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On Symmetric Generalized *n*-Derivations on Prime Ideals

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Abstract: Throughout this paper, we will establish a comprehensive theoretical foundation and rigorously develop the methodology to investigate the structure of quotient ring $\mathfrak{S}/\mathfrak{P}$ under the influence of symmetric generalized n-derivations, where $\mathfrak{S}$ represents an arbitrary ring, and $\mathfrak{P}$ signifies a prime ideal of $\mathfrak{S}$ satisfying certain algebraic identities acting on prime ideal $\mathfrak{P}$ without relying on the assumption of primeness or semi-primeness of the ring.

Keywords: prime ring; prime ideal; derivation; symmetric n-derivation; symmetric generalized n-derivation.

MSC: 16W25, 16R50, 16N60

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1. Introduction

At the heart of ring theory lies the concept of derivations, over the years, several researchers have extended the notion of derivations in various directions such as generalized derivations, (α,β)-derivations, bi-derivations, higher derivations, symmetric n-derivations, etc. and have studied the structure of rings as well as the structure of additive mappings (refer to [3–6,12,13,22,23]). In this research article, we present a comprehensive investigation of symmetric generalized n-derivations, seeking to establish a theoretical connection between symmetric generalized n-derivations and other fundamental algebraic concepts. "Throughout the discussion, we will consider $\mathfrak{S}$ to be an associative ring with $Z(\mathfrak{S})$ being its center. A ring $\mathfrak{S}$ is said to be prime if, $\varrho\mathfrak{S}\xi = \{0\}$ implies that either $\varrho = 0$ or $\xi = 0$, and semiprime if, $\varrho\mathfrak{S}\varrho = \{0\}$ implies that $\varrho = 0$, where $\varrho,\xi \in \mathfrak{S}$. The symbols $[\varrho,\xi]$ and $\varrho\circ\xi$ denote the commutator, $\varrho\xi - \xi\varrho$ and the anti-commutator, $\varrho\xi + \xi\varrho$, respectively, for all $\varrho,\xi \in \mathfrak{S}$. A ring $\mathfrak{S}$ is said to be n-torsion free if $n\varrho = 0$ implies that $\varrho = 0$ for all $\varrho \in \mathfrak{S}$. If $\mathfrak{S}$ is $n!$-torsion free, then it is d-torsion free for every divisor d of $n!$. Recall that an ideal $\mathfrak{P}$ of $\mathfrak{S}$ is said to be prime if, $\mathfrak{P} \neq \mathfrak{S}$ and for $\varrho,\xi \in \mathfrak{S}$, $\varrho\mathfrak{S}\xi \subseteq \mathfrak{P}$ implies that $\varrho \in \mathfrak{P}$ or $\xi \in \mathfrak{P}$. An additive mapping $d : \mathfrak{S} \to \mathfrak{S}$ is called a derivation if $d(\varrho\xi) = d(\varrho)\xi + \varrho d(\xi)$ holds for all $\varrho,\xi \in \mathfrak{S}$. Following [14], an additive mapping $g : \mathfrak{S} \longrightarrow \mathfrak{S}$ is said to be a generalized derivation on $\mathfrak{S}$ if there exists a derivation $d : \mathfrak{S} \longrightarrow \mathfrak{S}$ such that $g(\varrho\xi) = g(\varrho)\xi + \varrho d(\xi)$ holds for all $\varrho,\xi \in \mathfrak{S}$. A bi-additive map $\mathfrak{D} : \mathfrak{S}\times\mathfrak{S} \to \mathfrak{S}$ is said to be symmetric if $\mathfrak{D}(\varrho,\xi) = \mathfrak{D}(\xi,\varrho)$ for all $\varrho,\xi \in \mathfrak{S}$. A symmetric bi-additive map is said to be symmetric bi-derivation if $\mathfrak{D}(\varrho\xi,z) = \varrho\mathfrak{D}(\xi,z) + \mathfrak{D}(\varrho,z)\xi$ for all $\varrho,\xi,z \in \mathfrak{S}$. The concept of symmetric bi-derivation in rings was introduced by G. Maksa [16]. Suppose n is a fixed positive integer and $\mathfrak{S}^n = \mathfrak{S}\times\mathfrak{S}\times\cdots\times\mathfrak{S}$. A map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be symmetric(permuting) if the relation $\mathfrak{D}(\varrho_1,\varrho_2,\ldots,\varrho_n) = \mathfrak{D}\left(\varrho_{\pi(1)},\varrho_{\pi(2)},\ldots,\varrho_{\pi(n)}\right)$ holds for all $\varrho_i \in \mathfrak{S}$ and for every permutation $\{\pi(1),\pi(2),\ldots,\pi(n)\}$. The concept of derivation and symmetric bi-derivation was generalized by Park [18] as follows: a permuting map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be a permuting n-derivation if $\mathfrak{D}$ is n-additive (i.e.; additive in each coordinate) and $\mathfrak{D}(\varrho_1,\varrho_2,\cdots,\varrho_i\varrho_i',\cdots,\varrho_n) = \varrho_i\mathfrak{D}(\varrho_1,\varrho_2,\cdots,\varrho_i',\cdots,\varrho_n) + \mathfrak{D}(\varrho_1,\varrho_2,\cdots,\varrho_i,\cdots,\varrho_n)\varrho_i'$ holds for all

