Article

Continuous Rankin Bound for Hilbert and Banach Spaces

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Abstract: Let (Ω, μ) be a measure space and $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a normalized continuous Bessel family for a real Hilbert space $\mathcal{H}$. If the diagonal $\Delta := \{(\alpha, \alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega \times \Omega$, then we show that $\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \langle \tau_\alpha, \tau_\beta \rangle \geq \frac{-(\mu \times \mu)(\Delta)}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)}$. We call Inequality (1) as continuous Rankin bound. It improves 76 years old result of Rankin [*Ann. of Math., 1947*]. It also answers one of the questions asked by K. M. Krishna in the paper [Continuous Welch bounds with applications, *Commun. Korean Math. Soc., 2023*]. We also derive Banach space version of Inequality (1).

Keywords: rankin bound; Continuous Bessel family

MSC: 42C15

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1. Introduction

In 1947, Rankin derived following result for a collection of unit vectors in $\mathbb{R}^d$.

Theorem 1. (*Rankin Bound*) [1–3] *If $\{\tau_j\}_{j=1}^n$ is a collection of unit vectors in $\mathbb{R}^d$, then*

$$\max_{1 \leq j,k \leq n, j \neq k} \langle \tau_j, \tau_k \rangle \geq \frac{-1}{n-1}. \tag{1}$$

In particular,

$$\min_{1 \leq j,k \leq n, j \neq k} \|\tau_j - \tau_k\|^2 \leq \frac{2n}{n-1}. \tag{2}$$

Striking feature of Inequalities (1) and (2) is that they do not depend upon the dimension d. Inequalities (1) and (2) play important roles in the study of packings of lines (which motivated to study the packings of planes) [4,5], Kepler conjecture [6,7], sphere packings [8,9] and the geometry of numbers [10].

After the derivation of continuous Welch bounds in most general form [11], author asked what is the version of Rankin bound for collections indexed by measure spaces. We are going to answer this in this paper.

2. Continuous Rankin bound

We start by recalling the notion of continuous frames which are introduced independently by Ali, Antoine and Gazeau [12] and Kaiser [13]. In the paper, $\mathcal{H}$ denotes a real Hilbert space (need not be finite dimensional).

Definition 1. [12–15] *Let (Ω, μ) be a measure space. A collection $\{\tau_\alpha\}_{\alpha \in \Omega}$ in a Hilbert space $\mathcal{H}$ is said to be a* **continuous frame** *(or generalized frame) for $\mathcal{H}$ if the following holds.*

(i) For each $h \in \mathcal{H}$, the map $\Omega \ni \alpha \mapsto \langle h, \tau_\alpha \rangle \in \mathbb{K}$ is measurable.

(ii) There are $a, b > 0$ such that

$$a\|h\|^2 \le \int_\Omega |\langle h, \tau_\alpha \rangle|^2 \, d\mu(\alpha) \le b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

*If we do not demand the first inequality in (ii), then we say it is a **continuous Bessel family** for $\mathcal{H}$. A continuous Bessel family $\{\tau_\alpha\}_{\alpha \in \Omega}$ is said to be normalized or unit norm if $\|\tau_\alpha\| = 1$, $\forall \alpha \in \Omega$.*

Given a continuous Bessel family, the analysis operator

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h \in \mathcal{L}^2(\Omega); \quad \theta_\tau h : \Omega \ni \alpha \mapsto \langle h, \tau_\alpha \rangle \in \mathbb{K}$$

is a well-defined bounded linear operator. Its adjoint, the synthesis operator is given by

$$\theta_\tau^* : \mathcal{L}^2(\Omega) \ni f \mapsto \int_\Omega f(\alpha) \tau_\alpha \, d\mu(\alpha) \in \mathcal{H}.$$

By combining analysis and synthesis operators, we get the frame operator, defined as

$$S_\tau := \theta_\tau^* \theta_\tau : \mathcal{H} \ni h \mapsto \int_\Omega \langle h, \tau_\alpha \rangle \tau_\alpha \, d\mu(\alpha) \in \mathcal{H}.$$

Note that the integrals are weak integrals (Pettis integrals [16]). With this machinery, we generalizes Theorem 1.

