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Generalized Rough Neighborhood Approximations and Related Topological Approaches

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Abstract: This work aims to generate some types of M_j –neighborhoods. Relationships among these M_j -neighborhoods and some other types of neighborhoods are discussed. Additionally, M_j -lower, M_j -upper approximations and M_j -accuracy in terms of M_j -neighborhoods are presented. Their properties, with some examples, are investigated. Finally, We addressed the comparisons among M_j approach and other approaches.

Keywords: topological approximation spaces; rough sets; Nj-neighborhood; Ej-neighborhood; M_j –neighborhood

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1. Introduction

Topology and rough sets are two distinct mathematical frameworks, but many con nections exist. Rough set theory [1–5] is a good non-statistical tool for big data. It depends essentially on binary relations that are equivalence. These types of relations are often hard to satisfy because of their restrictions and application limitations. The philosophy of rough set theory depends on two approximations of categories. This theory can extend to many other approaches related to topological notations, such as near-open sets and neighborhoods. Topology is an important branch of mathematics that is rich with logical relations. These relations can make use of many generalizations and useful applications. It has many near-open logical notations, such as alpha and beta open sets, that generate new real-life applications. More studies had done to use topology with rough set theory in different ways [6–11]. Topologically, rough approximations are motivated by different types of neighborhoods. Some studies [12–15] used generalized binary relations to generate new rough-set approaches. Others used topological neighborhood systems to generate granules used in many applications [16–19]. Also, some introduced a mixed neighborhood system for rough approximations and developed eight different types of neighborhoods. These types of neighborhood systems are studied and generalized by to P_j -neighborhoods, E_j -neighborhoods, C_j -neighborhoods, and recently S_j-neighborhoods. The connection between rough set theory and topology is studied in various directions depending on the suitable applications [8]. The following manuscript is formulated as follows: Sections 1&2 are the introduction and preliminaries of the work. Section 3 introduces the concepts of M_j neighborhoods systems. Section 4 introduces the relations between these neighborhoods, and Section 5 communicates and revision the notions M_j -lower and M_j -upper approximations, M_j -boundary region, M_j -positive and M_j -negative regions, and M_j -accuracy measure, and generate their M_j –topologies in section 6. We studied the relations among the generated topologies in Section 7. Some approximations operators induced by the topologies τ_{M_j} studied in Section 8, and we investigate the concepts from a topological concept, and its main properties are explored. Moreover, Section 9 compares our approaches and previous approaches concerning different j types. Finally, we give a conclusion in Section 10.

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2. Preliminaries

We rewrite some important definitions and notations for non-specific readers to help them read this work easily. These definitions spotlight on different types of neighborhoods and relations.

2.1. Pawlak Approximation Space

Definition 2.1.1. *[1,20] The relation Q on universe U is a subset of $U \times U$. Two elements, p and r, are in relation (in symbols pQr) when (p, r) $\in Q$. The relation Q is called:*

1. *Reflexive if pQp for each p $\in$ U.*
2. *Symmetric if pQr ⟷ rQp.*
3. *Transitive if pQn whenever pQr and rQn.*
4. *Equivalence is when it is reflexive, transitive, and symmetric.*

Definition 2.1.2. *[1,20] We assume Q is an equivalence relation. A pair (U, Q) is called an approximation space, where U is the universe, and Q is an equivalence relation on U. Let P be a subset of U, i.e., P $\subseteq$ U. Our goal is to characterize the set P concerning Q.*

- $\underline{Q}(P) = \cup\{ V \in U \mid V \subseteq P\}$.
- $\bar{Q}(P) = \cup \{ V \in U \mid V \cap P \neq \emptyset\}$.

The two sets $\underline{Q}(P)$ and $\bar{Q}(P)$ are respectively called lower and upper approximations of P, and P is said to be a rough set if $\underline{Q}(P) \neq \bar{Q}(P)$. Otherwise, it is definable (or exact).

The boundary and accuracy of Pawlak approximations are defined, respectively, by:

- $BND(P) = \bar{Q}(P) - \underline{Q}(P)$ and $\mu(P) = \frac{|\underline{Q}(P)|}{|\bar{Q}(P)|}$, Where $\bar{Q}(P) \neq \emptyset$.

Proposition 2.1.1. *[1,20] Pawlak supposes that (U, Q) be an approximation space, $\emptyset$ represents an empty set and P^c represents a complement of P in U. Then, the lower approximations and the upper approximations have the followings properties:*

(L1) $\underline{Q}(P) \subseteq P$.
(L2) $\underline{Q}(\emptyset) = \emptyset$.
(L3) $\underline{Q}(U) = U$.
(L4) $\underline{Q}(P \cap V) = \underline{Q}(P) \cap \underline{Q}(V)$.
(L5) If $P \subseteq V$, *then* $\underline{Q}(P) \subseteq \underline{Q}(V)$.
(L6) $\underline{Q}(P) \cup \underline{Q}(V) \subseteq \underline{Q}(P \cup V)$.
(L7) $\underline{Q}(P^c) = (\bar{Q}(P))^c$.
(L8) $\underline{Q}\ (\underline{Q}(P)) = \underline{Q}(P)$.
(L9) $\underline{Q}\ ((\underline{Q}(P))^c) = (\underline{Q}(P))^c$.

(U1) $P \subseteq \bar{Q}(P)$.
(U2) $\bar{Q}(\emptyset) = \emptyset$.
(U3) $\bar{Q}(U) = U$.
(U4) $\bar{Q}(P \cup V) = \bar{Q}(P) \cup \bar{Q}(V)$.
(U5) If $P \subseteq V$, *then* $\bar{Q}(P) \subseteq \bar{Q}(V)$.
(U6) $\bar{Q}(P) \cap \bar{Q}(A) \supseteq \bar{Q}(P \cap V)$.
(U7) $\bar{Q}(P^c) = (\underline{Q}(P))^c$.
(U8) $\bar{Q}\ (\bar{Q}(P)) = \bar{Q}(P)$.
(U9) $\bar{Q}\ ((\bar{Q}(P))^c) = (\bar{Q}(P))^c$.

2.2. j-Neighborhood space

Definition 2.2.1. *[12] For any binary relation Q on U. The following j-neighborhood sets are defined for p $\in$ $\cup(N_j(p))$, j $\in$ {r, l, ‹ r ›, ‹ l ›, i, u, ‹ i ›, ‹ u ›} as follows:*

1. *The r-Neighborhood when,* $N_r(p) = \{v \in U \mid p\ Q\ v\}$.
2. *The l-Neighborhood when,* $N_l(p) = \{v \in U \mid v\ Q\ p\}$.
3. *The ‹ r ›-Neighborhood when,* $N_{\langle r \rangle}(p) = \cap_{p \in N_r(v)} N_r(v)$.
4. *The ‹ l ›-Neighborhood when,* $N_{\langle l \rangle}(p) = \cap_{p \in N_l(v)} N_l(v)$.
5. *The i-Neighborhood when,* $N_i(p) = N_r(p) \cap N_l(p)$.
6. *The u-Neighborhood when,* $N_u(p) = N_r(p) \cup N_r(p)$.
7. *The ‹ i ›-Neighborhood when,* $N_{\langle i \rangle}(p) = N_{\langle r \rangle}(p) \cap N_{\langle l \rangle}(p)$.

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8. The $\langle u \rangle$-Neighborhood when, $N_{\langle u \rangle}(p) = N_{\langle r \rangle}(p) \cup N_{\langle l \rangle}(p)$.

Definition 2.2.2. *[12] Let Q be an arbitrary binary relation on U and* $\xi j : U \to P(U)$ *be a mapping which assigns for each* $p \in U$ *its* $N_j(p)$ *in the* $P(U)$. *Then, the triple* $(U, Q, \xi j)$ *called a j-neighborhood space (briefly, j-NS).*

Theorem 2.2.1. *[12] Let* $(U, Q, \xi j)$ *be a j-NS and* $P \subseteq U$. *Then, for each* $j \in \{r, l, \langle r \rangle, \langle l \rangle, i, u, \langle i \rangle, \langle u \rangle\}$ *the group* $\tau_j = \{P \subseteq U \mid \forall v \in P, N_j(v) \subseteq P\}$ *is a topology on U.*

Definition 2.2.3. *[12] Let* $(U, Q, \xi j)$ *be a j-NS. Then, a subset* $P \subseteq U$ *is called a j-open set if* $P \in \tau_j$. *The complement of a j-open set is called j-closed, and the family* Γ_j *of all j-closed sets is given by* $\Gamma_j = \{F \subseteq U \mid F^c \in \tau_j\}$ *and the j-interior (resp. j-closure) operator of A is given by*

- $int_j(P) = \cup\{G \in \tau_j : G \subseteq P\}$.
- $cl_j(P) = \cap\{H \in \tau^c{}_j : P \subseteq H\}$.

Definition 2.2.4. *[12] Let* $(U, Q, \xi j)$ *be a j-NS and* $P \subseteq U$. *Then, the (j-lower and j-upper) approximations, (j-boundary, j-positive and j-negative) regions, and j-accuracy of the approximations of* $P \subseteq U$ *are given, respectively, by*

- $\underline{Q}_j(P) = \cup\{G \in \tau_j : G \subseteq P\}$ = *j-interior of P.*
- $\bar{Q}_j(P) = \cap\{H \in \tau_j : P \subseteq H\}$ = *j-closure of P.*
- $B_j(P) = \bar{Q}_j(P) - \underline{Q}_j(P)$.
- $POS_j(P) = \underline{Q}_j(P)$.
- $NEG_j(P) = U - \underline{Q}_j(P)$.
- $\mu_j(P) = \frac{|\underline{Q}_j(P)|}{|\bar{Q}_j(P)|}$, *where* $|\bar{Q}_j(P)| \neq 0$.