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$\varrho_i, \varrho_i' \in \mathfrak{S}$. A 1-derivation is a derivation and a 2-derivation is a symmetric bi-derivation while a 3-derivation is known as permuting tri-derivation (viz., [2,7,12,17,23–25]). Let $n \geq 2$ be a fixed integer and a map $\delta : \mathfrak{S} \to \mathfrak{S}$ defined by $\delta(\varrho) = \mathfrak{D}(\varrho, \varrho, \ldots, \varrho)$ for all $\varrho \in \mathfrak{S}$, where $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is a permuting map, be the trace of $\mathfrak{D}$. If $\mathfrak{D}$ is symmetric and n-additive, then the trace d of $\mathfrak{D}$ satisfies the relation $d(\varrho+\xi) = d(\varrho) + d(\xi) + \sum_{t=1}^{n-1} {}^nC_t\, \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{\xi, \ldots, \xi}_{t-\text{times}})$ for all $\varrho, \xi \in \mathfrak{S}$."

Motivated by the concept of generalized derivation in ring, Ashraf et al. [12] introduced the notion of permuting generalized n-derivation in ring. Let $n \geq 1$ be a fixed positive integer. A permuting n-additive map $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ is known to be permuting generalized n-derivation if there exists a permuting n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ such that $\mathcal{G}(\varrho_1, \varrho_2, \ldots, \varrho_i\varrho_i', \ldots, \varrho_n) = \mathcal{G}(\varrho_1, \varrho_2, \ldots, \varrho_i, \ldots, \varrho_n)\, \varrho_i' + \varrho_i \mathfrak{D}(\varrho_1, \varrho_2, \ldots, \varrho_i', \ldots, \varrho_n)$ holds for all $\varrho_i, \varrho_i' \in \mathfrak{S}$. In fact, in [12], the authors proved that "for a fixed positive integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring admitting a permuting generalized n-derivation Ω with associated n-derivation $\mathfrak{D}$ such that the trace ω of Ω is centralizing on $\mathfrak{S}$. Then ω is commuting on $\mathfrak{S}$". Also, in [11], Ashraf et al. have characterized the traces of permuting generalized n-derivations. In fact, their result was motivated by the result due to Hvala [15]. Basically, they proved that "for a fixed positive integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free prime ring. Suppose that ω_1 and ω_2 are the traces of permuting generalized n-derivations Ω_1, Ω_2 respectively and $\delta_1 \neq 0$; δ_2 are the traces of associated derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. If $\omega_1(\varrho)\omega_2(\xi) = \omega_2(\varrho)\omega_1(\xi)$ holds for all $\varrho, \xi \in \mathfrak{S}$, then there exists $\gamma \in C$, the extended centroid of $\mathfrak{S}$ such that $\delta_2(\varrho) = \gamma\delta_1(\varrho)$."