Theorem 2. (***Continuous Rankin Bound***) *Let (Ω, μ) be a measure space and $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a normalized continuous Bessel family for a real Hilbert space $\mathcal{H}$. If the diagonal $\Delta := \{(\alpha, \alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega \times \Omega$, then*

$$\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \langle \tau_\alpha, \tau_\beta \rangle \ge \frac{-(\mu \times \mu)(\Delta)}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)}. \tag{3}$$

In particular,

$$\inf_{\alpha, \beta \in \Omega, \alpha \neq \beta} \|\tau_\alpha - \tau_\beta\|^2 \le 2\left(1 + \frac{(\mu \times \mu)(\Delta)}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)}\right). \tag{4}$$

Proof. Since $\mu(\Omega) < \infty$ (see lemma 2.3 in [11]), $\chi_\Omega \in \mathcal{L}^2(\Omega)$ and

$$\int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau_\alpha, \tau_\beta \rangle| \, d(\mu \times \mu)(\alpha, \beta) \le \int_{(\Omega \times \Omega) \setminus \Delta} \|\tau_\alpha\| \|\tau_\beta\| \, d(\mu \times \mu)(\alpha, \beta) = (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta) < \infty.$$

Now by using Fubini's theorem, we get

$$0 \leq \|\theta_\tau^* \chi_\Omega\|^2 = \langle \theta_\tau^* \chi_\Omega, \theta_\tau^* \chi_\Omega \rangle = \left\langle \int_\Omega \chi_\Omega(\alpha) \tau_\alpha \, d\mu(\alpha), \int_\Omega \chi_\Omega(\beta) \tau_\beta \, d\mu(\beta) \right\rangle$$

$$= \left\langle \int_\Omega \tau_\alpha \, d\mu(\alpha), \int_\Omega \tau_\beta \, d\mu(\beta) \right\rangle = \int_\Omega \int_\Omega \langle \tau_\alpha, \tau_\beta \rangle \, d\mu(\alpha) \, d\mu(\beta) = \int_{\Omega \times \Omega} \langle \tau_\alpha, \tau_\beta \rangle \, d(\mu \times \mu)(\alpha, \beta)$$

$$= \int_\Delta \langle \tau_\alpha, \tau_\beta \rangle \, d(\mu \times \mu)(\alpha, \beta) + \int_{(\Omega \times \Omega) \setminus \Delta} \langle \tau_\alpha, \tau_\beta \rangle \, d(\mu \times \mu)(\alpha, \beta)$$

$$= \int_\Delta \langle \tau_\alpha, \tau_\alpha \rangle \, d(\mu \times \mu)(\alpha, \beta) + \int_{(\Omega \times \Omega) \setminus \Delta} \langle \tau_\alpha, \tau_\beta \rangle \, d(\mu \times \mu)(\alpha, \beta)$$

$$= (\mu \times \mu)(\Delta) + \int_{(\Omega \times \Omega) \setminus \Delta} \langle \tau_\alpha, \tau_\beta \rangle \, d(\mu \times \mu)(\alpha, \beta)$$

$$\leq (\mu \times \mu)(\Delta) + \left(\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \langle \tau_\alpha, \tau_\beta \rangle \right) (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta).$$

Now writing inner product using norm, we get

$$\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \langle \tau_\alpha, \tau_\beta \rangle = \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \left(\frac{\|\tau_\alpha\|^2 + \|\tau_\beta\|^2 - \|\tau_\alpha - \tau_\beta\|^2}{2} \right) = \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \left(\frac{2 - \|\tau_\alpha - \tau_\beta\|^2}{2} \right)$$

$$= 1 - \frac{\inf_{\alpha, \beta \in \Omega, \alpha \neq \beta} \|\tau_\alpha - \tau_\beta\|^2}{2}.$$