Definition 2.2.5. *[18] For any binary relation Q on U. The group of* E_j *-neighborhoods of* $P \subseteq U$ *are defined as follows:*

1. $E_r(P) = \{v \in U: N_r(v) \cap N_r(P) \neq \emptyset\}$.
2. $E_l(P) = \{v \in U: N_l(v) \cap N_l(P) \neq \emptyset\}$.
3. $E_i(P) = E_r(P) \cap E_l(P)$.
4. $E_u(P) = E_r(P) \cup E_l(P)$.
5. $E_{\langle r \rangle}(P) = \{v \in U: N_{\langle r \rangle}(v) \cap N_{\langle r \rangle}(P) \neq \emptyset\}$.
6. $E_{\langle l \rangle}(P) = \{v \in U: N_{\langle l \rangle}(v) \cap N_{\langle l \rangle}(P) \neq \emptyset\}$.
7. $E_{\langle i \rangle}(P) = E_{\langle r \rangle}(P) \cap E_{\langle l \rangle}(P)$.
8. $E_{\langle u \rangle}(P) = E_{\langle r \rangle}(P) \cup E_{\langle l \rangle}(P)$.

3. M_j -Neighborhoods in the j -neighborhood space

In this section, we define the concepts of M_j neighborhoods By inclusion relations between N_j -neighborhoods and E_j -neighborhoods. An illustrative example is given to support the obtained results and relationships. The study of M_j -neighborhoods aims to increase the accuracy of approximations.

Definition 3.1. *For any binary relation Q on U. The class of* M_j *-neighborhoods of* $P \in U$ $(M_j(P))$ *are defined, for each* $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$ *as follows:*

1. $M_r(P) = \{v \in U: N_r(P) \cap E_r(P)\}$.
2. $M_l(P) = \{v \in U: N_L(P) \cap E_l(P)\}$.
3. $M_i(P) = \{v \in U: M_r(P) \cap M_l(P)\}$.
4. $M_u(P) = \{v \in U: M_r(P) \cup M_l(P)\}$.
5. $M_{\langle r \rangle}(P) = \{v \in U: N_{\langle r \rangle}(P) \cap E_{\langle r \rangle}(P)\}$.

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6. $M_{\langle l \rangle}(P) = \{ v \in U: N_{\langle l \rangle}(P) \cap E_{\langle l \rangle}(P)\}$.
7. $M_{\langle i \rangle}(P) = \{ v \in U: M_{\langle r \rangle}(P) \cap M_{\langle l \rangle}(P)\}$.
8. $M_{\langle u \rangle}(P) = \{ v \in U: M_{\langle r \rangle}(P) \cup M_{\langle l \rangle}(P)\}$.

The following example will show the behavior of these neighborhoods, help us show the relationships among them, and illustrates the method of Mj-neighborhood.

Example 3.1. *Suppose the universe of some types of characters in a story* $U = \{ Protagonist\ (Pr), Antagonist\ (An), Sidekick\ (Si), Confidante\ (Co) \}$ *as an initial set of objects. We suppose the relation* $Q = \{(Pr, Si), (An, An), (Si, Pr), (Co, Pr)\}$ *that defined on U. The classes* N_j *neighborhoods,* E_j *-neighborhoods and* M_j *-neighborhoods are shown in Table 1.*

Table 1. Classes of N_j -neighborhoods, E_j -neighborhoods and M_j -neighborhoods.

Neighborhoods	Protagonist	Antagonist	Sidekick	Confidante
N_r	{Si}	{An}	{Pr}	{Pr}
N_l	{Si,Co}	{An}	{Pr}	∅
N_i	{Si}	{An}	{Pr}	∅
N_u	{Si,Co}	{An}	{Pr}	{Pr}
$N_{\langle r \rangle}$	{Pr}	{An}	{Si}	∅
$N_{\langle l \rangle}$	{Pr}	{An}	{Si,Co}	{Si,Co}
$N_{\langle i \rangle}$	{Pr}	{An}	{Si}	∅
$N_{\langle u \rangle}$	{Pr}	{An}	{Si,Co}	{Si,Co}
E_r	{Pr}	{An}	{Si,Co}	{Si,Co}
E_l	{Pr}	{An}	{Si}	∅
E_i	{Pr}	{An}	{Si}	∅
E_u	{Pr}	{An}	{Si,Co}	{Si,Co}
$E_{\langle r \rangle}$	{Pr}	{An}	{Si}	∅
$E_{\langle l \rangle}$	{Pr}	{An}	{Si,Co}	{Si,Co}
$E_{\langle i \rangle}$	{Pr}	{An}	{Si}	∅
$E_{\langle u \rangle}$	{Pr}	{An}	{Si,Co}	{Si,Co}
M_r	∅	{An}	∅	∅
M_l	∅	{An}	∅	∅
M_i	∅	{An}	∅	∅
M_u	∅	{An}	∅	∅
$M_{\langle r \rangle}$	{Pr}	{An}	{Si}	∅
$M_{\langle l \rangle}$	{Pr}	{An}	{Si,Co}	{Si,Co}
$M_{\langle i \rangle}$	{Pr}	{An}	{Si}	∅
$M_{\langle u \rangle}$	{Pr}	{An}	{Si,Co}	{Si,Co}

4. The Relationships Among M_j –neighborhoods

The following results show the relationships between the different types of Mj-neighborhoods.

Proposition 4.1. *Let (U, Q, ξj) be a j-approximation space. Then for each* $v \in U$*, the following relations hold.*

1. $v \in M_j(v)$ *for each j.*
2. $M_i(v) \subseteq M_r(v) \subseteq M_u(v)$.

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3. *$M_i(v) \subseteq M_l(v) \subseteq M_u(v)$.*
4. *$M_{\langle i \rangle}(v) \subseteq M_{\langle r \rangle}(v) \subseteq M_{\langle u \rangle}(v)$.*
5. *$M_{\langle i \rangle}(v) \subseteq M_{\langle l \rangle}(v) \subseteq M_{\langle u \rangle}(v)$.*
6. *If Q is symmetric, then $M_r(v) = M_l(v) = M_i(v) = M_u(v)$ and $M_{\langle r \rangle}(v) = M_{\langle l \rangle}(v) = M_{\langle i \rangle}(v) = M_{\langle u \rangle}(v)$.*

Proof. The proof of (1) comes from the fact $N_j(v) \subseteq N_j(v)$ and $E_j(v) \subseteq E_j(v)$ for $j \in \{r, l, \langle r \rangle, \langle l \rangle\}$

The proof of (2)-(5) come from the Definition 3.1.

The proof of (6): Since Q is symmetric, then $N_r(v) = N_l(v)$. This leads to $N_i(v) = N_u(v)$ as well and $E_r(v) = E_l(v)$. This leads to $E_i(v) = E_u(v)$. Therefore, $M_r(v) = M_l(v) = M_i(v) = M_u(v)$. Similarly, one can prove that and $M_{\langle r \rangle}(v) = M_{\langle l \rangle}(v) = M_{\langle i \rangle}(v) = M_{\langle u \rangle}(v)$. □

5. Generalized Rough Approximations Based on M_j–neighborhoods.

In this section, New concepts of M_j-lower approximations and M_j-upper approximations, M_j-boundary region, M_j–positive regions, M_j-negative regions and M_j-accuracy measure of a subset based on M_j–neighborhoods. We reveal the main properties with the help of some examples.

Definition 5.1. *Let (U, Q, ξj) be a j-approximation Space, given a set $P \subseteq U$. The lower $ƀ_{M_j}(P)$ and the upper $ƀ^{M_j}(P)$ approximations based on M_j-neighborhoods defined as follows:*

- $ƀ_{M_j}(P) = \{ v \in U, M_j(v) \subseteq P \}$ Called M_j lower approximation of P.
- $ƀ^{M_j}(P) = \{ v \in U, M_j(v) \cap P \neq \emptyset \}$ Called M_j upper approximation of P.

Definition 5.2. *Let (U, Q, ξj) be a j-approximation Space, given a set $P \subseteq U$. The M_j-boundary region, M_j-positive region, M_j-negative region and M_j-accuracy of P is defined as follows:*

- $B_{M_j}(P) = ƀ^{M_j}(P) - ƀ_{M_j}(P)$
- $POS_{M_j}(P) = ƀ_{M_j}(P)$
- $NEG_{M_j}(P) = U - ƀ^{M_j}(P)$
- $\delta_{M_j}(P) = \frac{|ƀ_{M_j}(P)|}{|ƀ^{M_j}(P)|}$, where $|ƀ^{M_j}(P)| \neq 0$.

In the next example, we calculate the M_j lower approximation of P, M_j upper approximation of P, M_j-boundary region, M_j-positive region, M_j-negative region and M_j-accuracy measure of P.

Example 5.1. *Suppose the universe U= {e, f, g, h} as an initial set of objects and $P \subseteq U$. We suppose the relation Q = {(e,e),(e,h),(f,e),(f,g),(g,f),(g,g),(g,h),(h,e)} that defined on U.*

Let P ={e,f,h}. Then we calculate $ƀ_{M_j}(P)$, $ƀ^{M_j}(P)$, $F_{M_j}(P)$ and $\delta_{M_j}(P)$ for j=(r,l,i,u) as follows:

- For j = r, we have $ƀ_{M_r}(P)$ ={e,h} , $ƀ^{M_r}(P)$ = U, $F_{M_r}(P)$ ={f,g} and $\delta_{M_r}(P) = ½$.
- For j = l, we have $ƀ_{M_l}(P)$ ={e} , $ƀ^{M_l}(P)$ = {e,g,h}, $F_{M_l}(P)$ ={g,h} and $\delta_{M_l}(P) = 1/3$.
- For j = i, we have $ƀ_{M_i}(P)$ ={e,h} , $ƀ^{M_i}(P)$ = {e,g,h}, $F_{M_i}(P)$ ={g} and $\delta_{M_i}(P) = 2/3$
- For j = u, we have $ƀ_{M_u}(P)$ ={e} , $ƀ^{M_u}(P)$ = U, $F_{M_u}(P)$ ={f,g,h} and $\delta_{M_u}(P) = 1/3$.

