $\mathbf{x}(\alpha,\beta,\gamma)$, $\mathbf{x}'(\alpha',\beta',\gamma')$. We will omit them for simplicity and explicitly put them when needed. Let us present the quantization of a few elementary functions $v(\mathbf{p})$.

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- $v(l,m,n) = m$ We have:

$$\begin{aligned}
\delta^v(\mathbf{x},\mathbf{x}') &= \sum_{\mathbf{p}} v(\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')} \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l} \frac{2l+1}{8\pi^2} D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\, m\, \overline{D^l_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma')) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l} \frac{2l+1}{8\pi^2} D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\, \mathrm{i}\frac{\partial}{\partial\alpha'} \overline{D^l_{m\,n}(\mathbf{x}'(\alpha',\beta',\gamma'))} \\
&= \mathrm{i}\frac{\partial}{\partial\alpha'}\delta(\mathbf{x}(\alpha,\beta,\gamma)), \mathbf{x}'^{-1}(\alpha',\beta',\gamma')\,.
\end{aligned}$$

The kernel is then given by:

$$A^{\varpi}_m(\mathbf{x},\mathbf{x}') = \left[\mathrm{i}\frac{\partial}{\partial\alpha'}\delta(\mathbf{x},\mathbf{x}')\right]\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,.$$

The action of the quantum version of m on $\psi \in \mathcal{K}$ is then obtained through integration by part and use of (91) and notation (92):

$$\begin{aligned}
\left(A^{\varpi}_m\psi\right)(\mathbf{x}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\,\delta(\mathbf{x},\mathbf{x}')\left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right](-\mathrm{i})\frac{\partial}{\partial\alpha'}\psi(\mathbf{x}') \\
&\quad -\mathrm{i}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}'\,\delta(\mathbf{x},\mathbf{x}')\left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\frac{\partial}{\partial\alpha'}\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right]\psi(\mathbf{x}') \\
&= \left(-\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\Omega^{(1)}_{\alpha}\right)\psi(\mathbf{x}) = \left(L_z - \mathrm{i}\Omega^{(1)}_{\alpha}\right)\psi(\mathbf{x})\,. \qquad (114)
\end{aligned}$$

Under mild conditions on the weight function, we have $\Omega^{(1)}_{\alpha} = 0$, and so we recover exactly the angular momentum operator component L_z. A similar result holds with the quantization of $v(l,m,n) = n$:

$$A^{\varpi}_m = -\mathrm{i}\frac{\partial}{\partial\gamma} - \mathrm{i}\Omega^{(1)}_{\gamma}\,. \qquad (115)$$

- $v(l,m,n) = m^2$

  We have:

$$\begin{aligned}
\delta^v(\mathbf{x},\mathbf{x}') &= \sum_{\mathbf{p}} v(\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')} \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l} \frac{2l+1}{8\pi^2} D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\, m^2\overline{D^l_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma')) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l} \frac{2l+1}{8\pi^2} D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\left[-\frac{\partial^2}{\partial\alpha'^2}\overline{D^l_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma'))\right] \\
&= -\frac{\partial^2}{\partial\alpha'^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\sum_{n=-l}^{n=+l} \frac{2l+1}{8\pi^2}\overline{D^l_{m\,n}}(\mathbf{x}'(\alpha',\beta',\gamma'))D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma)) \\
&= -\frac{\partial^2}{\partial\alpha'^2}\delta(\mathbf{x}'^{-1}(\alpha',\beta',\gamma')\mathbf{x}(\alpha,\beta,\gamma)) = -\frac{\partial^2}{\partial\alpha'^2}\delta(\mathbf{x},\mathbf{x}')\,.
\end{aligned}$$

There results for the kernel:

$$A^{\varpi}_{m^2}(\mathbf{x},\mathbf{x}') = \left[-\frac{\partial^2}{\partial\alpha'^2}\delta(\mathbf{x},\mathbf{x}')\right]\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,, \qquad (116)$$

and for the quantum operator:

$$\begin{aligned}\left(A^{\varpi}_{m^2}\psi\right)(\mathbf{x}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \left[-\frac{\partial^2}{\partial\alpha'^2}\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right]_{\mathbf{x}'=\mathbf{x}} \psi(\mathbf{x}) \\ &+ 2\mathrm{i}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \left[\frac{\partial}{\partial\alpha'}\widetilde{\varpi}(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\right]_{\mathbf{x}'=\mathbf{x}} \left(-\mathrm{i}\frac{\partial}{\partial\alpha}\psi(\mathbf{x})\right) \\ &+ \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \widetilde{\varpi}(\mathrm{e},\mathbf{q}) \left(-\frac{\partial^2}{\partial\alpha^2})\psi(\mathbf{x})\right).\end{aligned}$$

i.e.,

$$A^{\varpi}_{m^2} = L_z^2 + 2\mathrm{i}\Omega^{(1)}_{\alpha}(\mathrm{e})L_z - \Omega^{(2)}_{\alpha}(\mathrm{e})\,. \tag{117}$$

- $v(l,m,n) = l(l+1)$. We have just to use the eigenvalue property of the functions $D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))$, or, equivalently, of the functions $d^l_{m\,n}(\beta)$ [46]:

$$\begin{aligned}&l(l+1)\, D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma)) \\ &= \left[-\frac{\partial^2}{\partial\beta^2} - \cot\beta\frac{\partial}{\partial\beta} + \frac{m^2+n^2-2mn\cos\beta}{\sin^2\beta}\right] D^l_{m\,n}(\mathbf{x}(\alpha,\beta,\gamma))\,,\end{aligned} \tag{118}$$

and the results given the above examples. The corresponding kernel is given by:

$$\mathcal{A}^{\varpi}_{l(l+1)}(\mathbf{x},\mathbf{x}') = L^2(\delta(\mathbf{x},\mathbf{x}'))\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \widetilde{\omega}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q})\,, \tag{119}$$

where[46]:

$$L^2 = \left[-\frac{\partial^2}{\partial\beta'^2} - \cot\beta'\frac{\partial}{\partial\beta'} - \frac{1}{\sin^2\beta'}\left(\frac{\partial^2}{\partial\alpha'^2} + \frac{\partial'^2}{\partial\gamma'2} - 2\cos\beta'\frac{\partial^2}{\partial\alpha'\partial\gamma'}\right)\right]. \tag{120}$$

