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Keywords: integrable equations; Lax pairs; symmetry; Hamiltonian structure; zero curvature equations

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Similarity Transformations and Nonlocal Reduced Integrable Nonlinear Schrödinger Type Equations

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Abstract: We present three reduced integrable hierarchies of nonlocal integrable NLS type equations from a vector integrable hierarchy associated with a matrix Lie algebra, not being A type. Three similarity transformations are taken to keep the invariance of the transformed zero curvature equations. The key step is to formulate a solution to a reduced stationary zero curvature equation so that the zero curvature formulation works for a reduced case.

Keywords: matrix spectral problem; zero curvature equation; similarity transformation; integrable hierarchy; NLS equations

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1. Introduction

Lax pairs of matrix spectral problems are primary objects in the study of nonlinear integrable partial differential equations. It is important to formulate a matrix spectral problem from a given matrix Lie algebra [1,2]. Many integrable hierarchies of nonlinear partial differential equations are generated from the special linear algebras [3–5], and the special orthogonal algebras (see, e.g., [6,7]). Hamiltonian structures could be constructed via the trace identity [8] if the associated matrix Lie algebra is semisimple, and the variational identity [9] if the associated matrix Lie algebra is non-seimsimple. Combined with a recursion structure that an integrable hierarchy possesses, a bi-Hamiltonian structure can often be presented, thereby its Liouville integrability being guaranteed.

Similarity transformations are used to present integrable reductions from given integrable hierarchies, which keep the original zero curvature equations invariant. Many local and nonlocal reduced integrable NLS and mKdV equations have been generated from the Ablowitz-Kaup-Newell-Segur (AKNS) spectral problems (see, e.g., [10–12] and [13]-[19] for local and nonlocal reductions, respectively). Recently, it has been shown that if taking pairs of similarity transformations, one can generate novel kinds of reduced integrable partial differential equations, both

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local and nonlocal [15,20–23]. All such studies bring many new interesting problems to solve in the theory of partial differential equations.

This paper aims to present an application of similarity transformations to present reduced nonlocal integrable NLS type equations. In Section 2, we recall a vector integrable Hamiltonian hierarchy [24]. In Section 3, we propose three classes of similarity transformations for the involved spectral matrix, and work out three reduced integrable hierarchies. The first two typical examples of nonlocal reduced integrable NLS type equations in the presented reduced hierarchies are

$$ir_t = r_{xx} - 2r\gamma\delta r^T(-x,-t)r + r\gamma r^T r(-x,-t)\delta,$$

and

$$ir_t = r_{xx} + 2r\gamma\delta r^T(x,-t)r - r\gamma r^T r(x,-t)\delta,$$

where γ and δ are two constant commuting symmetric and orthogonal matrices and r^T denotes the matrix transpose of the potential r. The third one is

$$ir_t = r_{xx} + 2r\gamma\delta r^\dagger(-x,t)r - r\gamma r^T r^*(-x,t)\delta,$$

where γ and δ are two real commuting symmetric and orthogonal matrices, and $r^\dagger$ and r^* denote the Hermitian transpose and the complex conjugate of the potential r, respectively. In the last section, a conclusion and a few concluding remarks will be given.

2. A vector integrable Hamiltonian hierarchy

We recall a vector integrable Hamiltonian hierarchy, presented in [24], for subsequent analysis. The integrable hierarchy was generated from a matrix spectral problem associated with a non-special linear algebra [24]. Let $n \in \mathbb{N}$ be a given number, and γ be a given symmetric and orthogonal matrix of order n, and λ denote the spectral parameter. Assume that the potential vector reads:

$$u = u(x,t) = (r, s^T)^T,\ r = r(x,t) = (r_1, \cdots, r_n),\ s = s(x,t) = (s_1, \cdots, s_n)^T. \tag{2.1}$$

Starting from the spatial matrix spectral problem:

$$-i\phi_x = M\phi = M(u,\lambda)\phi,\ M = \begin{bmatrix} -\lambda & r & 0 \\ s & 0 & \gamma^T r^T \\ 0 & s^T\gamma^T & \lambda \end{bmatrix}, \tag{2.2}$$

a counterpart of the AKNS spectral problem [3], we solve the associated stationary zero curvature equation

$$-iZ_x = [M, Z], \tag{2.3}$$

 doi:10.20944/preprints202308.1021.v1

and assume a Laurent series solution:

$$Z = \begin{bmatrix} -e & f & 0 \\ g & h & \gamma^T f^T \\ 0 & g^T \gamma^T & e \end{bmatrix} = \sum_{l \geq 0} \lambda^{-l} Z^{[l]}, \tag{2.4}$$

with $Z^{[l]}$'s being given by

$$Z^{[l]} = \begin{bmatrix} -e^{[l]} & f^{[l]} & 0 \\ g^{[l]} & h^{[l]} & \gamma^T f^{[l]T} \\ 0 & g^{[l]T} \gamma^T & e^{[l]} \end{bmatrix}, \; l \geq 0. \tag{2.5}$$

