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Homogenization of Smoluchowski Equations in Thin Heterogeneous Porous Domains

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Abstract: We carry out in a thin heterogeneous porous layer, the multiscale analysis of a set of Smoluchowski's discrete diffusion-coagulation equations describing the evolution density of diffusion particles that are prone to coagulate in pairs. Assuming that the thin heterogeneous layer is made of microstructures that are uniformly distributed inside, we obtain in the limit an upscaled model in lower space dimension. To achieve our goal, we use the concept of two-scale convergence adapted to thin heterogeneous media.

Keywords: Homogenizatio; Smoluchowski equation; two-scale convergence; thin domains

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1. Introduction and main result

The Smoluchowski equation modeling Alzheimzer's disease (AD) is a system of partial differential equations that describes the evolving densities of diffusing particles that are prone to coagulate in pairs. Recently, the important role of Smoluchowski equation in modeling the evolution of AD at different scales has been investigated in [1,12,14,29] in which the authors presented a mathematical model for the aggregation and diffusion of β-amyloid (Aβ) in the brain affected by AD at a microscopic scale (the size of a single neuron) and at the early stage of the disease when small amyloid fibrils are free to move and to coalesce. We also refer to [2,3,10,15,22–24,26] for some other works in the same direction. In the model proposed in [12], a very small portion of the cerebral tissue is described by a bounded smooth region $\Omega \subset \mathbb{R}^3$ which is perforated by removing from it a set of periodically distributed holes of size ε (the neurons). Moreover the production of Aβ in monomeric form at the level of neuron membranes is modeled by a non homogeneous Neumann condition on the boundary of the porosities.

In the current work, we consider the model stated in [12], but this time in a thin porous layer. This is motivated by the fact that Alzheimer's disease particularly affects the cerebral cortex (responsible for language and information processing) and hippocampus (essential for memory), which represent very thin layers of brain tissue and contain thousands millions of neurons. Here we describe a very small layer of the brain tissue by a highly heterogeneous thin porous layer in which the heterogeneities are due to the number of millions of neurons that the brain tissue can contain. To be more precise, our model problem at the micro level is stated below.

Let Ω be a bounded open Lipschitz connected subset in $\mathbb{R}^2$. For $0 < \varepsilon < 1$ be freely fixed, we set

$$\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon) = \left\{ (\overline{x}, x_3) \in \mathbb{R}^3 : \overline{x} \in \Omega \text{ and } -\varepsilon < x_3 < \varepsilon \right\}.$$

We denote by $Z = Y \times I$ the reference layer cell, where $Y = (0,1)^2$ and $I = (-1,1)$. Let $Z_f \subset Z$ be a compact set in Z with smooth boundary, which represents a generic neuron, and let $Z_s = Z \backslash Z_f$ be the supporting cerebral tissue (often call the solid part in the literature of porous media). Next, let $K_\varepsilon = \{k \in \mathbb{Z}^2 \times \{0\} : \varepsilon(k+Z) \subset \Omega_\varepsilon\}$, and set $T^\varepsilon = \cup_{k \in K_\varepsilon} \varepsilon(k + Z_f)$. We define the thin porous layer by

$$\Omega^\varepsilon = \Omega_\varepsilon \backslash T^\varepsilon \text{ (points in } \Omega_\varepsilon \text{ lying off } T^\varepsilon).$$

The boundary of Ω^ε is divided into two parts: the outer boundary $\partial_D \Omega^\varepsilon = \partial\Omega_\varepsilon$ and the inner boundary $\Gamma^\varepsilon = \partial T^\varepsilon$. We also denote by $\Gamma = \partial Z_f$, so that $\Gamma^\varepsilon = \cup_{k \in K_\varepsilon} \varepsilon(k + \Gamma)$. Finally we denote by ν the outward unit normal to Γ^ε. We assume that Ω^ε is connected and that $|Z_s| > 0$, where $|Z_s|$ stands for the Lebesgue measure of Z_s in $\mathbb{R}^3$. The ε-model reads as follows: for $m = 1$, u_1^ε solves the PDE

$$\begin{cases} \frac{\partial u_1^\varepsilon}{\partial t} - \operatorname{div}(d_1 \nabla u_1^\varepsilon) + u_1^\varepsilon \sum_{j=1}^{M} a_{1,j} u_j^\varepsilon = 0 \text{ in } Q_\varepsilon = (0,T) \times \Omega^\varepsilon \\ \frac{\partial u_1^\varepsilon}{\partial \nu} = 0 \text{ on } (0,T) \times \partial\Omega_\varepsilon \\ \frac{\partial u_1^\varepsilon}{\partial \nu} = \varepsilon \psi^\varepsilon \text{ on } (0,T) \times \Gamma^\varepsilon \\ u_1^\varepsilon(0,x) = 0 \text{ in } \Omega^\varepsilon; \end{cases} \tag{1}$$

for $1 < m < M$, u_m^ε solves the PDE

$$\begin{cases} \frac{\partial u_m^\varepsilon}{\partial t} - \operatorname{div}(d_m \nabla u_m^\varepsilon) + u_m^\varepsilon \sum_{j=1}^{M} a_{m,j} u_j^\varepsilon = f_m^\varepsilon \text{ in } Q_\varepsilon \\ \frac{\partial u_m^\varepsilon}{\partial \nu} = 0 \text{ on } (0,T) \times \partial\Omega^\varepsilon \\ u_m^\varepsilon(0,x) = 0 \text{ in } \Omega^\varepsilon; \end{cases} \tag{2}$$

and for $m = M$, u_M^ε solves the equation

$$\begin{cases} \frac{\partial u_M^\varepsilon}{\partial t} - \operatorname{div}(d_M \nabla u_M^\varepsilon) = g_\varepsilon \text{ in } Q_\varepsilon \\ \frac{\partial u_M^\varepsilon}{\partial \nu} = 0 \text{ on } (0,T) \times \partial\Omega^\varepsilon \\ u_M^\varepsilon(0,x) = 0 \text{ in } \Omega^\varepsilon, \end{cases} \tag{3}$$

where

$$f_m^\varepsilon = \frac{1}{2} \sum_{j=1}^{m-1} a_{j,m-j} u_j^\varepsilon u_{m-j}^\varepsilon, \; g_\varepsilon = \frac{1}{2} \sum_{\substack{j+k \geq M \\ j<M,\, k<M}} a_{j,k} u_j^\varepsilon u_k^\varepsilon \text{ and } \psi^\varepsilon(t,x) = \psi(t, \overline{x}, \frac{x}{\varepsilon}) \tag{4}$$

for $(t,x) \in Q_\varepsilon$.

We assume that:

H1. *the coefficients $a_{i,j}$ are positive constants and satisfy $a_{i,j} = a_{j,i}$ $(1 \leq i,j \leq M)$ with $a_{M,M} = 0$, and that the diffusion coefficients d_i are positive constants that become smaller as j is large.*

H2. *The function ψ^ε is defined by $\psi^\varepsilon(t,x) = \psi(t,\overline{x},\frac{x}{\varepsilon})$ $((t,x) \in Q_\varepsilon)$, where $\psi \in \mathcal{C}^1([0,T];\mathcal{C}^1(\overline{\Omega};\mathcal{C}^1_{per}(Y;\mathcal{C}^1(I))))$ with $0 \leq \psi \leq 1$ and $\psi(0,\overline{x},y) = 0$ for $(\overline{x},y) \in \Omega \times Z$.*

In (H2), $\mathcal{C}^1_{per}(Y;\mathcal{C}^1(I))$ denotes the space of functions in $\mathcal{C}^1(\mathbb{R}^2;\mathcal{C}^1(I))$ that are Y-periodic. In (1)-(3), ∇ stands for the usual gradient operator while div denotes the divergence operator with respect to the variable x; T is a positive number representing the final time. The unknowns are the vectors value functions $\boldsymbol{u}^\varepsilon : Q_\varepsilon \to \mathbb{R}^M$, $\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, ..., u_M^\varepsilon)$ where the coordinate $u_m^\varepsilon \geq 0$ $(1 \leq m < M)$ stands for the concentration of m-clusters, that is clusters made of m identical elementary particles, while u_M^ε takes into account aggregation of more than $M - 1$ monomers. It is worth noting that the meaning of u_M^ε is different from that of u_m^ε $(m < M)$ as it aims at describing the sum of densities of all the large assemblies. It is assumed that the large assemblies exhibit all the same coagulation properties and do not coagulate with each other. We also assume that the only reaction allowing clusters to form large clusters is a binary coagulation mechanism, while the movement of clusters leading to aggregation arises only from a diffusion process described by the constant diffusion coefficient d_m $(1 \leq m \leq M)$. The kinetic coefficient $a_{i,j}$ arises from a reaction in which an $(i+j)$-cluster is formed from an i-cluster and a j-cluster. Therefore, they can be interpreted as coagulation rates. Finally, f_m^ε $(1 < m < M)$ accounts for the formation of m-clusters by coalescence of smaller clusters and g_ε

accounts for the formation of a large clusters by coalescence of others large one that have the same coagulation properties.