6. Semi-classical portrait

Given a function $\varpi(\mathbf{q},\mathbf{p})$ on the phase space Γ, normalised at $\varpi(\mathrm{e},(0,0,0)) = 1$, and yielding a non-negative unit trace operator, *i.e.*, a density operator, M^{ϖ}, the quantum phase space portrait of an operator A on $L^2(\Gamma,\mathrm{d}\gamma)$ is defined as:

$$\check{A}(\mathbf{q},\mathbf{p}) := \mathrm{Tr}\left(A\, U(\mathbf{q},\mathbf{p})\,\mathsf{M}^{\varpi}U^{\dagger}(\mathbf{q},\mathbf{p})\right) = \mathrm{Tr}\left(A\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\right). \tag{121}$$

The most interesting aspect of this notion in terms of probabilistic interpretation holds when the operator A is precisely the integral quantized version A^{ϖ}_f of a classical $f(\mathbf{q},\mathbf{p})$ with the same function ϖ (actually we could define the transform with 2 different ones, one for the "analysis" and the other for the "reconstruction"). Then, with the use of the composition rule let us compute the transform:

$$f(\mathbf{q},\mathbf{p}) \mapsto \check{f}(\mathbf{q},\mathbf{p}) \equiv \check{A}^{\varpi}_f(\mathbf{q},\mathbf{p}) = \mathrm{Tr}\left(A^{\varpi}_f\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\right). \tag{122}$$

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We successively have:

$$
\begin{aligned}
&\mathrm{Tr}\left(A_f^{\varpi}\,\mathsf{M}^{\varpi}(\mathbf{q},\mathbf{p})\right) = \sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,f(\mathbf{q}',\mathbf{p}')\varpi(\mathbf{q}'',\mathbf{p}'')\varpi(\mathbf{q}''',\mathbf{p}''')\times\\
&\mathrm{Tr}\left(U(\mathbf{q}',\mathbf{p}')U(\mathbf{q}'',\mathbf{p}'')U^{\dagger}(\mathbf{q}',\mathbf{p}')\,U(\mathbf{q},\mathbf{p})U(\mathbf{q}''',\mathbf{p}''')U^{\dagger}(\mathbf{q},\mathbf{p})\right)\\
&= \sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,f(\mathbf{q}',\mathbf{p}')\varpi(\mathbf{q}'',\mathbf{p}'')\varpi(\mathbf{q}''',\mathbf{p}''')\times\\
&\sum_{\mathbf{b}}\langle U(\mathbf{q}',\mathbf{p}')U^{\dagger}(\mathbf{q}'',\mathbf{p}'')U^{\dagger}(\mathbf{q}',\mathbf{p}')\,e_{\mathbf{b}}|\,U(\mathbf{q},\mathbf{p})U(\mathbf{q}''',\mathbf{p}''')U^{\dagger}(\mathbf{q},\mathbf{p})e_{\mathbf{b}}\rangle\\
&= \sum_{\mathbf{p}',\mathbf{p}'',\mathbf{p}'''}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,f(\mathbf{q}',\mathbf{p}')\varpi(\mathbf{q}'',\mathbf{p}'')\varpi(\mathbf{q}''',\mathbf{p}''')\times\\
&\sum_{\mathbf{b}}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}''}(\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\overline{e_{\mathbf{b}}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})}\times\\
&e_{\mathbf{p}}(\mathbf{x})\,e_{\mathbf{p}'''}(\mathbf{q}^{-1}\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})}e_{\mathbf{b}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})\,.
\end{aligned}
$$

Using the partial Fourier transform $\widetilde{\varpi}_p$ of ϖ, we get:

$$
\begin{aligned}
\check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'',\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}''',\mathbf{q}^{-1}\mathbf{x})\times\\
\sum_{\mathbf{b}}\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\;\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})}\,\overline{e_{\mathbf{b}}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x})\,e_{\mathbf{b}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})\,.
\end{aligned}
$$

Summing on $\mathbf{b}$ gives:

$$
\begin{aligned}
\check{f}(\mathbf{q},\mathbf{p}) &= \sum_{\mathbf{p}'}\iiiint_{(\mathrm{SO}(3))^4}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{q}'''\,\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'',\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}''',\mathbf{q}^{-1}\mathbf{x})\times\\
&\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\;\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{x})}\,\delta(\mathbf{q}'''^{-1}\mathbf{q}^{-1}\mathbf{q}'\mathbf{q}''^{-1}\mathbf{q}'^{-1}\mathbf{q})\\
&= \sum_{\mathbf{p}'}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{q}''\,\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'',\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{q}'\mathbf{q}''^{-1}\mathbf{q}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times\\
&\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})\;\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}'\mathbf{q}''\mathbf{q}'^{-1}\mathbf{x})}\\
&= \sum_{\mathbf{p}'}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{y},\mathbf{y}\mathbf{q}'^{-1}\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{q}'\mathbf{y}^{-1}\mathbf{q}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times\\
&\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{q}'\mathbf{y}\mathbf{q}'^{-1}\mathbf{x})\;\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{q}'\mathbf{y}\mathbf{q}'^{-1}\mathbf{x})}\,,\quad(\mathbf{x}'=\mathbf{y})\\
&= \sum_{\mathbf{p}'}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{y}'\,\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{y}'\mathbf{q},\mathbf{q}'^{-1}\mathbf{y}'\mathbf{x})\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{y}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times\\
&\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{y}'\mathbf{x})\;\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{y}'\mathbf{x})}\,,\quad(\mathbf{y}'=\mathbf{q}'\mathbf{y}\mathbf{q}'^{-1})\\
&= \sum_{\mathbf{p}'}\iiint_{(\mathrm{SO}(3))^3}\mathrm{d}\mathbf{q}'\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\,f(\mathbf{q}',\mathbf{p}')\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'\mathbf{x}^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x}')\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\times\\
&\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{x}')\;\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\,,\quad(\mathbf{x}'=\mathbf{y}'\mathbf{x})\,,\quad(\mathbf{y}'=\mathbf{x}'\mathbf{x}^{-1})\,.
\end{aligned}
$$

Hence we can write:

$$
\check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'}\int_{(\mathrm{SO}(3)}\mathrm{d}\mathbf{q}'\,K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\,f(\mathbf{q}',\mathbf{p}')\,,
$$

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where the kernel is given by

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = \iint_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\, \widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'\mathbf{x}^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x}')\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{x}')\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\,.$$

Using the adjointness condition for M^{ϖ}, one gets:

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = \iint_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,,\mathrm{d}\mathbf{x}'\, \overline{\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x})}\widetilde{\varpi}_p(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x})\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}'}(\mathbf{x}')\;e_{\mathbf{p}}(\mathbf{x})\overline{e_{\mathbf{p}}(\mathbf{x}')}\,,$$

where:

$$\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'\mathbf{x}^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x}') = \widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}'(\mathbf{q}'^{-1}\mathbf{x})^{-1},\mathbf{q}'^{-1}\mathbf{x}') = \overline{\widetilde{\varpi}_p(\mathbf{q}'^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x})}$$

