Article

Not peer-reviewed version

On the Life-Line Game of the Inertial Players with Integral and Geometric Constraints

Bahrom Samatov , Gafurjan Ibragimov , Bahodirjon Juraev , Massimiliano Ferrara *

Posted Date: 18 August 2023

doi: 10.20944/preprints202308.1311.v1

Keywords: Differential games; integral constraint; geometric constraint; pursuers; evaders; strategy; life-line games

Article

On the Life-Line Game of the Inertial Players with Integral and Geometric Constraints

Bahrom Samatov [1], Gafurjan Ibragimov [2], Bahodirjon Juraev [3] and Massimiliano Ferrara [4,*]

1 Department of Mathematics, Namangan State University, Namangan, 116019, Uzbekistan; samatov57@gmail.com
2 Department of General and Exact Subjects, Tashkent State University of Economics, Tashkent, 100006, Uzbekistan; gofurjon.ibragimov@tsue.uz
3 Faculty of Physics-Mathematics, Andijan State University, Andijon, 170100, Uzbekistan; jbahodirjon@bk.ru
4 Department of Law, Economics and Human Sciences, University Mediterranea of Reggio Calabria, & ICRIOS - The Invernizzi Centre for Research in Innovation, Organization, Strategy and Entrepreneurship Bocconi University, Department of Management and Technology, Italy; massimiliano.ferrara@unirc.it, massimiliano.ferrara@unibocconi.it
* Correspondence: massimiliano.ferrara@unirc.it, massimiliano.ferrara@unibocconi.it

Abstract: In this paper, we consider a pursuit-evasion game of inertial players, where pursuer's control is subject to integral constraint, and evader's control is subject to geometric constraint. In the pursuit problem, the main tool is the strategy of parallel pursuit. Sufficient conditions are obtained for the solvability of pursuit-evasion problems. Also, the main lemma describing the monotonicity of an attainability domain of the evader is proved and an explicit analytical formula for this domain is given. As one of the important results of our paper, the Isaacs Life-line game is solved for a special case.

Keywords: differential game; integral constraint; geometric constraint; pursuer; evader; strategy; guaranteed capture time; attainability domain; Life-line game

1. Introduction

In the middle of the twentieth century, the term "Differential games" was first apparent in Isaacs's monograph [1] including exclusive conflict and game problems and also, additional open problems for future studies. In view of the basic approaches in the theory of differential games constructed by Pontryagin [2], Krasovskii [3], Pshenichnii [4], Chikrii [5] and others, a differential game is regarded as an optimal control problem from the standpoint of either a pursuer or an evader, and in consequence, the differential game comes to study a pursuit problem and to an evasion problem, respectively.

A number of the problems in [1] were solved by Petrosjan, Azamov and et al. In particular, in Petrosjan [6] completely solved the Life-line problem when players' controls obey to geometric constraints, developing the strategy of parallel approach. In the middle of the eighteenth Azamov [7] proposed for the first time an analytical solution of the Life-line game of multiple pursuers and one evader by using support function of the multi-valued mapping. Azamov [8] investigated the structure of phase space of differential pursuit-evasion games for the case where an evader has the discriminated information and here, an alternative for differential pursuit-evasion games in $[0, \infty)$ was established by the transfinite iteration method of Pshenichnii's operator.

In addition, in Munts and Kumkov [9,10] examined the classic time-optimal differential games of Life-line in the formalization of Krasovskii [3]. Thereafter, for the cases in which controls of both objects adhering to integral, linear, Grönwall type or mixed constraints, the pursuit-evasion problem and the Life-line game have been thoroughly considered in the works of Samatov et al.[11–13].

In the theory of differential games, it is known that it is not easy to construct players' optimal strategies and to determine the game value. The works [14,15] are especially devoted to establish the existence of the game value by constructing the players' optimal strategies.

 doi:10.20944/preprints202308.1311.v1

In order to implement mathematical models to real-life processes, the researches on differential games with various type of restrictions on controls are getting a great interest. For example, the works [16,17] are devoted to investigate such type of game problems. It is essential to mention that the games with various type constraints on controls are not enough explored yet. In the applying sense, differential games under integral constraints on players' controls are much more difficult than the games with geometric constraint.

The method of resolving functions for the games with integral constraints on control functions was developed by Belousov [18] to obtain a sufficient condition for the pursuit differential game. The solution was further extended to the case of convex integral constraints [19]. The pursuit-evasion and Life-line problems under integral constraints on controls were widely considered by Satimov at al. [20], Azamov [7], Azamov and Samatov [21]. Such differential game problems with integral constraints have been considered for differential-difference equations as well. For instance, the works of Mamadaliev [22–24] are devoted to linear pursuit games under integral constraints on players' controls with delay information. Thereafter, the games under linear, linear-geometric, integro-geometric, and mixed constraints on controls were comprehensively solved using the Π-strategy in the works of Samatov [25–29].

Differential games, of inertial players are of great significance owing to many applications to technical, air processes. Ibragimov et al. [30–32] studied the fixed duration differential game of countable number inertial players in Hilbert space with integral constraints. The works [33,34] studied the pursuit-evasion differential games of many pursuers and one evader with geometric constraints for the infinite system of second order differential equations in Hilbert space.

