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Article

The Convergence and Boundedness of Solutions to SFDEs with the G-Framework

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Abstract: Generally, stochastic functional differential equations (SFDEs) pose a challenge as they often lack explicit exact solutions. Consequently, it becomes necessary to seek certain favorable conditions under which numerical solutions can converge towards the exact solutions. This article aims to delve into the convergence analysis of solutions for stochastic functional differential equations by employing the framework of G-Brownian motion. To establish the goal, we find a set of useful monotone type conditions and work within the space $C_r((-\infty, 0]; R^n)$. The investigation conducted in this article confirms the mean square boundedness of solutions. Furthermore, this study enables us to compute both L_G^2 and exponential estimates.

Keywords: G-Brownian motion; exponential and L_G^2 estimates; boundedness; convergence

1. Introduction

Stochastic functional differential equations (SFDEs) find applications in various fields of engineering and science, such as neural networks [15], financial assets [4,26,30], population dynamics [1,20], and gene expression [21]. A vast literature exists on moment estimates, convergence, stability, and existence of solutions for SFDEs [17–19,22,23]. The study of SFDEs driven by G-Brownian motion is relatively new, dating back to the invention of G-Brownian theory in 2006 [24]. In [4,25], the classical Lipschitz condition and linear growth condition were used to establish the existence-uniqueness theorem for SFDEs in the space $BC((-\infty, 0]; R^n)$. The study of SFDEs under the G-framework with non-Lipschitz conditions and mean square stability was investigated in [8]. The work in [7,9–11] provides insights into pth moment estimates, the Cauchy-Maruyama approximation scheme, and exponential estimates for solutions to SFDEs within the framework of G-Brownian motion. Asymptotic estimates were studied in [28], while SFDEs under the G-Lévy processes were investigated in [5]. In this article, we introduce some useful monotone type conditions to study SFDEs under the G-framework within the space $C_r((-\infty, 0]; R^n)$. Our findings contribute to the growing body of research on SFDEs driven by G-Brownian motion and deepen our understanding of the role of G-framework in stochastic analysis. We study the convergence of solutions for a SFDEs using the framework of G-Brownian motion. Our analysis results in the mean square boundedness of solutions and allows us to compute both L_G^2 and exponential estimates. Consider a matrix A ; its transpose is denoted by A^T . Let $C((-\infty, 0]; R^n)$ denotes the set of continuous mappings from $(-\infty, 0]$ to R^n . Define the space $C_r((-\infty, 0]; R^n)$, $r > 0$ as

$$C_r((-\infty, 0]; R^n) = \{\alpha \in C((-\infty, 0]; R^n) : \lim_{\sigma \rightarrow -\infty} e^{r\sigma} \alpha(\sigma) \text{ exists in } R^n\},$$

Associated with norm $\|\alpha\|_r = \sup_{-\infty < \sigma \leq 0} e^{r\sigma} |\alpha(\sigma)| < \infty$, the space $C_r((-\infty, 0]; R^n)$ is a Banach space of bounded continuous mappings. For each $0 < r_1 \leq r_2 < \infty$, $C_{r_1} \subseteq C_{r_2}$ [14,29]. Represent the σ -algebra of C_r by $\mathcal{B}(C_r)$ and $C_r^0 = \{\alpha \in C_r : \lim_{\sigma \rightarrow -\infty} e^{r\sigma} \alpha(\sigma) = 0\}$. Let $\mathbb{L}^2(C_r)$ denote the space of all \mathcal{F} -measurable stochastic processes ψ taking values in C_r , such that $\hat{\mathbb{E}}|\alpha|_r^2 < \infty$. Similarly, let $\mathbb{L}^2(C_r^0)$ denote the space of all \mathcal{F} -measurable stochastic processes α taking values in C_r^0 , such that $\mathbb{E}|\alpha|_r^2 < \infty$. Let (Ω, \mathcal{F}, P) be a complete probability space, where \mathcal{F} is a sigma-algebra of subsets. The natural

filtration \mathcal{F}_t on (Ω, \mathcal{F}, P) is defined as the sigma-algebra, denoted by $\mathcal{F}_t = \sigma\mathcal{B}(v) : 0 \leq v \leq t$, where $\mathcal{B}(v)$ represents the Borel sigma-algebra of \mathbb{C}_r . We use \mathcal{P} to represent the set of all probability measures on $(\mathbb{C}_r, \mathcal{B}(\mathbb{C}_r))$. Additionally, $\mathbb{L}_b(\mathbb{C}_r)$ denotes the collection of continuous bounded functionals on \mathbb{C}_r . Finally, let Λ_0 be the collection of probability measures on $(-\infty, 0]$ satisfying $\int_{-\infty}^0 \mu(d\sigma) = 1$ for every $\mu \in \Lambda_0$. We define

$$\Lambda_k = \{\mu \in \Lambda_0 : \mu^{(k)} = \int_{-\infty}^0 e^{-k\sigma} \mu(d\sigma) < \infty\}, \quad (1)$$

where $\Lambda_{k_0} \subset \Lambda_k \subset \Lambda_0$ for any $k \in (0, k_0)$ [29]. Let $\kappa : \mathbb{C}_r((-\infty, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$, $\eta : \mathbb{C}_r((-\infty, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ and $\gamma : \mathbb{C}_r((-\infty, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ be Borel measurable. Consider the SFDEs driven by G-Brownian motion of the form

