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# An Endpoint Functional Continuous Uncertainty Principle

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Posted Date: 15 August 2023

doi: 10.20944/preprints202308.1108.v1

Keywords: Uncertainty Principle, Parseval Frame, Banach space.

 doi:10.20944/preprints202308.1108.v1



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# An Endpoint Functional Continuous Uncertainty Principle

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**Abstract:** Let $(\Omega,\mu)$, $(\Delta,\nu)$ be measure spaces. Let $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ and $(\{g_\beta\}_{\beta\in\Delta},\{\omega_\beta\}_{\beta\in\Delta})$ be continuous 1-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x\in\mathcal{X}\setminus\{0\}$, we show that

$$\mu(\operatorname{supp}(\theta_f x))\geq\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(\omega_\beta)|},\quad \nu(\operatorname{supp}(\theta_g x))\geq\frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|g_\beta(\tau_\alpha)|}.$$

where

$$\theta_f:\mathcal{X}\ni x\mapsto\theta_f x\in\mathcal{L}^1(\Omega,\mu);\quad \theta_f x:\Omega\ni\alpha\mapsto(\theta_f x)(\alpha):=f_\alpha(x)\in\mathbb{K},$$
$$\theta_g:\mathcal{X}\ni x\mapsto\theta_g x\in\mathcal{L}^1(\Delta,\nu);\quad \theta_g x:\Delta\ni\beta\mapsto(\theta_g x)(\beta):=g_\beta(x)\in\mathbb{K}.$$

This solves a problem asked by K. M. Krishna in the paper 'Functional Continuous Uncertainty Principle' *[arXiv:2308.00312v1]*.

**Keywords:** Uncertainty Principle; parseval frame; banach space

**MSC:** 42C15

## 1. Introduction

Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), let

$$\theta_\tau:\mathcal{H}\ni h\mapsto\theta_\tau h:=(\langle h,\tau_j\rangle)_{j=1}^n\in\mathbb{K}^n.$$

Following is the most general form of discrete uncertainty principle for finite dimensional Hilbert spaces.

**Theorem 1** (**(Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle)** [2,3,7]). *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0+\|\theta_\omega h\|_0}{2}\right)^2\geq\|\theta_\tau h\|_0\|\theta_\omega h\|_0\geq\frac{1}{\max\limits_{1\leq j,k\leq n}|\langle\tau_j,\omega_k\rangle|^2},\quad\forall h\in\mathcal{H}\setminus\{0\}.$$

Recently, Theorem 1 has been greatly improved to continuous families in Banach spaces (even infinite dimensions). To state the result, we need a notion.

**Definition 1** ([4]). *Let $(\Omega,\mu)$ be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a* ***continuous p-Schauder frame*** *for $\mathcal{X}$ ($1<p<\infty$) if the following holds.*

(i) *For every $x\in\mathcal{X}$, the map $\Omega\ni\alpha\mapsto f_\alpha(x)\in\mathbb{K}$ is measurable.*

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(ii) *For every* $x \in \mathcal{X}$,

$$\|x\|^p = \int_{\Omega} |f_\alpha(x)|^p \, d\mu(\alpha).$$

(iii) *For every* $x \in \mathcal{X}$, *the map* $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ *is weakly measurable.*
(iv) *For every* $x \in \mathcal{X}$,

$$x = \int_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

*where the integral is weak integral.*

Note that condition (i) in Definition 1 says that the map

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

is a linear isometry.