In the present paper we consider the pursuit-evasion differential game problems of one inertial pursuer and one inertial evader. The control of the pursuer is subject to integral constraint and that of the evader is subject to geometric constraint. In the pursuit problem, we propose the parallel approach strategy(Π-strategy) for the pursuer, and obtain sufficient solvability condition of pursuit. In the evasion problem, a special admissible control is implemented for the evader and sufficient solvability condition of evasion is obtained. An estimate from the below for the distance between the pursuer and evader is derived. Further, the main lemma describing the monotonicity of an attainability domain of the evader is proved and an explicit analytical formula for this domain is given. As one of the important results of our paper, we obtain a condition under which pursuer wins in the Isaacs Life-line game.

2. Statement of problems

Suppose that a player P, namely Pursuer, with a control u goes after another player E, namely Evader, with a control v in $\mathbb{R}^n$. Let x represent the position of Pursuer, and let y represent that of Evader. Then we are going to discuss a differential game where players' motions take place in accordance with the following second order differential equations:

$$P: \ \ddot{x} = u, \ \ x(0) = x_0, \ \ \dot{x}(0) = x_1, \tag{1}$$

$$E: \ \ddot{y} = v, \ \ y(0) = y_0, \ \ \dot{y}(0) = y_1, \tag{2}$$

where $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$; x_0, y_0 are players' initial positions, and x_1, y_1 are their initial velocities, respectively. Assume that game is considered under the conditions $x_0 \neq y_0$ and $x_1 = y_1$.

The control parameter u is the acceleration vector, and this is considered as a measurable function $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, and on that function, impose the *integral constraint* in the form

$$\int_0^t (t-s)|u(s)|^2 ds \leq \rho_0, \ \ t \geq 0, \tag{3}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

where ρ_0 is a given positive number. In the sequel, by $\mathbb{U}$ we infer the class of all control functions $u(\cdot)$ fulfilling the constraint (3).

In the same way, the control parameter v is the acceleration vector, and this is taken as a measurable function $v(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, and on that function, impose the *geometrical constraint* of the form

$$|v(t)| \le \beta, \quad t \ge 0, \tag{4}$$

where β is a given positive number. In the sequel, by $\mathbb{V}$ we infer the class of all control functions $v(\cdot)$ fulfilling the constraint (4).

Definition 1. *We say that a measurable function $u(\cdot)$ (respectively, $v(\cdot)$) the admissible control of Pursuer (respectively, of Evader) if $u(\cdot) \in \mathbb{U}$ (respectively, $v(\cdot) \in \mathbb{V}$).*

Definition 2. *For each pair $(\rho_0, u(\cdot))$, $u(\cdot) \in \mathbb{U}$, we call the quantity*

$$\rho(t) = \rho_0 - \int_0^t (t-s)|u(s)|^2 ds, \;\; \rho(0) = \rho_0 \tag{5}$$

the residual resource of Pursuer at the current time t, $t \ge 0$.

Let $D_\beta = \{d \in \mathbb{R}^n : \; |d| \le \beta\}$.

Definition 3. *We call a mapping $\boldsymbol{u} : D_\beta \to \mathbb{R}^n$ the strategy of Pursuer if the following conditions are satisfied:*
1) $\boldsymbol{u}(v)$ is a Borel measurable function of v, $v \in D_\beta$;
2) For arbitrary $v(\cdot) \in \mathbb{V}$, the inclusion $\boldsymbol{u}(v(\cdot)) \in \mathbb{U}$ is satisfied on some time interval $[0, t]$.

Let

$$z(t) = x(t) - y(t), \; z_0 = x_0 - y_0, \; \dot{z}(0) = z_1 = x_1 - y_1.$$

From equations (1)–(2) we then derive the initial value problem

$$\ddot{z} = u - v, \;\; z(0) = z_0, \;\; \dot{z}(0) = 0.$$

Definition 4. *We call a strategy $\boldsymbol{u}(v)$ the parallel convergence strategy (Π- strategy) if for arbitrary $v(\cdot) \in \mathbb{V}$ the solution $z(t)$ of initial value problem*

$$\ddot{z} = \boldsymbol{u}(v(t)) - v(t), \;\; z(0) = z_0, \;\; \dot{z}(0) = 0,$$

has the form

$$z(t) = z_0\Theta(t, v(\cdot)), \;\; \Theta(0, v(\cdot)) = 1, \;\; t \ge 0,$$

where $\Theta(t, v(\cdot))$ is a scalar function, and this is generally called the convergence function of Pursuer and Evader in the pursuit problem.

Definition 5. *We say that the Pursuer wins by using the Π-strategy on a finite time interval $[0, T]$ if, for any $v(\cdot) \in \mathbb{V}$:*
a) $z(t^*) = 0$ at some instant $t^* \in [0, T]$;
b) $\boldsymbol{u}(v(t)), 0 \le t \le t^*$, belongs to $\mathbb{U}$ on the interval $[0, t^*]$. In this case, the number T is called the guaranteed capture time.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

Definition 6. *We say that Evader wins by using a control $\boldsymbol{v}^*(\cdot) \in \mathbb{V}$ if for arbitrary $u(\cdot) \in \mathbb{U}$ the solution $z(t)$ of the initial value problem*

$$\ddot{z} = u(t) - \boldsymbol{v}^*(t), \;\; z(0) = z_0, \;\; \dot{z}(0) = 0,$$

satisfies the condition $z(t) \neq 0$ for all $t \geq 0$.