Therefore

$$1 - \frac{\inf_{\alpha, \beta \in \Omega, \alpha \neq \beta} \|\tau_\alpha - \tau_\beta\|^2}{2} \geq \frac{-(\mu \times \mu)(\Delta)}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)}$$

which gives

$$\frac{\inf_{\alpha, \beta \in \Omega, \alpha \neq \beta} \|\tau_\alpha - \tau_\beta\|^2}{2} \leq 1 + \frac{(\mu \times \mu)(\Delta)}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)}.$$

□

Corollary 1. *Theorem 1 follows from Theorem 2.*

Proof. Take $\Omega = \{1, \ldots, n\}$ and μ as the counting measure. □

A remarkable feature of Inequality (3) is that it allows to derive Inequality (4). We can not do this by using first order continuous Welch bound [11].

Given a measure space (Ω, μ) with measurable diagonal and a normalized continuous Bessel family $\{\tau_\alpha\}_{\alpha \in \Omega}$ for a real Hilbert space $\mathcal{H}$, we define

$$\mathcal{M}(\{\tau_\alpha\}_{\alpha \in \Omega}) := \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} \langle \tau_\alpha, \tau_\beta \rangle$$

and

$$\mathcal{N}(\{\tau_\alpha\}_{\alpha \in \Omega}) := \inf_{\alpha, \beta \in \Omega, \alpha \neq \beta} \|\tau_\alpha - \tau_\beta\|^2.$$

Similar to the problem of Grassmannian frames (see [17]), we propose following problem.

Question 3. *Given a measure space (Ω, μ) with measurable diagonal and a real Hilbert space $\mathcal{H}$, find normalized continuous Bessel family $\{\tau_\alpha\}_{\alpha \in \Omega}$ for $\mathcal{H}$, such that*

$$\mathcal{M}(\{\tau_\alpha\}_{\alpha \in \Omega}) = \inf\{\mathcal{M}(\{\omega_\alpha\}_{\alpha \in \Omega}) : \{\omega_\alpha\}_{\alpha \in \Omega} \text{ is a normalized continuous Bessel family for } \mathcal{H}\}. \quad (5)$$

Equivalently, find normalized continuous Bessel family $\{\tau_\alpha\}_{\alpha \in \Omega}$ for $\mathcal{H}$, such that

$$\mathcal{N}(\{\tau_\alpha\}_{\alpha \in \Omega}) = \sup\{\mathcal{N}(\{\omega_\alpha\}_{\alpha \in \Omega}) : \{\omega_\alpha\}_{\alpha \in \Omega} \text{ is a normalized continuous Bessel family for } \mathcal{H}\}.$$

Further, for which measure spaces (Ω, μ) and real Hilbert spaces $\mathcal{H}$, solution to (5) exists?

3. Continuous Rankin bound for Banach spaces

In this section, we derive continuous Rankin bound for Banach spaces. First we need a notion.

Definition 2. *[18] Let (Ω, μ) be a measure space and $p \in [1, \infty)$. Let $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha \in \Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ is said to be a **continuous p-Bessel family** for $\mathcal{X}$ if the following conditions are satisfied.*

(i) *For each $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ is measurable.*
(ii) *For each $u \in \mathcal{L}^p(\Omega, \mu)$, the map $\Omega \ni \alpha \mapsto u(\alpha)\tau_\alpha \in \mathcal{X}$ is measurable.*
(iii) *The map (**continuous analysis operator**)*

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

is a well-defined bounded linear operator.
1. *The map (**continuous synthesis operator**)*

$$\theta_\tau : \mathcal{L}^p(\Omega, \mu) \ni u \mapsto \theta_\tau u := \int_\Omega u(\alpha)\tau_\alpha \, d\mu(\alpha) \in \mathcal{X}$$

is a well-defined bounded linear operator.