Many researchers have extensively examined a wide range of identities involving traces of n-derivations, leading to the discovery of various interesting results (see, for example [1,2,10–12,19] and the associated references). Very recently, Ali et al. [2], explored some algebraic identities associated with the trace of symmetric n-derivations acting on prime ideal $\mathfrak{P}$ of $\mathfrak{S}$, but without imposing the assumption of primeness on the ring under consideration. In fact, apart from proving some other interesting results, they extended the famous result [20, Theorem 2] for the trace of symmetric n-derivations which involves prime ideals. Precisely, they proved that for any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If there exists a non-zero symmetric n-derivation $\mathfrak{D}$ with trace $\mathscr{d}$ on $\mathfrak{S}$ such that $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$, for all $\varrho \in \mathfrak{S}$, then either $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$.

The main purpose of our current research is to delve into the structure of the quotient ring $\mathfrak{S}/\mathfrak{P}$ where $\mathfrak{S}$ is any ring and $\mathfrak{P}$ is a prime ideal of $\mathfrak{S}$ which admits symmetric generalized n-derivations satisfying certain algebraic identities acting on prime ideals $\mathfrak{P}$. In particular, we prove that if a ring $\mathfrak{S}$ admits a symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying any one of the following functional identities:

(i) $[g(\varrho), \varrho] \in \mathfrak{P}$
(ii) $g([\varrho, \xi]) \pm [\mathscr{d}(\varrho), \xi] \in \mathfrak{P}$
(iii) $g(\varrho \circ \xi) \pm [\mathscr{d}(\varrho), \xi] \in \mathfrak{P}$
(iv) $g([\varrho, \xi]) \pm \mathscr{d}(\varrho) \circ \xi \in \mathfrak{P}$
(v) $[g(\varrho), g(\xi)] \pm [g(\varrho), \xi] \pm [\varrho, g(\xi)] \in \mathfrak{P}$
(vi) $[g(\varrho), g(\xi)] \pm g([\varrho, \xi]) \pm [\mathscr{d}(\varrho), \xi] \in \mathfrak{P}$,

for all $\varrho, \xi \in \mathfrak{S}$, then one of the following holds:

(1) $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$
(2) $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

2. The Results

The following auxiliary results are essential for proving the above mentioned results:

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Lemma 1. " ([19, Lemma 2.3]) For a fixed positive integer n, let $\mathcal{R}$ be a ring and $\mathcal{P}$ be a prime ideal of $\mathcal{R}$, such that $\mathfrak{S}/\mathcal{P}$ is $n!$-torsion free. Suppose that $\ell_1, \ell_2, \ldots, \ell_n \in \mathcal{R}$ satisfy $\alpha\ell_1 + \alpha^2\ell_2 + \cdots + \alpha^n\ell_n \in \mathcal{P}$ for $\alpha = 1, 2, \ldots, n$. Then $\ell_t \in \mathcal{P}$ for $t = 1, 2, \ldots, n$.

Lemma 2. [21] Let $\mathcal{R}$ be a ring and $\mathcal{P}$ be a prime ideal of $\mathcal{R}$. If one of the following conditions is satisfied, then $\mathcal{R}/\mathcal{P}$ is a commutative integral domain.

1. $[\varrho, \xi] \in \mathcal{P} \ \forall \ \varrho, \xi \in \mathcal{R}$
2. $\varrho \circ \xi \in \mathcal{P} \ \forall \ \varrho, \xi \in \mathcal{R}$.

Lemma 3. ([2, Theorem 1.4]) For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a nonzero symmetric n-derivation on $\mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$. If $[d(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$, then $d(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain."

Our first main result establishes a link between the derivation and symmetric generalized n-derivation. In simpler terms, we demonstrate the following result:

Theorem 4. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. Let $\mathcal{G} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ be a nonzero symmetric generalized n-derivation with associated symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with traces $g : \mathfrak{S} \longrightarrow \mathfrak{S}$ of $\mathcal{G}$ and $d : \mathfrak{S} \longrightarrow \mathfrak{S}$ of $\mathfrak{D}$. Next, let $\delta : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a derivation on $\mathfrak{S}$. If $[\delta(\varrho), \varrho] - g(\varrho) \in \mathfrak{P} \ \forall \ \varrho \in \mathfrak{S}$, then we have one of the following assertions:*

1. *$\delta(\mathfrak{S}) \subseteq \mathfrak{P}$*
2. *$\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. By the assumption, we have

$$[\delta(\varrho), \varrho] - g(\varrho) \in \mathfrak{P} \ \forall \ \varrho \in \mathfrak{S}. \tag{1}$$