Proposition 5.1. *For any j and each non-empty subset P of U, we have $\delta_{M_j}(P) \in [0,1]$.*

Proof. Let P be a non-empty subset of U. It follows from (1) of Proposition 4.1 that $v \in M_j(v)$ for each j. Then $M_j(v) \cap P \neq \emptyset$ for each $v \in P$; consequently, $ƀ^{M_j}(P) \neq \emptyset$. Thus, $|ƀ^{M_j}(P)| > 0$. □

Now, we have two cases:

Case 1: $ƀ_{M_j}(P) = \emptyset$. Then $\frac{|ƀ_{M_j}(P)|}{|ƀ^{M_j}(P)|} = \frac{|\emptyset|}{|ƀ^{M_j}(P)|} = 0$.

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Case 2: $ƀ_{M_j}(P) \neq \emptyset$. Then $v \in ƀ_{M_j}(P)$. Then $v \in M_j(v) \cap P$. Therefore, $v \in ƀ^{M_j}(P)$. Thus $ƀ_{M_j}(P) \subseteq ƀ^{M_j}(P)$. This means that $0 \leq \frac{|ƀ_{M_j}(P)|}{|ƀ^{M_j}(P)|} \leq 1$. Hence, we obtain from the two cases above that $0 \leq \delta_{M_j}(P) \leq 1$ as required.

In the following results, we present the main properties of M_j-lower approximations and M_j-upper approximations for each j.

Theorem 5.2. *Let (U, Q, ξj) be a j-approximation space, given a set P ⊆ U. Then the following properties hold for each j.*

1. $ƀ_{M_j}(P) \subseteq P \subseteq ƀ^{M_j}(P)$.
2. $ƀ_{M_j}(\emptyset) = \emptyset$ and $ƀ^{M_j}(\emptyset) = \emptyset$.
3. $ƀ_{M_j}(U) = U$ and $ƀ^{M_j}(U) = U$.
4. If $P \subseteq W$, then $ƀ_{M_j}(P) \subseteq ƀ_{M_j}(W)$.
5. If $P \subseteq W$, then $ƀ^{M_j}(P) \subseteq ƀ^{M_j}(W)$.
6. $ƀ_{M_j}(P \cap W) = ƀ_{M_j}(P) \cap ƀ_{M_j}(W)$.
7. $ƀ^{M_j}(P \cup W) = ƀ^{M_j}(P) \cup ƀ^{M_j}(W$.
8. $ƀ_{M_j}(P^c) = (ƀ^{M_j}(P))^C$.
9. $ƀ^{M_j}(P^c) = (ƀ_{M_j}(P))^C$.

Proof. We prove (1) ,(2),(3),(4),(6) and (8) and the other cases similarly.

1. Let $v \in ƀ_{M_j}(P)$. Then $M_j(v) \subseteq P$. It follows from the item (1) of Proposition 4.1 that $v \in M_j(v)$ therefore, $v \in P$. Thus, $ƀ_{M_j}(P) \subseteq P$.
2. Since $M_j(v) \neq \emptyset$. For each $v \in U$, then $ƀ_{M_j}(\emptyset) = \emptyset$.
3. Since $v \in M_j(v) \subseteq U$ for each $v \in U$, then $\cup_{v \in U} \{v\} \subseteq \cup_{v \in U} M_j(v) \subseteq U$ thus $ƀ_{M_j}(U) = U$.
4. Since $P \subseteq W$, then $ƀ_{M_j}(P) = \{v \in Q : M_j(v) \subseteq P\} \subseteq \{v \in Q : M_j(v) \subseteq W\} = ƀ_{M_j}(W)$.
5. It follows from (iv) that $ƀ_{M_j}(P \cap W) = ƀ_{M_j}(P) \cap ƀ_{M_j}(W)$. Conversely, let $v \in ƀ_{M_j}(P) \cap ƀ_{M_j}(W)$. Then $v \in ƀ_{M_j}(P)$ and $v \in ƀ_{M_j}(W)$ so that $M_j(v) \subseteq P$ and $M_j(v) \subseteq W$ thus $M_j(v) \subseteq P \cap W$, Hence $v \in ƀ_{M_j}(P \cap W)$.
6. $v \in ƀ_{M_j}(P^c) \leftrightarrow M_j(v) \subseteq P^c \leftrightarrow M_j(v) \cap P = \emptyset \leftrightarrow$ v not belong $ƀ^{M_j}(P) \leftrightarrow v \in (ƀ^{M_j}(P))^C$.

Corollary 5.3. *Let (U, Q, ξj) be a j-approximation space and P, V ⊆ Q. Then* $ƀ_{M_\star}(P) \cup ƀ_{M_\star}(V) \subseteq ƀ_{M_\star}(P \cup V)$. *The next example shows that the converse of the above corollary need not be true.*

Example 5.2. *According to Example 5.1. Let P = {e}, V = {f, h} and P ∪ V = {e,f,h}. Then* $ƀ_{M_\star}(P \cup V)$ *={e},* $ƀ_{M_\star}(P)$ *={e} and* $ƀ_{M_\star}(V) = \emptyset$. $ƀ_{M_\star}(P \cup V) \nsubseteq ƀ_{M_\star}(P) \cup ƀ_{M_\star}(V)$.

Remark 5.1. *The converse of Theorem 5.2. that* $ƀ_{M_\star}(P) \subseteq P$ *need not be true. The following example shows this remark.*

Example 5.3. *According to Example 5.1. Let P = {f, g}, we have* $ƀ_{M_\star}(p)$*= {g} then* $P \nsubseteq ƀ_{M_\star}(P)$.

Remark 5.2. *The converse of the item (4) of Theorem 5.2. that If P ⊆ W, then* $ƀ_{M_\star}(P) \subseteq ƀ_{M_\star}(W)$ *need not be true as shown in Example 5.4.*

Example 5.4. *According to Example 5.1. Let P = {e,f,h} and W = {e,g,h}. Then* $ƀ_{M_\star}(P) = \{e,h\} \subseteq ƀ_{M_\star}(W) = \{e,f,h\}$ *although* $P \nsubseteq W$.

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Some properties of Pawlak's lower approximations are not realized for M_j-lower approximation $ƃ_{M_j}(P)$ such as $ƃ_{M_j}(ƃ_{M_j}(P)) = ƃ_{M_j}(P)$. To clarify this fact, consider a subset $P = \{e, f, h\}$. Then $ƃ_{M_u}(P) = \{e\}$, But $ƃ_{M_j}(ƃ_{M_j}(P)) = \emptyset$ given in Example 5.1.

Corollary5.4. *Let $(U, Q, \xi j)$ be a j-approximation space and $P, V \subseteq Q$. Then $ƃ^{M_j}(P \cap V) \subseteq ƃ^{M_j}(P) \cap ƃ^{M_j}(V)$. The next example show that the converse of the above corollary need not be true.*

Example 5.5. *According to Example 5.1. Let $P = \{f, h\}$, $V = \{e, g, h\}$ and $P \cap V = \{h\}$. Then $ƃ^{M_l}(P) = \{e, g\}$, $ƃ^{M_l}(V) = U$ and $ƃ^{M_l}(P \cap V) = \{e\}$. $ƃ^{M_l}(P) \cap ƃ^{M_l}(V) \nsubseteq ƃ^{M_l}(P \cap V)$.*

Remark 5.3. *The converse or the item (1) of Theorem 5.2. that $P \subseteq ƃ^{M_j}(P)$ Need not be true as shown in the following example.*

Example 5.6. *According to Example 5.1. Let $P = \{f, g\}$, we have $ƃ^{M_r}(p) = \{g\}$ then $P \nsubseteq ƃ^{M_r}(p)$.*

Remark 5.4. *The converse of item (5) of Theorem 5.2. need not be true as shown in Example 5.7.*

Example 5.7. *According to Example 5.1. Let $P = \{g, h\}$ and $V = \{e,g,h\}$. Then $ƃ^{M_u}(P) = \{e,f,h\} \subseteq ƃ^{M_u}(V) = U$ although $P \nsubseteq V$.*

Some properties of Pawlak's upper approximations are not realized for M_j -Upper approximation $ƃ^{M_j}(P)$ such as $ƃ^{M_j}(ƃ^{M_j}(P)) = ƃ^{M_j}(P)$. To clarify this fact, consider a subset $P = \{f\}$. Then $ƃ^{M_u}(P) = \{g\}$ But $ƃ^{M_u}(ƃ^{M_u}(P)) = \{f, g, h\}$ given in example 5.1.

Proposition 5.5. *Let $(U, Q, \xi j)$ be a j-approximation space, given a set $P \subseteq U$. Then the following proposition gives the relationships between the M_j -lower approximations $ƃ_{M_j}(P)$ and M_j -upper approximations $ƃ^{M_j}(P)$ where $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$.*

1. $ƃ_{M_u}(P) \subseteq ƃ_{M_r}(P) \subseteq ƃ_{M_i}(P)$.
2. $ƃ_{M_u}(P) \subseteq ƃ_{M_l}(P) \subseteq ƃ_{M_i}(P)$.
3. $ƃ_{M_{\langle u \rangle}}(P) \subseteq ƃ_{M_{\langle r \rangle}}(P) \subseteq ƃ_{M_{\langle i \rangle}}(P)$
4. $ƃ_{M_{\langle u \rangle}}(P) \subseteq ƃ_{M_{\langle l \rangle}}(P) \subseteq ƃ_{M_{\langle i \rangle}}(P)$
5. $ƃ^{M_i}(P) \subseteq ƃ^{M_r}(P) \subseteq ƃ^{M_u}(P)$.
6. $ƃ^{M_i}(P) \subseteq ƃ^{M_l}(P) \subseteq ƃ^{M_u}(P)$.
7. $ƃ^{M_{\langle i \rangle}}(P) \subseteq ƃ^{M_{\langle r \rangle}}(P) \subseteq ƃ^{M_{\langle u \rangle}}(P)$.
8. $ƃ^{M_{\langle i \rangle}}(P) \subseteq ƃ^{M_{\langle l \rangle}}(P) \subseteq ƃ^{M_{\langle u \rangle}}(P)$.