Obviously, the stationary zero curvature equation with such a solution form determines the recursion relation for the solution Z:

$$\begin{cases} e_x^{[0]} = 0, \; f^{[0]} = 0, \; g^{[0]} = 0, \; h_x^{[0]} = 0, \\ f^{[l+1]} = i f_x^{[l]} + r h^{[l]} + e^{[l]} r, \; g^{[l+1]} = -i g_x^{[l]} + h^{[l]} s + s e^{[l]}, \\ e_x^{[l+1]} = i(f^{[l+1]} s - r g^{[l+1]}) = -f_x^{[l]} s - r g_x^{[l]}, \\ h_x^{[l+1]} = i(s f^{[l+1]} - g^{[l+1]} r - \gamma^T f^{[l+1]T} s^T \gamma^T + \gamma^T r^T g^{[l+1]T} \gamma^T), \end{cases} \tag{2.6}$$

where $l \geq 0$. Once choosing

$$e^{[0]} = 1, \; h^{[0]} = 0, \tag{2.7}$$

and taking the constant of integration to be zero,

$$e^{[l]}|_{u=0} = 0, \; h^{[l]}|_{u=0} = 0, \; l \geq 1, \tag{2.8}$$

a sequence of $\{e^{l]}, f^{l]}, g^{l]}, h^{l]} \,|\, l \geq 1\}$ can be worked out.

Upon taking the temporal matrix spectral problems:

$$-i\phi_t = N^{[k]}\phi = N^{[k]}(u, \lambda)\phi, \; N^{[k]} = (\lambda^k Z)_+ = \sum_{l=0}^{k} \lambda^l Z^{[k-l]}, \; k \geq 0, \tag{2.9}$$

the compatibility conditions of the two matrix spectral problems in (2.2) and (2.9), namely, the zero curvature equations:

$$N_t - N_x^{[k]} + i[M, N^{[k]}] = 0, \; k \geq 0, \tag{2.10}$$

yield the vector integrable hierarchy:

$$u_t = X^{[k]} = (i f^{[k+1]}, -i g^{[k+1]T})^T, \text{ i.e., } r_t = i f^{[k+1]}, \; s_t = -i g^{[k+1]}, \; k \geq 0. \tag{2.11}$$

The first example of nonlinear integrable equations is the generalized vector nonlinear Schrödinger equations

$$\begin{cases} i r_t = r_{xx} + 2 r s r - r \gamma r^T s^T \gamma, \\ i s_t = -s_{xx} - 2 s r s + \gamma r^T s^T \gamma s, \end{cases} \tag{2.12}$$

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where γ is an arbitrary symmetric and orthogonal matrix.

The Hamiltonian structure for the integrable hierarchy (2.11), established by the trace identity, is given by

$$u_t = X_k = J\frac{\delta\mathcal{H}^{[k]}}{\delta u},\ \ k \geq 1, \tag{2.13}$$

where J is a Hamiltonian operator defined by

$$J = i\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{2.14}$$

with I_n being the identity matrix of order n, and $\mathcal{H}^{[k]}$ is defined by

$$\mathcal{H}^{[l]} = -\int \frac{e^{[l+1]}}{l}\,dx,\ l \geq 1, \tag{2.15}$$

with $e^{[l]}$ being determined by the stationary zero curvature equation. The Hamiltonian structure presents a relation between symmetries and conserved quantities [25,26]. Infinitely many symmetries commute

$$[\![X_k, X_l]\!] = \frac{d}{d\varepsilon}\left[(X_k(u+\varepsilon X_l) - X_l(u+\varepsilon X_k)\right]\big|_{\varepsilon=0} = 0,\ k,l \geq 0, \tag{2.16}$$

which is guaranteed by a Lax operator:

$$[\![V^{[k]}, V^{[l]}]\!] = \frac{d}{d\varepsilon}\left[(V^{[k]}(u+\varepsilon X_l) - V^{[l]}(u+\varepsilon X_k)\right]\big|_{\varepsilon=0} + [V^{[k]}, V^{[l]}] = 0,\ k,l \geq 0. \tag{2.17}$$

Moreover, the Hamiltonian structure in (2.13) guarantees that infinitely many conserved functionals commute under the Poisson bracket associated with the Hamiltonian operator J:

$$\{\mathcal{H}^{[k]}, \mathcal{H}^{[l]}\}_J = \int (\frac{\delta\mathcal{H}^{[k]}}{\delta u})^T J \frac{\delta\mathcal{H}^{[l]}}{\delta u}\,dx = 0,\ k,l \geq 0. \tag{2.18}$$