Replacing ϱ by $\varrho + m\xi$ for $1 \leq m \leq n-1$, $\xi \in \mathfrak{S}$, we get

$$[\delta(\varrho + m\xi), \varrho + m\xi] - g(\varrho + m\xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{2}$$

Continuing to solve, we obtain

$$\begin{aligned}[\delta(\varrho), \varrho] + [\delta(\varrho), m\xi] + [\delta(m\xi), \varrho] + [\delta(m\xi), m\xi] - g(\varrho) - \\ g(m\xi) - \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{m\xi, \ldots, m\xi}_{t-\text{times}}) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}.\end{aligned} \tag{3}$$

Application of relation (1) yields that

$$m[\delta(\varrho), \xi] + m[\delta(\xi), \varrho] - \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{m\xi, \ldots, m\xi}_{t-\text{times}}) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}.$$

This can be written as

$$mA_1(\varrho, \xi) + m^2A_2(\varrho, \xi) + \cdots + m^{n-1}A_{n-1}(\varrho, \xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S},$$

where $A_t(\varrho, \xi)$ represents the term in which ξ appears t-times.

On taking account of Lemma 1, we get

$$[\delta(\varrho), \xi] + [\delta(\xi), \varrho] - n\mathcal{G}(\varrho, \ldots, \varrho, \xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{4}$$

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Substitute $\xi\varrho$ for ξ, we see that

$$[\delta(\varrho), \xi\varrho] + [\delta(\xi\varrho), \varrho] - n\mathcal{G}(\varrho, \ldots, \varrho, \xi\varrho) \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}$$

which on solving, we get

$$2\xi[\delta(\varrho), \varrho] - n\xi\mathscr{g}(\varrho) + [\xi, \varrho]\delta(\varrho) + \{[\delta(\varrho), \xi] + [\delta(\xi), \varrho] - n\mathcal{G}(\varrho, \ldots, \varrho, \xi)\}\varrho - n\xi\mathscr{g}(\varrho) \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

By (4), we have

$$2\xi[\delta(\varrho), \varrho] - n\xi\mathscr{g}(\varrho) + [\xi, \varrho]\delta(\varrho) - n\xi\mathscr{g}(\varrho) \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}. \tag{5}$$

Replace ξ by $r\xi$ in (5) and use (5) to get

$$[r, \varrho]\xi\delta(\varrho) \in \mathfrak{P} \,\forall\, \varrho, \xi, r \in \mathfrak{S}$$

or

$$[r, \varrho]\mathfrak{S}\delta(\varrho) \in \mathfrak{P} \,\forall\, \varrho, r \in \mathfrak{S}.$$

Considering the primeness of the ideal $\mathfrak{P}$, we get for all $\varrho \in \mathfrak{S}$

$$\text{either } [r, \varrho] \in \mathfrak{P} \text{ or } \delta(\varrho) \in \mathfrak{P}.$$

Consequently, $\mathfrak{S}$ is a union of two of its proper subgroups H_1 and H_2, where

$$H_1 = \{\varrho \in \mathfrak{S} \mid [r, \varrho] \in \mathfrak{P}\} \text{ and } H_2 = \{\varrho \in \mathfrak{S} \mid \delta(\mathfrak{S}) \subseteq \mathfrak{P}\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to conclude that either $\mathfrak{S} = H_1$ or $\mathfrak{S} = H_2$. Consider the first case, $\mathfrak{S} = H_1$, i.e., $[r, \varrho] \in \mathfrak{P}$. Using Lemma 2, we conclude that $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain. By the second case, we have $\delta(\mathfrak{S}) \subseteq \mathfrak{P}$. □

Theorem 5. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{g} : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $[g(\varrho), \varrho] \in \mathfrak{P} \,\forall\, \varrho \in \mathfrak{S}$, then one of the following holds:*

1. $\mathscr{g}(\mathfrak{S}) \subseteq \mathfrak{P}$
2. *$\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.*

Proof. We have

$$[g(\varrho), \varrho] \in \mathfrak{P} \,\forall\, \varrho \in \mathfrak{S}. \tag{6}$$

Replace ϱ by $\varrho + m\xi$ for $\xi \in \mathfrak{S}$ and $1 \leq m \leq n-1$ leads to

$$[g(\varrho + m\xi), (\varrho + m\xi)] \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}.$$