Our main aim in this work is to investigate the limiting behaviour as $\varepsilon \to 0$, of the solution $\boldsymbol{u}^\varepsilon$ to (1)-(3) under the assumptions (H1)-(H2). This falls within the scope of the homogenization theory in thin porous domains.

There is a huge literature on homogenization in fixed or porous media. A few works deal with the homogenization theory in thin heterogeneous domains; see e.g. [4–6,8,9,11,16–18,21,25]. As for the homogenization in thin heterogeneous porous media, very few results are known up to now. We may cite [4–6,8,11]. Concerning the Smoluchowski equation as stated in this work, to the best of our knowledge, the only work dealing with its homogenization is the paper [12] in which the considered domain is a uniformly perforated one that is not thin. Because our domain is thin and porous, the homogenization process is not an easy task. Indeed, we make use of the partial mean integral operator M_ε (see below for its definition) associated to the extension operator, while in [12], even the extension operator is not used. So, the main novelty in our work arises from the fact that the domain Ω^ε is a thin heterogeneous porous layer. This leads to a dimension reduction problem in the limit as shown here below in the main result, which reads as follows.

Theorem 1. *Assume that* (H1)-(H2) *hold. For any $\varepsilon > 0$, let $\boldsymbol{u}^\varepsilon = (u_m^\varepsilon)_{1\le m\le M}$ be the unique solution of* (1)-(3) *in the class $(\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q_\varepsilon))^M$, ($\alpha \in (0,1)$). Let also M_ε and E_ε denote respectively the partial mean integral operator and the extension operator defined by* (34) *(see Section 3) and in Lemma 3 (see Section 2). Then, as $\varepsilon \to 0$, one has, for any $1 \le m \le M$,*

$$M_\varepsilon E_\varepsilon u_m^\varepsilon \to u_m \text{ in } L^2(Q)\text{-strong,} \tag{5}$$

$$M_\varepsilon \nabla E_\varepsilon u_m^\varepsilon \to \nabla_{\overline{x}} u_m \text{ in } L^2(Q)^2\text{-weak,} \tag{6}$$

$$M_\varepsilon E_\varepsilon \frac{\partial u_m^\varepsilon}{\partial t} \to \frac{\partial u_m}{\partial t} \text{ in } L^2(Q)\text{-weak,} \tag{7}$$

where $\boldsymbol{u} = (u_m)_{1\le m\le M} \in [L^\infty(Q) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))]^M$ is the unique solution of the system (8)-(10) *below:*

$$\begin{cases} \theta\frac{\partial u_1}{\partial t} - \mathrm{div}_{\overline{x}}\left(d_1 A\nabla_{\overline{x}} u_1\right) + \theta u_1 \sum_{j=1}^{M} a_{1,j}u_j = d_1\widetilde{\psi} \text{ in } Q = (0,T)\times\Omega \\ A\nabla_{\overline{x}} u_1 \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_1(0,\overline{x}) = 0 \text{ in } \Omega; \end{cases} \tag{8}$$

If $1 < m < M$,

$$\begin{cases} \theta\frac{\partial u_m}{\partial t} - \mathrm{div}_{\overline{x}}\left(d_m A\nabla_{\overline{x}} u_m\right) + \theta u_m \sum_{j=1}^{M} a_{m,j}u_j - \frac{\theta}{2}\sum_{j=1}^{M} a_{j,m-j}u_j u_{m-j} = 0 \text{ in } Q \\ A\nabla_{\overline{x}} u_m \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_m(0,\overline{x}) = 0 \text{ in } \Omega; \end{cases} \tag{9}$$

and

$$\begin{cases} \theta\frac{\partial u_M}{\partial t} - \mathrm{div}_{\overline{x}}\left(d_M A\nabla_{\overline{x}} u_m\right) - \frac{\theta}{2}\sum_{\substack{j+k\ge M \\ j<M,\, k<M}} a_{j,k}u_j u_k = 0 \text{ in } Q \\ A\nabla_{\overline{x}} u_M \cdot n = 0 \text{ on } (0,T)\times\partial\Omega \\ u_M(0,\overline{x}) = 0 \text{ in } \Omega. \end{cases} \tag{10}$$

Moreover $\boldsymbol{u} \in (\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q))^M$ and is such that

$$u_m > 0 \text{ in } Q,\ m = 1, ..., M. \tag{11}$$

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In (8)-(10), n is the outward unit normal to $\partial\Omega$ and the matrix $A = I_2 + \nabla_{\overline{y}}\omega$, where I_2 is the 2×2 identity matrix and $\omega = (\omega_i)_{i=1,2}$ with ω_i being the unique solution in $H^1_\#(Z_s) = \{u \in H^1(Z_s) : u$ is Y-periodic and $\int_{Z_s} u dy = 0\}$ of the cell problem

$$\begin{cases} \operatorname{div}_y(e_i + \nabla_y \omega_i) = 0 \text{ in } Z_s, \quad (e_i + \nabla_y \omega_i) \cdot \nu = 0 \text{ on } \Gamma, \\ \omega_i(., y_3) \text{ is } Y\text{-periodic,} \end{cases}$$

where here, ν stands for the outward unit normal to Γ and e_i is the ith vector of the canonical basis in $\mathbb{R}^3$; the function $\widetilde{\psi}$ and θ are defined respectively by $\widetilde{\psi}(t,\overline{x}) = \int_\Gamma \psi(t,\overline{x},y)d\sigma(y)$, $(t,\overline{x}) \in Q$ and $\theta = |Z_s|$ (the Lebesgue measure of Z_s in $\mathbb{R}^3$.

The partial mean integral M_ε considered in Theorem 1 is defined, for a function ϕ by

$$M_\varepsilon \phi(t,\overline{x}) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \phi(t,\overline{x},\zeta)d\zeta \text{ for } (t,\overline{x}) \in Q.$$

The system (8)-(10) is the upscaled model arising from the ε-model (1)-(3). It is posed in a 2 dimensions space, leading to an expected dimension reduction problem as it is usually the case for the homogenization theory in thin domains. Moreover the information given on the microscale by the Neumann boundary condition in (1) is transferred (in the limit) into the source term in the leading equation in (8), so that, in the case of (1), the limiting equation does not have the same form as the original equation posed in the ε-model. For (9) and (10), apart from the diffusion term, they are similar to the ε-equations in (2) and (3).

The rest of the paper is organized as follows. In Section 2, we investigate the well posedness of (1)-(3) and provide useful uniform estimates. Section 3 deals with the treatment of the concept of two-scale convergence for thin heterogeneous domains. We prove therein some compactness results that will be used in the homogenization process. With the help of the results obtained in Section 3, we pass to the limit in (1)-(3) in Section 4 where we prove the main result, viz. Theorem 1.

2. Well posedness and uniform estimates

The current section deals with the existence and uniqueness of the solution to (1)-(3), together with some uniform estimates that will be useful in the sequel. The following result holds true.

Theorem 2. *Assume that* (H1)-(H2) *hold true. For any $\varepsilon > 0$, the system* (1)-(3) *possesses a unique weak solution $\boldsymbol{u}^\varepsilon = (u^\varepsilon_m)_{1\leq m\leq M} \in (\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}(Q_\varepsilon))^M$ ($\alpha \in (0,1)$ be fixed) such that*

$$u^\varepsilon_m(t,x) > 0 \text{ for } (t,x) \in Q_\varepsilon,\ m = 1,...,M.$$

Furthermore there exists $\varepsilon_0 > 0$ such that, for all $1 \leq m \leq M$,

$$\|u^\varepsilon_m\|_{L^\infty(Q_\varepsilon)} \leq C, \tag{12}$$

$$\|\nabla u^\varepsilon_m\|_{L^2(Q_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}, \tag{13}$$

$$\left\|\frac{\partial u^\varepsilon_m}{\partial t}\right\|_{L^2(Q_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}, \tag{14}$$

and

$$\|\psi^\varepsilon\|_{L^2((0,T)\times\Gamma^\varepsilon)} \leq C\,\|\psi\|_{L^2(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))}, \tag{15}$$

for all $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is independent of m and ε.