Hence we can conclude with the following result:

Proposition 6.1. *The semi-classical portrait of the operator A^{ϖ}_f: with respect to the weight ϖ is given by:*

$$f \to A^{\varpi}_f \to \check{f}(\mathbf{q},\mathbf{p}) = \mathrm{Tr}(A^{\varpi}_f\,\mathrm{M}^{\varpi}(\mathbf{q},\mathbf{p})) = \sum_{\mathbf{p}'}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, f(\mathbf{q}',\mathbf{p}')K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\,, \tag{123}$$

where the kernel is given by:

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = \int_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}\,e_{\mathbf{p}'}(\mathbf{x}')\,\overline{\widetilde{\varpi}(\mathbf{q}'^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q}',\mathbf{q}'^{-1}\mathbf{x})}\,\widetilde{\varpi}(\mathbf{q}^{-1}\mathbf{x}\mathbf{x}'^{-1}\mathbf{q},\mathbf{q}^{-1}\mathbf{x}). \tag{124}$$

This kernel satisfies the property:

$$K(\mathbf{q}',\mathbf{p}';\mathbf{q},\mathbf{p}) = \overline{K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')}. \tag{125}$$

Below we give the kernels and semi-classical portraits for two specific weights.

(i) For the unit weight $\varpi(\mathbf{q},\mathbf{p}) = 1$ the kernel reads:

$$\begin{aligned}K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') &= \int_{(\mathrm{SO}(3))^2} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\,\overline{e_{\mathbf{p}'}(\mathbf{x})}\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}\,e_{\mathbf{p}'}(\mathbf{x}')\\ &\times\left\{\sum_{\mathbf{b}'}\overline{e_{\mathbf{b}'}(\mathbf{q}'^{-1}\mathbf{x})}\right\}\left\{\sum_{\mathbf{b}}e_{\mathbf{b}}(\mathbf{q}^{-1}\mathbf{x})\right\}\\ &= \delta_{\mathbf{p}\mathbf{p}'}\int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,|\,e_{\mathbf{p}}(\mathbf{x})|^2\left\{\sum_{\mathbf{b}'}\overline{e_{\mathbf{b}'}(\mathbf{q}'^{-1}\mathbf{x})}\right\}\left\{\sum_{\mathbf{b}}e_{\mathbf{b}}(\mathbf{q}^{-1}\mathbf{x})\right\}\\ &= \delta_{\mathbf{p}\mathbf{p}'}\mathrm{h}(\mathbf{q},\mathbf{q}'),\end{aligned} \tag{126}$$

where:

$$\mathrm{h}(\mathbf{q},\mathbf{q}') = \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{x}\,|\,e_{\mathbf{p}}(\mathbf{x})|^2\left\{\sum_{\mathbf{b}'}\overline{e_{\mathbf{b}'}(\mathbf{q}'^{-1}\mathbf{x})}\right\}\left\{\sum_{\mathbf{b}}e_{\mathbf{b}}(\mathbf{q}^{-1}\mathbf{x})\right\}. \tag{127}$$

Finally:

$$\check{f}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)}\mathrm{d}\mathbf{q}'\,f(\mathbf{q}',\mathbf{p})\mathrm{h}(\mathbf{q},\mathbf{q}')\,. \tag{128}$$

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(ii) For the squaring rotation map weight $\varpi(\mathbf{q},\mathbf{p}) = \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})} = e_{\mathbf{p}^t}(\mathbf{q})$,

$$K(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = \delta(\mathbf{q},\mathbf{q}') \int_G \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}'}(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x})\, e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}) \overline{e_{\mathbf{p}'}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})} \tag{129}$$

and:

$$\check{f}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'} f(\mathbf{q},\mathbf{p}')\{ \int_G \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})} e_{\mathbf{p}'}(\mathbf{x})\, e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}) \overline{e_{\mathbf{p}'}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})} \}. \tag{130}$$

For the univariate functions $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$ and $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$:

$$\check{u}(\mathbf{q}) = u(\mathbf{q}) \tag{131}$$

$$\check{v}(\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}) \left\{ \sum_{\mathbf{p}'} v(\mathbf{p}') e_{\mathbf{p}'}(\mathbf{x}) \overline{e_{\mathbf{p}'}(\mathbf{x}\mathbf{q}^{-1}\mathbf{x})} \right\}.$$

A third example of weight, namely that one corresponding to coherent states, is given in the next section.

7. Quantization and semi-classical portraits with coherent states with non-unit fiducial states

7.1. CS quantization

In this section, we consider the quantization yielded by the weight ϖ^ϕ which corresponds through Proposition 8.2 to the one-rank density operator $|\phi\rangle\langle\phi| = \mathsf{M}^{\varpi^\phi}$, $\phi \in \mathcal{K}$, $\|\phi\|^2 = 1$.

Proposition 7.1. *Given a fiducial vector ϕ and the projector $|\phi\rangle\langle\phi|$, the trace of the operator $\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})|\phi\rangle\langle\phi|$ is given by (with the notations of (59)):*

$$\varpi^\phi(\mathbf{q},\mathbf{p}) \equiv \mathrm{Tr}\left[\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})|\phi\rangle\langle\phi|\right] = {}_\phi\langle \mathbf{q},\mathbf{p}|\phi\rangle\,. \tag{132}$$

In addition, we have:

(i) the partial inverse Fourier transform of $\varpi^\phi(\mathbf{q},\mathbf{p})$ with respect to $\mathbf{p}$ is given by:

$$\widetilde{\varpi^\phi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) = \overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q})\,, \tag{133}$$

(ii) the kernel of the related quantum operator is given by:

$$\mathcal{A}_f^{\varpi^\phi}(\mathbf{x},\mathbf{x}') = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \delta^f(\mathbf{x}\mathbf{q}^{-1};(\mathbf{x},\mathbf{x}'))\, \overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q})\,, \tag{134}$$

with the notations of (88).