$$dz(t) = \kappa(z_t)dt + \eta(z_t)d\langle B, B \rangle(t) + \gamma(z_t)dB(t), \quad (2)$$

on $t \geq 0$ where $z_t = \{z(t+\theta) : -\infty < \theta \leq 0\}$. Equation (2) has the starting value $z_0 = \zeta \in \mathbb{C}_r((-\infty, 0]; \mathbb{R}^n)$. Let $\langle B, B \rangle(t)$ denote the quadratic variation process of the G-Brownian motion $B(t)$ defined on a complete probability space (Ω, \mathcal{F}, P) , where $B(t)$ is a one-dimensional process under the filtration $\mathcal{F}_{t \geq 0}$ satisfying the usual conditions. The remaining paper is arranged as follows. Section 2 presents the basic results. In section 3, some useful lemmas are given. Section 4 investigates the mean square boundedness and convergence of solutions. In section 5, we first study the L_G^2 estimate and then derive the exponential estimate. Section 6 contains conclusions.

2. Basic Notions and Results

This section presents some fundamental concepts and results that we utilize in the forthcoming research [2,6,16,27]. Let \mathcal{H} be a space of real mappings defined on a non-empty set Ω .

Definition 1. $\forall x, y \in \mathcal{H}$, a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ assuring the below given features is called a G-expectation

1. $\hat{\mathbb{E}}[y] \geq \hat{\mathbb{E}}[x]$ whenever $y \geq x$.
2. $\hat{\mathbb{E}}[m_1] = m_1$, for any $m_1 \in \mathbb{R}$.
3. $\hat{\mathbb{E}}[m_2 y] = m_2 \hat{\mathbb{E}}[y]$, for any $m_2 \in \mathbb{R}^+$.
4. $\hat{\mathbb{E}}[y + x] \leq \hat{\mathbb{E}}[y] + \hat{\mathbb{E}}[x]$.

Suppose that Ω be the space of \mathbb{R}^n -valued continuous paths $(w(t))_{t \geq 0}$ such that $w(0) = 0$ associated with the norm

$$\mathbb{D}(w^1, w^2) = \sum_{j=1}^{\infty} 2^{-j} \left(\max_{t \in [0, j]} |w^1(t) - w^2(t)| \wedge 1 \right).$$

Choose $\Omega_T = \{\omega \wedge T : \omega \in \Omega\}$. Assume the canonical process $B(t) = B(t, w)$ where $t \geq 0$ and $w \in \Omega$. Let $\lambda \in C_{b.Lip}(\mathbb{R}^n \times d)$ and $t_1, t_2, \dots, t_n \in [0, T]$ then

$$\mathbb{L}_{ip}^0(\Omega_T) = \left\{ \lambda(B(t_1), B(t_2), \dots, B(t_n)) : n \geq 1 \right\}.$$

Notice that $\mathbb{L}_{ip}^0(\Omega_t) \subseteq \mathbb{L}_{ip}^0(\Omega_T)$, $\mathbb{L}_{ip}^0(\Omega) = \cup_{m=1}^{\infty} \mathbb{L}_{ip}^0(\Omega_m)$ and the completion of $\mathbb{L}_{ip}^0(\Omega)$ associated with $\hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$, $p \geq 1$ is $\mathbb{L}_G^p(\Omega)$. Related to $\{B(t)\}_{t \geq 0}$, we can express the filtration as $\mathcal{F}_t = \sigma\{B(u), 0 \leq u \leq t\}$ where $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Let $0 \leq t_0 \leq t_1 \leq \dots \leq t_N < \infty$ and $\Delta_T = \{t_0, t_1, \dots, t_N\}$ be a partition of $[0, T]$. Let $p \geq 1$ then $\mathbb{M}_G^{p,0}(0, T)$ is given by

$$\mathbb{M}_G^{p,0}(0, T) = \left\{ \rho_t(w) = \sum_{o=0}^{N-1} \vartheta_j(w) I_{[t_o, t_{o+1}]}(t) : \vartheta_o \in \mathbb{L}_G^p(\Omega_{t_o}), o = 0, 1, \dots, N-1 \right\},$$

The space $\mathbb{M}_G^p(0, T)$ is the completion of $\mathbb{M}_G^{p,0}(0, T)$ under the norm $\|\rho\| = \left\{ \int_0^T \mathbb{E}[|\rho_s|^p] ds \right\}^{1/p}$, $p \geq 1$.

Definition 2. Let $\rho_t \in \mathbb{M}_G^{2,0}(0, T)$. The G-Itô integral $\mathcal{I}(\rho)$ is defined as the stochastic integral of a function ρ with respect to G-Brownian motion given by

$$\mathcal{I}(\rho) = \int_0^T \rho(s) dB^a(s) = \sum_{k=0}^{N-1} \xi_k \left(B^a(t_{k+1}) - B^a(t_k) \right),$$