This paper is dedicated to study the following game problems where the controls $u(\cdot)$ and $v(\cdot)$ of the players are subject to the constraints (3) and (4) respectively:

Problem 1. *Construct Π-strategy to ensure the completion of the pursuit possible (Pursuit Game).*

Problem 2. *Set a special admissible control for Evader and determine conditions guaranteeing the escape (Evasion Game).*

Problem 3. *Find sufficient conditions of completion pursuit in Life-line game.*

3. Pursuit Game

We call the function

$$\mathbf{u}(v) = v - \theta(v)\xi_0 \tag{6}$$

the Π-*strategy* of Pursuer, where

$$\theta(v) = \langle v, \xi_0 \rangle + \frac{\eta_0}{2} + \sqrt{\left(\langle v, \xi_0 \rangle + \frac{\eta_0}{2}\right)^2 - |v|^2}, \;\; \xi_0 = \frac{z_0}{|z_0|}, \;\; \eta_0 = \frac{\rho_0}{|z_0|},$$

and $\langle v, \xi_0 \rangle$ is the inner product of the vectors v and ξ_0 in $\mathbb{R}^n$. Here $\theta(v)$ is usually called the *resolving function*.

Let us present the following important property of the strategy (6) and the resolving function $\theta(v)$.

Proposition 1. *If $\rho_0 \geq 4\beta|z_0|$, then, for all $v \in D_\beta$,*
a) $\theta(v)$ is well-defined and continuous in D_β,
b) the following is true

$$\theta_1 \leq \theta(v) \leq \theta_2, \tag{7}$$

where $\theta_1 = \frac{\eta_0}{2} - \beta + \sqrt{\frac{\eta_0^2}{4} - \eta_0\beta}, \;\; \theta_2 = \frac{\eta_0}{2} + \beta + \sqrt{\frac{\eta_0^2}{4} + \eta_0\beta},$
c) the identity

$$|\boldsymbol{u}(v)|^2 = \eta_0\theta(v), \;\; t \geq 0, \tag{8}$$

holds.

Proof. a) From the conditions $\rho_0 \geq 4\beta|z_0|$ and $\eta_0 = \frac{\rho_0}{|z_0|}$ we infer $\eta_0\left(\frac{\eta_0}{4} - \beta\right) \geq 0$. Since $|v| \leq \beta$ (see (4)), therefore

$$\begin{aligned} 0 \;\; &\leq \;\; \eta_0\left(\frac{\eta_0}{4} - \beta\right) \leq \frac{\eta_0^2}{4} - \eta_0|v| \\ &= \;\; \left(-|v| + \frac{\eta_0}{2}\right)^2 - |v|^2 \leq \left(\langle v, \xi_0 \rangle + \frac{\eta_0}{2}\right)^2 - |v|^2. \end{aligned}$$

b) Letting $\tau = \langle v, \xi_0 \rangle$ in $\theta(v)$ consider the function

$$f(\tau) = \tau + \frac{\eta_0}{2} + \sqrt{\left(\tau + \frac{\eta_0}{2}\right)^2 - |v|^2}, \quad -\beta \leq \tau \leq \beta.$$

Clearly, $\frac{df(\tau)}{d\tau} > 0$. As a consequence, using (4) it is not hard to obtain that $\min f(\tau) = f(-\beta) = \theta_1$ and $\max f(\tau) = f(\beta) = \theta_2$.

c) From (6) we get

$$|\mathbf{u}(v)|^2 = \langle \mathbf{u}(v), \mathbf{u}(v) \rangle = \Big\langle v - \theta(v)\xi_0, v - \theta(v)\xi_0 \Big\rangle$$
$$= |v|^2 + \theta(v)\Big[\theta(v) - 2\langle v, \xi_0 \rangle\Big] = \eta_0 \theta(v).$$

and this completes the proof. □

Thanks to equations (1)-(2), for arbitrary $v(\cdot) \in \mathbb{V}$ and for the function $\mathbf{u}(v(\cdot)) \in \mathbb{U}$, Pursuer's trajectory is

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)\mathbf{u}(v(s))ds, \tag{9}$$

and Evader's trajectory is

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s)ds. \tag{10}$$

In this case, the goal of Pursuer P is to capture Evader E, i.e. to achieve the equation $x(t) = y(t)$. Evader E strives to avoid an encounter, i.e. to maintain the inequality $x(t) \neq y(t)$ for all $t \geq 0$, and if this can't be done, to delay the encounter time as long as possible.

If $\rho_0 \geq 4\beta|z_0|$, then the scalar function

$$\Theta(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t (t-s)\theta(v(s))ds, \quad t \geq 0, \tag{11}$$

is called the *convergence function* of the players in Pursuit Game.

Lemma 1. *Let $\rho_0 \geq 4\beta|z_0|$. Then:*

a) for any $v(\cdot) \in \mathbb{V}$ the function $\Theta(t, v(\cdot))$ is monotonically decreasing in t, $t \geq 0$;

b) for all $t \in [0, T]$,

$$\Theta^*(t) \leq \Theta(t, v(\cdot)) \leq \Theta^{**}(t), \tag{12}$$

where $\Theta^*(t) = 1 - \frac{\theta_2}{2|z_0|}t^2$ and $\Theta^{**}(t) = 1 - \frac{\theta_1}{2|z_0|}t^2$.