Proof.

- We prove (1): Let $v \in ƃ_{M_u}(P)$ then $v \in M_u(v) \subseteq P$. Since $M_r(v) \subseteq M_u(v)$, then $v \in ƃ_{M_r}(P)$ therefore $ƃ_{M_u}(P) \subseteq ƃ_{M_r}(P)$. □

  similarly, we prove that $ƃ_{M_r}(P) \subseteq ƃ_{M_i}(P)$ and the other cases (2),(3) and (4) can be proved similarly.

- We prove (v): let $v \in ƃ^{M_i}(P)$ then $v \in M_i(v)$. Such that $M_i(v) \cap P \neq \emptyset$ Since $M_i(v) \subseteq M_r(v)$. Then $M_r(v) \cap P \neq \emptyset$, therefore $ƃ^{M_i}(P) \subseteq ƃ^{M_r}(P)$. □

  similarly, we prove that $ƃ^{M_r}(P) \subseteq ƃ^{M_u}(P)$ and the other cases (6),(7) and (8) can be proved similarly.

Proposition 5.6. *Let $(U, Q, \xi j)$ be a j-approximation space, given a non-empty set $P \subseteq U$*

1. $\delta_{M_u}(P) \leq \delta_{M_r}(P) \leq \delta_{M_i}(P)$.
2. $\delta_{M_u}(P) \leq \delta_{M_l}(P) \leq \delta_{M_i}(P)$.
3. $\delta_{M_{\langle u \rangle}}(P) \leq \delta_{M_{\langle r \rangle}}(P) \leq \delta_{M_{\langle i \rangle}}(P)$.
4. $\delta_{M_{\langle u \rangle}}(P) \leq \delta_{M_{\langle l \rangle}}(P) \leq \delta_{M_{\langle i \rangle}}(P)$.

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Proof. We prove (1) and one can prove the other cases similarly.

Since $ƀ_{M_u}(P) \subseteq ƀ_{M_r}(P) \subseteq ƀ_{M_i}(P)$ then $|ƀ_{M_u}(P)| \leq |ƀ_{M_r}(P)| \leq |ƀ_{M_i}(P)|$
Since $ƀ^{M_i}(P) \subseteq ƀ^{M_r}(P) \subseteq ƀ^{M_u}(P)$ then $|ƀ^{M_i}(P)| \leq |ƀ^{M_r}(P)| \leq |ƀ^{M_u}(P)|$

By hypothesis, P is non-empty, so that $|ƀ^{M_j}(P)| > 0$ for each j. Therefore:

$$\frac{1}{|ƀ^{M_u}(P)|} \leq \frac{1}{|ƀ^{M_r}(P)|} \leq \frac{1}{|ƀ^{M_i}(P)|}. \text{Then } \frac{|ƀ_{M_u}(P)|}{|ƀ^{M_u}(P)|} \leq \frac{|ƀ_{M_r}(P)|}{|ƀ^{M_r}(P)|} \leq \frac{|ƀ_{M_i}(P)|}{|ƀ^{M_i}(P)|}. \square$$

To support the above results, we present the following example.

Example 5.8. *According to Example 5.1, we calculate the approximations and their accuracy measure for* $j \in \{r, l, i, u\}$ *as in Table 2, and we calculate the approximations and their accuracy measure* $j \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$ *as in Table 3.*

Table 2. The approximations and accuracy measure in cases of $j \in \{r, l, I, u\}$.

P(E)	$ƀ_{M_u}(E)$	$ƀ^{M_u}(E)$	δ_{M_u}	$ƀ_{M_r}(E)$	$ƀ^{M_r}(E)$	δ_{M_r}	$ƀ_{M_l}(E)$	$ƀ^{M_l}(E)$	δ_{M_l}	$ƀ_{M_i}(E)$	$ƀ^{M_i}(E)$	δ_{M_i}
{e}	∅	{e,f,h}	0	{e}	{e,f,h}	1/3	∅	{e,h}	0	{h}	{e,h}	½
{f}	∅	{g}	0	∅	{g}	0	∅	{g}	0	∅	{g}	0
{g}	∅	{f,g,h}	0	∅	{f,g}	0	{f}	{f,g,h}	1/3	{f}	{f,g}	½
{h}	∅	{e}	0	∅	{e}	0	∅	{e}	0	∅	{e}	0
{e,f}	∅	U	0	{h}	U	¼	∅	{e,g,h}	0	{h}	{e,g,h}	1/3
{e,g}	{f,h}	U	½	{f,h}	U	½	{f,h}	U	½	{f,h}	U	½
{e,h}	{e}	{e,f,h}	1/3	{e,h}	{e,f,h}	2/3	{e}	{e,h}	½	{e,h}	{e,h}	1
{f,g}	{g}	{f,g,h}	1/3	{g}	{f,g}	½	{f,g}	{f,g,h}	2/3	{f,g}	{f,g}	1
{f,h}	∅	{e,g}	0	∅	{e,g}	0	∅	{e,g}	0	∅	{e,g}	0
{g,h}	∅	U	0	∅	{e,g}	0	{f}	U	¼	{f}	{e,f,g}	1/3
{e,f,g}	{f,g,h}	U	¾	{f,g,h}	U	¾	{f,g,h}	U	¾	{f,g,h}	U	¾
{e,f,h}	{e}	U	¼	{e,h}	U	½	{e}	{e,g,h}	1/3	{e,h}	{e,g,h}	2/3
{e,g,h}	{e,f,h}	U	¾	{e,f,h}	U	¾	{e,f,h}	U	¾	{e,f,h}	U	¾
{f,g,h}	{g}	U	¼	{g}	{e,f,g}	1/3	{f,g}	U	½	{f,g}	{e,f,g}	2/3
U	U	U	1	U	U	1	U	U	1	U	U	1

Table 3. The approximations and accuracy measure in cases of $j \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$.

P(E)	$ƀ_{M_{\langle u \rangle}}(E)$	$ƀ^{M_{\langle u \rangle}}(E)$	$\delta_{M_{\langle u \rangle}}$	$ƀ_{M_{\langle r \rangle}}(E)$	$ƀ^{M_{\langle r \rangle}}(E)$	$\delta_{M_{\langle r \rangle}}$	$ƀ_{M_{\langle l \rangle}}(E)$	$ƀ^{M_{\langle l \rangle}}(E)$	$\delta_{M_{\langle l \rangle}}$	$ƀ_{M_{\langle i \rangle}}(E)$	$ƀ^{M_{\langle i \rangle}}(E)$	$\delta_{M_{\langle i \rangle}}$
{e}	{e}	{e,h}	½	{e}	{e}	1	{e}	{e,h}	½	{e}	{e}	1
{f}	∅	{f,h}	0	∅	{f}	0	{f}	{f,h}	½	{f}	{f}	1
{g}	{g}	{f,g}	½	{g}	{f,g}	½	{g}	{g}	1	{g}	{g}	1
{h}	∅	{h}	0	{h}	{h}	1	∅	{h}	0	{h}	{h}	1
{e,f}	{e}	{e,f,h}	1/3	{e}	{e,f}	½	{e,f}	{e,f,h}	2/3	{e,f}	{e,f}	1
{e,g}	{e,g}	U	½	{e,g}	{e,f,g}	2/3	{e,g}	{e,g,h}	2/3	{e,g}	{e,g}	1
{e,h}	{e}	{e,h}	½	{e,h}	{e,h}	1	{e}	{e,h}	½	{e,h}	{e,h}	1
{f,g}	{f,g}	{f,g,h}	2/3	{f,g}	{f,g}	1	{f,g}	{f,g,h}	2/3	{f,g}	{f,g}	1
{f,h}	∅	{f,h}	0	{h}	{f,h}	½	{f}	{f,h}	½	{f,h}	{f,h}	1
{g,h}	{g}	{f,g,h}	1/3	{g,h}	{f,g,h}	2/3	{g}	{g,h}	½	{g,h}	{g,h}	1
{e,f,g}	{e,f,g}	U	¾	{e,f,g}	{e,f,g}	1	{e,f,g}	U	¾	{e,f,g}	{e,f,g}	1

{e,f,h}	{e,h}	{e,f,h}	2/3	{e,h}	{e,f,h}	2/3	{e,f,h}	{e,f,h}	1	{e,f,h}	{e,f,h}	1
{e,g,h}	{e,g}	U	½	{e,g,h}	U	¾	{e,g}	{e,g,h}	2/3	{e,g,h}	{e,g,h}	1
{f,g,h}	{f,g}	{f,g,h}	2/3	{f,g,h}	{f,g,h}	1	{f,g}	{f,g,h}	2/3	{f,g,h}	{f,g,h}	1
U	U	U	1	U	U	1	U	U	1	U	U	1

6. M_j – Toplogies

Using the following theorem, we can generate eight different topologies via M_j -neighborhoods.

Theorem 6.1. *If $(U, Q, \xi j)$ is a j-approximation space, Then the collection*

$$\tau_{M_j} = \{B \subseteq U \mid M_j(v) \subseteq B \text{ for all } v \in B\}, \text{ for } j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\} \text{ is a topology On } U.$$

Proof.