One can extend $\mathcal{I} : \mathbb{M}_G^{2,0}(0, T) \mapsto \mathbb{L}_G^2(\mathcal{F}_T)$ to $\mathcal{I} : \mathbb{M}_G^2(0, T) \mapsto \mathbb{L}_G^2(\mathcal{F}_T)$, where for $\rho \in \mathbb{M}_G^2(0, T)$ we have

$$\int_0^T \rho(s) dB^a(s) = \mathcal{I}(\rho).$$

Definition 3. Let $\langle B^a \rangle(0) = 0$. The G-quadratic variation process $\{\langle B^a \rangle(t)\}_{t \geq 0}$ is given as follows

$$\langle B^a \rangle(t) = \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} \left(B^a(t_{j+1}^N) - B^a(t_j^N) \right)^2 = B^a(t)^2 - 2 \int_0^t B^a(s) dB^a(s).$$

Consider a function $\mathcal{K}_{0,T} : \mathbb{M}_G^{0,1}(0, T) \mapsto \mathbb{L}_G^2(\mathcal{F}_T)$ given as

$$\mathcal{K}_{0,T}(\rho) = \int_0^T \rho(s) d\langle B^a \rangle(s) = \sum_{i=0}^{N-1} \xi_i \left(\langle B^a \rangle(t_{i+1}) - \langle B^a \rangle(t_i) \right).$$

One can extend $\mathcal{K}_{0,T}$ to $\mathbb{M}_G^1(0, T)$. For $\rho \in \mathbb{M}_G^1(0, T)$, it is given by

$$\int_0^T \rho(s) d\langle B^a \rangle(s) = \mathcal{K}_{0,T}(\rho).$$

Lemma 1. Assume that $\gamma \in \mathbb{M}_G^p(0, T)$ and $p \geq 2$. Then

$$\hat{\mathbb{E}} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \gamma(s) dB(s) \right|^p \right] \leq a_3 \hat{\mathbb{E}} \left[\int_0^t |\gamma(s)|^2 ds \right]^{\frac{p}{2}},$$

where $a_3 \in (0, \infty)$ is a p dependent constant.

Lemma 2. Assume that $\gamma \in \mathbb{M}_G^p(0, T)$, $p \geq 1$. Then

$$\hat{\mathbb{E}} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \gamma(s) d\langle B, B \rangle(s) \right|^p \right] \leq a_2 \hat{\mathbb{E}} \left[\int_0^t |\gamma(s)|^2 ds \right]^{\frac{p}{2}},$$

where $a_2 \in (0, \infty)$ depends on p .

Definition 4. The G-Brownian motion is an n -dimensional stochastic process $\{B(t)\}_{t \geq 0}$ fulfilling the following characteristics

- (i) $B(0) = 0$.
- (ii) $B(t+v) - B(t)$ is G-normally distributed and independent of $B(t_1), B(t_2), \dots, B(t_n)$ for any $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$.

Lemma 3. Let $p \geq 0$, $z \in \mathbb{L}_G^p$. For each $m_1 > 0$,

$$\hat{\nu}(|z| > m_1) \leq \frac{\hat{\mathbb{E}}[|z|^p]}{m_1},$$

where $\hat{\mathbb{E}}|z|^p < \infty$.

The following two basic lemmas can also be utilized in forthcoming sections of this paper [17].

Lemma 4. Let $c, l \geq 0$ and $\delta \in (0, 1)$. Then

$$(c + l)^2 \leq \frac{c^2}{\delta} + \frac{l^2}{1 - \delta}.$$

Lemma 5. Let $p \geq 2$ and $\hat{\delta}, c, l > 0$. Then

$$\begin{aligned} \text{(i)} \quad c^{p-1}l &\leq \frac{(p-1)\hat{\delta}c^p}{p} + \frac{l^p}{p\hat{\delta}^{p-1}}. \\ \text{(ii)} \quad c^{p-2}l^2 &\leq \frac{(p-2)\hat{\delta}c^p}{p} + \frac{2l^p}{p\hat{\delta}^{\frac{p-2}{2}}}. \end{aligned}$$

3. Some Useful Results

In this section, we introduce and discuss some important assumptions and establish two lemmas. We consider the following hypotheses

(H) Let $a_i > 0$, $i = 1, 2, \dots, 5$ and $\alpha(\sigma) - \beta(\sigma) = \Psi(\sigma)$. For any $\alpha, \beta \in C_r((-\infty, 0]; R^n)$ and for any probability measure $\mu_1, \mu_2, \mu_3 \in \Lambda_{2r}$ the following inequalities hold

$$[\Psi(0)]^T [\kappa(\alpha) - \kappa(\beta)] \leq -a_1 |\Psi(0)|^2 + a_2 \int_{-\infty}^0 |\Psi(\sigma)|^2 \mu_1(d\sigma), \quad (3)$$

$$[\Psi(0)]^T [\eta(\alpha) - \eta(\beta)] \leq -a_3 |\Psi(0)|^2 + a_4 \int_{-\infty}^0 |\Psi(\sigma)|^2 \mu_2(d\sigma), \quad (4)$$

$$|\gamma(\alpha) - \gamma(\beta)|^2 \leq a_5 \int_{-\infty}^0 |\Psi(\sigma)|^2 \mu_3(d\sigma). \quad (5)$$

Lemma 6. Let $a < pr$ and $p \geq 1$. Then

$$\|z_t\|_r^p \leq e^{-at} \|\zeta\|_r^p + \sup_{0 < u \leq t} |z(u)|^p,$$

where $\zeta \in C_r((-\infty, 0]; R^n)$.