**Theorem 2** (**(Functional Continuous Uncertainty Principle)** [4]). *Let* $(\Omega, \mu)$, $(\Delta, \nu)$ *be measure spaces. Let* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *and* $(\{g_\beta\}_{\beta\in\Delta}, \{\omega_\beta\}_{\beta\in\Delta})$ *be continuous p-Schauder frames for a Banach space* $\mathcal{X}$. *Then for every* $x \in \mathcal{X} \setminus \{0\}$, *we have*

$$\mu(\operatorname{supp}(\theta_f x))^{\frac{1}{p}} \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|}, \quad \nu(\operatorname{supp}(\theta_g x))^{\frac{1}{p}} \mu(\operatorname{supp}(\theta_f x))^{\frac{1}{q}} \geq \frac{1}{\sup\limits_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|},$$

*where q is the conjugate index of p.*

**Corollary 1** (**(Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle)** [5]). *Let* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *and* $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ *be p-Schauder frames for a finite dimensional Banach space* $\mathcal{X}$. *Then for every* $x \in \mathcal{X} \setminus \{0\}$, *we have*

$$\|\theta_f x\|_0^{\frac{1}{p}} \|\theta_g x\|_0^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0^{\frac{1}{p}} \|\theta_f x\|_0^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|},$$

*where q is the conjugate index of p.*

In paper [4], it is asked that whether we have version of Theorem 2 for $p = 1$ and $p = \infty$. In this paper, we solve the problem for $p = 1$.

## 2. Functional Continuous Uncertainty Principle for Continuous 1-Schauder Frames

We clearly have the following definition from Definition 1.

**Definition 2.** *Let* $(\Omega, \mu)$ *be a measure space. Let* $\{\tau_\alpha\}_{\alpha\in\Omega}$ *be a collection in a Banach space* $\mathcal{X}$ *and* $\{f_\alpha\}_{\alpha\in\Omega}$ *be a collection in* $\mathcal{X}^*$. *The pair* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *is said to be a* ***continuous 1-Schauder frame*** *for* $\mathcal{X}$ *if the following holds.*

*i For every* $x \in \mathcal{X}$, *the map* $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ *is measurable.*
*ii For every* $x \in \mathcal{X}$,

$$\|x\| = \int_{\Omega} |f_\alpha(x)| \, d\mu(\alpha).$$

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*iii For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ is weakly measurable.*
*iv For every $x \in \mathcal{X}$,*

$$x = \int\limits_{\Omega} f_\alpha(x)\tau_\alpha \, d\mu(\alpha),$$

*where the integral is weak integral.*

We note that condition (i) in Definition 2 says that the map

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^1(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

is a linear isometry.

**Theorem 3 (Functional Continuous Uncertainty Principle for Continuous 1-Schauder Frames).** *Let $(\Omega, \mu)$, $(\Delta, \nu)$ be measure spaces. Let $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ and $(\{g_\beta\}_{\beta \in \Delta}, \{\omega_\beta\}_{\beta \in \Delta})$ be continuous 1-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we have*

$$\mu(\operatorname{supp}(\theta_f x)) \geq \frac{1}{\sup\limits_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)|}, \quad \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)|}.$$

*In particular,*

$$\mu(\operatorname{supp}(\theta_f x))\nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\left(\sup\limits_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)|\right)\left(\sup\limits_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)|\right)}$$

*and*

$$\mu(\operatorname{supp}(\theta_f x)) + \nu(\operatorname{supp}(\theta_g x)) \geq \frac{1}{\sup\limits_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)|} + \frac{1}{\sup\limits_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)|}.$$

**Proof.** Let $x \in \mathcal{X} \setminus \{0\}$. First using $\theta_f$ is an isometry and later using $\theta_g$ is an isometry, we get