In what follows, we compute the kernels $\mathcal{A}_f^{\varpi^\phi}$ or/and the corresponding operators for various simple cases of $f(\mathbf{q},\mathbf{p})$.

a) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})v(\mathbf{p})$,

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = u(\mathbf{x}\mathbf{q}^{-1})\,\widetilde{v}(\mathbf{x},\mathbf{x}')\,, \quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{x}')}$$

$$\mathcal{A}_f^{\varpi^\phi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}') \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{x}\mathbf{q}^{-1}) \overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q})\,.$$

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b) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$,

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')\, u(\mathbf{x}\mathbf{q}^{-1})$$
$$\mathcal{A}^{\varpi\phi}_{f}(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{x}\mathbf{q}^{-1})|\phi(\mathbf{q})|^2\,.$$

Hence, the quantized of $u(\mathbf{q})$ is the multiplication operator.

$$\left(A^{\varpi\phi}_{u(\mathbf{q})}\psi\right)(\mathbf{x}) = \left[\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, u(\mathbf{x}\mathbf{q}^{-1})\,|\phi(\mathbf{q})|^2\right]\psi(\mathbf{x})\,.$$

c) For $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = \widetilde{v}(\mathbf{x},\mathbf{x}')\,;\quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\, e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}$$
$$\mathcal{A}^{\varpi}_{f}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')\phi(\mathbf{q})}\,.$$

7.2. *Semi-classical portraits through CS*

For the coherent state weight $\varpi^{\phi}(\mathbf{q},\mathbf{p}) = {}_{\phi}\langle \mathbf{q},\mathbf{p}|\phi\rangle$ the kernel is the probability distribution on the phase space:

$$K^{\phi}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}') = |\langle_{\phi}\langle \mathbf{q},\mathbf{p}|\mathbf{q}',\mathbf{p}'\rangle_{\phi}|^2\,. \tag{135}$$

Hence $\check{f}(\mathbf{q},\mathbf{p})$ is the local averaging of the original $f(\mathbf{q},\mathbf{p})$:

$$\check{f}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}'\, f(\mathbf{q}',\mathbf{p}')\,|\langle_{\phi}\langle \mathbf{q},\mathbf{p}|\mathbf{q}',\mathbf{p}'\rangle_{\phi}|^2\,. \tag{136}$$

8. Squaring rotation operator for Wigner function

In preamble to this section, let us consider the following phase portrait of a state $\psi \in \mathcal{K}$,

$$W^{\widehat{\mathcal{O}}}_{\psi}(\mathbf{q},\mathbf{p}) := \langle \psi|U(\mathbf{q},\mathbf{p})\widehat{\mathcal{O}}U^{\dagger}(\mathbf{q},\mathbf{p})|\psi\rangle\,, \tag{137}$$

where the operator $\widehat{\mathcal{O}}$ is requested to yield marginality properties *à la* Wigner for $W^{\widehat{\mathcal{O}}}_{\psi}$:

$$\sum_{\mathbf{p}} W^{\widehat{\mathcal{O}}}_{\psi}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{q})|^2\,,\quad \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W^{\widehat{\mathcal{O}}}_{\psi}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{p})|^2\,. \tag{138}$$

Let us assume that $\widehat{\mathcal{O}}$ acts on $\mathcal{K}$ through some differentiable transformation of the group manifold $\mathrm{SO}(3) \ni \mathbf{x} \mapsto \zeta(\mathbf{x}) \in \mathrm{SO}(3)$, namely

$$(\widehat{\mathcal{O}}\psi)(\mathbf{x}) = \mathsf{m}(\mathbf{x})\psi(\zeta(\mathbf{x}))\,, \tag{139}$$

where the factor $\mathsf{m}(\mathbf{x})$ has also to be determined.

With (54), (55), and the above definitions, we have

$$\begin{aligned} W^{\widehat{\mathcal{O}}}_{\psi}(\mathbf{q},\mathbf{p}) &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}e_{\mathbf{p}}(\mathbf{x})\,\left(\widehat{\mathcal{O}}\,U^{\dagger}(\mathbf{q},\mathbf{p})\psi\right)\left(\mathbf{q}^{-1}\mathbf{x}\right) \\ &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}e_{\mathbf{p}}(\mathbf{x})\,\mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right)\left(U^{\dagger}(\mathbf{q},\mathbf{p})\psi\right)\left(\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right) \\ &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{\psi(\mathbf{x})}e_{\mathbf{p}}(\mathbf{x})\,\mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right)\,\overline{e_{\mathbf{p}}\left(\mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right)}\psi\left(\mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right)\,. \end{aligned}$$

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The completeness relation (46) combined with the above integral and change of variable $\mathbf{x} \mapsto \mathbf{y} = \mathbf{q}^{-1}\mathbf{x}$ allows to write

$$\sum_{\mathbf{p}} W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, |\psi(\mathbf{x})|^2\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right)\, \delta\left(\mathbf{x}, \mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right)$$
$$= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{y}\, |\psi(\mathbf{q}\mathbf{y})|^2\, \mathsf{m}\left(\mathbf{y}\right)\, \delta\left(\mathbf{q}\mathbf{y}, \mathbf{q}\zeta\left(\mathbf{y}\right)\right).$$

The condition for getting marginality with regard to summing on $\mathbf{p}$, namely

$$\sum_{\mathbf{p}} W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{q})|^2, \tag{140}$$

thus imposes on the action ζ and the function m the following conditions

$$\delta\left(\mathbf{q}\mathbf{y}, \mathbf{q}\zeta\left(\mathbf{y}\right)\right) = \delta(\mathbf{y}, \mathrm{e}), \quad \mathsf{m}(\mathrm{e}) = 1.$$

Besides the constraint $\mathsf{m}(\mathrm{e}) = 1$ possible solutions for ζ are

$$\zeta(\mathbf{y}) = \mathbf{y}^n, \quad n \in \mathbb{Z}, \tag{141}$$

since the group structure imposes that $\mathbf{q}\mathbf{y} = \mathbf{q}\mathbf{y}^n \Rightarrow \mathbf{y}^{n-1} = \mathrm{e}$, *i.e.*, $\mathbf{y}$ should be one of the $n-1$ roots of the unity in $\mathrm{SO}(3)$. The most natural choice for n is obviously $n = 2$:

$$\zeta(\mathbf{y}) = \mathbf{y}^2, \tag{142}$$

and we will keep it in the sequel. Let us now examine the condition for getting marginality with regard to integrating on $\mathbf{q}$, namely

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = |\widehat{\psi}(\mathbf{p})|^2, \tag{143}$$

where we remind that $\widehat{\psi}$ is the Fourier transform of ψ:

$$\widehat{\psi}(\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{p}}(\mathbf{x})}\psi(\mathbf{x}) \equiv \mathcal{F}[\psi](\mathbf{p}),$$

With the solution (142) at hand, let us evaluate the l.h.s. of (143), assuming that inverting the order of integrations is legitimate.