Proof. The well posedness of (1)-(3) has been addressed in [1,12,13,29]. We are concerned here only with the uniform estimates (12)-(14), the estimate (15) being a classical result arising from the trace

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result. We just emphasize that since $|\Gamma^\varepsilon| = O(1)$ ($|\Gamma^\varepsilon|$ stands for the Lebesgue measure of Γ^ε), no scaling is needed in the left-hand side of (15). Now, as for (12), we follows exactly the same lines of reasoning as in [12] to obtain it. It remains to check (13) and (14). We first consider (13). We distinguish the cases $m = 1$ and $1 < m \leq M$.

We start with $m = 1$. Multiplying $(1)_1$ by u_1^ε and integrating over Ω^ε, next using the divergence theorem, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|u_1^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + d_1\left\|\nabla u_1^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \int_{\Omega^\varepsilon}\left(\left|u_1^\varepsilon\right|^2\sum_{j=1}^M a_{1,j}u_j^\varepsilon\right)dx \\ &= \varepsilon d_1\int_{\Gamma^\varepsilon}\psi(t,\overline{x},\tfrac{x}{\varepsilon})u_1^\varepsilon(t,x)d\sigma_\varepsilon(x) \\ &\leq \frac{\varepsilon d_1}{2}\left\|\psi^\varepsilon(t)\right\|^2_{L^2(\Gamma^\varepsilon)} + \frac{\varepsilon d_1}{2}\left\|u_1^\varepsilon(t)\right\|^2_{L^2(\Gamma^\varepsilon)}, \end{aligned} \tag{16}$$

where the last inequality above stems from Hölder's and Young's inequalities. We use a well-known trace inequality to deduce the existence of a positive constant C_1 independent of ε such that

$$\varepsilon\left\|u_1^\varepsilon(t)\right\|^2_{L^2(\Gamma^\varepsilon)} \leq C_1\left(\int_{\Omega^\varepsilon}\left|u_1^\varepsilon(t)\right|^2dx + \varepsilon^2\int_{\Omega^\varepsilon}\left|\nabla u_1^\varepsilon(t)\right|^2dx\right). \tag{17}$$

Therefore, integrating (16) over $(0,t)$ ($t \in (0,T]$) and taking into account (15) and (17), we are led to

$$\begin{aligned} &\left\|u_1^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)} + d_1(2-\varepsilon^2C_1)\int_0^t\left\|\nabla u_1^\varepsilon(s)\right\|^2_{L^2(\Omega^\varepsilon)}ds \\ \leq\ & C_1d_1\int_0^t\left\|u_1^\varepsilon(s)\right\|^2_{L^2(\Omega^\varepsilon)}ds + \varepsilon d_1C\left\|\psi\right\|_{L^2(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))}. \end{aligned} \tag{18}$$

We therefore infer the boundedness of u_1^ε in $L^\infty(Q_\varepsilon)$ associated to (18) that there exists $\varepsilon_0 > 0$ such that (13) holds for $m = 1$ and $\left\|u_1^\varepsilon\right\|^2_{L^\infty(0,T;L^2(\Omega^\varepsilon))} \leq C\varepsilon^{\frac{1}{2}}$ for all $0 < \varepsilon \leq \varepsilon_0$, where ε_0 is chosen such that $2 - \varepsilon^2C_1 \geq 1$, that is, $\varepsilon_0 \leq C_1^{-\frac{1}{2}}$.

For $1 < m < M$, we proceed as for $m = 1$ and multiply $(2)_1$ by u_m^ε and integrate over Ω^ε; then one obtains

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|u_m^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)} + d_m\left\|\nabla u_m^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \int_{\Omega^\varepsilon}\left(\left|u_m^\varepsilon\right|^2\sum_{j=1}^M a_{m,j}u_j^\varepsilon\right)dx \\ =\ & \int_{\Omega^\varepsilon}f_m^\varepsilon u_m^\varepsilon dx \leq \left\|f_m^\varepsilon\right\|_{L^2(\Omega^\varepsilon)}\left\|u_m^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}. \end{aligned}$$

Integrating over $(0,t)$ for $t \in (0,T]$, we get

$$\left\|u_m^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)} + 2d_m\int_0^t\left\|\nabla u_m^\varepsilon(s)\right\|^2_{L^2(\Omega^\varepsilon)}ds \leq 2\left\|f_m^\varepsilon\right\|_{L^2(Q_\varepsilon)}\left\|u_m^\varepsilon\right\|^2_{L^2(Q_\varepsilon)}.$$

Using (12), we get at once

$$\left\|u_m^\varepsilon\right\|^2_{L^\infty(0,T;L^2(\Omega^\varepsilon))} + \left\|\nabla u_m^\varepsilon\right\|^2_{L^2(Q_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}.$$

Finally, the proof of (13) for $m = M$ is obtained exactly as the one for the case $1 < m < M$ mutatis mutandis (replace f_m^ε by g_ε).

Let us now prove (14). We proceed as above by distinguishing three cases.

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For $m = 1$, we multiply $(1)_1$ by $\partial u_1^\varepsilon / \partial t$ and use $(1)_2$-$(1)_3$ to get

$$\int_{\Omega^\varepsilon} \left| \frac{\partial u_1^\varepsilon}{\partial t} \right|^2 dx + \frac{d_1}{2} \frac{\partial}{\partial t} \int_{\Omega^\varepsilon} |\nabla u_1^\varepsilon|^2 \, dx = \varepsilon d_1 \int_{\Gamma^\varepsilon} \psi^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} d\sigma_\varepsilon(x) - \int_{\Omega^\varepsilon} \left(u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right) dx.$$

But

$$\begin{aligned} \int_{\Omega^\varepsilon} \left(u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right) dx &\leq \left\| \frac{\partial u_1^\varepsilon}{\partial t} \right\|_{L^2(\Omega^\varepsilon)} \left\| u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right\|_{L^2(\Omega^\varepsilon)} \\ &\leq \frac{1}{2} \left\| \frac{\partial u_1^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)} + \frac{1}{2} \left\| u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} . \end{aligned}$$

Thus

$$\begin{aligned} &\left\| \frac{\partial u_1^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)} + d_1 \frac{\partial}{\partial t} \|\nabla u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \\ &\leq 2\varepsilon d_1 \int_{\Gamma^\varepsilon} \psi^\varepsilon \frac{\partial u_1^\varepsilon}{\partial t} d\sigma_\varepsilon(x) + \left\| u_1^\varepsilon \sum_{j=1}^M a_{1,j} u_j^\varepsilon \right\|^2_{L^2(\Omega^\varepsilon)} . \end{aligned} \tag{19}$$

Integrating (19) over $(0, t)$ and using the boundedness property (12), we obtain after integration by parts,

$$\begin{aligned} &\int_0^t \left\| \frac{\partial u_1^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds + d_1 \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} \leq C\varepsilon \\ &+ 2\varepsilon d_1 \int_{\Gamma^\varepsilon} \psi^\varepsilon u_1^\varepsilon d\sigma_\varepsilon(x) - 2\varepsilon d_1 \int_0^t \int_{\Gamma^\varepsilon} \frac{\partial \psi^\varepsilon}{\partial s}(s) u_1^\varepsilon(s) d\sigma_\varepsilon(x) ds, \end{aligned} \tag{20}$$

where we have used the fact that $\psi(0, \overline{x}, y) = 0$. Now, we use the inequality (17); then (20) becomes

$$\begin{aligned} &\int_0^t \left\| \frac{\partial u_1^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds + d_1 \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} \\ &\leq C\varepsilon + \varepsilon d_1 \left(\|\psi^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \|u_1^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \right) \\ &+ \varepsilon d_1 \int_0^t \left(\left\| \frac{\partial \psi^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Gamma^\varepsilon)} + \|u_1^\varepsilon(s)\|^2_{L^2(\Gamma^\varepsilon)} \right) ds \\ &\leq C\varepsilon + C\varepsilon \left(\|\psi\|^2_{L^\infty(0,T;\mathcal{C}(\overline{\Omega} \times \Gamma))} + \left\| \frac{\partial \psi}{\partial t} \right\|^2_{L^2(0,T;\mathcal{C}(\overline{\Omega} \times \Gamma))} \right) \\ &+ C \|u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + C d_1 \varepsilon^2 \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} + C \|u_1^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + C\varepsilon^2 \|\nabla u_1^\varepsilon\|^2_{L^2(Q_\varepsilon)} . \end{aligned}$$

It follows that

$$\int_0^t \left\| \frac{\partial u_1^\varepsilon}{\partial s}(s) \right\|^2_{L^2(\Omega^\varepsilon)} ds + d_1(1 - C\varepsilon^2) \|\nabla u_1^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)} \leq C\varepsilon, \tag{21}$$

where in (21), we took advantage of (12) and (13). Hence, choosing $\varepsilon \leq \varepsilon_0$ sufficiently small so that $1 - C\varepsilon^2 \geq 0$, we get (14) for $m = 1$.