$$
\begin{aligned}
\|x\| &= \|\theta_f x\| = \int_{\Omega} |f_\alpha(x)|\,d\mu(\alpha) = \int_{\mathrm{supp}(\theta_f x)} |f_\alpha(x)|\,d\mu(\alpha)\\
&= \int_{\mathrm{supp}(\theta_f x)} \left| f_\alpha\left(\int_{\Delta} g_\beta(x)\omega_\beta\,d\nu(\beta)\right)\right| d\mu(\alpha) = \int_{\mathrm{supp}(\theta_f x)} \left|\int_{\Delta} g_\beta(x) f_\alpha(\omega_\beta)\,d\nu(\beta)\right| d\mu(\alpha)\\
&= \int_{\mathrm{supp}(\theta_f x)} \left|\int_{\mathrm{supp}(\theta_g x)} g_\beta(x) f_\alpha(\omega_\beta)\,d\nu(\beta)\right| d\mu(\alpha) \leq \int_{\mathrm{supp}(\theta_f x)} \int_{\mathrm{supp}(\theta_g x)} |g_\beta(x) f_\alpha(\omega_\beta)|\,d\nu(\beta)\,d\mu(\alpha)\\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \int_{\mathrm{supp}(\theta_f x)} \int_{\mathrm{supp}(\theta_g x)} |g_\beta(x)|\,d\nu(\beta)\,d\mu(\alpha)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \mu(\mathrm{supp}(\theta_f x)) \int_{\mathrm{supp}(\theta_g x)} |g_\beta(x)|\,d\nu(\beta)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \mu(\mathrm{supp}(\theta_f x))\|\theta_g x\| = \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|\right) \mu(\mathrm{supp}(\theta_f x))\|x\|.
\end{aligned}
$$

Therefore

$$
\frac{1}{\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(\omega_\beta)|} \leq \mu(\mathrm{supp}(\theta_f x)).
$$

On the other way, first using $\theta_g$ is an isometry and $\theta_f$ is an isometry, we get

$$
\begin{aligned}
\|x\| &= \|\theta_g x\| = \int_{\Delta} |g_\beta(x)|\,d\nu(\beta) = \int_{\mathrm{supp}(\theta_g x)} |g_\beta(x)|\,d\nu(\beta)\\
&= \int_{\mathrm{supp}(\theta_g x)} \left| g_\beta\left(\int_{\Omega} f_\alpha(x)\tau_\alpha\,d\mu(\alpha)\right)\right| d\nu(\beta) = \int_{\mathrm{supp}(\theta_g x)} \left|\int_{\Omega} f_\alpha(x) g_\beta(\tau_\alpha)\,d\mu(\alpha)\right| d\nu(\beta)\\
&= \int_{\mathrm{supp}(\theta_g x)} \left|\int_{\mathrm{supp}(\theta_f x)} f_\alpha(x) g_\beta(\tau_\alpha)\,d\mu(\alpha)\right| d\nu(\beta) \leq \int_{\mathrm{supp}(\theta_g x)} \int_{\mathrm{supp}(\theta_f x)} |f_\alpha(x) g_\beta(\tau_\alpha)|\,d\mu(\alpha)\,d\nu(\beta)\\
&\leq \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \int_{\mathrm{supp}(\theta_g x)} \int_{\mathrm{supp}(\theta_f x)} |f_\alpha(x)|\,d\mu(\alpha)\,d\nu(\beta)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\mathrm{supp}(\theta_g x)) \int_{\mathrm{supp}(\theta_f x)} |f_\alpha(x)|\,d\mu(\alpha)\\
&= \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\mathrm{supp}(\theta_g x))\|\theta_f x\| = \left(\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|\right) \nu(\mathrm{supp}(\theta_g x))\|x\|.
\end{aligned}
$$

Therefore

$$
\frac{1}{\sup_{\alpha\in\Omega,\beta\in\Delta} |g_\beta(\tau_\alpha)|} \leq \nu(\mathrm{supp}(\theta_g x)).
$$

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□

**Corollary 2 (Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle for 1-Schauder Frames).** *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be 1-Schauder frames for a finite dimensional Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we have*

$$\|\theta_f x\|_0 \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0 \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|}.$$

*In particular,*

$$\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{1}{\left(\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|\right)\left(\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)}$$

*and*

$$\|\theta_f x\|_0 + \|\theta_g x\|_0 \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} + \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|}.$$

**Remark 1.** *We note that the $\mathcal{L}^1$-norm uncertainty principles derived in [1,6,8] differ from our result.*

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