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right)\, \overline{e_{\mathbf{p}}\left(\mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right)}\psi\left(\mathbf{q}\zeta\left(\mathbf{q}^{-1}\mathbf{x}\right)\right)$$
$$= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})} e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, \mathsf{m}\left(\mathbf{q}^{-1}\mathbf{x}\right)\, \overline{e_{\mathbf{p}}\left(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}\right)}\psi\left(\mathbf{x}\mathbf{q}^{-1}\mathbf{x}\right).$$

Now, after changing the variable in the second integral $\mathbf{q} \mapsto \mathbf{q}' = \mathbf{x}\mathbf{q}^{-1}\mathbf{x}$ and using the invariance of the measure $\mathrm{d}\mathbf{q} \mapsto \mathrm{d}\mathbf{q}'$, one obtains

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\widehat{\mathcal{O}}}(\mathbf{q},\mathbf{p}) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{\psi(\mathbf{x})}\, e_{\mathbf{p}}(\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}'\, \mathsf{m}\left(\mathbf{x}^{-1}\mathbf{q}'\right)\, \overline{e_{\mathbf{p}}\left(\mathbf{q}'\right)}\, \psi(\mathbf{q}').$$

In order to get marginality with regard to integrating on $\mathbf{q}$ the only possibility is that $\mathsf{m}(\mathbf{x}) = 1$. Then, we get (143), and $\widehat{\mathcal{O}}$ is precisely the squaring rotation operator introduce in (82),

$$\widehat{\mathcal{O}} \equiv \hat{\mathrm{I}}_{\mathrm{sq}}. \tag{144}$$

In summary, and extending the above properties to mixed states in $\mathcal{K}$, we state the following.

Proposition 8.1. *To any density operator $\hat{\rho}$ in $\mathcal{K}$ the squaring rotation operator $\hat{I}_{sq}$ associates its Wigner-like function on the phase space defined by*

$$W_{\hat{\rho}}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p}) := \mathrm{Tr}\left(\hat{\rho}\, U(\mathbf{q},\mathbf{p})\hat{I}_{sq} U^{\dagger}(\mathbf{q},\mathbf{p})\right) . \tag{145}$$

This function obeys the marginality properties:

(i)

$$\sum_{\mathbf{p}} W_{\hat{\rho}}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}'\mathbf{p}''} \hat{\rho}_{\mathbf{p}'\mathbf{p}''}\, e_{\mathbf{p}'}(\mathbf{q})\, \overline{e_{\mathbf{p}''}(\mathbf{q})}\,, \tag{146}$$

where the $\hat{\rho}_{\mathbf{p}'\mathbf{p}''}$ are the matrix elements of $\hat{\rho}$ in the basis $\{e_{\mathbf{p}}(\mathbf{q})\}$.

(ii)

$$\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\hat{\rho}}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p}) = \hat{\rho}_{\mathbf{p}\mathbf{p}}\,. \tag{147}$$

(iii) For a pure state $\hat{\rho} = [\psi\rangle\langle\psi|$ these formulae simplify to

$$\sum_{\mathbf{p}} W_{\psi}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p}) = |\psi(\mathbf{q})|^2\,, \quad \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\psi}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p}) = |\widehat{\psi}(\mathbf{p})|^2\,. \tag{148}$$

(iv) It results from these two marginal properties the normalisation of $W_{\psi}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p})$ as a complex-valued quasi-distribution on the phase space:

$$\sum_{\mathbf{p}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\, W_{\hat{\rho}}^{\hat{I}_{sq}}(\mathbf{q},\mathbf{p}) = 1\,. \tag{149}$$

Let us tell more about the properties of the squaring rotation operator.

Proposition 8.2. *(i) The operator $\hat{I}_{sq}$ is unit trace.*

$$\mathrm{Tr}(\hat{I}_{sq}) = 1$$

(ii) The weight function $\varpi^{sq}(\mathbf{q},\mathbf{p})$ giving rise to $\hat{I}_{sq}$ through (75), i.e., $\mathsf{M}^{\varpi^{sq}} = \hat{I}_{sq}$, is given by the trace of the operator $\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\hat{I}_{sq}$:

$$\varpi^{sq}(\mathbf{q},\mathbf{p}) = \mathrm{Tr}(\mathsf{U}^{\dagger}(\mathbf{q},\mathbf{p})\hat{I}_{sq}) = \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})}\,. \tag{150}$$

(iii) The inverse partial Fourier transform of the weight with respect to the momentum is given by:

$$\widetilde{\varpi^{sq}}_p(\mathbf{q},\mathbf{x}) = \sum_{\mathbf{p}} e_{\mathbf{p}}(\mathbf{x})\, \varpi^{sq}(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{p}} e_{\mathbf{p}}(\mathbf{x})\, \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})} = \delta(\mathbf{q}^{-1},\mathbf{x})\,. \tag{151}$$

and:

$$\widetilde{\varpi^{sq}}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) = \delta(\mathbf{x}'^{-1}\mathbf{x}\mathbf{q})\,, \tag{152}$$

and the kernel of the related quantum operator is given by:

$$\mathcal{A}_f^{\varpi^{\phi}}(\mathbf{x},\mathbf{x}') = \delta^f(\mathbf{x}';(\mathbf{x},\mathbf{x}'))\,, \tag{153}$$

with the notations of (88).

Proof. (i)

$$\mathrm{Tr}\,(\hat{I}_{sq}) = \sum_{\mathbf{b}} \langle e_{\mathbf{p}}|\,\hat{I}_{sq}|e_{\mathbf{p}}\rangle = \sum_{\mathbf{b}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\ \overline{e_{\mathbf{p}}(\mathbf{x})}\,(\hat{I}_{sq}e_{\mathbf{p}})(\mathbf{x}) = \sum_{\mathbf{b}} \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \overline{e_{\mathbf{b}}(\mathbf{x})}\, e_{\mathbf{b}}(\mathbf{x}^2)$$

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Using the completeness property (46) and (36), we get:

$$\mathrm{Tr}\left(\hat{\mathrm{I}}_{\mathrm{sq}}\right) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\, \delta(\mathbf{x},\mathbf{x}^2) = 1\,.$$

(ii)

$$\begin{aligned}
\mathrm{Tr}\left(\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})\right)\hat{\mathrm{I}}_{\mathrm{sq}}\Big) &= \sum_{\mathbf{b}}\langle e_{\mathbf{b}}|\,\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})\,\hat{\mathrm{I}}_{\mathrm{sq}}|e_{\mathbf{b}}\rangle = \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\,\left(\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})\,\hat{\mathrm{I}}_{\mathrm{sq}}\,e_{\mathbf{b}}\right)(\mathbf{x}) \\
&= \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\left(\mathsf{U}^\dagger(\mathbf{q},\mathbf{p})\,\left(\hat{\mathrm{I}}_{\mathrm{sq}}e_{\mathbf{b}}\right)\right)(\mathbf{x}) = \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})}\,\left(\hat{\mathrm{I}}_{\mathrm{sq}}e_{\mathbf{b}}\right)(\mathbf{q}\mathbf{x}) \\
&= \sum_{\mathbf{b}}\int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\overline{e_{\mathbf{b}}(\mathbf{x})}\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})}e_{\mathbf{b}}((\mathbf{q}\mathbf{x})^2) = \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{x}\,\delta\left(\mathbf{x},(\mathbf{q}\mathbf{x})^2\right)\,\overline{e_{\mathbf{p}}(\mathbf{q}\mathbf{x})} \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{y}\,\delta\left(\mathbf{q}^{-1}\mathbf{y},\mathbf{y}^2\right)\,\overline{e_{\mathbf{p}}(\mathbf{y})} = \overline{e_{\mathbf{p}}(\mathbf{q}^{-1})} = e_{\mathbf{p}^t}(\mathbf{q})\,,
\end{aligned}$$

where we again have used the completeness property (46) and (36) after changing the variable $\mathbf{x} \mapsto \mathbf{y} = \mathbf{q}\mathbf{x}$.