Proof. Assuming that $pr > a$, we can obtain the following using the definition of the norm $|\cdot|_r$:

$$\begin{aligned} \|z_t\|_r^p &= \left[\sup_{-\infty < \sigma \leq 0} e^{r\sigma} |z(t + \sigma)| \right]^p \leq \sup_{-\infty < \sigma \leq 0} e^{a\sigma} |z(t + \sigma)|^p \\ &\leq \sup_{0 < u \leq t} e^{-a(t-u)} |z(u)|^p + \sup_{-\infty < u \leq 0} e^{-a(t-u)} |z(u)|^p \\ &= e^{-at} \|\zeta\|_r^p + e^{-at} \sup_{0 < u \leq t} e^{as} |z(u)|^p \\ &\leq \sup_{0 < u \leq t} |z(u)|^p + e^{-at} \|\zeta\|_r^p. \end{aligned}$$

The proof stands completed. \square

Throughout this paper we let that for any $p \geq 1$, $a < pr$.

Lemma 7. Let $a < pr$, $p \geq 2$ and $\mu_i \in \Lambda_k, \forall i \in N$. Then

$$\int_0^t \int_{-\infty}^0 |z(\sigma + s)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr} \|\zeta\|_r^p + \int_0^t |z(s)|^p ds, \quad (6)$$

$$\int_0^t \int_{-\infty}^0 e^{as} |z(\sigma + s)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr - a} \|\zeta\|_r^p + \mu_i^{(pr)} \int_0^t e^{as} |z(s)|^p ds, \quad (7)$$

where $\zeta \in C_r((-\infty, 0]; R^n)$.

Proof. As for each $i \in Z^+$, $\mu_i \in \Lambda_{pr}$ and $\zeta \in C_r((-\infty, 0]; R^n)$, by using the Fubini theorem and the definition of norm, it follows

$$\begin{aligned} & \int_0^t \int_{-\infty}^0 |z(\sigma + s)|^p \mu_i(d\sigma) ds \\ &= \int_0^t \left[\int_{-\infty}^{-s} e^{pr(\sigma+s)} |z(\sigma + s)|^p e^{-pr(s+\sigma)} \mu_i(d\sigma) + \int_{-s}^0 |z(\sigma + s)|^p \mu_i(d\sigma) \right] ds \\ &\leq \|\zeta\|_r^p \int_0^t e^{-pr s} ds \int_{-\infty}^0 e^{-pr\sigma} \mu_i(d\sigma) + \int_{-\infty}^0 \mu_i(d\sigma) \int_0^t |z(s)|^p ds. \end{aligned}$$

Observing that $\int_{-\infty}^0 \mu_i(d\sigma) = 1$ and $\int_{-\infty}^0 e^{-pr\sigma} \mu_i(d\sigma) = \mu_i^{(pr)}$, $i \in N$, it follows

$$\int_0^t \int_{-\infty}^0 |z(s + \sigma)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr} \|\zeta\|_r^p + \int_0^t |z(s)|^p ds.$$

The proof of (6) is complete. Using similar arguments as used above we determine

$$\begin{aligned} & \int_0^t \int_{-\infty}^0 e^{as} |z(\sigma + s)|^p \mu_i(d\sigma) ds \\ &= \int_0^t e^{as} ds \left[\int_{-\infty}^{-s} |z(\sigma + s)|^p \mu_i(d\sigma) + \int_{-s}^0 |z(\sigma + s)|^p \mu_i(d\sigma) \right] \\ &= \int_0^t e^{as} ds \int_{-\infty}^{-s} |z(\sigma + s)|^p \mu_i(d\sigma) + \int_{-t}^0 \mu_i(d\sigma) \int_{-\sigma}^t e^{as} |z(\sigma + s)|^p ds \\ &\leq \int_0^t e^{as} ds \int_{-\infty}^{-s} e^{pr(\sigma+s)} |z(s + \sigma)|^p e^{-pr(\sigma+s)} \mu_i(d\sigma) + \int_{-\infty}^0 \mu_i(d\sigma) \int_0^t e^{a(s-\sigma)} |z(s)|^p ds \\ &\leq \|\zeta\|_r^p \int_0^t e^{-(pr-a)s} ds \int_{-\infty}^0 e^{-pr\sigma} \mu_i(d\sigma) + \int_{-\infty}^0 e^{-a\sigma} \mu_i(d\sigma) \int_0^t e^{as} |z(s)|^p ds. \end{aligned}$$

With reference to equation (1), and taking note that $pr > a$, we can conclude that

$$\int_0^t \int_{-\infty}^0 e^{as} |z(\sigma + s)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr - a} \|\zeta\|_r^p + \mu_i^{(pr)} \int_0^t e^{as} |z(s)|^p ds.$$

The proof of (7) is complete. \square

4. Convergence and Mean Square Boundedness

Firstly, let us derive the mean square boundedness for solutions to equation (2).