1. U and Ø belong to τ_{M_j} .
2. If $\{B_i \mid i \in I\}$ is a family of elements in τ_{M_j} and $v \in \cup_i B_i$. Then there exists $i_0 \in I$ such that $v \in B_{i_0}$. Thus, $M_j(v) \subseteq B_{i_0}$ this lead to $M_j(v) \subseteq \cup_i B_i$ and $\cup_i B_i \in \tau_{M_j}$.
3. If $B_1, B_2 \in \tau_{M_j}$ and $P \in B_1 \cap B_2$.Then, $v \in B_1$ and $v \in B_2$ which lead to $M_j(v) \subseteq B_1$ and $M_j(v) \subseteq B_2$ Thus, $M_j(v) \subseteq B_1 \cap B_2$ and hence $B_1, B_2 \in \tau_{M_j}$. Then τ_{M_j} is a topology on U. □

Example 6.1. *According to Example 3.1, the classes of $M_{\curlyvee}$-topologies are given as follows:*

$\tau_{M_r} = P(U)$,
$\tau_{M_l} = P(U)$,
$\tau_{M_i} = P(U)$,
$\tau_{M_u} = P(U)$,
$\tau_{M_{\langle r \rangle}} = P(U)$,
$\tau_{M_{\langle l \rangle}} = \{U, Ø, \{Pr\}, \{An\}, \{Si, Co\}, \{Pr, An\}, \{Pr, Si, Co\}, \{An, Si, Co\}\}$,
$\tau_{M_{\langle i \rangle}} = P(U)$,
$\tau_{M_{\langle u \rangle}} = \{U, Ø, \{Pr\}, \{An\}, \{Si, Co\}, \{Pr, An\}, \{Pr, Si, Co\}, \{An, Si, Co\}\}$.

7. The Relations Among τ_{M_j} Topologies.

We study the relationships among the topologies $\tau_{M_r}, \tau_{M_l}, \tau_{M_i}, \tau_{M_u}, \tau_{M_{\langle r \rangle}}, \tau_{M_{\langle l \rangle}}, \tau_{M_{\langle i \rangle}}$ and $\tau_{M_{\langle u \rangle}}$.

Proposition 7.1. *In a j-NS (U, R, ξj), $\tau_{M_{\curlyvee}}$ is the dual topology of $\tau_{M_{\curlyvee}}$.*

Proof. Firstly, let $A \in \tau_{M_r}$ then $\forall v \in A$, $M_r(v) \subseteq A$. Suppose that $q \in B$, then $M_l(q) = \{y \in U \mid yRq\}$.

Now, let $M_l(q) \cap A \neq \emptyset$ then $x \in U$ such that $x \in A$ and $x \in M_l(q) \Rightarrow x \in A$ and $xRq \Rightarrow x \in A$ and $q \in M_r(x)$ Since $x \in A \Rightarrow M_r(x) \subseteq A$ and so $q \in A$ which contradicts with $q \in B$. Thus, $M_l(q) \cap A = \emptyset$ this implies $M_l(q) \subseteq B$. Now, we have $\forall q \in B$, $M_l(q) \subseteq B$. Hence, $B \in \tau_{M_l}$. Similarly, we can prove that if $B \in \tau_{M_l}$. Then, $B^c \in \tau_{M_r}$. Consequently, τ_{M_r} is the dual topology of τ_{M_l}. □

The following example shows Proposition 7.1

Exemple7.1. *Let U = {e, f, g, h} and R = {(e, e),(e, h),(f, e),(f, g),(g, f),(g, g),(g, h),(h, e)}.*

Then, we have
$\tau_{M_r} = \{U, Ø, \{e, h\}\}$,
$\tau_{M_l} = \{U, Ø, \{f, g\}\}$.

Obviously, $\tau_{M_r}$is the dual topology of τ_{M_l}.

Proposition 7.2. *Let (U, Q, ξj) is a j-approximation space , then*

$\tau_{M_r} \subseteq \tau_{M_i}$.
$\tau_{M_l} \subseteq \tau_{M_i}$.

Proof. Let B $\in \tau_{M_r}$, then ∀ v ∈ B, $M_r(v) \subseteq B$. Thus $M_r(v) \cap M_l(v) \subseteq B$ and so ∀ v ∈ B, $M_i(v) \subseteq B$,Hence B $\in \tau_{M_i}$and $\tau_{M_r} \subseteq \tau_{M_i}$, similarly, we can deduce that $\tau_{M_l} \subseteq \tau_{M_i}$. □

Remark 7.1. *Let (U, R, ξj) be a j-NS and then the following are not necessarily true.*

$$\tau_{M_r} = \tau_{M_i}$$
$$\tau_{M_l} = \tau_{M_i}$$

Proposition 7.3. *Let (U, Q, ξj) is a j-approximation space. Then*

$$\tau_{M_u} \subseteq \tau_{M_r}$$
$$\tau_{M_u} \subseteq \tau_{M_l}$$
$$\tau_{M_u} \subseteq \tau_{M_i}$$

Proof. Let B $\in \tau_{M_u}$, then ∀ v ∈ B, $M_u(v) \subseteq B$. Thus, ∀p ∈ B, $M_r(v)$ U $M_l(v) \subseteq B$ and so ∀ v ∈ B, $M_r(v) \subseteq B$, $M_l(v) \subseteq B$. Then, B $\in \tau_{M_r}$ and B $\in \tau_{M_l}$ and $\tau_{M_u} \subseteq \tau_{M_r}$ and $\tau_{M_u} \subseteq \tau_{M_l}$. Using Proposition 7.2. we have $\tau_{M_u} \subseteq \tau_{M_i}$. □

Remark 7.2: *Let (U, R, ξj) be a j-NS, then the following is not necessarily true.*

$$\tau_{M_u} = \tau_{M_r}$$
$$\tau_{M_u} = \tau_{M_l}$$
$$\tau_{M_u} = \tau_{M_i}$$

The following example explains Remark 7.1 and Remark 7.2.

Example 7.2. *Let U = {e, f, g, h, k} and R = {(e, e), (f, g), (f, h), (g, h), (g, k), (h, e), (h, k), (k, f), (k, k)}.*

Then, we have τ_{M_r}= {U, Ø, {e},{k},{e, k},{e, h, k},{e, g, h, k}},
τ_{M_l}={U,Ø,{e},{g},{g, h},{g, h, k},{e, g, h, k} {f, g, h, k}}
τ_{M_i} = P(U) and τ_{M_u}= {U, Ø , {e},{e, g, h, k}}.

Proposition 7.4. *Let (U, Q, ξj) is a j-approximation space. Then*

$$\tau_{M_{\langle r \rangle}} \subseteq \tau_{M_{\langle i \rangle}}$$
$$\tau_{M_{\langle l \rangle}} \subseteq \tau_{M_{\langle i \rangle}}$$

Proof. Let B $\in \tau_{M_{\langle r \rangle}}$, then ∀ v ∈ B, $M_{\langle r \rangle}(v) \subseteq B$. Thus ∀ v ∈ B, $M_{\langle r \rangle}(v) \cap M_{\langle l \rangle}(v) \subseteq B$ and so ∀ v ∈ B, $M_{\langle i \rangle}(v) \subseteq B$,Then B $\in \tau_{M_{\langle i \rangle}}$ and $\tau_{M_{\langle r \rangle}} \subseteq \tau_{M_{\langle i \rangle}}$. we can prove that $\tau_{M_{\langle l \rangle}} \subseteq \tau_{M_{\langle i \rangle}}$similarly. □

Remark 7.3. *Let (U, R, ξj) be a j-NS, then the following is not necessarily true.*

$$\tau_{M_{\langle r \rangle}} = \tau_{M_{\langle i \rangle}}$$
$$\tau_{M_{\langle l \rangle}} = \tau_{M_{\langle i \rangle}}$$

Remark 7.4. *In a j-NS, $\tau_{M_{\langle r \rangle}}$and $\tau_{M_{\langle l \rangle}}$ are not necessarily comparable.*

The following example explains remarks Remark 7.3 and Remark 7.4.

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Example 7.3. *Let U = {e, f, g, h} and R = {(e, h), (f, f), (f, g), (g, f), (h, e), (h, g)}.*

Then, we have
$\tau_{M_{\langle r \rangle}}$= {U, Ø, {f},{g}, {h}, {e, g},{f, g},{f, h},{g, h},{e, f, g},{e, g, h},{f, g, h}},
$\tau_{M_{\langle l \rangle}}$={U,Ø,{e},{f},{h},{e, f},{e, h},{f, g},{f, h},{e, f, g},{e, f, h},{f, g, h}} and
$\tau_{M_{\langle i \rangle}}$ = P(U).

Proposition 7.5. *Let (U, Q, ξj) is a j-approximation space. Then*

$\tau_{M_{\langle u \rangle}} \subseteq \tau_{M_{\langle r \rangle}}$
$\tau_{M_{\langle u \rangle}} \subseteq \tau_{M_{\langle l \rangle}}$
$\tau_{M_{\langle u \rangle}} \subseteq \tau_{M_{\langle i \rangle}}$

Proof. Let B ∈ $\tau_{M_{\langle u \rangle}}$, then ∀ v ∈ B, $M_{\langle u \rangle}(v) \subseteq B$. Thus, ∀ v ∈ B , $M_{\langle r \rangle}(v) \cup M_{\langle l \rangle}(v) \subseteq B$. and so ∀ v ∈ B, $M_{\langle r \rangle}(v) \subseteq B$ and $M_{\langle l \rangle}(v) \subseteq B$. Then, B ∈ $\tau_{M_{\langle r \rangle}}$ and B ∈ $\tau_{M_{\langle l \rangle}}$ and hence $\tau_{M_{\langle u \rangle}} \subseteq \tau_{M_{\langle r \rangle}}$ and $\tau_{M_{\langle u \rangle}} \subseteq \tau_{M_{\langle l \rangle}}$. Similarly, we can prove that $\tau_{M_{\langle u \rangle}} \subseteq \tau_{M_{\langle i \rangle}}$. □

Remark 7.5. *Let (U, R, ξj) be a j-NS; then the following are not necessarily true.*

$$\tau_{M_{\langle u \rangle}} = \tau_{M_{\langle r \rangle}}$$
$$\tau_{M_{\langle u \rangle}} = \tau_{M_{\langle l \rangle}}$$

$\tau_{M_{\langle u \rangle}} = \tau_{M_{\langle i \rangle}}$
The following example shows Remark 7.5.