(iii) The proof is direct.

□

In what follows, we compute the kernels $\mathcal{A}_f^{\varpi\mathrm{sq}}$ or/and the corresponding operators for various simple cases of $f(\mathbf{q},\mathbf{p})$.

a) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})v(\mathbf{p})$,

$$\widetilde{f}(\mathbf{x}',(\mathbf{x},\mathbf{x}')) = u(\mathbf{x}')\,\widetilde{v}(\mathbf{x},\mathbf{x}')\,, \quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}$$

$$\mathcal{A}_f^{\varpi\phi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')u(\mathbf{x}').$$

b) For $f(\mathbf{q},\mathbf{p}) = u(\mathbf{q})$,

$$\widetilde{f}(\mathbf{x}',(\mathbf{x},\mathbf{x}')) = \delta(\mathbf{x},\mathbf{x}')\,u(\mathbf{x}') = \mathcal{A}_f^{\varpi\phi}(\mathbf{x},\mathbf{x}')\,.$$

Hence, the quantized of $u(\mathbf{q})$ is the multiplication operator.

$$\left(A_{u(\mathbf{q})}^{\varpi\phi}\psi\right)(\mathbf{x}) = u(\mathbf{x})\psi(\mathbf{x})\,.$$

c) For $f(\mathbf{q},\mathbf{p}) = v(\mathbf{p})$

$$\widetilde{f}(\mathbf{x}\mathbf{q}^{-1},(\mathbf{x},\mathbf{x}')) = \widetilde{v}(\mathbf{x},\mathbf{x}')\,; \quad \widetilde{v}(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}} v(\mathbf{p})\,e_{\mathbf{p}}(\mathbf{x})\,\overline{e_{\mathbf{p}}(\mathbf{x}')}$$

$$\mathcal{A}_f^{\varpi}(\mathbf{x},\mathbf{x}') = \widetilde{v}(\mathbf{x},\mathbf{x}')\,.$$

9. Conclusion

In this paper, we have established a covariant integral quantization for systems whose phase space is the so-called semi-discrete hypercylinder $\mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$, *i.e.*, whose configuration space is $SO(3)$, and where $\widehat{\mathrm{SO}(3)} = \{(l,m,n), l \in \mathbb{N}, m = -l:1:l, n = -l:1:l\}$. This extends our previous work on the discrete cylinder $\mathrm{SO}(2) \times \mathbb{Z}$ for the motion of particle on circle $\mathrm{S}^1 = \mathrm{SO}(2)$. Although the phase space $\mathrm{SO}(3) \times \widehat{\mathrm{SO}(3)}$ is not a coset arising from a group, we have shown that the Weyl-Gabor formalism applies.

First, we have established the concomitant resolution of the identity and subsequent properties such as the Gabor transform on SO(3) and its inversion, the reproducing kernel and the fact that any square integrable function on the groupe SO(3) is fiducial vector. The decomposition of the identity allows to define an integral operator M^{ϖ} from weight function ϖ defined on the phase space.

There are noticeable results related to the quantization of a point

$$(\mathbf{q}, \mathbf{p}) = ((\alpha, \beta, \gamma), (l, m, n))$$

in the phase space according to two standard choices of the weight.

- With the squaring rotation weight which yields the Wigner distribution, the quantization of the projection of momentum on third axis is the expected angular momentum operator L_z.

$$m \mapsto \hat{m} = L_z\,, \quad L_z\psi(\alpha, \beta, \gamma) = -\mathrm{i}\frac{\partial}{\partial\alpha}\psi(\alpha, \beta, \gamma)\,. \tag{154}$$

  while the quantization of the angle yields the multiplication operator by the angle,

$$\theta_i \mapsto \hat{\theta}_i\,, \quad \hat{\theta}_i\psi(\theta) = \theta_i\psi(\theta)\,, \quad \theta = (\alpha, \beta, \gamma)\,. \tag{155}$$

  This is of course not acceptable due to the discontinuity of the periodized angle function, since there is no regularisation of this discontinuity.
- On the other hand, with the squaring rotation weight, we derive a Wigner distribution that is more tractable than the ones derived in [37][40]. We will show this in future works with concrete examples.
- With the coherent state weight one obtains the quantization of the momentum as the usual L_z plus an additional term, *i.e.*, a kind of covariant derivative along the circle $\mathbb{S}^1$, in SO(3), whose topology is now taken into account,

$$A^{\varpi}_m = -\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\Omega^{(1)}_{\alpha}\,. \tag{156}$$

  whereas the quantization of the periodized function in α variable leads to its smooth regularisation.

In a follow up studies, we plan to extend the results of this work in several directions.

- Extend the work to all rotation groups SO(n), and also to the related Spheres $\mathbb{S}^n$. Also look into the quantization of continuation phase space related to the Euclidean groups $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$, ≥ 2, [51,52].
- Extend the work to full configuration space of the rigid body that is the Euclidean motion group in three dimensions $\mathrm{E}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$.
- Extend the formalism to the case where the configuration space is a non-compact group, for example, $\mathrm{SO}_0(2,1) \simeq \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$.
- Explore the possibility of covariant integral quantization in the situation where the phase space is $T^*\mathrm{SO}(3)$. This phase space is used in the context of quantum loop gravity [53,54].
- Apply to signal analysis on SO(3). We will investigate the robustness of the phase space of representation $(\mathbf{q}, \mathbf{p}) \to S(\mathbf{q}, \mathbf{p})$ of signal $x \to s(\mathbf{x})$ in capturing salient futures in the signal. Various tools will be used. These include: visualization of various partial energy densities of the Gabor transform, quantum operators related to various fiducial vectors, Husimi distributions, the Wigner distribution, entropy, and sampling/frames on SO(3) [48,55].