The proof of (14) in the case when $1 < m \leq M$ follows the same lines of reasoning as above, but is much easier. It is therefore left to the reader. This completes the proof. □

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The following result whose proof can be found in [19, Theorem 3] will be useful in the sequel.

Lemma 3. *There exists a bounded linear operator $E_\varepsilon : H^1(\Omega^\varepsilon) \to H^1(\Omega_\varepsilon)$ such that, for all $v \in H^1(\Omega^\varepsilon)$, $E_\varepsilon v = v$ in Ω^ε and*

$$\|E_\varepsilon v\|_{L^2(\Omega_\varepsilon)} \le C\left(\|v\|_{L^2(\Omega^\varepsilon)} + \varepsilon\|\nabla v\|_{L^2(\Omega^\varepsilon)}\right),$$

and

$$\|\nabla E_\varepsilon v\|_{L^2(\Omega_\varepsilon)} \le C\|\nabla v\|_{L^2(\Omega^\varepsilon)}$$

for a positive constant independent of both ε and v.

Based on Lemma 3, we define the extension $E_\varepsilon v$ of a function $v \in L^2(0,T;H^1(\Omega^\varepsilon))$ to $L^2(0,T;H^1(\Omega_\varepsilon))$ as follows:

$$(E_\varepsilon v)(t) = E_\varepsilon(v(t)) \ \text{ a.e. } t \in (0,T).$$

Then accounting of Lemma 3 and Theorem 2, we have

$$\sup_{1\le m\le M}\left(\|E_\varepsilon u_m^\varepsilon\|_{L^\infty(\Omega_\varepsilon^T)} + \|E_\varepsilon u_m^\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))}\right) \le C\varepsilon^{\frac{1}{2}}, \tag{22}$$

where $C > 0$ is independent of ε and

$$\Omega_\varepsilon^T = (0,T)\times\Omega_\varepsilon. \tag{23}$$

We also need an estimate on $\partial u_m^\varepsilon/\partial t$ in $L^2(\Omega_\varepsilon^T)$. To that end, we procced as in [20] and consider the restriction operator $R_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega^\varepsilon)$, $R_\varepsilon v = v|_{\Omega^\varepsilon}$ (the restriction of v to Ω^ε). Then it is a fact that R_ε is a bounded linear operator as

$$\|R_\varepsilon v\|_{L^2(\Omega^\varepsilon)} \le \|v\|_{L^2(\Omega_\varepsilon)} \quad \forall v \in L^2(\Omega_\varepsilon).$$

Now, if $R^* : L^2(\Omega^\varepsilon) \to L^2(\Omega_\varepsilon)$ denotes the adjoint operator of R_ε, then for $v \in L^2(0,T;L^2(\Omega^\varepsilon)) = L^2(Q_\varepsilon)$, we define $R_\varepsilon^* v$ as follows:

$$(R_\varepsilon^* v)(t) = R_\varepsilon^*(v(t)) \text{ a.e. } t \in (0,T),$$

and we have

$$\langle R_\varepsilon^* u, v\rangle = \int_0^T \langle R_\varepsilon^*(u(t)), v(t)\rangle\, dt = \int_0^T \langle u(t), R_\varepsilon(v(t))\rangle\, dt$$

for all $u \in L^2(Q_\varepsilon)$ and $v \in L^2(\Omega_\varepsilon^T)$. It is therefore easy to see that $R_\varepsilon^* v = \chi_{\Omega^\varepsilon} v$ for all $v \in L^2(Q_\varepsilon)$, or equivalently

$$R_\varepsilon^* v = \chi_{\Omega^\varepsilon} E_\varepsilon v \text{ for all } v \in L^2(Q_\varepsilon). \tag{24}$$

Lemma 4. *Let the assumptions of Theorem 2 hold. It holds that*

$$\left\|\chi_{\Omega^\varepsilon}\frac{\partial E_\varepsilon u_m^\varepsilon}{\partial t}\right\|_{L^2(\Omega_\varepsilon^T)} \le C\varepsilon^{\frac{1}{2}} \text{ for all } 0 < \varepsilon \le \varepsilon_0, \tag{25}$$

where $C > 0$ is independent of ε, and ε_0 is defined in Theorem 2.

Proof. First, we have $R_\varepsilon^* \partial_t u_m^\varepsilon = \chi_{\Omega^\varepsilon}\partial_t E_\varepsilon u_m^\varepsilon$, where $\partial_t = \partial/\partial t$. Thus it is sufficient to show that

$$\|R_\varepsilon^* \partial_t E_\varepsilon u_m^\varepsilon\|_{L^2(\Omega_\varepsilon^T)} \le C\varepsilon^{\frac{1}{2}}.$$

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So, let $\varphi \in L^2(\Omega_\varepsilon^T)$; then

$$\begin{aligned} |\langle R_\varepsilon^* \partial_t E_\varepsilon u_m^\varepsilon, \varphi\rangle| &= |\langle \partial_t E_\varepsilon u_m^\varepsilon, R_\varepsilon \varphi\rangle| \leq \|\partial_t E_\varepsilon u_m^\varepsilon\|_{L^2(Q_\varepsilon)} \|R_\varepsilon \varphi\|_{L^2(Q_\varepsilon)} \\ &\leq \|\partial_t u_m^\varepsilon\|_{L^2(Q_\varepsilon)} \|\varphi\|_{L^2(\Omega_\varepsilon^T)} \leq C\varepsilon^{\frac{1}{2}} \|\varphi\|_{L^2(\Omega_\varepsilon^T)} . \end{aligned}$$

Whence the result. □

3. Two-scale convergence in thin heterogeneous domains

The two-scale convergence for thin heterogeneous domains has been introduced in [25] and extended to thin porous surfaces in [8,19]. The notations used in this section are the same as in the previous ones. Especially, the domain Ω_ε is defined as above, that is, $\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$. When $\varepsilon \to 0$, Ω_ε shrinks to the "interface" $\Omega_0 = \Omega \times \{0\} \equiv \Omega$. We know that $Q_\varepsilon = (0,T) \times \Omega^\varepsilon$ and $\Omega_\varepsilon^T = (0,T) \times \Omega_\varepsilon$, and we set $Q = (0,T) \times \Omega_0$, $I = (-1,1)$, $Y = (0,1)^2$ and finally $Z = Y \times I$. Let $1 \leq p < \infty$; by $L^p_{per}(Y; L^p(I))$ we denote the space of functions in $L^p_{loc}(\mathbb{R}^2; L^p(I))$ that are Y-periodic. Accordingly we define $W^{1,p}_{per}(Y; W^{1,p}(I))$ as the subspace of $W^{1,p}_{loc}(Y; W^{1,p}(I))$ made of periodic Y-periodic functions, and we set

$$W^{1,p}_{\#}(Y; W^{1,p}(I)) = \left\{ u \in W^{1,p}_{per}(Y; W^{1,p}(I)) : \int_Z u(\overline{y}, y_3) dy = 0 \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{\#} = \left(\int_Z |\nabla u|^p \, dy \right)^{1/p}, \ u \in W^{1,p}_{\#}(Y; W^{1,p}(I)).$$

Any x in $\mathbb{R}^3$ writes $(\overline{x}, x_3)$ or $(\overline{x}, \zeta)$ where $\overline{x} = (x_1, x_2)$. We identify Ω_0 with Ω so that the generic element in Ω_0 is also denoted by $\overline{x}$ instead of $(\overline{x}, 0)$.