Theorem 1. Let the hypothesis H holds. Assume the equation (2) with initial condition $\zeta \in C_r((-\infty, 0]; R^n)$ has just one solution $z(t)$. Let a_i , $i = 1, 2, \dots, 5$ assure $2a_1 > 2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)} - 2b_1a_3$. Then there is $a \in (0, (2a_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \wedge 2r)$ so that

$$\hat{\mathbb{E}}[|z(t)|^2] \leq c_1 + c_2 e^{-at}, \quad (8)$$

where

$$c_1 = \frac{1}{a} \left(\frac{1}{\delta} |\kappa(0)|^2 + \frac{b_1}{\delta_1} |\eta(0)|^2 + \frac{b_1}{\delta_2} |\gamma(0)|^2 \right)$$

and

$$c_2 = \hat{\mathbb{E}}|z(0)|^2 + \frac{2a_2\mu_1^{(2r)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{2b_1a_4\mu_2^{(2r)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{b_1a_5\mu_3^{(2r)}}{(2r-a)(1-\delta_2)}\hat{\mathbb{E}}\|\zeta\|_r^2.$$

The values of δ, δ_1 and δ_2 are sufficiently small so that

$$2a_1 - \delta - a - b_1\delta_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - \frac{b_1a_5}{1-\delta_2}\mu_3^{(2r)} > 0.$$

Proof. By using the G-Itô formula, G-Itô integral and Lemma 2, it follows

$$\begin{aligned} \hat{\mathbb{E}}[e^{at}|z(t)|^2] &\leq \hat{\mathbb{E}}|z(0)|^2 + \hat{\mathbb{E}} \int_0^t e^{as} \left[a|z(s)|^2 + 2z^T(s)\kappa(z_s) \right] ds \\ &\quad + b_1 \hat{\mathbb{E}} \int_0^t e^{as} \left[2z^T(s)\eta(z_s) + |\gamma(z_s)|^2 \right] ds. \end{aligned} \quad (9)$$

Utilizing (3), (4) and Lemma 5 we determine

$$\begin{aligned} z^T(t)\kappa(z_t) &\leq \left(\frac{\delta}{2} - a_1\right)|z(t)|^2 + \frac{1}{2\delta}|\kappa(0)|^2 + a_2 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_1(d\sigma), \\ z^T(t)\eta(z_t) &\leq \left(\frac{\delta_1}{2} - a_3\right)|z(t)|^2 + \frac{1}{2\delta_1}|\eta(0)|^2 + a_4 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_2(d\sigma). \end{aligned}$$

It follows from Lemma 4 and the condition given in (5) that

$$|\gamma(z_t)|^2 \leq \frac{1}{\delta_2}|\gamma(0)|^2 + \frac{a_5}{1-\delta_2} \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_3(d\sigma).$$

Using the above inequalities, (9) becomes

$$\begin{aligned} \hat{\mathbb{E}}[e^{at}|z(t)|^2] &\leq \hat{\mathbb{E}}|z(0)|^2 + \frac{1}{a} \left(\frac{1}{\delta}|\kappa(0)|^2 + \frac{b_1}{\delta_1}|\eta(0)|^2 + \frac{b_1}{\delta_2}|\gamma(0)|^2 \right) (e^{at} - 1) \\ &\quad + (\delta + a - 2a_1 + b_1\delta_1 - 2b_1a_3) \hat{\mathbb{E}} \int_0^t e^{as} |z(s)|^2 ds \\ &\quad + 2a_2 \hat{\mathbb{E}} \int_0^t e^{as} \int_{-\infty}^0 |z(s+\sigma)|^2 \mu_1(d\sigma) ds \\ &\quad + 2b_1a_4 \hat{\mathbb{E}} \int_0^t e^{as} \int_{-\infty}^0 |z(s+\sigma)|^2 \mu_2(d\sigma) ds \\ &\quad + b_1 \frac{a_5}{1-\delta_2} \hat{\mathbb{E}} \int_0^t e^{as} \int_{-\infty}^0 |z(s+\sigma)|^2 \mu_3(d\sigma) ds \end{aligned} \quad (10)$$

In view of Lemma 7, it follows

$$\int_0^t \int_{-\infty}^0 e^{as} |z(s+\sigma)|^2 \mu_i(d\sigma) ds \leq \frac{1}{2r-a} \|\zeta\|_r^2 \mu_i^{(2r)} + \mu_i^{(2r)} \int_0^t e^{as} |z(s)|^2 ds. \quad (11)$$

Substituting (11) in (10) we derive

$$\begin{aligned} \mathbb{E}[e^{at}|z(t)|^2] &\leq \mathbb{E}|z(0)|^2 + \frac{2a_2\mu_1^{(2r)}}{2r-a} \mathbb{E}\|\zeta\|_r^2 + \frac{2b_1a_4\mu_2^{(2r)}}{2-ra} \mathbb{E}\|\zeta\|_r^2 + \frac{b_1a_5\mu_3^{(2r)}}{(2r-a)(1-\delta_2)} \mathbb{E}\|\zeta\|_r^2 \\ &+ \frac{1}{\lambda} \left(\frac{1}{\delta} |\kappa(0)|^2 + \frac{b_1}{\delta_1} |\eta(0)|^2 + \frac{b_1}{\delta_2} |\gamma(0)|^2 \right) (e^{at} - 1) \\ &- (2a_1 - \delta - a - b_1\delta_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - \frac{b_1a_5}{1-\delta_2} \mu_3^{(2r)}) \mathbb{E} \int_0^t e^{as} |z(s)|^2 ds. \end{aligned}$$