Proof. a) According to (7) and (11), we have

$$\frac{d\Theta(t, v(\cdot))}{dt} = -\frac{1}{|z_0|}\int_0^t \theta(v(s))ds \leq -\frac{\theta_2}{|z_0|}t < 0, \quad t > 0.$$

b) Clearly [35],

$$\Theta(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|}\min_{v(\cdot)\in\mathbb{V}}\int_0^t (t-s)\theta(v(s))ds$$
$$\leq 1 - \frac{t^2}{2|z_0|}\min_{|v|\leq\beta}\theta(v) = \Theta^{**}(t).$$

Also, from (7) we get

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

$$\begin{aligned}\Theta(t,v(\cdot)) &\geq 1-\frac{1}{|z_0|}\max_{v(\cdot)\in\mathbb{V}}\int_0^t (t-s)\theta(v(s))ds \\ &= 1-\frac{t^2}{2|z_0|}\max_{|v|\leq\beta}\theta(v)=\Theta^*(t).\end{aligned}$$

This finishes the proof. □

Theorem 3.1. If $\rho_0 \geq 4\beta|z_0|$, then Pursuer wins by using Π-strategy (6) on the time interval $[0,T]$, where $T=\sqrt{2|z_0|/\theta_1}$ and θ_1 is defined by (7).

Proof. Let $v(\cdot)$ be an arbitrary admissible control of Evader E, and let Pursuer P employ strategy (6). Then given (9), (10), $z(t)=x(t)-y(t)$, $x_1=y_1$, we have

$$z(t)=z_0+\int_0^t (t-s)[\mathbf{u}(v(s))-v(s)]ds,\ \ z(0)=z_0.$$

Taking into account (6) and (11) we obtain

$$z(t)=z_0\Theta(t,v(\cdot)). \tag{13}$$

Since in view of (12), $\Theta(t,v(\cdot))\leq\Theta^{**}(t)$ and $\Theta^{**}(T)=0$, we find that there exists time $t^*\in(0,T]$ depending on $v(\cdot)$ such that $\Theta(t^*,v(\cdot))=0$. Hence, by virtue of (13) the desired result $z(t^*)=0$, i.e. $x(t^*)=y(t^*)$ is obtained.

It remains only to verify that Π-strategy (6) is admissible for each $t\in[0,t^*]$. By (11) $\Theta(t^*,v(\cdot))=0$ implies that

$$\int_0^{t^*}(t^*-s)\theta(v(s))ds=|z_0|,$$

and so

$$\begin{aligned}\int_0^t (t-s)|\mathbf{u}(v(s))|^2 ds &\leq \int_0^{t^*}(t^*-s)|\mathbf{u}(v(s))|^2 ds \\ &= \eta_0\int_0^{t^*}(t^*-s)\theta(v(s))ds=\eta_0|z_0|=\rho_0.\end{aligned}$$

Thus, strategy (6) is admissible. The proof of Theorem 3.1 is complete. □

Observe if $\rho_0\geq4\beta|z_0|$ and Pursuer applies strategy (6) then, for all $t\in[0,t^*]$, for the function (5) by (8) we have $\rho(t)=\Theta(t,v(\cdot))\rho_0$.

Indeed

$$\begin{aligned}\rho(t) &= \rho_0-\int_0^t (t-s)\eta_0\theta(v(s))ds \\ &= \rho_0\left(1-\frac{1}{|z_0|}\int_0^t (t-s)\theta(v(s))ds\right)=\rho_0\Theta(t,v(\cdot)).\end{aligned} \tag{14}$$

4. Evasion Game

In this section, an admissible control will be offered for Evader, to establish that T is the optimal time of pursuit.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

Let Evader employ the following control

$$\mathbf{v}^*(t) = -\beta\xi_0, \;\; \xi_0 = \frac{z_0}{|z_0|}. \tag{15}$$

In accordance with equations (1)–(2), for an arbitrary $u(\cdot) \in \mathbb{U}$ and for the control $\mathbf{v}^*(t)$, we get the following trajectories of the players:

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s)ds,$$

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)\mathbf{v}^*(s)ds.$$

We prove the following statement.

Theorem 4.1. a) If $\rho_0 \geq 4\beta|z_0|$, then Evader wins by control (15) in the interval $[0, T)$, where $T = \sqrt{2|z_0|/\theta_1}$.

b) If $\rho_0 < 4\beta|z_0|$, then Evader wins by control (15) in the time interval $[0, \infty)$ and

$$|z(t)| = |y(t) - x(t)| \geq |z_0| - \frac{\rho_0}{4\beta}.$$

Proof. a) Let $\rho_0 \geq 4\beta|z_0|$. If Evader utilizes control (15), then for any control of Pursuer $u(\cdot) \in \mathbb{U}$, in view of $x_1 = y_1$, we have

$$z(t) = z_0 + \int_0^t (t-s)u(s)ds - \int_0^t (t-s)\mathbf{v}^*(s)ds.$$

By (15), we have

$$|z(t)| = \left| z_0 + \beta\xi_0 \int_0^t (t-s)ds + \int_0^t (t-s)u(s)ds \right|$$
$$\geq |z_0| + \frac{\beta}{2}t^2 - \int_0^t (t-s)|u(s)|ds. \tag{16}$$