We are now able to define the two-scale convergence for thin heterogeneous domains and for thin boundaries.

Definition 5. (a) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega_\varepsilon^T)$ ($1 \leq p < \infty$) is said to

(i) weakly two-scale converge in $L^p(\Omega_\varepsilon^T)$ to $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$ if as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon} \int_{\Omega_\varepsilon^T} u_\varepsilon(t,x) f\left(t, \overline{x}, \frac{x}{\varepsilon}\right) dx dt \to \int_Q \int_Z u_0(t, \overline{x}, y) f(t, \overline{x}, y) dy d\overline{x} dt$$

for any $f \in L^{p'}(Q; \mathcal{C}_{per}(Y; L^{p'}(I)))$ $(1/p' = 1 - 1/p)$; we denote this by "$u_\varepsilon \to u_0$ in $L^p(\Omega_\varepsilon^T)$-weak 2$s$";

(ii) strongly two-scale converge in $L^p(\Omega_\varepsilon^T)$ to $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$ if it is weakly two-scale convergent and further

$$\varepsilon^{-\frac{1}{p}} \|u_\varepsilon\|_{L^p(Q_\varepsilon)} \to \|u_0\|_{L^p(Q; L^p_{per}(Y; L^p(I)))}; \tag{26}$$

we denote this by "$u_\varepsilon \to u_0$ in $L^p(\Omega_\varepsilon^T)$-strong 2$s$".

(b) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p((0,T) \times \Gamma^\varepsilon)$ is said to weakly two-scale converge in $L^p((0,T) \times \Gamma^\varepsilon)$ to $u_0 \in L^p(Q \times \Gamma)$ if, as $\varepsilon \to 0$,

$$\int_{(0,T)\times\Gamma^\varepsilon} u_\varepsilon(t,x) f(t, \overline{x}, \frac{x}{\varepsilon}) d\sigma_\varepsilon(x) dt \to \iint_{Q\times\Gamma} u_0(t, \overline{x}, y) d\sigma(y) d\overline{x} dt$$

for all $f \in L^{p'}(0,T; \mathcal{C}(\overline{\Omega} \times \Gamma))$ that is Y-periodic in $\overline{y}$.

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Remark 6. It is easy to see that if $u_0 \in L^p(Q;\mathcal{C}_{per}(Y;L^p(I)))$ then (26) is equivalent to

$$\varepsilon^{-\frac{1}{p}} \|u_\varepsilon - u_0^\varepsilon\|_{L^p(\Omega_\varepsilon^T)} \to 0 \text{ as } \varepsilon \to 0, \tag{27}$$

where $u_0^\varepsilon(t,x) = u_0(t,\overline{x},x/\varepsilon)$ for $(t,x) \in \Omega_\varepsilon^T$.

We start with the following important result that should be used in the sequel; see [7, Lemma 3.2.3] for the proof.

Lemma 7. *Let $\psi \in L^p(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))$ that is Y-periodic in $\overline{y}$. Then, letting $\psi^\varepsilon(t,x) = \psi(t,\overline{x},x/\varepsilon)$ for $(t,x) \in (0,T)\times\Gamma^\varepsilon$, we have*

(i) $\|\psi^\varepsilon\|_{L^p((0,T)\times\Gamma^\varepsilon)} \leq \|\psi\|_{L^p(0,T;\mathcal{C}(\overline{\Omega}\times\Gamma))}$;
(ii) $\int_0^T \int_{\Gamma^\varepsilon} \psi(t,\overline{x},x/\varepsilon)d\sigma_\varepsilon(x)dt \to \iint \psi(t,\overline{x},y)d\sigma(y)dt.$

Throughout the work, the letter E will stand for any ordinary sequence $(\varepsilon_n)_{n\geq 1}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ when $n \to \infty$. The generic term of E will be merely denote by ε and $\varepsilon \to 0$ will mean $\varepsilon_n \to 0$ as $n \to \infty$. This being so, we have the following compactness results.

Theorem 8. (i) *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(\Omega_\varepsilon^T)$ $(1 < p < \infty)$ such that*

$$\sup_{\varepsilon\in E} \varepsilon^{-1/p} \|u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)} \leq C$$

where C is a positive constant independent of ε. Then there exists a subsequence E' of E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly two-scale converges in $L^p(\Omega_\varepsilon^T)$ to some $u_0 \in L^p(Q;L^p_{per}(Y;L^p(I)))$.

(ii) *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p((0,T)\times\Gamma^\varepsilon)$ such that*

$$\|u_\varepsilon\|_{L^p((0,T)\times\Gamma^\varepsilon)} \leq C,$$

$C > 0$ being independent of ε. Then there exist a subsequence E' of E and a function $u_0 \in L^p(Q\times\Gamma)$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p((0,T)\times\Gamma^\varepsilon)\text{-weak } 2s.$$

Proof. The proof of part (i) can be found in [16] while the proof of part (ii) can be found in [7] (see also [8,19]). □

Theorem 9. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(0,T;W^{1,p}(\Omega_\varepsilon))$ $(1 < p < \infty)$ such that*

$$\sup_{\varepsilon\in E} \left(\varepsilon^{-1/p} \|u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)} + \varepsilon^{-1/p} \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)}\right) \leq C$$

where $C > 0$ is independent of ε. Then there exist a subsequence E' of E and a couple (u_0,u_1) with $u_0 \in L^p(0,T;W^{1,p}(\Omega))$ and $u_1 \in L^p(Q;W^{1,p}_\#(Y;W^{1,p}(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s,$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s,\ i = 1,2, \tag{28}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_3} \to \frac{\partial u_1}{\partial y_3} \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{29}$$

Proof. See [16] for the proof. □

Remark 10. If we set

$$\nabla_{\overline{x}} u_0 = \left(\frac{\partial u_0}{\partial x_1}, \frac{\partial u_0}{\partial x_2}, 0 \right),$$

then (28) and (29) are equivalent to

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u_0 + \nabla_y u_1 \text{ in } L^p(\Omega_\varepsilon^T)^3\text{-weak } 2s.$$

The following result is sharper than its homologue in Theorem 9.

Theorem 11. *Let $(u_\varepsilon)_{\varepsilon \in E}$ be a sequence in $L^2(0,T;H^1(\Omega_\varepsilon))$ such that*

$$\sup_{\varepsilon \in E} \varepsilon^{-\frac{1}{2}} \left(\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \|u_\varepsilon\|_{H^1(0,T;L^2(\Omega_\varepsilon))} \right) \leq C, \tag{30}$$

where C is a positive constant independent of ε. Finally, suppose that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Then there exist a subsequence E' of E and a couple $(u, u_1) \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))) \times L^2(Q;H^1_\#(Y;H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u \textit{ in } L^2(\Omega_\varepsilon^T)\textit{-strong } 2s, \tag{31}$$

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u + \nabla_y u_1 \textit{ in } L^2(\Omega_\varepsilon^T)^3\textit{-weak } 2s, \tag{32}$$

and

$$\partial_t u_\varepsilon \to \partial_t u \textit{ in } L^2(\Omega_\varepsilon^T)\textit{-weak } 2s. \tag{33}$$

Proof. First, owing to Theorem 9, there exist a subsequence E' of E and a couple $(u, u_1) \in L^2(0,T;H^1(\Omega))) \times L^2(Q;H^1_\#(Y;H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s,$$

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u + \nabla_y u_1 \text{ in } L^2(\Omega_\varepsilon^T)^3\text{-weak } 2s,$$

and

$$\partial_t u_\varepsilon \to \partial_t u \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s.$$

It remains to prove (31). To that end, we set

$$M_\varepsilon u_\varepsilon(t, \overline{x}) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_\varepsilon(t, \overline{x}, x_3) dx_3 \text{ for } (t, \overline{x}) \in Q. \tag{34}$$

Then we easily see that $M_\varepsilon u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ with

$$\sup_{\varepsilon \in E} \left(\|M_\varepsilon u_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|M_\varepsilon u_\varepsilon\|_{H^1(0,T;L^2(\Omega))} \right) \leq C. \tag{35}$$

Then from (35), we derive the existence of a subsequence of E' still denoted by E' and of a function $u_0 \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that, as $E' \ni \varepsilon \to 0$,

$$M_\varepsilon u_\varepsilon \to u_0 \text{ in } L^2(0,T;L^2(\Omega))\text{-strong}. \tag{36}$$

We recall that (36) stems from the compactness of the embedding $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$.