3. Nonlocal reduced integrable NLS type equations

Let us take another symmetric and orthogonal matrix δ of order n, which commute with the previous symmetric and orthogonal matrix γ. and introduce a higher-order orthogonal matrix Θ by

$$\Theta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{3.1}$$

by which one can see that a transformed matrix reads

$$\Theta M(\lambda)\Theta^{-1} = \begin{bmatrix} \lambda & s^T\gamma\delta & 0 \\ \gamma\delta r^T & 0 & \delta s \\ 0 & r\delta & -\lambda \end{bmatrix}. \tag{3.2}$$

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3.1. Similarity transformation 1

Let us first take a similarity transformation for the spectral matrix M:

$$\Theta M(x,t,\lambda)\Theta^{-1} = -M(-x,-t,\lambda). \tag{3.3}$$

Based on the result in (3.2), the above transformation equivalently yields

$$s = -\gamma\delta r^T(-x,-t) \text{ or } r = -s^T(-x,-t)\gamma\delta. \tag{3.4}$$

Further, we can directly observe that

$$\Theta Z(x,t,\lambda)\Theta^{-1} = -Z(-x,-t,\lambda). \tag{3.5}$$

This is because both Laurent series matrices $\Theta Z(x,t\lambda)\Theta^{-1}$ and $Z(-x,-t,\lambda)$ of λ solve the stationary zero curvature equation (2.3) with $M(x,t,\lambda)$ being replaced by $-M(-x,-t,\lambda)$ and possess the opposite initial values at $\lambda = \infty$. Thus, noting $N^{[k]} = (\lambda^k Z)_+$, we can have

$$\Theta N^{[k]}(x,t,\lambda)\Theta^{-1} = -N^{[k]}(-x,-t,\lambda),\ k \geq 0. \tag{3.6}$$

This tells us the relation

$$\begin{aligned} &\Theta\big(M_t(x,t,\lambda) - N_x^{[k]}(x,t,\lambda) + i[M(x,t,\lambda), N^{[k]}(x,t,\lambda)]\big)\Theta^{-1} \\ &= M_t(-x,-t,\lambda) - N_x^{[k]}(-x,-t,\lambda) + i[M(-x,-t,\lambda), N^{[k]}(-x,-t,\lambda)],\ k \geq 0, \end{aligned} \tag{3.7}$$

between the transformed and untransformed zero curvature equations. A consequence of this fact is that we obtain a reduced integrable hierarchy

$$r_t = i f^{[k+1]}|_{s=-\gamma\delta r^T(-x,-t)},\ k \geq 0. \tag{3.8}$$

Each equation in this reduced hierarchy possesses infinitely many symmetries and conserved functionals inherited from the original ones, under the potential reduction (3.4). The nonlocal reduced integrable NLS type equation in this hierarchy reads

$$i r_t = r_{xx} - 2r\gamma\delta r^T(-x,-t)r + r\gamma r^T r(-x,-t)\delta, \tag{3.9}$$

where γ and δ are two commuting symmetric and orthogonal matrices.

3.2. Similarity transformation 2

Let us second take the following similarity transformation for the spectral matrix M:

$$\Theta M(x,t,\lambda)\Theta^{-1} = M(x,-t,-\lambda), \tag{3.10}$$

where Θ is defined by (3.1).

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Upon observing (3.2), the above transformation on the spectral matrix generates

$$s(x,t) = \gamma\delta r^T(x,-t) \text{ or } r(x,t) = s^T(x,-t)\gamma\delta. \tag{3.11}$$

Under this potential reduction, we can have

$$\Theta Z(x,t,\lambda)\Theta^{-1} = -Z(x,-t,-\lambda). \tag{3.12}$$

The reason is similar, that is, it is because both Laurent series matrices $\Theta Z(x,t\lambda)\Theta^{-1}$ and $Z(x,-t,-\lambda)$ of λ present solutions to the stationary zero curvature equation (2.3), with $M(x,t,\lambda)$ being replaced by $M(x,-t,-\lambda)$, and have the opposite initial values at $\lambda=\infty$. This relation guarantees that

$$\theta N^{[2l]}(x,t,\lambda)\Theta^{-1} = -N^{[2l]}(x,-t,-\lambda),\ l\geq 0, \tag{3.13}$$

and then there follows

$$\begin{aligned}&\Theta\big(M_t(x,t,\lambda) - N_x^{[2l]}(x,t,\lambda) + i[M(x,t,\lambda),N^{[2l]}(x,t,\lambda)]\big)\Theta^{-1}\\ &= -\big(M_t(x,-t,-\lambda) - N_x^{[2l]}(x,-t,-\lambda) + i[M(x,-t,-\lambda),N^{[2l]}(x,-t,-\lambda)]\big),\ l\geq 0.\end{aligned} \tag{3.14}$$