As $2a_1 > 2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)} - 2b_1a_3$ and $a \in (0, (2a_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \wedge 2r)$. Selecting δ, δ_1 and δ_2 sufficiently small so that

$$2a_1 - \delta - a - b_1\delta_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - \frac{b_1a_5}{1-\delta_2} \mu_3^{(2r)} > 0,$$

we obtain the desired result

$$\mathbb{E}[|z(t)|^2] \leq c_1 + c_2 e^{-at},$$

where

$$c_1 = \frac{1}{a} \left(\frac{1}{\delta} |\kappa(0)|^2 + \frac{b_1}{\delta_1} |\eta(0)|^2 + \frac{b_1}{\delta_2} |\gamma(0)|^2 \right)$$

and

$$c_2 = \mathbb{E}|z_0|^2 + \frac{2a_2\mu_1^{(2q)}}{2r-a} \mathbb{E}\|\zeta\|_r^2 + \frac{2b_1a_4\mu_2^{(2r)}}{2r-a} \mathbb{E}\|\zeta\|_r^2 + \frac{b_1a_5\mu_3^{(2r)}}{(2r-a)(1-\delta_2)} \mathbb{E}\|\zeta\|_r^2.$$

□

Theorem 1 describes that equation (2) has a mean square bounded solution. The following Theorem 2 expresses that any two distinct solutions of equation (2) are convergent.

Theorem 2. Assuming that all hypotheses of Theorem 1 are satisfied, let $z(t)$ and $y(t)$ be two solutions of equation (2) associated with initial values ζ and ξ , respectively. Then, we have:

$$\mathbb{E}[|z(t) - y(t)|^2] \leq c_3 \mathbb{E}|\xi - \zeta|_r^2 e^{-at}, \quad (12)$$

where $c_3 = 1 + \frac{1}{2r-a} (2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)})$.

Proof. Define $\phi(t) = z(t) - y(t)$, $\hat{\gamma}(t) = \gamma(z_t) - \gamma(y_t)$, $\hat{\kappa}(t) = \kappa(z_t) - \kappa(y_t)$, and $\hat{\eta}(t) = \eta(z_t) - \eta(y_t)$. Utilizing the G-Itô integral, Lemma 2, and G-Itô formula, it follows

$$\begin{aligned} \mathbb{E}[e^{at}|\phi(t)|^2] &\leq \mathbb{E}|\xi(0) - \zeta(0)|^2 + \mathbb{E} \int_0^t e^{as} [a|\phi(s)|^2 + 2\phi^T(s)\hat{\kappa}(s)] ds \\ &+ b_1 \mathbb{E} \int_0^t e^{\phi s} [2\phi^T(s)\hat{\eta}(s) + |\hat{\gamma}(s)|^2] ds. \end{aligned} \quad (13)$$

From hypothesis H, we derive

$$\begin{aligned} \phi^T(t)\hat{\kappa}(t) &\leq -a_1|\phi(t)|^2 + a_2 \int_{-\infty}^0 \phi(\sigma+t)\mu_1(d\sigma), \\ \phi^T(t)\hat{\eta}(t) &\leq -a_3|\phi(t)|^2 + a_4 \int_{-\infty}^0 \phi(\sigma+t)\mu_2(d\sigma) \end{aligned}$$

and

$$|\hat{\gamma}(t)|^2 \leq a_5 \int_{-\infty}^0 \phi(t + \sigma) \mu_3(d\sigma).$$

Utilizing the above inequalities, (13) becomes

$$\begin{aligned} \hat{\mathbb{E}}[e^{at} |\phi(t)|^2] &\leq \hat{\mathbb{E}}|\zeta(0) - \xi(0)|^2 + (a - 2a_1 - 2b_1a_3) \hat{\mathbb{E}} \int_0^t e^{as} |\phi(s)|^2 ds \\ &\quad + 2a_2 \hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 e^{as} \phi(s + \sigma) \mu_1(d\sigma) ds + 2b_1a_4 \hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 e^{as} \phi(s + \sigma) \mu_2(d\sigma) ds \\ &\quad + k_1a_5 \hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 e^{as} \phi(s + \sigma) \mu_3(d\sigma) ds. \end{aligned} \quad (14)$$

From Lemma 7 for $i = 1, 2, 3$ it follows

$$\int_0^t \int_{-\infty}^0 e^{as} |\phi(s + \sigma)|^2 \mu_i(d\sigma) ds \leq \frac{1}{2q - a} \|\zeta - \xi\|_r^2 \mu_i^{(2r)} + \mu_i^{(2r)} \int_0^t e^{as} |\phi(s)|^2 ds \quad (15)$$

Plugging (15) in (14) we determine

$$\begin{aligned} e^{at} \hat{\mathbb{E}}|\phi(t)|^2 &\leq \hat{\mathbb{E}}|\zeta(0) - \xi(0)|^2 + \frac{1}{2r - a} [2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)}] \hat{\mathbb{E}}\|\zeta - \xi\|_r^2 \\ &\quad - (2a_1 + 2b_1a_3 - a - 2a_2\mu_1^{(2r)} - 2k_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \hat{\mathbb{E}} \int_0^t e^{as} |\phi(s)|^2 ds. \end{aligned}$$

As $2a_1 > 2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)} - 2b_1a_3$ and $a \in (0, (2a_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \wedge 2r)$ it follows

$$\hat{\mathbb{E}}|\phi(t)|^2 \leq [1 + \frac{1}{2r - a} (2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)})] \hat{\mathbb{E}}\|\zeta - \xi\|_r^2 e^{-at},$$

consequently, we derive the following required result

$$\hat{\mathbb{E}}|z(t) - y(t)|^2 \leq c_3 \hat{\mathbb{E}}\|\zeta - \xi\|_r^2 e^{-at},$$

where $c_3 = 1 + \frac{1}{2r - a} (2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)})$. \square

If $\kappa(0) = \eta(0) = \gamma(0) = 0$, then from Theorem 2 we can obtain that the trivial solution of equation (2) is mean square exponentially stable.