Now, from the Poincaré-Wirtinger inequality, it holds that

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))},$$

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so that

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \to 0 \text{ as } E' \ni \varepsilon \to 0. \tag{37}$$

Thus the inequality

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)} \leq \varepsilon^{-\frac{1}{2}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^2(\Omega_\varepsilon^T)} + \varepsilon^{-\frac{1}{2}} \|M_\varepsilon u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)}$$

associated to the equality

$$\varepsilon^{-\frac{1}{2}} \|M_\varepsilon u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)} = \sqrt{2} \|M_\varepsilon u_\varepsilon - u_0\|_{L^2(Q)}$$

yield (with the help of (36) and (37))

$$\varepsilon^{-\frac{1}{2}} \|u_\varepsilon - u_0\|_{L^2(\Omega_\varepsilon^T)} \to 0 \text{ as } E' \ni \varepsilon \to 0.$$

This shows that $u_\varepsilon \to u_0$ in $L^2(\Omega_\varepsilon^T)$-strong $2s$, and so $u_0 = u$. The proof is complete. □

The next result and its corollary are proved exactly as their homologues in [27, Theorem 6 and Corollary 5] (see also [28]).

Theorem 12. *Let $1 < p, q < \infty$ and $r \geq 1$ be such that $1/r = 1/p + 1/q \leq 1$. Assume $(u_\varepsilon)_{\varepsilon \in E} \subset L^q(\Omega_\varepsilon^T)$ is weakly two-scale convergent in $L^q(\Omega_\varepsilon^T)$ to some $u_0 \in L^q(Q; L^q_{per}(Y; L^q(I)))$, and $(v_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega_\varepsilon^T)$ is strongly two-scale convergent in $L^p(\Omega_\varepsilon^T)$ to some $v_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$. Then the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon \in E}$ is weakly two-scale convergent in $L^r(\Omega_\varepsilon^T)$ to $u_0 v_0$.*

Corollary 13. *Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega_\varepsilon^T)$ and $(v_\varepsilon)_{\varepsilon \in E} \subset L^{p'}(\Omega_\varepsilon^T) \cap L^\infty(\Omega_\varepsilon^T)$ ($1 < p < \infty$ and $p' = p/(p-1)$) be two sequences such that:*

(i) $u_\varepsilon \to u_0$ in $L^p(Q_\varepsilon)$-weak $2s$;
(ii) $v_\varepsilon \to v_0$ in $L^{p'}(Q_\varepsilon)$-strong $2s$;
(iii) $(v_\varepsilon)_{\varepsilon \in E}$ is bounded in $L^\infty(Q_\varepsilon)$.

Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(Q_\varepsilon)$-weak $2s$.

4. Derivation of the homogenized system

4.1. Preliminary results

In this subsection, we aim at providing further important convergence results that will be very useful in the sequel. In that order, it is to be noted that Ω^ε can alternatively be defined as follows: $\Omega^\varepsilon = \cup_{k \in K_\varepsilon} \varepsilon(k + Z_s)$, where $K_\varepsilon = \{k \in \mathbb{Z}^2 \times \{0\} : \varepsilon(k + Z) \subset \Omega_\varepsilon\}$ with $\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$. We set $\Lambda_\varepsilon = \cup_{k \in K_\varepsilon}(k + Z_s)$, a periodic repetition of the set Z_s. We denote by χ_ε the characteristic function function of Λ_ε in Ω_ε: $\chi_\varepsilon \equiv \chi_{\Lambda_\varepsilon}$. Then it holds that

$$\Omega^\varepsilon = \{x \in \Omega_\varepsilon : \chi_\varepsilon(\frac{x}{\varepsilon}) = 1\},$$

so that $\chi_{\Omega^\varepsilon}(x) = \chi_\varepsilon(\frac{x}{\varepsilon})$ for $x \in \Omega_\varepsilon$.

Lemma 14. *Let $(u_\varepsilon)_{\varepsilon > 0}$ be a sequence in $L^p(\Omega_\varepsilon^T)$ ($1 < p < \infty$) that weakly two-scale converges in $L^p(\Omega_\varepsilon^T)$ towards $u_0 \in L^p(Q; L^p_{per}(Y; L^p(I)))$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \chi_\varepsilon \to u_0 \chi_{Z_s} \text{ in } L^p(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{38}$$

If further the two-scale convergence is strong, then (38) holds in the strong two-scale sense.

Proof. Set $v_\varepsilon(t,\overline{x},\zeta) = u_\varepsilon(t,\overline{x},\varepsilon\zeta)$ for $(t,\overline{x},\zeta) \in \Omega_1^T$. Then since $u_\varepsilon \to u_0$ in $L^p(\Omega_\varepsilon^T)$-weak 2$s$, it holds that $\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^T)} \leq C\varepsilon^{1/2}$ ($C > 0$ being independent of ε), so that $\|v_\varepsilon\|_{L^p(\Omega_1^T)} \leq C$. Hence, up to a subsequence, $v_\varepsilon \to v_0$ in $L^p(\Omega_1^T)$ in the usual classical two-scale weak sense, where $v_0 \in L^p(Q \times I; L^p_{per}(Y))$. Next, let $f \in \mathcal{C}(\overline{Q}; \mathcal{C}_{per}(Y; \mathcal{C}(\overline{I})))$. Passing to the limit (in the subsequence determined above) in the obvious equality

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} u_\varepsilon(t,x) f(t,\overline{x},\frac{x}{\varepsilon})dxdt = \int_{\Omega_1^T} v_\varepsilon(t,\overline{x},\zeta) f(t,\overline{x},\frac{\overline{x}}{\varepsilon},\zeta)d\overline{x}d\zeta dt,$$

we get at once $u_0 = v_0$.

This being so, choosing f as above, one has

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} u_\varepsilon(t,x)\chi_\varepsilon(\frac{x}{\varepsilon}) f(t,\overline{x},\frac{x}{\varepsilon})dxdt &= \int_{\Omega_1^T} v_\varepsilon(t,\overline{x},\zeta)\chi_{\Lambda_1}(\frac{\overline{x}}{\varepsilon},\zeta) f(t,\overline{x},\frac{\overline{x}}{\varepsilon},\zeta)d\overline{x}d\zeta dt \\ &\equiv J_\varepsilon.\end{aligned}$$

Owing to the usual two-scale concept, we obtain, as $\varepsilon \to 0$,

$$J_\varepsilon \to \iint_{\Omega_1^T \times Y} u_0(t,\overline{x},\overline{y},\zeta)\chi_{Z_s}(\overline{y},\zeta) f(t,\overline{x},\overline{y},\zeta)d\overline{x}d\overline{y}d\zeta dt, \tag{39}$$

where in (39) we have used the fact that $u_0 = v_0$ proved above. This concludes the proof. □

The following result will be crucial in the homogenization process. From now on, we set $\chi_s = \chi_{Z_s}$, the characteristic function of Z_s in Z.