Example 7.4. *Let U = {e, f, g, h} and R = {(e, e), (e, f), (f, g), (f, h), (g, e), (h, e)}.*

Then, we have
$\tau_{M_{<r>}}$= {U, Ø, {e},{e, f},{g, h},{e, g, h}},
$\tau_{M_{<l>}}$={U,Ø,{e},{f},{e, f},{e, g, h}},
$\tau_{M_{<i>}}$ = { U,Ø,{e},{f},{e, f},{g, h},{e, g, h},{f, g, h}} and
$\tau_{M_{<u>}}$={U,Ø,{e},{e, f},{e, g, h}}.

Remark 7.6. *In a j-NS,* $\tau_{M_{\curlyvee}}$ *and* $\tau_{M_{<r>}}$ *are not necessarily comparable. Also,* $\tau_{M_{\curlyvee}}$ *and* $\tau_{M_{<i>}}$ *are not necessarily comparable, as the following example illustrates.*

Example 7.5. *Let U = {e, f, g, h} and R = {(e, e), (f, f), (g, g), (g, h), (h, e)}.*

Then, we have
τ_{M_r} = {U, Ø, {e},{f},{g},{e, f},{e, g},{e, h},{f, g},{e, f, g},{e, f, h},{e, g, h}},
$\tau_{M_{<r>}}$={U,Ø,{e},{f},{e, f},{g, h},{e, g, h},{f, g, h}},
τ_{M_l} = {U, Ø, {e},{f},{g},{e, f},{e, g},{f, h},{g, h},{e, f, g},{e, g, h},{f, g, h}} and
$\tau_{M_{<l>}}$={U,Ø,{f},{g},{e, h},{f, g},{e, f, h},{e, g, h}}.

Proposition 7.6. *Let (U, Q, ξj) is a j-approximation space. Then*

$\tau_{M_u} \subseteq \tau_{M_r} \subseteq \tau_{M_i}$
$\tau_{M_u} \subseteq \tau_{M_l} \subseteq \tau_{M_i}$
$\tau_{M_{<u>}} \subseteq \tau_{M_{<r>}} \subseteq \tau_{M_{<i>}}$
$\tau_{M_{<u>}} \subseteq \tau_{M_{<l>}} \subseteq \tau_{M_{<i>}}$

Proof. By using Propositions 7.2, 7.3, 7.4, and 7.5, the proof is obvious. □

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8. Rough Approximations Generated by M_j -Topologies.

We introduced the rough approximations by using M_j -neighborhoods. Let (U, Q, ξj) be j-approximation space. We employ M_j-neighborhoods to generate classes of topologies $\tau_{M_j} = \{B \subseteq U \mid M_j(v) \subseteq B$ for all $v \in B\}$ (called M_j-topologies).

Rough approximations using interior and closure operators in the topology τ_{M_j}, for all $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$ are defined.

Definition 8.1. *Let (U, Q, ξj) be a j-approximation space. The subset $B \subseteq U$ is said to be M_j-open set if $B \in \tau_{M_j}$ and the supplement of M_j-open sets are called M_j-closed sets. The family Γ_{M_j} of all M_j-closed sets are defined as $\Gamma_{M_j} = \{F \subseteq U \mid F^c \in \tau_{M_j}\}$.*

Definition 8.2. *Let (U, Q, ξj) be a j-approximation space and $P \subseteq U$. The M_j-lower approximations $ℏ_{M_j}(P)$ and M_j-upper approximations $ℏ^{M_j}(P)$ are defined by:*

- $ℏ_{M_j}(P) = \cup\{G \in \tau_{M_j} : G \subseteq P\}$ = M_j-interior of P.
- $ℏ^{M_j}(P) = \cap\{H \in \tau_{M_j} : P \subseteq H\}$ = M_j- closure of P.

Definition 8.3. *Let (U, Q, ξj) be a j-approximation space and $P \subseteq U$. The M_j-boundary region, M_j-positive region and M_j-negative region of P are defined as follows:*

- $B_{M_j}(P) = ℏ^{M_j}(P) - ℏ_{M_j}(P)$.
- $POS_{M_j}(P) = ℏ_{M_j}(P)$.
- $NEG_{M_j}(P) = U - ℏ^{M_j}(P)$.

Definition 8.4. *Let (U, Q, ξj) be a j-approximation space. The M_j-accuracy of the approximations of $P \subseteq U$ is defined by:*

$$\delta_{M_j}(P) = \frac{|ℏ_{M_j}(P)|}{|ℏ^{M_j}(P|}, \text{ where } \left|ℏ^{M_j}(P)\right| \neq 0.$$

It is clear that $0 \leq \delta_{M_j} \leq 1$. if $\delta_{M_j}(P) = 1$, then P is called M_j-exact set. Otherwise, P called M_j-rough.

According to the data given in Example 3.1, we get two tables that give M_j -lower approximations and M_j- upper approximations for $j \in \{r, l, i, u\}$ as shown in Table 4 and for $j \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$as shown in Table 5. The M_j-accuracy for $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$ as shown in Table 6.

Table 4. The M_j-approximations for each $j \in \{r, l, i, u\}$.

P(U)	τ_{M_r}		τ_{M_l}		τ_{M_i}		τ_{M_u}	
	$ℏ_{M_j}(P)$	$ℏ^{M_j}(P)$	$ℏ_{M_j}(P)$	$ℏ^{M_j}(P)$	$ℏ_{M_j}(P)$	$ℏ^{M_j}(P)$	$ℏ_{M_j}(P)$	$ℏ^{M_j}(P)$
{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}
{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}
{Si}	{Si}	{Si}	{Si}	{Si}	{Si}	{Si}	{Si}	{Si}
{Co}	{Co}	{Co}	{Co}	{Co}	{Co}	{Co}	{Co}	{Co}
{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}
{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr,Si}
{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr,Co}
{An,Si}	{An,Si}	{An,Si}	{An,Si}	{An,Si}	{An,Si}	{An,Si}	{An,Si}	{An,Si}

{An,Co}	{An,Co}	{An,Co}	{An,Co}	{An,Co}	{An,Co}	{An,Co}	{An,Co}	{An,Co}
{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}
{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}
{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}	{Pr,An,Co}
{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}
{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}
U	U	U	U	U	U	U	U	U
∅	∅	∅	∅	∅	∅	∅	∅	∅

Table 5. The M_j-approximations for each $j \in \{\langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$.

P(U}	$\tau_{M_{<r>}}$		$\tau_{M_{<l>}}$		$\tau_{M_{<i>}}$		$\tau_{M_{<u>}}$	
	$\text{łƀ}_{M_j}(P)$	$\text{łƀ}^{M_j}(P)$	$\text{łƀ}_{M_j}(P)$	$\text{łƀ}^{M_j}(P)$	$\text{łƀ}_{M_j}(P)$	$\text{łƀ}^{M_j}(P)$	$\text{łƀ}_{M_j}(P)$	$\text{łƀ}^{M_j}(P)$
{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}	{Pr}
{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}
{Si}	{Si}	{Si}	∅	{Si,Co}	{Si}	{Si}	∅	{Si,Co}
{Co}	{Co}	{Co}	∅	{Si,Co}	{Co}	{Co}	∅	{Si,Co}
{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}	{Pr,An}
{Pr,Si}	{Pr,Si}	{Pr,Si}	{Pr}	{Pr,Si,Co}	{Pr,Si}	{Pr,Si}	{Pr}	{Pr,Si,Co}
{Pr,Co}	{Pr,Co}	{Pr,Co}	{Pr}	{Pr,Si,Co}	{Pr,Co}	{Pr,Co}	{Pr}	{Pr,Si,Co}
{An,Si}	{An,Si}	{An,Si}	{An}	{An,Si,Co}	{An,Si}	{An,Si}	{An}	{An,Si,Co}
{An,Co}	{An,Co}	{An,Co}	{An}	{Si,Co}	{An,Co}	{An,Co}	{An}	{Si,Co}
{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}	{Si,Co}
{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An}	U	{Pr,An,Si}	{Pr,An,Si}	{Pr,An}	U
{Pr,An,Co}	{Pr,An,Co }	{Pr,An,Co }	{Pr,An}	U	{Pr,An,Co }	{Pr,An,Co }	{Pr,An}	U
{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}
{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}	{An,Si,Co}
U	U	U	U	U	U	U	U	U
∅	∅	∅	∅	∅	∅	∅	∅	∅

Table 6. The M_j-accuracy for $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$.

P(U)	δ_r	δ_l	δ_i	δ_u	$\delta_{\langle r \rangle}$	$\delta_{\langle l \rangle}$	$\delta_{\langle i \rangle}$	$\delta_{\langle u \rangle}$
{Pr}	1	1	1	1	1	1	1	1
{An}	1	1	1	1	1	1	1	1
{Si}	1	1	1	1	1	0	1	0
{Co}	1	1	1	1	1	0	1	0
{Pr,An}	1	1	1	1	1	1	1	1
{Pr,Si}	1	1	1	1	1	1/3	1	1/3
{Pr,Co}	1	1	1	1	1	1/3	1	1/3
{An,Si}	1	1	1	1	1	1/3	1	1/3
{An,Co}	1	1	1	1	1	½	1	1/3

{Si,Co}	1	1	1	1	1	1	1	1
{Pr,An,Si}	1	1	1	1	1	½	1	½
{Pr,An,Co}	1	1	1	1	1	½	1	½
{Pr,Si,Co}	1	1	1	1	1	1	1	1
{An,Si,Co}	1	1	1	1	1	1	1	1
U	1	1	1	1	1	1	1	1
∅	0	0	0	0	0	0	0	0

9. Comparison between our approach and previous approaches.

According to data given in Example 3.1. Tables 7 and 8 provide a comparison between Yao's method [21], Abd El-Monsef et al.'s method [12](τ_j), and the current methods (τ_{M_j}) in case of $j = r, l, i$ and u, respectively.

Table 7. Comparison between Yao, τ_j and M_j approaches for j = (r, l).