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As a consequence obtaining

$$[g(\varrho),\varrho]+[g(\varrho),m\xi]+[g(m\xi),\varrho]+[g(m\xi),m\xi]+ \left[\sum_{t=1}^{n-1}{}^{n}C_t\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),\varrho\right]+ \left[\sum_{t=1}^{n-1}{}^{n}C_t\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),m\xi\right]\in\mathfrak{P}$$

$\forall\ \varrho,\xi\in\mathfrak{S}$. Using the relation (6), we obviously find that

$$m[g(\varrho),\xi]+m^n[g(\xi),\varrho]+\left[\sum_{t=1}^{n-1}{}^{n}C_t\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),\varrho\right]+ \left[\sum_{t=1}^{n-1}{}^{n}C_t\mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{m\xi,\ldots,m\xi}_{t-\text{times}}),m\xi\right]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$$

and thus,

$$mA_1(\varrho;\xi)+m^2A_2(\varrho;\xi)+\cdots+m^nA_n(\varrho;\xi)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S},$$

where $A_t(\varrho;\xi)$ represents the term in which ξ appears t-times.

The application of Lemma 1 yields

$$[g(\varrho),\xi]+n[\mathcal{G}(\varrho,\ldots,\varrho,\xi),\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}. \tag{7}$$

Replacing ξ by $\xi\varrho$, we can see that

$$[g(\varrho),\xi\varrho]+n[\mathcal{G}(\varrho,\ldots,\varrho,\xi\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

After additional computation

$$\{[g(\varrho),\xi]+n[\mathcal{G}(\varrho,\ldots,\varrho,\xi),\varrho]\}\varrho+\xi[g(\varrho),\varrho]+n\xi[d(\varrho),\varrho]+n[\xi,\varrho]d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

By using (7) and using the hypothesis, we get

$$n\xi[d(\varrho),\varrho]+n[\xi,\varrho]d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we get

$$\xi[d(\varrho),\varrho]+[\xi,\varrho]d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}. \tag{8}$$

Again replacing ξ by ξz and using (8), we obtain

$$[z,\varrho]\xi d(\varrho)\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}. \tag{9}$$

Next, replace ϱ by $\varrho+mw$ for $1\leq m\leq n-1$, $w\in\mathfrak{S}$, to get

$$[z,\varrho+mw]zd(\varrho+mw)\in\mathfrak{P}\ \forall\ \varrho,\xi,w,z\in\mathfrak{S}.$$

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After simplification, we find that

$$[z,\varrho]\xi d(\varrho)+[z,mw]\xi d(\varrho)+[z,\varrho]\xi d(mw)+[z,mw]\xi d(mw)+ [z,\varrho]\xi\sum_{t=1}^{n-1}{}^{n}C_{t}\mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mw,\ldots,mw}_{t-\text{times}})+ [z,mw]\xi\sum_{t=1}^{n-1}{}^{n}C_{t}\mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mw,\ldots,mw}_{t-\text{times}})\in\mathfrak{P}$$

$\forall\ \varrho,\xi,w,z\in\mathfrak{S}$. Application of (9) and Lemma 1 gives

$$[z,w]\xi d(\varrho)+[z,\varrho]\xi\mathfrak{D}(\varrho,\ldots,\varrho,w)\in\mathfrak{P}\ \forall\ \varrho,\xi,w\in\mathfrak{S}. \tag{10}$$

Putting wz instead of z in (10) and using (10), we can see that

$$[w,\varrho]zy\mathfrak{D}(w,\varrho,\ldots,\varrho)\in\mathfrak{P}\ \forall\ \varrho,y,w,z\in\mathfrak{S},$$

or

$$[w,\varrho]\mathfrak{S}\mathfrak{D}(w,\varrho,\ldots,\varrho)\subseteq\mathfrak{P}\ \forall\ \varrho,w\in\mathfrak{S}.$$

On taking account of primeness of $\mathfrak{P}$, we get for all $w\in\mathfrak{S}$

$$\text{either } [w,\varrho]\in\mathfrak{P} \text{ or } \mathfrak{D}(w,\varrho,\ldots,\varrho)\in\mathfrak{P}.$$