Proposition 15. *Let $(u_m^\varepsilon)_{1\leq m\leq M}$ be the solution of (1)-(3). Given any ordinary sequence E, there exist a subsequence E' of E and functions $(u_m, u_m^1)_{1\leq m\leq M}$ with $u_m \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $u_m^1 \in L^2(Q; H^1_\#(Y; H^1(I)))$, such that, as $E' \ni \varepsilon \to 0$,*

$$\chi_\varepsilon u_m^\varepsilon \to \chi_s u_m \text{ in } L^2(\Omega_\varepsilon^T)\text{-strong } 2s, \tag{40}$$

$$\chi_\varepsilon \nabla u_m^\varepsilon \to \chi_s(\nabla_{\overline{x}} u_m + \nabla_y u_m^1) \text{ in } L^2(\Omega_\varepsilon^T)^3\text{-weak } 2s, \tag{41}$$

and

$$\chi_\varepsilon \partial_t u_m^\varepsilon \to \chi_s \partial_t u_m \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{42}$$

Proof. Since $E_\varepsilon u_m^\varepsilon = u_m^\varepsilon$ in Q_ε, we have

$$\chi_\varepsilon u_m^\varepsilon = \chi_\varepsilon E_\varepsilon u_m^\varepsilon. \tag{43}$$

Next, appealing to (22) and (25), we are in a condition to apply Theorem 11: Given an ordinary sequence E, there exist a subsequence E' of E and a couple $(u_m, u_m^1) \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))) \times L^2(Q; H^1_\#(Y; H^1(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$E_\varepsilon u_m^\varepsilon \to u_m \text{ in } L^2(\Omega_\varepsilon^T)\text{-strong } 2s, \tag{44}$$

$$\nabla E_\varepsilon u_m^\varepsilon \to \nabla_{\overline{x}} u_m + \nabla_y u_m^1 \text{ in } L^2(\Omega_\varepsilon^T)^3\text{-weak } 2s, \tag{45}$$

and

$$E_\varepsilon \partial_t u_m^\varepsilon \to \partial_t u_m \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s. \tag{46}$$

Applying Lemma 14 and accounting of (43), we are done. □

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4.2. Passage to the limit: Proof of the main result

Assume that the functions u_m and u_m^1 are as in Proposition 15. Let $\varphi \in \mathcal{C}^1(\overline{Q})$ and $\varphi_1 \in \mathcal{C}^1(\overline{Q} \times \overline{I}; \mathcal{C}^1_{per}(Y))$, and define

$$\Phi_\varepsilon(t,x) = \varphi(t,\overline{x}) + \varepsilon\varphi_1(t,\overline{x},\frac{x}{\varepsilon}) \text{ for } (t,x) \in \Omega_\varepsilon^T.$$

We Φ_ε as test function in the variational form of (1)-(3):

$$\begin{cases} \frac{1}{\varepsilon}\int_{Q_\varepsilon} \frac{\partial u_1^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_1}{\varepsilon}\int_{Q_\varepsilon} \nabla u_1^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt + \frac{1}{\varepsilon}\int_{Q_\varepsilon} u_1^\varepsilon \sum_{j=1}^{M} a_{1,j}u_j^\varepsilon \Phi_\varepsilon dxdt \\ = \int_0^T \int_{\Gamma^\varepsilon} \psi(t,\overline{x},\frac{x}{\varepsilon})\Phi_\varepsilon(t,x)dtd\sigma_\varepsilon(x); \end{cases} \tag{47}$$

For $1 < m < M$,

$$\begin{cases} \frac{1}{\varepsilon}\int_{Q_\varepsilon} \frac{\partial u_m^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_m}{\varepsilon}\int_{Q_\varepsilon} \nabla u_m^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt + \frac{1}{\varepsilon}\int_{Q_\varepsilon} u_m^\varepsilon \sum_{j=1}^{M} a_{m,j}u_j^\varepsilon \Phi_\varepsilon dxdt \\ = \frac{1}{2\varepsilon}\int_{Q_\varepsilon} \sum_{j=1}^{m-1} a_{j,m-j}u_j^\varepsilon u_{m-j}^\varepsilon \Phi_\varepsilon dtdx; \end{cases} \tag{48}$$

and

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon} \frac{\partial u_M^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_M}{\varepsilon}\int_{Q_\varepsilon} \nabla u_M^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt = \frac{1}{2}\sum_{j+k\geq M, j<M, k<M} \frac{1}{\varepsilon}\int_{Q_\varepsilon} a_{j,k}u_j^\varepsilon u_k^\varepsilon \Phi_\varepsilon dxdt. \tag{49}$$

Let us first deal with (47). We note that it is equivalent to

$$\begin{cases} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} \chi_\varepsilon \frac{\partial u_1^\varepsilon}{\partial t}\Phi_\varepsilon dxdt + \frac{d_1}{\varepsilon}\int_{\Omega_\varepsilon^T} \chi_\varepsilon \nabla u_1^\varepsilon \cdot \nabla\Phi_\varepsilon dxdt + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^T} \chi_\varepsilon u_1^\varepsilon \sum_{j=1}^{M} a_{1,j}u_j^\varepsilon \Phi_\varepsilon dxdt \\ = \int_0^T \int_{\Gamma^\varepsilon} \psi(t,\overline{x},\frac{x}{\varepsilon})\Phi_\varepsilon(t,x)dtd\sigma_\varepsilon(x). \end{cases} \tag{50}$$

We have that

$$\nabla\Phi_\varepsilon(t,x) = \nabla_{\overline{x}}\varphi(t,\overline{x}) + \nabla_y\varphi_1((t,\overline{x},\frac{x}{\varepsilon}) + \varepsilon\nabla_{\overline{x}}\varphi_1((t,\overline{x},\frac{x}{\varepsilon}).$$

Thus we may apply Proposition 15 to pass to the limit in the first two terms on the left-hand side of (50), using Φ_ε as test function in the two-scale concept. As for the term on the right-hand side of (50), we use Lemma 7 to pass to the limit therein. We end up with the last term on the left-hand side where the limit passage therein is more involved. Indeed, we use there the strong two-scale convergence of $\chi_\varepsilon u_1^\varepsilon$ towards $\chi_s u_1$ associated to the weak two-scale convergence of $\chi_\varepsilon u_j^\varepsilon$ $(1 \leq j \leq M)$ towards $\chi_s u_j$ to get from Corollary 13 that, for $1 \leq j \leq M$, we have, as $E' \ni \varepsilon \to 0$,

$$\chi_\varepsilon u_1^\varepsilon u_j^\varepsilon = (\chi_\varepsilon u_1^\varepsilon)(\chi_\varepsilon u_j^\varepsilon) \to \chi_s u_1 u_j \text{ in } L^2(\Omega_\varepsilon^T)\text{-weak } 2s.$$

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Therefore, using in that term the test function Φ_ε and taking into account all the process described above after (50), we are led, as $E' \ni \varepsilon \to 0$ in (50), to

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_1}{\partial t} \varphi d\overline{x} dy dt + d_1 \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot (\nabla_{\overline{x}} \varphi + \nabla_y \varphi_1) d\overline{x} dy dt \\ + \iint_{Q\times Z} \chi_s u_1 \sum_{j=1}^{M} a_{1,j} u_j \varphi d\overline{x} dy dt = \iint_{Q\times \Gamma} \psi \varphi d\overline{x} d\sigma(y) dt \\ \forall (\varphi, \varphi_1) \in \mathcal{C}^1(\overline{Q}) \times \mathcal{C}^1(\overline{Q} \times \overline{I}; \mathcal{C}^1_{per}(Y)). \end{cases} \tag{51}$$

We use the same process as for (50) to pass to the limit in (48) and in (49), and we obtain:
For $1 < m < M$,

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_m}{\partial t} \varphi d\overline{x} dy dt + d_m \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_m + \nabla_y u_m^1) \cdot (\nabla_{\overline{x}} \varphi + \nabla_y \varphi_1) d\overline{x} dy dt \\ + \iint_{Q\times Z} \chi_s u_m \sum_{j=1}^{M} a_{m,j} u_j \varphi d\overline{x} dy dt = \frac{1}{2} \iint_{Q\times Z} \chi_s \sum_{j=1}^{m-1} a_{j,m-j} u_j u_{m-j} \varphi d\overline{x} dy dt \\ \text{for all } (\varphi, \varphi_1) \in \mathcal{C}^1(\overline{Q}) \times \mathcal{C}^1(\overline{Q} \times \overline{I}; \mathcal{C}^1_{per}(Y)); \end{cases} \tag{52}$$

and

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_M}{\partial t} \varphi d\overline{x} dy dt + d_M \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_M + \nabla_y u_M^1) \cdot (\nabla_{\overline{x}} \varphi + \nabla_y \varphi_1) d\overline{x} dy dt \\ = \frac{1}{2} \sum_{j+k\geq M, j<M, k<M} \iint_{Q\times Z} \chi_s a_{j,k} u_j u_k \varphi d\overline{x} dy dt \\ \text{for all } (\varphi, \varphi_1) \in \mathcal{C}^1(\overline{Q}) \times \mathcal{C}^1(\overline{Q} \times \overline{I}; \mathcal{C}^1_{per}(Y)). \end{cases} \tag{53}$$

We have proved the following result.