Therefore, the reduced zero curvature equations lead to an integrable hierarchy of nonlocal reduced NLS type equations:

$$r_t = if^{[2l+1]}|_{s=\gamma\delta r^T(x,-t)},\ l\geq 0, \tag{3.15}$$

whose infinitely many symmetries and conserved functionals are similarly inherited from the original ones, under the potential reduction (3.11). In this reduced hierarchy, the nonlocal reduced integrable NLS type equation is

$$ir_t = r_{xx} + 2r\gamma\delta r^T(x,-t)r - r\gamma r^T r(x,-t)\delta, \tag{3.16}$$

where γ and δ are two commuting symmetric and orthogonal matrices.

3.3. Similarity transformation 3

Let us third take another similarity transformation for the spectral matrix M:

$$\Theta M(x,t,\lambda)\Theta^{-1} = M^*(-x,t,-\lambda^*), \tag{3.17}$$

where Θ is again defined by (3.1) and $*$ stands for the complex conjugate.

Upon recognizing the result in (3.2), the above transformation on the spectral matrix exactly engenders

$$s(x,t) = \gamma\delta r^\dagger(-x,t) \text{ or } r(x,t) = s^\dagger(-x,t)\gamma\delta, \tag{3.18}$$

where γ and δ are assumed to be real and $\dagger$ stands for the Hermitian transpose. Under such a potential reduction, we can obtain

$$\Theta Z(x,t,\lambda)\Theta^{-1} = -Z^*(-x,t,-\lambda^*), \tag{3.19}$$

since both Laurent series matrices $\Theta Z(x,t\lambda)\Theta^{-1}$ and $Z^*(-x,t,-\lambda^*)$ of λ solve the stationary zero curvature equation (2.3), with $M(x,t,\lambda)$ be replaced by $M^*(-x,t,-\lambda^*)$, and take the opposite initial values at $\lambda=\infty$. This ensures that

$$\Theta N^{[2l]}(x,t,\lambda)\Theta^{-1} = -N^{[2l]*}(x,-t,-\lambda^*),\ l\geq 0. \tag{3.20}$$

Then there follows

$$\begin{aligned}&\theta\big(M_t(x,t,\lambda)-N_x^{[2l]}(x,t,\lambda)+i[M(x,t,\lambda),N^{[2l]}(x,t,\lambda)]\big)\Theta^{-1}\\ &=\big(M_t(-x,t,-\lambda^*)-N_x^{[2l]}(-x,t,-\lambda^*)+i[M(-x,t,-\lambda^*),N^{[2l]}(-x,t,-\lambda^*)]\big)^*,\ l\geq 0,\end{aligned} \tag{3.21}$$

and thus, we obtain an integrable hierarchy of nonlocal reduced NLS type equations:

$$r_t = if^{[2l+1]}|_{s=\gamma\delta r^\dagger(-x,t)},\ l\geq 0, \tag{3.22}$$

whose infinitely many symmetries and conserved functionals are similarly inherited from the original ones, under the potential reduction (3.18). The nonlocal reduced integrable NLS type equation in the hierarchy reads

$$ir_t = r_{xx}+2r\gamma\delta r^\dagger(-x,t)r - r\gamma r^T r^*(-x,t)\delta, \tag{3.23}$$

where γ and δ are two real commuting symmetric and orthogonal matrices, and $r^\dagger$ and r^* denote the Hermitian transpose and the complex conjugate of r, respectively.

All three nonlocal reduced integrable hierarchies of NLS type equations presented above are different from the ones previously presented from the multi-component AKNS hierarchy (see, e.g., [14,21,22]).

4. Concluding remarks

Three reduced integrable hierarchies of nonlocal NLS type equations have been computed from a new vector integrable Hamiltonian hierarchy. The presented three similarity transformations are the key in the formulation of nonlocal reduced integrable equations.

It would be very interesting to search for new nonlocal integrable equations under similarity transformations from other matrix spectral problems. A further question is whether soliton solutions to reduced integrable equations could be guaranteed by Darboux transformations [27] or the Riemann-Hilbert technique [14]. Other interesting solutions include lump solutions [28,29], complexitons [30], rogue waves [31,32], Grammian solutions [33,34] and algebro-geometric solutions [35,36]. Reduced Lax pairs of matrix spectral problems and the Hirota bilinear method should be helpful.

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