Theorem 4. *(**Functional Continuous Rankin Bound**) Let (Ω, μ) be a finite measure space and $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ be a continuous p-approximate Bessel family for a real Banach space $\mathcal{X}$ satisfying the following.*

(i) *$f_\alpha(\tau_\alpha) = 1$ for all $\alpha \in \Omega$.*
(ii) *$\|f_\alpha\| \leq 1$, $\|\tau_\alpha\| \leq 1$ for all $1 \leq \alpha \in \Omega$.*
(iii) *$\theta_f \theta_\tau \chi_\Omega \geq 0$.*

If the diagonal $\Delta := \{(\alpha, \alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega \times \Omega$, then

$$\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} f_\alpha(\tau_\beta) \geq \frac{-(\mu \times \mu)(\Delta)}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)}.$$

Proof. Since $\mu(\Omega) < \infty$, we have

$$\int_{(\Omega \times \Omega) \setminus \Delta} |f_\alpha(\tau_\beta)| \, d(\mu \times \mu)(\alpha, \beta) \leq \int_{(\Omega \times \Omega) \setminus \Delta} \|f_\alpha\| \|\tau_\beta\| \, d(\mu \times \mu)(\alpha, \beta) \leq (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta) < \infty.$$

Now by using Fubini's theorem, we get

$$
\begin{aligned}
0 \leq \int_{\Omega} (\theta_f \theta_\tau \chi_\Omega)(\alpha)\, d\mu(\alpha) &= \int_{\Omega} f_\alpha(\theta_\tau \chi_\Omega)\, d\mu(\alpha) = \int_{\Omega} f_\alpha \left(\int_{\Omega} \chi_\Omega(\beta)\tau_\beta\, d\mu(\alpha) \right) d\mu(\alpha) \\
&= \int_{\Omega} f_\alpha \left(\int_{\Omega} \tau_\beta\, d\mu(\beta) \right) d\mu(\alpha) = \int_{\Omega} \int_{\Omega} f_\alpha(\tau_\beta)\, d\mu(\beta)\, d\mu(\alpha) = \int_{\Omega \times \Omega} f_\alpha(\tau_\beta)\, d(\mu \times \mu)(\alpha, \beta) \\
&= \int_{\Delta} f_\alpha(\tau_\beta)\, d(\mu \times \mu)(\alpha, \beta) + \int_{(\Omega \times \Omega) \setminus \Delta} f_\alpha(\tau_\beta)\, d(\mu \times \mu)(\alpha, \beta) \\
&= \int_{\Delta} f_\alpha(\tau_\alpha)\, d(\mu \times \mu)(\alpha, \beta) + \int_{(\Omega \times \Omega) \setminus \Delta} f_\alpha(\tau_\beta)\, d(\mu \times \mu)(\alpha, \beta) \\
&= (\mu \times \mu)(\Delta) + \int_{(\Omega \times \Omega) \setminus \Delta} f_\alpha(\tau_\beta)\, d(\mu \times \mu)(\alpha, \beta) \\
&\leq (\mu \times \mu)(\Delta) + \left(\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} f_\alpha(\tau_\beta) \right) (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta).
\end{aligned}
$$

□

Corollary 2. *Let $\{\tau_j\}_{j=1}^n$ be a collection in a real Banach space $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathcal{X}^*$ satisfying the following.*

(i) $f_j(\tau_j) = 1$ *for all* $1 \leq j \leq n$.
(ii) $\|f_j\| \leq 1$, $\|\tau_j\| \leq 1$ *for all* $1 \leq j \leq n$.
(iii) $\sum_{1 \leq j,k \leq n} f_j(\tau_k) \geq 0$.

Then

$$
\max_{1 \leq j,k \leq n, j \neq k} f_j(\tau_k) \geq \frac{-1}{n-1}.
$$

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