Theorem 16. *The functions $(u_m, u_m^1)_{1\leq m\leq M}$ determined by Proposition 15 solve the variational problems (51), (52) and (53).*

Our next goal is to derive the system whose $(u_m)_{1\leq m\leq M}$ is solution to. To that end, we start by uncoupling each of the equations (51)-(53). We first consider (51) and we see that it is equivalent to the following system consisting of (54) and (55) below:

$$\iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot \nabla_y \varphi_1 d\overline{x} dy dt = 0 \ \ \forall \varphi_1 \in \mathcal{C}^1(\overline{Q} \times \overline{I}; \mathcal{C}^1_{per}(Y)), \tag{54}$$

$$\begin{cases} \iint_{Q\times Z} \chi_s \frac{\partial u_1}{\partial t} \varphi d\overline{x} dy dt + d_1 \iint_{Q\times Z} \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot \nabla_{\overline{x}} \varphi d\overline{x} dy dt \\ + \iint_{Q\times Z} \chi_s u_1 \sum_{j=1}^{M} a_{1,j} u_j \varphi d\overline{x} dy dt = \iint_{Q\times \Gamma} \psi \varphi d\overline{x} d\sigma(y) dt \ \ \forall \varphi \in \mathcal{C}^1(\overline{Q}). \end{cases} \tag{55}$$

Let us first consider Eq. (54) and choose therein φ_1 under the form $\varphi_1(t, \overline{x}, y) = \phi(t, \overline{x}) \eta(y)$ with $\phi \in \mathcal{C}_0^\infty(Q)$ and $\eta \in \mathcal{C}^\infty_{per}(Y) \otimes \mathcal{C}^1(\overline{I})$; then (54) becomes

$$\int_Z \chi_s (\nabla_{\overline{x}} u_1 + \nabla_y u_1^1) \cdot \nabla_y \eta dy = 0 \ \ \forall \eta \in \mathcal{C}^\infty_{per}(Y) \otimes \mathcal{C}^1(\overline{I}). \tag{56}$$

To solve (56), we rather consider the variation problem

$$\int_Z \chi_s(e_j + \nabla_y \omega_j) \cdot \nabla_y \eta dy = 0 \;\; \forall \eta \in \mathcal{C}^\infty_{per}(Y) \otimes \mathcal{C}^1(\bar{I}), \tag{57}$$

where e_j ($j = 1, 2, 3$) denotes the jth vector of the canonical basis of $\mathbb{R}^3$. Then (57) is equivalent to the cell problem

$$\begin{cases} -\operatorname{div}_y(e_j + \nabla_y \omega_j) = 0 \text{ in } Z_s, \;\; (e_j + \nabla_y \omega_j) \cdot \nu = 0 \text{ on } \Gamma \\ \omega_j(., y_3) \text{ is } Y\text{-periodic,} \end{cases} \tag{58}$$

where ν stands for the outward unit normal to Γ. It is well known that (58) possesses a unique solution in the space

$$H^1_\#(Z_s) = \left\{ u \in H^1(Z_s) : u \text{ is } Y\text{-periodic and } \int_{Z_s} u dy = 0 \right\}.$$

Now, multiplying (57) by $\partial u_1/\partial x_j$ ($j = 1, 2$) and summing up the resulting equations, then comparing the latter sum with (56) yields at once

$$u_1^1(t, \bar{x}, y) = \sum_{j=1}^{2} \omega_j(y) \frac{\partial u_1}{\partial x_j}(t, \bar{x}) \equiv \omega(y) \cdot \nabla_{\bar{x}} u_1(t, \bar{x}), \tag{59}$$

where $\omega = (\omega_1, \omega_2)$.

Next, going back to (55) and replacing there u_1^1 by the expression obtained in (59), we get

$$\begin{cases} \int_Q \left(\int_Z \chi_s dy \right) \frac{\partial u_1}{\partial t} \varphi d\bar{x} dt + d_1 \int_Q \left(\int_Z \chi_s (I_2 + \nabla_{\bar{y}} \omega) dy \right) \nabla_{\bar{x}} u_1 \cdot \nabla_{\bar{x}} \varphi d\bar{x} dt \\ + \int_Q \left(\int_Z \chi_s dy \right) u_1 \sum_{j=1}^{M} a_{1,j} u_j \varphi d\bar{x} dt = \int_Q \left(\int_\Gamma \psi(., ., y) d\sigma(y) \right) \varphi d\bar{x} dt \\ \text{for all } \varphi \in \mathcal{C}^1(\overline{Q}), \end{cases} \tag{60}$$

where I_2 is the identity 2×2 matrix.

This being so, we set

$$\theta = \int_Z \chi_s dy = |Z_s| > 0, \; A = I_2 + \nabla_{\bar{y}} \omega \text{ and } \widetilde{\psi}(t, \bar{x}) = \int_\Gamma \psi((t, \bar{x}, y) d\sigma(y). \tag{61}$$

Then A is a 2×2 symmetric positive definite matrix. Indeed it is a fact that the entries of A have the form

$$A_{ij} = \int_{Z_s} (e_i + \nabla_y \omega_i) \cdot (e_j + \nabla_y \omega_j) dy, \;\; 1 \leq i, j \leq 2;$$

this stems from (57) where we show that it is still valid for $\eta \in H^1_\#(Y; H^1(I))$ and the choose therein $\eta = \omega_i$. With the above notations in (61), we see that (60) is equivalent to the problem

$$\begin{cases} \theta \frac{\partial u_1}{\partial t} - \operatorname{div}_{\bar{x}}(d_1 A \nabla_{\bar{x}} u_1) + \theta u_1 \sum_{j=1}^{M} a_{1,j} u_j = d_1 \widetilde{\psi} \text{ in } Q \\ A \nabla_{\bar{x}} u_1 \cdot n = 0 \text{ on } (0, T) \times \partial\Omega \\ u_1(0, \bar{x}) = 0 \text{ in } \Omega. \end{cases} \tag{62}$$

Proceeding as we did for (51), we easily show that (52) and (53) are equivalent to the variational formulations of the following PDEs:

For $1 < m < M$, (52) is equivalent to

$$\begin{cases} \theta \frac{\partial u_m}{\partial t} - \text{div}_{\overline{x}}(d_m A \nabla_{\overline{x}} u_m) + \theta u_m \sum_{j=1}^{M} a_{m,j} u_j - \frac{\theta}{2} \sum_{j=1}^{m-1} a_{j,m-j} u_j u_{m-j} = 0 \text{ in } Q \\ A \nabla_{\overline{x}} u_m \cdot n = 0 \text{ on } (0,T) \times \partial\Omega \\ u_m(0, \overline{x}) = 0 \text{ in } \Omega; \end{cases} \tag{63}$$

and for $m = M$, (53) is equivalent to

$$\begin{cases} \theta \frac{\partial u_M}{\partial t} - \text{div}_{\overline{x}}(d_M A \nabla_{\overline{x}} u_M) - \frac{\theta}{2} \sum_{j+k \geq M, j<M, k<M} a_{j,k} u_j u_k = 0 \text{ in } Q \\ A \nabla_{\overline{x}} u_M \cdot n = 0 \text{ on } (0,T) \times \partial\Omega \\ u_M(0, \overline{x}) = 0 \text{ in } \Omega. \end{cases} \tag{64}$$

The system (62)-(64) is the homogenized model arising from the microscale ε-problem (1)-(3). It is posed in a 2 dimensional space, leading to a dimension reduction problem. We see from [12] that (62)-(64) possesses a unique solution. We are now able to prove the main result of the work.

4.3. *Proof of Theorem 1*

The proof of (5)-(7) follows easily from (44)-(46) associated to the properties of the operator M_ε. The fact that $(u_m)_{1 \leq m \leq M}$ solves (8)-(10) has been shown here above in Subsection 4.2. Now, if we proceed as in [1] (see also [12]), we get the wellposedness of (8)-(10) in the space $(\mathcal{C}^{1+\frac{\alpha}{2}, 2+\alpha}(Q))^M$, and especially, (11) holds true. Finally, the fact that the whole sequence $[(u_m^\varepsilon)_{1 \leq m \leq M}]_{\varepsilon > 0}$ converges towards $(u_m)_{1 \leq m \leq M}$ follows from the uniqueness of the solution (8)-(10). This concludes the proof.

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