Applying the Cauchy-Schwartz inequality to the last integral in (16) we find

$$\int_0^t (t-s)|u(s)|ds \leq \left(\int_0^t (t-s)ds\right)^{1/2} \left(\int_0^t (t-s)|u(s)|^2 ds\right)^{1/2} \leq \sqrt{\frac{\rho_0}{2}}\, t. \tag{17}$$

As a consequence of (16) and (17), we conclude that

$$|z(t)| \geq \Gamma(t), \quad \Gamma(t) = \frac{\beta}{2}t^2 - \sqrt{\frac{\rho_0}{2}}t + |z_0|. \tag{18}$$

It is not difficult to verify that the smallest positive root of the equation $\Gamma(t) = 0$ is exactly T. Thus, $\Gamma(t) > 0$ for all $t \in [0, T)$, and in consequence, it follows immediately from (18) that $|z(t)| > 0$ on that time interval.

b) Suppose $\rho_0 < 4\beta|z_0|$. Then, we come to the estimation $|z(t)| \geq \Gamma(t)$ again. It is obvious that

$$\min_{t \geq 0} \Gamma(t) = |z_0| - \rho_0/(4\beta).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

On account of the condition $\rho_0 < 4\beta|z_0|$, we get $\Gamma(t) > 0$ for all $t \geq 0$, and therefore by (18) we see that $z(t) \neq 0$, i.e. $x(t) \neq y(t)$, $t \geq 0$. The proof is complete. □

5. Life-line game

The current section is devoted to investigate the dynamics of the attainability domain of Evader and the Life-line problem of R. Isaacs.

Let a non-empty and closed subset L of the space $\mathbb{R}^n$ be given. Here and subsequently, the area L is called the Life-line.

In the Life-line game L, Pursuer P intends to intercept Evader E, to accomplish $x(t_*) = y(t_*)$ at a finite time $t_* > 0$ when Evader E is in $\mathbb{R}^n \setminus L$. Evader E aims to get the area L maintaining the condition $x(t) \neq y(t)$, $t \geq 0$; and if there is no chance of doing this, then Evader E strives to maximize the moment of the encounter with Pursuer P. It should be noted that the area L doesn't limit the motion of Pursuer P. Further, it is required that the conditions $x_0 \neq y_0$ and $y_0 \notin L$ are satisfied for the initial positions x_0 and y_0.

Definition 7. Π-*strategy* (6) *is said to be winning on the time interval* $[0, T]$ *in the Life-line game L, if, for Evader's arbitrary control* $v(\cdot) \in \mathbb{V}$, *there exists an instant* $t_* \in [0, T]$ *such that:*

1) $x(t_*) = y(t_*)$;

2) $y(t) \notin L$ *at each* $t \in [0, t_*]$.

Definition 8. *We say that Evader wins in the Life-line game L by a control* $v(\cdot) \in \mathbb{V}$ *if for every* $u(\cdot) \in \mathbb{U}$:

1) there exists some moment $\bar{t}$, $\bar{t} > 0$, *that* $y(\bar{t}) \in L$ *and* $x(t) \neq y(t)$ *while* $t \in [0, \bar{t})$; *or*

2) $x(t) \neq y(t)$ *for all* $t \geq 0$.

In the theory of differential games, constructing the attainability domain of Evader in the Pursuit Game is considered as the main step to solve the game with a Life-line, and therefore we will first study the dynamics of the attainability domain.

If $\rho_0 \geq 4\beta|z_0|$ in game (1)-(4), then Theorem 3.1 asserts that by virtue of Π-strategy (6) Pursuer is able to capture Evader. The players P and E will meet at various points according to the choice of the control $v(\cdot) \in \mathbb{V}$.

Let $M(x, y, \rho)$ be the set consisting of all points μ where Pursuer moving from the position x and consuming the resource ρ should encounter Evader moving from the position y, i.e.

$$M(x, y, \rho) = \left\{ \mu : \; |\mu - x|^2 \geq \frac{\rho}{\beta} |\mu - y| \right\}.$$

When $\rho \neq \beta$, the set $M(x, y, \rho)$ is bounded by the curve

$$\partial M(x, y, \rho) = \left\{ \mu : \; |\mu - x|^2 = \frac{\rho}{\beta} |\mu - y| \right\}. \tag{19}$$

Set (19) in the plane is *Descartes' Oval* or *Pascal's Snail* [36].

Next, let Pursuer hold Π-strategy (6) while Evader employs arbitrary control $v(\cdot) \in \mathbb{V}$. Then t^*, $0 < t^* \leq T$ is players' meeting time, i.e. $x(t^*) = y(t^*)$. Then for each triad $\Big(x(t), y(t), \rho(t)\Big)$, $t \in [0, t^*]$, we build the following sets:

$$M\Big(x(t), y(t), \rho(t)\Big) = \left\{ \mu : \; |\mu - x(t)|^2 \geq \frac{\rho(t)}{\beta} |\mu - y(t)| \right\}, \tag{20}$$

$$M(x_0, y_0, \rho_0) = \left\{ \mu : \; |\mu - x_0|^2 \geq \frac{\rho_0}{\beta} |\mu - y_0| \right\}. \tag{21}$$

We can now formulate the following essential statement for the set (20).