P(U)	**Yao**		τ_r		τ_{M_r}		τ_l		τ_{M_l}	
	$\underline{Apr}$(P)	$\overline{Apr}$(P)	$\underline{R}$(P)	$\bar{R}$(P)	ƀ$_{M_r}$(P)	ƀ^{M_r}(P)	$\underline{R}$(P)	$\bar{R}$(P)	ƀ$_{M_l}$(P)	ƀ^{M_l}(P)
{Pr}	{Si,Co}	{Si,Co}	∅	{Pr,Si,Co}	{Pr}	{Pr}	∅	{Pr,Si}	{Pr}	{Pr}
{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}
{Si}	{Pr}	{Pr}	∅	{Pr,Si,Co}	{Si}	{Si}	∅	{Pr,Si}	{Si}	{Si}
{Co}	∅	∅	∅	{Co}	{Co}	{Co}	{Co}	{Pr,Si,Co}	{Co}	{Co}
{Pr,An}	{An,Si,Co}	{An,Si,Co}	{An}	U	{Pr,An}	{Pr,An}	{An}	{Pr,An,Si}	{Pr,An}	{Pr,An}
{Pr,Si}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr, Si}	{Pr,Si,Co}	{Pr,Si}	{Pr,Si}	∅	{Pr,Si}	{Pr,Si}	{Pr,Si}
{Pr,Co}	{Si,Co}	{Si,Co}	∅	{Pr,Si,Co}	{Pr,Co}	{Pr,Co}	{Co}	{Pr,Si,Co}	{Pr,Co}	{Pr,Co}
{An,Si}	{Pr,An}	{Pr,An}	{An}	U	{An,Si}	{An,Si}	{An}	U	{An,Si}	{An,Si}
{An,Co}	{An}	{An}	{An}	{An,Co}	{An,Co}	{An,Co}	{An,Co}	U	{An,Co}	{An,Co}
{Si,Co}	{Pr}	{Pr}	∅	{Pr,Si,Co}	{Si,Co}	{Si,Co}	{Co}	{Pr,Si,Co}	{Si,Co}	{Si,Co}
{Pr,An,Si}	U	U	{Pr,An,Si}	U	{Pr,An,Si}	{Pr,An,Si}	{An}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}
{Pr,An,Co}	{An,Si,Co}	{An,Si,Co}	{An}	U	{Pr,An,Co}	{Pr,An,Co}	{An,Co}	U	{Pr,An,Co}	{Pr,An,Co}
{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}
{An,Si,Co}	{Pr,An}	{Pr,An}	{An}	U	{An,Si,Co}	{An,Si,Co}	{An,Co}	U	{An,Si,Co}	{An,Si,Co}

U	U	U	U	U	U	U	U	U	U	U
∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅

Table 8. Comparison between Yao, τ_j and M_j approaches for j = (i, u).

P(U)	**Yao**		τ_i		τ_{M_*}		τ_u		τ_{M_*}	
	$\underline{Apr}$(P)	$\overline{Apr}$(P)	$\underline{R}$(P)	$\bar{R}$(P)	łb_{M_r}(P)	łb^{M_r}(P)	$\underline{R}$(P)	$\bar{R}$(P)	łb_{M_l}(P)	łb^{M_l}(P)
{Pr}	{Si,Co}	{Si,Co}	∅	{Pr,Si}	{Pr}	{Pr}	∅	{Pr,Si,Co}	{Pr}	{Pr}
{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}	{An}
{Si}	{Pr}	{Pr}	∅	$\{Pr, Si\}$	{Si}	{Si}	∅	{Pr,Si,Co}	{Si}	{Si}
{Co}	∅	∅	{Co}	{Co}	{Co}	{Co}	∅	{Pr,Si,Co}	{Co}	{Co}
{Pr,An}	{An,Si,Co}	{An,Si,Co}	{An}	{Pr,An,Si}	{Pr,An}	{Pr,An}	{An}	U	{Pr,An}	{Pr,An}
{Pr,Si}	{Pr,Si,Co}	{Pr,Si,Co}	$\{Pr, Si\}$	{Pr,Si}	{Pr,Si}	{Pr,Si}	∅	{Pr,Si,Co}	{Pr,Si}	{Pr,Si}
{Pr,Co}	{Si,Co}	{Si,Co}	{Co}	{Pr,Si,Co}	{Pr,Co}	{Pr,Co}	∅	{Pr,Si,Co}	{Pr,Co}	{Pr,Co}
{An,Si}	{Pr,An}	{Pr,An}	{An}	{Pr,An,Si}	{An,Si}	{An,Si}	{An}	U	{An,Si}	{An,Si}
{An,Co}	{An}	{An}	{An,Co}	{An,Co}	{An,Co}	{An,Co}	{An}	U	{An,Co}	{An,Co}
{Si,Co}	{Pr}	{Pr}	{Co}	{Pr,Si,Co}	{Si,Co}	{Si,Co}	∅	{Pr,Si,Co}	{Si,Co}	{Si,Co}
{Pr,An,Si}	U	U	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{Pr,An,Si}	{An}	U	{Pr,An,Si}	{Pr,An,Si}
{Pr,An,Co}	{An,Si,Co}	{An,Si,Co}	{An,Co}	U	{Pr,An,Co}	{Pr,An,Co}	{An}	U	{Pr,An,Co}	{Pr,An,Co}
{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}	{Pr,Si,Co}
{An,Si,Co}	{Pr,An}	{Pr,An}	{An,Co}	U	{An,Si,Co}	{An,Si,Co}	{An}	U	{An,Si,Co}	{An,Si,Co}
U	U	U	U	U	U	U	U	U	U	U
∅	∅	∅	∅	∅	∅	∅	∅	∅	∅	∅

From the Tables 7 and 8, it is clear that the best of these methods given by using $\tau_{M_r}, \tau_{M_l}, \tau_{M_i}, \tau_{M_{<r>}}, \tau_{M_{<i>}}$. The boundary regions, in this case, are canceled. The results are so important for eliminating the imprecision of rough sets.

Example 9.1. *We give the following example to illuminate some consequences.*

Suppose that our universe is a set of cars $U = \{Sedan\ (Se), Sports\ Car\ (Sp), Coupe\ (Co), Minivan\ (Mi)\}$ *and our relationship is given as* $Q =$

$\{(Se,Se),(Mi,Mi),(Se,Co),(Se,Mi),(Mi,Sp),(Sp,Mi)\}$. *The different types of neighborhoods are given in Table 9 as follows:*

Table 9. Different types of neighborhoods.

U	Sedan	Sports Car	Coupe	Minivan
N_r	{Se,Co,Mi}	{Mi}	∅	{Sp,Mi}
N_l	{Se}	{Mi}	{Se}	{Se,Sp,Mi}
N_i	{Se}	{Mi}	∅	{Sp,Mi}
N_u	{Se,Co,Mi}	{Mi}	{Se}	{Se,Sp,Mi}
$N_{\langle r \rangle}$	{Se,Co,Mi}	{Sp,Mi}	{Se,Co,Mi}	{Mi}
$N_{\langle l \rangle}$	{Se}	{Se,Sp,Mi}	∅	{Mi}
$N_{\langle i \rangle}$	{Se}	{Sp,Mi}	∅	{Mi}
$N_{\langle u \rangle}$	{Se,Co,Mi}	{Se,Sp,Mi}	{Se,Co,Mi}	{Mi}
E_r	{Se,Sp,Mi}	{Se,Sp,Mi}	∅	{Se,Sp,Mi}
E_l	{Se,Co,Mi}	{Sp,Mi}	{Se,Co,Mi}	U
E_i	{Se,Mi}	{Sp,Mi}	∅	{Se,Sp,Mi}
E_u	U	{Se,Sp,Mi}	{Se,Co,Mi}	U
$E_{\langle r \rangle}$	U	U	U	U
$E_{\langle l \rangle}$	{Se,Sp}	{Se,Sp,Mi}	∅	{Sp,Mi}
$E_{\langle i \rangle}$	{Se,Sp}	{Se,Sp,Mi}	∅	{Sp,Mi}
$E_{\langle u \rangle}$	U	U	U	U
M_r	{Se,Mi}	{Mi}	∅	{Sp,Mi}
M_l	{Se}	{Mi}	{Se}	{Se,Sp,Mi}
M_i	{Se}	{Mi}	∅	{Sp,Mi}
M_u	{Se,Mi}	{Mi}	{Se}	{Se,Sp,Mi}
$M_{\langle r \rangle}$	{Se,Co,Mi}	{Sp,Mi}	{Se,Co,Mi}	{Mi}
$M_{\langle l \rangle}$	{Se}	{Se,Sp,Mi}	∅	{Mi}
$M_{\langle i \rangle}$	{Se}	{Sp,Mi}	∅	{Mi}
$M_{\langle u \rangle}$	{Se,Co,Mi}	{Se,Sp,Mi}	{Se,Co,Mi}	{Mi}

The M_j-Topologies as follows:

$$\tau_{M_r} = \{U, \varphi, \{Co\}, \{Sp, Mi\}, \{Sp, Co, Mi\}, \{Se, Sp, Mi\}\}.$$

$$\tau_{M_l} = \left\{U, \varphi, \{Se\}, \{Se, Co\}, \{Se, Sp, Mi\}\right\}.$$

$$\tau_{M_i} = \{U, \varphi, \{Se\}, \{Co\}, \{Sp, Mi\}, \{Se, Co\}, \{Se, Sp, Mi\}, \{Sp, Co, Mi\}\}.$$

$$\tau_{M_u} = \{U, \varphi, \{Se, Sp, Mi\}\}.$$

$$\tau_{M_{\langle r \rangle}} = \{U, \varphi, \{Mi\}, \{Sp, Mi\}, \{Se, Co, Mi\}\}.$$

$$\tau_{M_{\langle l \rangle}} = \left\{U, \varphi, \{Se\}, \{Co\}, \{Mi\}, \{Se, Co\}, \{Se, Mi\}, \{Co, Mi\}, \{Se, Sp, Mi\}, \{Se, Co, Mi\}\right\}$$

$$.\tau_{M_{\langle i \rangle}} = \left\{\begin{matrix} U, \varphi, \{Co\}, \{Se\}, \{Mi\}, \{Se, Co\}, \{Sp, Mi\}, \{Se, Mi\}, \\ \{Co, Mi\}, \{Se, Sp, Mi\}, \{Se, Co, Mi\}, \{Sp, Co, Mi\} \end{matrix}\right\}.$$

$$\tau_{M_{\langle u \rangle}} = \{U, \varphi, \{Mi\}, \{Se, Co, Mi\}\}.$$

Table 10 give M_j-lower approximations and M_j- upper approximations for each $j \in \{r, l, i, u\}$. Table 11 shows the comparison between Tareq M. Al-shami [18] (τ_{E_j}) approach and the current approach (τ_{M_j}) in case of $j \in \{r,i\}$ as given in Example 9.1. The M_j-accuracy for each $j \in \{r, l, i, u\}$ are given in

Table 10. Approximations operators based on τ_{M_j} for some subsets of U.