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Abbreviations

The following abbreviations are used in this manuscript:

POVM	Positive operator-valued measure
UIR	Unitary irreducible representation
CS	Coherent state

Appendix A. Some formulas for CS quantization

In this appendix, we compute $\Omega^{(j)}_{\alpha}(\mathbf{x},\mathbf{x}) = \frac{\partial^j}{\partial\alpha'^j}(\Omega(\mathbf{x},\mathbf{x}'))\Big|_{\mathbf{x}'=\mathbf{x}}$, for the coherent state weight $\varpi(\mathbf{q},\mathbf{p}) = \langle \mathrm{U}(\mathbf{q},\mathbf{p})\phi|\phi\rangle$, for $j=1,2$, using the following summation formulae [56],

$$\sum_{m'} m'|\mathrm{d}^l_{m'\,m}(\beta)|^2 = m\cos(\beta); \sum_{m'} m'^2|\mathrm{d}^l_{m'\,m}(\beta)|^2 = \frac{1}{2}\left\{l(l+1)\sin^2\beta - m^2(3\cos\beta - 1)\right\}.$$

$$\begin{aligned}
\Omega(\mathbf{x},\mathbf{x}') &= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\widetilde{\varpi}_p(\mathbf{q}\mathbf{x}'^{-1}\mathbf{x}\mathbf{q}^{-1},\mathbf{q}) \\
&= \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\phi(\mathbf{q}) = \sum_{(l,m,n)} \widehat{\phi}(l,m,n) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{q}\,\overline{\phi(\mathbf{q}\mathbf{x}^{-1}\mathbf{x}')}\sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^l_{m\,n}(\mathbf{q}) \\
&= \sum_{(l,m,n)} \widehat{\phi}(l,m,n) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z}\,\overline{\phi(\mathbf{z})}\sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^l_{m\,n}(\mathbf{z}\mathbf{x}'^{-1}\mathbf{x}) \\
&= \sum_{(l,m,n)}\sum_k \widehat{\phi}(l,m,n)\mathrm{D}^l_{k\,n}(\mathbf{x}'^{-1}\mathbf{x}) \int_{\mathrm{SO}(3)} \mathrm{d}\mathbf{z}\,\overline{\phi(\mathbf{z})}\sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^l_{m\,k}(\mathbf{z}) \\
&= \sum_{(l,m,n)}\sum_k \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\mathrm{D}^l_{k\,n}(\mathbf{x}'^{-1}\mathbf{x}) \\
&= \sum_{(l,m,n)}\sum_k\sum_{k'} \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\mathrm{D}^l_{k\,k'}(\mathbf{x}'^{-1})\mathrm{D}^l_{k'\,n}(\mathbf{x}) \\
&= \sum_{(l,m,n)}\sum_k\sum_{k'} \widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x}')}\mathrm{D}^l_{k'\,n}(\mathbf{x})
\end{aligned}$$

We therefore have:

$$\Omega^{(j)}_{\alpha}(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_k\sum_{k'}(-\mathrm{i})^{\mathrm{j}}k'^{\mathrm{j}}\widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x})}\mathrm{D}^l_{k'\,n}(\mathbf{x}). \tag{A.1}$$

$$\Omega^{(j)}_{\gamma}(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_k\sum_{k'}(-\mathrm{i})^{\mathrm{j}}k^{\mathrm{j}}\widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\;\overline{\mathrm{D}^l_{k'\,k}(\mathbf{x})}\mathrm{D}^l_{k'\,n}(\mathbf{x}). \tag{A.2}$$

Let us now two types of fiducial vectors.

- Free rotor and highest weight fiducial vectors.

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For the free rotor $\phi(\mathbf{x}) = \sqrt{\frac{2l_0+1}{8\pi^2}}\mathrm{D}^{l_0}_{m_0\,n_0}(\mathbf{x})$ with momentum (l_0, m_0, n_0), we get: $\widehat{\phi}(l,m,n) = \frac{2l_0+1}{8\pi^2}\delta_{l\,l_0}\delta_{m\,m_0}\delta_{n\,n_0}$, $\widehat{\phi}(l,m,k) = \frac{2l_0+1}{8\pi^2}\delta_{l\,l_0}\delta_{m\,m_0}\delta_{k\,n_0}$, and:

$$\begin{aligned}\Omega^{(j)}_{\alpha}(\mathbf{x},\mathbf{x}) &= \delta_{l_0\,l_0}\delta_{m_0\,m_0}\frac{2l_0+1}{8\pi^2}(-\mathrm{i})^{\mathrm{j}}\sum_{k'}k'^{j}|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2 \\ &= \frac{2l_0+1}{8\pi^2}(-\mathrm{i})^{\mathrm{j}}\sum_{k'}k'^{j}|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2.\end{aligned} \tag{A.3}$$

$$\begin{aligned}\Omega^{(j)}_{\gamma}(\mathbf{x},\mathbf{x}) &= \delta_{l_0\,l_0}\delta_{m_0\,m_0}\frac{2l_0+1}{8\pi^2}(-\mathrm{i})^{\mathrm{j}}{n_0}^{j}\sum_{k'}|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2 \\ &= \frac{2l_0+1}{8\pi^2}(-\mathrm{i})^{\mathrm{j}}{n_0}^{j}\sum_{k'}|\mathrm{d}^{l_0}_{k'\,n_0}(\beta)|^2.\end{aligned} \tag{A.4}$$

Using summation formula in (A.1) for j=1 and j=2, we get:

$$\begin{aligned}\Omega^{(1)}_{\alpha}(\mathbf{x},\mathbf{x}) &= -\mathrm{i}\,\frac{2l_0+1}{8\pi^2}\,n_0\cos(\beta), \\ \Omega^{(2)}_{\alpha}(\mathbf{x},\mathbf{x}) &= -\frac{1}{2}\frac{2l_0+1}{8\pi^2}\left\{l_0(l_0+1)\sin^2\beta - n_0^2(3\cos\beta-1)\right\}\end{aligned} \tag{A.5}$$

$$\begin{aligned}\Omega^{(1)}_{\gamma}(\mathbf{x},\mathbf{x}) &= -\mathrm{i}\,\frac{2l_0+1}{8\pi^2}\,n_0. \\ \Omega^{(2)}_{\gamma}(\mathbf{x},\mathbf{x}) &= -\frac{1}{2}\frac{2l_0+1}{8\pi^2}{n_0}^2.\end{aligned} \tag{A.6}$$