Lemma 2. *Let Pursuer apply strategy (6). Then, for any $v(\cdot) \in V$,*

$$M\Big(x(t), y(t), \rho(t)\Big) = x(t) + \Theta(t, v(\cdot))\Big[M(x_0, y_0, \rho_0) - x_0\Big], \quad t \in [0, t^*]. \tag{22}$$

Proof. By the set (20) it can be seen that the relation

$$\mu \in M\Big(x(t), y(t), \rho(t)\Big) - x(t)$$

is identical to

$$|\mu|^2 \geq \frac{\rho(t)}{\beta}|\mu + z(t)|. \tag{23}$$

Clearly, it suffices to analyse (23) for $t \in [0, t^*)$ when $\Theta(t, v(\cdot)) > 0$. As $z(t) = z_0\Theta(t, v(\cdot))$, where $\Theta(t, v(\cdot))$ is defined by (11), in view of (14) we rewrite (23) in the form

$$\left|\Theta^{-1}\Big(t, v(\cdot)\Big)\mu\right|^2 \geq \frac{\rho_0}{\beta}\left|\Theta^{-1}\Big(t, v(\cdot)\Big)\mu + z_0\right|.$$

From this we infer

$$\Theta^{-1}\Big(t, v(\cdot)\Big)\mu \in M(x_0, y_0, \rho_0) - x_0,$$

or, equivalently,

$$\mu \in \Theta(t, v(\cdot))\Big[M(x_0, y_0, \rho_0) - x_0\Big].$$

Accordingly, we arrive at the equation

$$\begin{aligned} M\Big(x(t), y(t), \rho(t)\Big) - x(t) &= \left\{\mu : |\mu|^2 \geq \frac{\rho_0}{\beta}|\mu + z_0|\right\} \\ &= \Theta(t, v(\cdot))\Big[M(x_0, y_0, \rho_0) - x_0\Big], \end{aligned}$$

which is the desired result. The proof is complete. □

Lemma 3. *The multi-valued mapping $M\Big(x(t), y(t), \rho(t)\Big) - tx_1$, $t \in [0, t^*]$, is monotonically decreasing with respect to the inclusion, i.e. if t_1, $t_2 \in [0, t^*]$ and $t_1 < t_2$, then $M\Big(x(t_1), y(t_1), \rho(t_1)\Big) - t_1x_1 \supset M\Big(x(t_2), y(t_2), \rho(t_2)\Big) - t_2x_1$.*

Proof. By virtue of (6) and (8), for any $v(\cdot) \in \mathbb{V}$ we have

$$\Big|v(t) - \theta(v(t))\xi_0\Big|^2 = \eta_0\theta(v(t)).$$

Hence in accordance with (4),

$$\Big|v(t) - \theta(v(t))\xi_0\Big|^2 \geq \frac{\eta_0|v(t)|}{\beta}\theta(v(t)). \tag{24}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

Multiplying both sides of (24) by $|z_0|^2/\theta^2(v(t))$ and using $\eta_0 = \rho_0/|z_0|$, yields

$$\left|\left(\frac{v(t)|z_0|}{\theta(v(t))} + y_0\right) - x_0\right|^2 \geq \frac{\rho_0}{\beta}\left|\left(\frac{v(t)|z_0|}{\theta(v(t))} + y_0\right) - y_0\right|.$$

From this letting $\mu = \frac{v(t)|z_0|}{\theta(v(t))} + y_0$ we obtain (21), and so

$$\frac{v(t)|z_0|}{\theta(v(t))} + y_0 \in M(x_0, y_0, \rho_0),$$

hence

$$v(t)|z_0| + \theta(v(t))y_0 \in \theta(v(t))M(x_0, y_0, \rho_0). \tag{25}$$

For arbitrary $\psi \in \mathbb{R}^n$, $|\psi| = 1$, the multi-valued mapping $M\Big(x(t), y(t), \rho(t)\Big)$ has the support function

$$F\Big(M(x(t), y(t), \rho(t)), \psi\Big) = \sup_{\mu \in M(x(t), y(t), \rho(t))} \langle \mu, \psi \rangle.$$

Due to the properties of the support function $F\Big(M(x(t), y(t), \rho(t)), \psi\Big)$ (see, Property 6 in [37]), the relation (25) implies that

$$\langle v(t)|z_0|, \psi \rangle - \theta(v(t))F\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big) \leq 0,$$

and so

$$\langle v(t), \psi \rangle - \frac{1}{|z_0|}\theta(v(t))F\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big) \leq 0 \tag{26}$$

for all $\psi \in \mathbb{R}^n$, $|\psi| = 1$.