Example 1. Consider $z(t)$ and $y(t)$ be two solutions of the equation

$$dz(t) = z_t dt + \sin(z_t) d\langle B \rangle(t) + z_t dB(t)$$

with initial values ζ and ξ respectively. Define $\phi(t) = z(t) - y(t)$, $\hat{\eta}(t) = \hat{\gamma}(t) = z_t - y_t$ and $\hat{\kappa}(t) = \sin(z_t) - \sin(y_t)$. Under the given hypothesis one can easily derive that $y(t)$ is mean square convergent to $z(t)$.

5. The Exponential Estimate

Firstly, let us determine the \mathcal{L}_G^2 estimates. Let equation (2) with initial condition $\zeta \in C_r((-\infty, 0]; R^n)$ has just one solution $z(t)$ on $t \in [0, \infty)$.

Theorem 3. Assume that the hypothesis H holds and $E\|\zeta\|_r^2 < \infty$. For every $t \geq 0$,

$$\hat{\mathbb{E}} \left[\sup_{-\infty < s \leq t} |z(t)|^2 \right] \leq [\hat{\mathbb{E}}\|\zeta\|_r^2 + m_1] e^{m_2 t},$$

where $m_1 = c + \frac{2}{r}[r + a_2\mu_1^{(2r)} + b_1(a_5\mu_3^{(2r)} + \mu_2^{(2r)}) + 2b_3a_5\mu_3^{(2r)}]\hat{\mathbb{E}}\|\zeta\|_r^2$, $c = 2[|\kappa(0)|^2 + b_1(|\eta(0)|^2 + 2|\gamma(0)|^2) + 4b_3|\gamma(0)|^2]T$ and $m_2 = 2[2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3a_5]$.

Proof. Using the G-Itô formula and properties of the G-expectation, it follows

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right] &\leq \hat{\mathbb{E}}|z(0)|^2 + 2\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s z^T(s)\kappa(z_s)ds\right] \\ &\quad + \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s (2z^T(s)\eta(z_s) + |\gamma(z_s)|^2)d\langle B, B \rangle(s)\right] \\ &\quad + 2\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s z^T(s)\gamma(z_s)dB(s)\right] \end{aligned} \quad (16)$$

From our assumption H , $2a_1a_2 \leq \sum_{i=1}^2 a_i^2$ and $(\sum_{i=1}^2 a_i)^2 \leq 2\sum_{i=1}^2 a_i^2$, it follows

$$z^T(t)\kappa(z_t) \leq -(a_1 - \frac{1}{2})|z(t)|^2 + \frac{1}{2}|\kappa(0)|^2 + a_2 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_1(d\sigma), \quad (17)$$

$$z^T(t)\eta(z_t) \leq -(a_3 - \frac{1}{2})|z(t)|^2 + \frac{1}{2}|\eta(0)|^2 + a_4 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_2(d\sigma), \quad (18)$$

$$|\gamma(z_t)|^2 \leq 2|\gamma(0)|^2 + 2a_5 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_3(d\sigma). \quad (19)$$

In view of (17) and (11), we determine

$$2\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s z^T(s)\kappa(z_s)ds\right] \leq |\kappa(0)|^2 T + \frac{a_2}{r}\hat{\mathbb{E}}\|\zeta\|_r^2 \mu_1^{(2r)} + (2a_2 - 2a_1 + 1)\hat{\mathbb{E}} \int_0^t |z(s)|^2 ds.$$

Utilizing (18), (19), (11) and Lemma 2, it follows

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s (2z^T(s)\eta(z_s) + |\gamma(z_s)|^2)d\langle B, B \rangle(s)\right] &\leq b_1\hat{\mathbb{E}}\left[\int_0^t (2z^T(s)\eta(z_s) + |\gamma(z_s)|^2)ds\right] \\ &\leq b_1[|\eta(0)|^2 + 2|\gamma(0)|^2]T + b_1\frac{1}{r}(a_5\mu_3^{(2r)} + \mu_2^{(2r)})\hat{\mathbb{E}}\|\zeta\|_r^2 \\ &\quad + b_1(2a_5 - 2a_3 + 3)\hat{\mathbb{E}} \int_0^t |z(s)|^2 ds. \end{aligned}$$

The inequality $a_1a_2 \leq \frac{1}{2}\sum_{i=1}^2 a_i$, Lemma 1 and (19) give

$$\begin{aligned} 2\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s z^T(s)\gamma(z_s)dB(s)\right] &\leq 2b_2\hat{\mathbb{E}}\left[\int_0^t |z^T(s)\gamma(z_s)|^2 ds\right]^{\frac{1}{2}} \\ &\leq \frac{1}{2}\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right] + 4b_2^2|\gamma(0)|^2 T + 4b_2^2a_5\hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 |z(s+\sigma)|^2 \mu_3(d\sigma)ds. \end{aligned}$$