U	τ_{M_r}		$\tau_{M_⚜}$		$\tau_{M_⚜}$		$\tau_{M_⚜}$	
	ƀ$_{M_r}$(P)	ƀ^{M_r}(P)	ƀ$_{M_l}$(P)	ƀ^{M_l}(P)	ƀ$_{M_i}$(P)	ƀ^{M_i}(P)	ƀ$_{M_u}$(P)	ƀ^{M_u}(P)
{Se}	∅	{Se}	{Se}	U	{Se}	{Se}	∅	U
{Sp}	∅	{Se,Sp,Mi}	∅	{Sp,Mi}	∅	{Sp,Mi}	∅	U
{Co}	{Co}	{Co}	∅	{Co}	{Co}	{Co}	∅	{Co}
{Mi}	∅	{Se,Sp,Mi}	∅	{Sp,Mi}	∅	{Sp,Mi}	∅	U
{Se,Sp}	∅	{Se,Sp,Mi}	{Se}	U	{Se}	{Se,Sp,Mi}	∅	U
{Se,Co}	{Co}	{Se,Co}	{Se,Co}	U	{Se,Co}	{Se,Co}	∅	U
{Se,Mi}	∅	{Se,Sp,Mi}	{Se}	U	{Se}	{Sp,Mi}	∅	U
{Sp,Co}	{Co}	U	∅	{Sp,Co,Mi}	{Co}	Sp,Co,Mi}	∅	U
{Sp,Mi}	{Sp,Mi}	{Se,Sp,Mi}	∅	{Sp,Mi}	{Sp,Mi}	{Sp,Mi}	∅	U
{Co,Mi}	{Co}	U	∅	{Sp,Co,Mi}	{Co}	Sp,Co,Mi}	∅	U
{Se,Sp,Co}	{Co}	U	{Se,Co}	U	{Se,Co}	U	∅	U
{Se,Sp,Mi}	{Se,Sp,Mi}	{Se,Sp,Mi}	{Se,Sp,Mi}	U	{Se,Sp,Mi}	{Se,Sp,Mi}	{Se, Sp, Mi	U
{Se,Co,Mi}	{Co}	U	{Se,Co}	U	{Se,Co}	U	∅	U
{Sp,Co,Mi}	{Sp,Co,Mi}	U	∅	{Sp,Co,Mi}	{Sp,Co,Mi}	Sp,Co,Mi}	∅	U
U	U	U	U	U	U	U	U	U
∅	∅	∅	∅	∅	∅	∅	∅	∅

Table 11. Comparison among E_i and M_i approaches for some subsets of U.

P(U}	$\tau_{E_⚜}$		$\tau_{M_⚜}$		$\tau_{E_⚜}$		$\tau_{M_⚜}$	
	Q(P)	$\bar{Q}$(P)	ƀ$_{M_r}$(P)	ƀ^{M_r}(P)	$\underline{Q}$(P)	$\bar{Q}$(P)	ƀ$_{M_i}$(P)	ƀ^{M_i}(P)
{Se}	∅	Se,Sp,Mi}	∅	{Se}	∅	Se,Sp,Mi}	{Se}	{Se}
{Sp}	∅	{Se,Sp,Mi}	∅	{Se,Sp,Mi}	∅	{Se,Sp,Mi}	∅	{Sp,Mi}
{Co}	{Co}	U	{Co}	{Co}	{Co}	U	{Co}	{Co}
{Mi}	∅	Se,Sp,Mi}	∅	{Se,Sp,Mi}	∅	Se,Sp,Mi}	∅	{SP,Mi}
{Se,Sp}	∅	{Se,Sp,Mi}	∅	{Se,Sp,Mi}	∅	{Se,Sp,Mi}	{Se}	{Se,Sp,Mi}
{Se,Co}	{Co}	U	{Co}	U	{Co}	U	{Se,Co}	{Se,Co}
{Se,Mi}	∅	Se,Sp,Mi}	∅	{Se,Sp,Mi}	∅	Se,Sp,Mi}	{Se}	{Se,Sp,Mi}
{Sp,Co}	{Co}	U	{Co}	U	{Co}	U	{Co}	{Sp,Co,Mi}
{Sp,Mi}	∅	Se,Sp,Mi}	{Sp,Mi}	{Se,Sp,Mi}	∅	Se,Sp,Mi}	{SP,Mi}	{SP,Mi}

{Co,Mi}	{Co}	U	{Co}	U	{Co}	U	Co}	{Sp,Co,Mi}
{Se,Sp,Co}	{Co}	U	{Co}	U	{Co}	U	{Se,Co}	U
{Se,Sp, Mi}	{Se,Sp, Mi}	{Se,Sp,Mi}	{Se,Sp, Mi}	{Se,Sp, Mi}	{Se,Sp, Mi}	{Se,Sp, Mi}	{Se,Sp, Mi}	{Se,Sp,Mi}
{Se,Co, Mi}	{Co}	U	{Co}	U	{Co}	U	{Se,Co}	U
{Sp,Co, Mi}	{Co}	U	{Sp,Co, Mi}	U	{Co}	U	{Sp,Co, Mi}	{Sp,Co,Mi}
U	U	U	U	U	U	U	U	U
∅	∅	∅	∅	∅	∅	∅	∅	∅

We complete this work by studying these concepts from a topological view and comparing them. In all comparisons, we obtain higher accurate approximations in our approach.

10. Conclusions

This paper generated some classes of neighborhood systems that we call them M_j -neighborhoods. We introduced some examples to discuss the main structures and their main concepts. We generalized these concepts to the topological case. We addressed many possible comparisons among them. We obtained refinement results that can be useful in real-life applications.

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References

1. Z. Pawlak, Rough sets, Int. J. Inf. Comput. Sci. 11 (5) (1982) 341–356.
2. Z. Pawlak, A. Skowron, Rough sets, Some extensions, Information Sciences, 177 (2007) 28–40.
3. K. Qin, J. Yang, Z. Pei, Generalized rough sets based on reflexive and transitive relations, Information Sciences, 178 (2008) 4138–4144.
4. W. Zhu, F. Wang, Reduction and axiomatization of covering generalized rough sets, Information Sciences, 152 (2003) 217–230.
5. W. Zhu, F. Wang, Binary relation based rough sets, Lecture Notes in Computer Sciences, 4223 (2006) 276–285.
6. A. S. Salama, Generalizations of Rough Sets Using two Topological Spaces with Medical Applications, INFORMATION, 19 (7A) (2016) 2425-2440.
7. A. S. Salama, Sequences of Topological Near Open and Near Closed Sets with Rough Applications, Filomat, 34(1) (2020), 51–58.
8. A. S. Salama, Bitopological Approximation Space with Application to Data Reduction in Multi-valued Information Systems, Filomat, 34 (1) (2020), 99–110.
9. A. S. Salama, Topological solution of missing attribute values problem in incomplete information tables, Information Sciences, 180 (2010) 631-639.
10. A.S. Salama, Topologies induced by relations with applications, Journal of Computer Science, 4(10) (2008), 877–887.
11. A. S. Salama, Abd El-Monsef, M. M. E., New topological approach of rough set generalizations, International Journal of Computer Mathematics, 88(7) (2011), 1347–1357.

 doi:10.20944/preprints202308.1036.v1

12. M.E. Abd El-Monsef, O.A. Embaby, M.K. El-Bably, Comparison between rough set approximations based on different topologies, International Journal of Granular Computing, Rough Set and Intelligent Systems, 3(4) (2014), 292–305.
13. M.E. Abd El-Monsef, A.S. Salama and O.A. Embaby, Granular computing using mixed neighborhood systems, Journal of Institute of Mathematics and Computer Sciences, 20 (2) (2009), 233–243. O.G.
14. El Barbary, A.S. Salama, Feature selection for document classification based on topology, Egyptian Informatics Journal, 19(2) (2018), 129–132.
15. Wiweger, On topological rough sets, Bulletin of the Polish Academy of Sciences Mathematics, 37 (1989) 89–93.
16. Tareq M. Al-shami, An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application. Inf Sci, 569 (2021),110–124
17. Tareq M. Al-shami, Ciucci D, Subset neighborhood rough sets. Knowl Based Syst, 237 (2022), 107868.
18. Tareq M. Al-shami, Fu WQ, Abo-Tabl E. A., New rough approximations based on E-neighborhoods. Complexity, 6666853 (2021), 1-6.
19. Mareay R, Generalized rough sets based on neighborhood systems and topological spaces, Journal of Egyptian Math Soc, 24 (2016) 603–608.
20. Z. Pawlak, Rough Sets, Theoretical Aspects of Reasoning About Data, Kluwer Acadmic Publishers Dordrecht, 1991.
21. Yao YY, Two views of the theory of rough sets in finite universes. Int J Approx Reason, (1996), 15:291–317.