By integrating both sides of (26) over $[0, t]$ and by the properties of the support function (see, Theorem 2 in [37]) we find

$$\left\langle \int_0^t v(s)ds, \psi \right\rangle - \frac{1}{|z_0|}\int_0^t \theta(v(s))dsF\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big) \leq 0. \tag{27}$$

We use (6), (9), (11), (22) to calculate the derivative of $F(M(x(t), y(t), \rho(t)), \psi)$ with t:

$$\frac{d}{dt}F\Big(M(x(t), y(t), \rho(t)), \psi\Big)$$
$$= \frac{d}{dt}F\left(x_0 + x_1 t + \int_0^t (t-s)\mathbf{u}(v(s))ds + \Theta(t, v(\cdot))\Big[M(x_0, y_0, \rho_0) - x_0\Big], \psi\right)$$

To transform the right hand side, we use the property of support function and equation (11)

$$\langle x_1, \psi \rangle + \left\langle \int_0^t \mathbf{u}(v(s))ds, \psi \right\rangle - \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right) F\Big(M(x_0, y_0, \rho_0) - x_0, \psi\Big)$$
$$= \langle x_1, \psi \rangle + \left\langle \int_0^t (v(s) - \theta(v(s))\xi_0 ds, \psi \right\rangle - \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right) F\Big(M(x_0, y_0, \rho_0) - x_0, \psi\Big).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 18 August 2023 doi:10.20944/preprints202308.1311.v1

In view of $z_0 = y_0 - x_0$ this expression takes the form

$$\langle x_1, \psi\rangle + \left\langle \int_0^t v(s)ds, \psi \right\rangle - \left(\frac{z_0}{|z_0|} \int_0^t \theta(v(s))ds, \psi \right) - \left(\frac{1}{|z_0|} \int_0^t \theta(v(s))ds \right) F\Big(M(x_0, y_0, \rho_0) - x_0, \psi \Big)$$

$$= \langle x_1, \psi\rangle + \left\langle \int_0^t v(s)ds, \psi \right\rangle - \left(\frac{1}{|z_0|} \int_0^t \theta(v(s))ds \right) F\Big(M(x_0, y_0, \rho_0) - y_0, \psi \Big).$$

Thus,

$$\frac{d}{dt} F\Big(M\big(x(t), y(t), \rho(t) \big), \psi \Big) = \langle x_1, \psi\rangle + \left\langle \int_0^t v(s)ds, \psi \right\rangle - \left(\frac{1}{|z_0|} \int_0^t \theta(v(s))ds \right) F\Big(M(x_0, y_0, \rho_0) - y_0, \psi \Big),$$

and so, for any $\psi \in \mathbb{R}^n$, $|\psi| = 1$, by (27)we get

$$\frac{d}{dt} F\Big(M\big(x(t), y(t), \rho(t) \big) - tx_1, \psi \Big) \le 0$$

which completes the proof of Lemma 3. □

Lemma 3 plays the key role in proving the following statements.

Corollary 1. *It can be directly inferred from Lemma 3 that:*

a) $M\big(x(t), y(t), \rho(t) \big) \subset M(x_0, y_0, \rho_0) + tx_1$ *at each* $t \in [0, t^*]$;

b) $y(t) \in M(x_0, y_0, \rho_0) + tx_1$ *for all* $t \in [0, t^*]$.

We call the set

$$M^*(x_0, y_0, \rho_0, T) = \bigcup_{t=0}^{T} \Big\{ M(x_0, y_0, \rho_0) + tx_1 \Big\}$$

the *attainability domain of the Evader* in the Pursuit Game.

Theorem 5.1. Suppose $\rho_0 \ge 4\beta|z_0|$ and $M^*(x_0, y_0, \rho_0, T) \cap L = \varnothing$. Then Π-strategy (6) is winning on the time interval $[0, T]$ in the the Life-line game L, where $T = \sqrt{2|z_0|/\theta_1}$.

Proof. The proof immediately follows from Theorem 3.1, Lemma 3 and Corollary 1. □

Theorem 5.2. Let $\rho_0 < 4\beta|z_0|$. Then there exists a control $v(\cdot) \in \mathbb{V}$ guaranteeing that the Evader wins in the Life-line game L.

Proof. The proof follows directly from Theorem 4.1. □

6. Conclusion

In the present paper, we have discussed the pursuit-evasion games of one inertial Pursuer and one inertial Evader with integral and geometrical constraints on the controls, respectively.

In the Pursuit Game, we have defined the Π-strategy for Pursuer, and we have found the sufficient solvability condition of pursuit. In addition, we have demonstrated that the Π-strategy is optimal, i.e.

Evader, using the control $\mathbf{v}^*(t)$, remains uncaught by the guaranteed capture time T. In the Evasion Game, we have obtained the sufficient solvability condition of escape by the control $\mathbf{v}^*(t)$ of Evader.

Moreover, the attainability domain of Evader $M^*(x_0, y_0, \rho_0, T)$ in the Pursuit Game has been constructed. For the case $M^*(x_0, y_0, \rho_0, T) \cap L = \varnothing$ (see Theorem 5.1) the sufficient solvability condition of the Life-line game for Pursuer has been determined, but for the case $M^*(x_0, y_0, \rho_0, T) \cap L \neq \varnothing$ the Life-line game is an open problem.