Consequently, $\mathfrak{S}$ is a union of two of its proper subgroups H_1 and H_2, where

$$H_1=\{w\in\mathfrak{S}\mid[w,\mathfrak{S}]\subseteq\mathfrak{P}\} \text{ and } H_2=\{w\in\mathfrak{S}\mid\mathfrak{D}(w,\varrho,\ldots,\varrho)\in\mathfrak{P}\}.$$

Since a group cannot be a union of two of its proper subgroups, we can only deduce that either $\mathfrak{S}=H_1$ or $\mathfrak{S}=H_2$. Consider the second case, $\mathfrak{S}=H_2$, i.e., $\mathfrak{D}(w,\varrho,\ldots,\varrho)\in\mathfrak{P}$. Replace w by $w\varrho$ to obtain $w\mathfrak{D}(\varrho,\ldots,\varrho)\in\mathfrak{P}$. This implies that $wd(\varrho)\in\mathfrak{P}$ for all $w,\varrho\in\mathfrak{S}$, i.e., $\mathfrak{S}d(\varrho)\subseteq\mathfrak{P}$. Since $\mathfrak{P}\neq\mathfrak{S}$, then $d(\varrho)\in\mathfrak{P}$ for all $\varrho\in\mathfrak{S}$, i.e., $d(\mathfrak{S})\subseteq\mathfrak{P}$. If $\mathfrak{S}=H_1$, then using Lemma 2, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. □

Theorem 6. *For any fixed integer $n\geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $g:\mathfrak{S}\to\mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\longrightarrow\mathfrak{S}$ with trace $d:\mathfrak{S}\to\mathfrak{S}$ satisfying any one of the following conditions:*

1. $g([\varrho,\xi])\pm[d(\varrho),\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$
2. $g(\varrho\circ\xi)\pm[d(\varrho),\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$

Then we have one of the following assertions:

1. $d(\mathfrak{S})\subseteq\mathfrak{P}$
2. *$\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. (i) It is given that

$$g([\varrho,\xi])\pm[d(\varrho),\xi]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.$$

Replacing ξ by $\xi+mz$ for $1\leq m\leq n-1$ and $z\in\mathfrak{S}$ in above, we get

$$g([\varrho,\xi+mz])\pm[d(\varrho),\xi+mz]\in\mathfrak{P}\ \forall\ \varrho,\xi,z\in\mathfrak{S}.$$

After simplifying the expression, we get

$$g([\varrho,\xi]) + g([\varrho,mz]) + \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{[\varrho,\xi],\ldots,[\varrho,\xi]}_{(n-t)-\text{times}},\underbrace{[\varrho,mz],\ldots,[\varrho,mz]}_{t-\text{times}}) \pm [\mathscr{d}(\varrho),\xi] \pm [\mathscr{d}(\varrho),mz] \in \mathfrak{P}$$

$\forall\ \varrho,\xi,z \in \mathfrak{S}$. On using the given condition, we obtain

$$\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{[\varrho,\xi],\ldots,[\varrho,\xi]}_{(n-t)-\text{times}},\underbrace{[\varrho,mz],\ldots,[\varrho,mz]}_{t-\text{times}}) \in \mathfrak{P}\ \forall\ \varrho,\xi,z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho,\xi,z) + m^2A_2(\varrho,\xi,z) + \cdots + m^{n-1}A_{n-1}(\varrho,\xi,z) \in \mathfrak{P}$$

$\forall\ \varrho,\xi,z \in \mathfrak{S}$ where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times.

In view of Lemma 1 and torsion restriction, we have

$$\mathcal{G}([\varrho,\xi],\ldots,[\varrho,\xi],[\varrho,z]) \in \mathfrak{P}\ \forall\ \varrho,\xi,z \in \mathfrak{S}.$$

Replacing z by ξ and using the given condition, we find that $[\mathscr{d}(\varrho),\xi] \in \mathfrak{P}$ for all $\varrho,\xi \in \mathfrak{S}$. In particular, $[\mathscr{d}(\varrho),\varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus by Lemma 3, $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$.