For the highest weight state for SO(3), that is, $\phi(\mathbf{x}) = \sqrt{\frac{2l+1}{8\pi^2}}\mathrm{D}^{l_0}_{l_0\,l_0}(\mathbf{x})$ with momentum (l_0, l_0, l_0), we get:

$$\begin{aligned}\Omega^{(1)}_{\alpha}(\mathbf{x},\mathbf{x}) &= -\mathrm{i}\left[\frac{2l_0+1}{8\pi^2}\right]l_0\cos(\beta), \\ \Omega^{(2)}_{\alpha}(\mathbf{x},\mathbf{x}) &= \frac{1}{2}\frac{2l_0+1}{8\pi^2}\left\{\cos\beta(\cos\beta+3)l_0^2 - l_0\sin^2\beta)\right\}.\end{aligned} \tag{A.7}$$

$$\begin{aligned}\Omega^{(1)}_{\gamma}(\mathbf{x},\mathbf{x}) &= -\mathrm{i}\,\frac{2l_0+1}{8\pi^2}\,l_0. \\ \Omega^{(2)}_{\gamma}(\mathbf{x},\mathbf{x}) &= \frac{1}{2}\frac{2l_0+1}{8\pi^2}l_0^2.\end{aligned} \tag{A.8}$$

- Radial fiducial vector

  We consider a radial vector ϕ depending only on the distance $|\mathbf{x}|$ to the origin e. We have $\widehat{\phi}(l,m,n) = f(l)\delta_{m\,n}$ and $\widehat{\phi}(l,m,k) = f(l)\delta_{m\,k}$ [22].

$$\Omega^{(j)}_{\alpha}(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)}\sum_{k}\sum_{k'}(-\mathrm{i})^{\mathrm{j}}k'^{\mathrm{j}}\widehat{\phi}(l,m,n)\overline{\widehat{\phi}(l,m,k)}\,\overline{\mathrm{D}^{l}_{k'\,k}(\mathbf{x})}\mathrm{D}^{l}_{k'\,n}(\mathbf{x}). \tag{A.9}$$

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$$\Omega^{(j)}_{\alpha}(\mathbf{x},\mathbf{x}) = \sum_{(l,m,n)} \sum_{k} \sum_{k'} (-\mathrm{i})^{\mathrm{j}} k'^{\mathrm{j}} |\, f(l)|^2 \delta_{m\,n} \delta_{m\,k}\, \overline{\mathrm{D}^{l}_{k'\,k}(\mathbf{x})} \mathrm{D}^{l}_{k'\,n}(\mathbf{x}), \qquad \text{(A.10)}$$
$$= \sum_{l} (2l+1) |\, f(l)|^2 \sum_{k} \sum_{k'} (-\mathrm{i})^{\mathrm{j}} k'^{\mathrm{j}}\, |\mathrm{d}^{l}_{k'\,k}(\beta)|^2.$$

Finally:

$$\Omega^{(1)}_{\alpha}(\mathbf{x},\mathbf{x}) = \left[-\mathrm{i} \sum_{l} (2l+1) |\, f(l)|^2 \right] (2l+1) \cos\beta \qquad \text{(A.11)}$$
$$= \left[-\mathrm{i} \sum_{l} (2l+1)^2 |\, f(l)|^2 \right] \cos\beta.$$
$$\Omega^{(2)}_{\alpha}(\mathbf{x},\mathbf{x}) = -\left[\frac{1}{2} \sum_{l} (2l+1)^2 |\, f(l)|^2 \right] \left\{ \cos\beta(\cos\beta + 3) l_0^2 - l_0 \sin^2\beta \right\}.$$

Appendix B. Plot of Von Mise fiducial vector and derivative

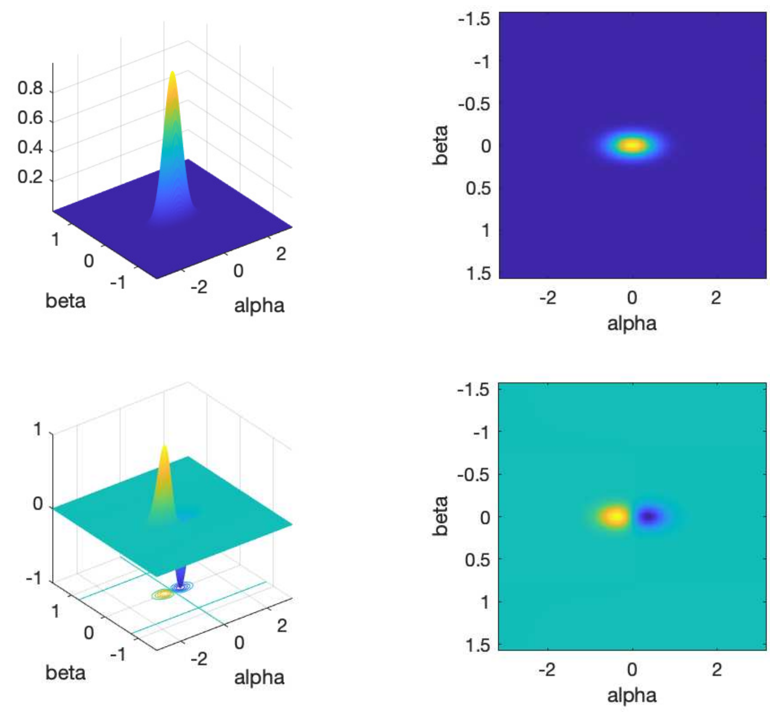

Figure B.1. Top left and right: Von Mises Fiducial vector in (α, β) variables at $\kappa = 30, \gamma = 0$; bottom left and right: derivative with respect to β of the Von Mises Fiducial vector in (α, β) variables at $\kappa = 30, \gamma = 0$

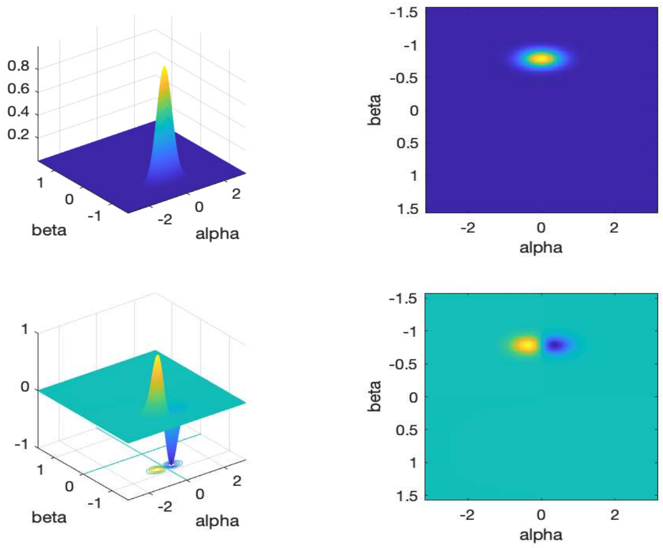

Figure B.2. Top left and right: Von Mises Fiducial vector in (α, β) variables at $\kappa = 30, \gamma = pi$; bottom left and right: derivative with respect to β of the Von Mises Fiducial vector in (α, β) variables at $\kappa = 30, \gamma = \pi$

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