Using Lemma 7, we derive

$$\begin{aligned} 2\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} \int_0^s z^T(s)\gamma(z_s)dB(s)\right] \\ \leq \frac{1}{2}\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right] + 4b_3|\gamma(0)|^2 T + \frac{4b_3a_5}{2r}\mu_3^{(2r)}\hat{\mathbb{E}}\|\zeta\|_r^2 + 4b_3a_5\hat{\mathbb{E}} \int_0^t |z(s)|^2 ds, \end{aligned} \quad (20)$$

where $b_3 = b_2^2$. By substituting the aforementioned inequalities into equation (16), and letting $m_1 = c + \frac{2}{r}[r + a_2\mu_1^{(2r)} + b_1(a_5\mu_3^{(2r)} + \mu_2^{(2r)}) + 2b_3a_5\mu_3^{(2r)}]E\|\zeta\|_r^2$, $m_2 = 2[2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3a_5]$, we can evaluate the result

$$\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(t)|^2\right] \leq m_1 + m_2 \int_0^t \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right] ds, \quad (21)$$

where $c = 2\left[|\kappa(0)|^2 + b_1(|\eta(0)|^2 + 2|\gamma(0)|^2) + 4b_3|\gamma(0)|^2\right]T$. By observing that

$$\hat{\mathbb{E}}\left[\sup_{-\infty < s \leq t} |z(s)|^2\right] \leq \hat{\mathbb{E}}\|\zeta\|_r^2 + \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right],$$

it follows

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{-\infty < s \leq t} |z(s)|^2\right] &\leq \hat{\mathbb{E}}\|\zeta\|_r^2 + m_1 + m_2 \int_0^t \hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right] ds \\ &\leq \hat{\mathbb{E}}\|\zeta\|_r^2 + m_1 + m_2 \int_0^t \hat{\mathbb{E}}\left[\sup_{-\infty < s \leq t} |z(s)|^2\right] ds. \end{aligned}$$

Finally, the required result is obtained by using the Gronwall inequality. \square

Theorem 4. Under the conditions of Theorem 3, it follows

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |z(t)| \leq \alpha,$$

and $\alpha = 2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3a_5$.

Proof. Assuming that $m_1 = m + \frac{2}{r}[r + a_2\mu_1^{(2r)} + b_1(a_5\mu_3^{(2r)} + \mu_2^{(2r)}) + 2b_3a_5\mu_3^{(2r)}]E\|\zeta\|_r^2$ and $m_2 = 2[2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3a_5]$ then from the inequality (21), we can conclude that:

$$\hat{\mathbb{E}}\left[\sup_{0 \leq s \leq t} |z(s)|^2\right] \leq m_1 e^{m_2 t}, \quad (22)$$

where $m = 2\left[|\kappa(0)|^2 + b_1(|\eta(0)|^2 + 2|\gamma(0)|^2) + 4b_3|\gamma(0)|^2\right]T$. For each $q = 1, 2, 3, \dots$, from (22) it follows

$$\hat{\mathbb{E}}\left[\sup_{q-1 \leq t \leq q} |z(t)|^2\right] \leq m_1 e^{m_2 q}.$$

By utilizing Lemma 3 for every given $\delta > 0$, we obtain:

$$\begin{aligned} \nu\left\{w : \sup_{q-1 \leq t \leq q} |z(t)|^2 > e^{(m_2 + \delta)q}\right\} &\leq \frac{\hat{\mathbb{E}}\left[\sup_{q-1 \leq t \leq q} |z(t)|^2\right]}{e^{(m_2 + \delta)q}} \\ &\leq \frac{m_1 e^{m_2 q}}{e^{(m_2 + \delta)q}} \\ &= m_1 e^{-\delta q}. \end{aligned}$$

But the Borel-Cantelli lemma gives that for almost every $w \in \Omega$ there is a random number $q_0 = q_0(w) \in \mathbb{Z}$ in a manner that when $q \geq q_0$, then

$$\sup_{q-1 \leq t \leq q} |z(t)|^2 \leq e^{(m_2 + \delta)q},$$

which implies

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{1}{t} \log |z(t)| &\leq \frac{m_2 + \delta}{2} \\ &= 2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3a_5 + \frac{\delta}{2},\end{aligned}$$

as δ is arbitrary and letting $\alpha = 2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3a_5$, we can conclude that

$$\alpha \geq \limsup_{t \rightarrow \infty} \frac{1}{t} \log |z(t)|.$$

The proof stands completed. \square

The lemma above expresses that the second moment of the Lyapunov exponent, as defined in [13] as $\lim_{t \rightarrow \infty} \sup \frac{1}{t} \log |z(t)|$, is bounded above by M.

6. Conclusion

Several stochastic functional differential equations (SFDEs) in financial mathematics do not hold the standard Lipschitz assumption such as the Cox-Ingersoll-Ross, Heston and Ait-Sahalia models. In this article some useful monotone type conditions have been introduced. We have proved that any two solutions of SFDEs in the G-framework under distinct initial conditions are convergent. The solutions are mean square bounded. The \mathcal{L}_G^2 and exponential estimates have been calculated. We anticipate that the findings presented in this article will offer valuable insights into the analysis of equations, even when not under the constraints of standard assumptions. This contribution is poised to have a substantial positive impact on the examination of various unresolved inquiries, including the investigation into the existence, uniqueness, convergence and stability of solutions for backward and forward stochastic dynamic systems driven by G-Brownian motion with conditions of a monotone nature.

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