References

1. Isaacs R. Differential games. John Wiley and Sons, New York (1965).
2. Pontryagin L.S. Izbrannye trudy. MAKS Press, Moscow (2004).
3. Krasovskii N.N. Control of a dynamical system. Nauka, Moscow (1985).
4. Pshenichnii B.N. Simple pursuit by several objects. Cybernetics and System Analysis. 12 (5), 484-485 (1976).
5. Chikrii A.A. Conflict-Controlled Processes. Kluwer, Dordrecht (1997).
6. Petrosjan L.A. Differential games of pursuit. Series on optimization. Vol.2. World Scientific Publishing, Singapore (1993).
7. Azamov A. On the quality problem for simple pursuit games with constraint, Serdica Bulgariacae math. Publ.Sofia 12 (1), 38-43 (1986).
8. Azamov A. On an alternative for pursuit-evasion games in an infinite time interval. Journal of Applied Mathematics and Mechanics. 50 (4), 428-432 (1986).
9. Munts N.V., Kumkov S.S. Numerical method for solving time-optimal differential games with lifeline. Mat. Teor. Igr Pril. 10 (3), 48-75 (2018).
10. Munts N.V., Kumkov S.S. On the Coincidence of the Minimax Solution and the Value Function in a Time-Optimal Game with a Lifeline. Proc. Steklov Inst. Math. 305, 125-139 (2019).
11. Samatov B.T., Soyibboev U.B. Differential game with a lifeline for the inertial movements of players, Ural Mathematical Journal. 7 (2), 94-109 (2021).
12. Samatov B.T., Ibragimov G.I, Khodjibayeva I.V. Pursuit-evasion differential games with the Grönwall type constraints on controls. Ural Mathematical Journal. 5 (1), 1-13 (2019).
13. Samatov B.T., Horilov M.A., Akbarov A.Ah. Differential game: "Life line" for non-stationary geometric constraints on controls. Lobachevskii Journal of Mathematics. 43 (1), 237-248 (2022).
14. Ibragimov G.I. Optimal pursuit time for a differential game in the Hilbert space l_2. ScienceAsia. 39S, 25-30 (2013).
15. Ivanov R.P., Ledyaev Yu.S. Time optimality for the pursuit of several objects with simple motion in a differential game. Proc. Steklov Inst. Math. 158, 93-103 (1983).
16. Kornev D.V., Lukoyanov N.Yu. On a minimax control problem for a positional functional under geometric and integral constraints on control actions. Proc. Steklov Inst. Math. 293, 85-100 (2016).
17. Dar'in A.N, Kurzhanskii A.B. Control Under Indeterminacy and Double Constraints. Differential Equations. 39 (11) 1554-1567 (2003).
18. Belousov A.A. Method of resolving functions for differential games with integral constraints. Theory of Optimal Solution. 9, 10-16 (2010).
19. Chikrii A.A., Belousov A.A. On linear differential games with convex integral constraints. Trudy Inst. Mat. i Mekh. UrO RAN, 19(4), 308-319 (2013).
20. Satimov N.Yu., Rikhsiev B.B., Khamdamov A.A. On a pursuit problem for n-person linear differential and discrete games with integral constraints. Mathematics of the USSR-Sbornik. 46 (4), 459—471 (1983).
21. Azamov A.A., Samatov B.T. The Π-Strategy: Analogies and Applications, The Fourth International Conference Game Theory and Management, St. Petersburg 4, 33-47 (2010).
22. Mamadaliev N.A. The Pursuit Problem for Linear Games with Integral Constraints on Players' Controls. Russ Math. 64, 9-24 (2020).
23. Mamadaliev N. On the pursuit problem for linear differential games with distinct constraints on the players' controls. Diff. Equat. 48, 867-880 (2012).
24. Mamadaliev N. On a pursuit problem with integral constraints on the players' controls. Sib Math J. 55, 107-124 (2015).

25. Samatov B.T. The pursuit-evasion problem under integral-geometric constraints on pursuer controls. Autom. Remote Control. 74 (7), 1072-1081 (2013).
26. Samatov B.T. The Π-strategy in a differential game with linear control constraints, J.Appl.Maths and Mechs. 78 (3) 258-263 (2014).
27. Samatov B.T, Juraev B.I. Pursuit-evasion problems with a constraint for energy expenditure of velocity. Uzbek Mathematical Journal. 66 (4), 145-15 (2022).
28. Samatov B.T, Juraev B.I. Pursuit-evasion problems under nonlinear increase of the pursuer's resource. Trudy Instituta Matematiki i Mekhaniki UrO RAN. 28 (3), 285-295 (2022).
29. Samatov B.T., Horilov M.A., Juraev B.I. Π-strategy for a differential game of pursuit with integral constraints of a generalized type. Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki. 33 (2), 293-311 (2023).
30. Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory. 50, 729–750 (2021).
31. Kuchkarov A.Sh, Ibragimov G.I., Ferrara M. Simple motion pursuit and evasion differential games with many pursuers on manifolds with Euclidean metric. Discrete Dynamics in Nature and Society. ID 1386242 (2016).
32. Ibragimov G.I, Kuchkarov A.Sh. Fixed duration pursuit-evasion differential game with integral constraints. Journal of Physics: Conference Series. 435 (012017), 1-12 (2013).
33. Ibragimov G.I., Salleh Y. Simple motion evasion differential game of many pursuers and one evader with integral constraints on control functions of players. Journal of Applied Mathematics. Article ID 748096 (2012).
34. Ibragimov G.I. Evasion Differential Game of One Evader and Many Slow Pursuers. Dynamic games and applications. 2023.
35. Alekseev V.M., Tikhomirov V.M., Fomin S.V. Optimal'noye upravleniye. Nauka, Moscow. (1979).
36. Azamov A.A., Samatov B.T. Π-Strategy, Tashkent: NUUz. (2000).
37. Blagodatskikh V.I. Introduction to Optimal Control. Moscow: Vysshaya Shkola. (2001).