(ii) Proceeding in the same way as in (i), we conclude. □

Theorem 7. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying any one of the following conditions:*

1. $g(\varrho \circ \xi) \pm \mathscr{d}(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}$
2. $g([\varrho,\xi]) \pm \mathscr{d}(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}$

Then we have one of the following assertions:

1. $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$
2. *$\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. (i) Assume that

$$g(\varrho \circ \xi) \pm \mathscr{d}(\varrho) \circ \xi \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}.$$

On replacing ξ by $\xi + mz$, for $1 \leq m \leq n-1$, we get

$$g(\varrho \circ \xi + mz) \pm \mathscr{d}(\varrho) \circ (\xi + mz) \in \mathfrak{P}\ \forall\ \varrho,\xi,z \in \mathfrak{S}.$$

By simplifying, we find

$$g(\varrho \circ \xi) + g(\varrho \circ mz) + \sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\varrho \circ \xi,\ldots,\varrho \circ \xi}_{(n-t)-\text{times}},\underbrace{\varrho \circ mz,\ldots,\varrho \circ mz}_{t-\text{times}}) \pm \mathscr{d}(\varrho) \circ \xi \pm \mathscr{d}(\varrho) \circ mz\ \in\ \mathfrak{P}$$

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$\forall\ \varrho, \xi, z \in \mathfrak{S}$. By applying the provided condition, we obtain the following:

$$\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\varrho\circ\xi,\ldots,\varrho\circ\xi}_{(n-t)-\text{times}},\underbrace{\varrho\circ mz,\ldots,\varrho\circ mz}_{t-\text{times}}) \in \mathfrak{P}\ \forall\ \varrho,\xi,z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho,\xi,z) + m^2A_2(\varrho,\xi,z) + \cdots + m^{n-1}A_{n-1}(\varrho,\xi,z) \in \mathfrak{P}$$

$\forall\ \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times.

Taking into account of Lemma 1 and torsion restriction, we find that

$$\mathcal{G}(\varrho\circ\xi,\ldots,\varrho\circ\xi,\varrho\circ z) \in \mathfrak{P}\ \forall\ \varrho,\xi,z \in \mathfrak{S}.$$

Replacing z by ξ, we get $g(\varrho\circ\xi) \in \mathfrak{P}$, then our hypothesis reduces to $\delta(\varrho)\circ\xi \in \mathfrak{P}$. Replace ξ by ξr to get $[\delta(\varrho),\xi]r + \xi(\delta(\varrho)\circ r) \in \mathfrak{P}$ and hence we get $[\delta(\varrho),\xi]r \in \mathfrak{P}$ for all $\varrho,\xi,r \in \mathfrak{S}$, i.e., $[\delta(\varrho),\xi]\mathfrak{S} \subseteq \mathfrak{P}$. Since $\mathfrak{P} \neq \mathfrak{S}$, then $[\delta(\varrho),\xi] \in \mathfrak{P}$. In particular, $[\delta(\varrho),\varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus by, [2, Theorem 1.4], $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain or $\delta(\mathfrak{S}) \subseteq \mathfrak{P}$.

(ii) Proceeding in the same way as in (i), we conclude. □

Theorem 8. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\delta : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $[g(\varrho), g(\xi)] \pm [g(\varrho), \xi] \pm [\varrho, g(\xi)] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$, then either $\delta(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. It is provided that

$$[g(\varrho),g(\xi)] \pm [g(\varrho),\xi] \pm [\varrho,g(\xi)] \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}.$$

Substitute $\xi + mz$ in place of ξ for $1 \leq m \leq n-1$ to get

$$\begin{aligned}[g(\varrho),g(\xi)] + [g(\varrho),g(mz)] + [g(\varrho),\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})] \\ \pm [g(\varrho),\xi] \pm [g(\varrho),mz] \pm [\varrho,g(\xi)] \pm [\varrho,g(mz)] \\ \pm [\varrho,\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}}) \in \mathfrak{P}\end{aligned}$$

$\forall\ \varrho, \xi, z \in \mathfrak{S}$. Through the utilization of the given condition, we get

$$[g(\varrho),\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})] \pm [\varrho,\sum_{t=1}^{n-1} {}^nC_t\mathcal{G}(\underbrace{\xi,\ldots,\xi}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}})] \in \mathfrak{P}$$

which implies that

$$mA_1(\varrho,\xi,z) + m^2A_2(\varrho,\xi,z) + \cdots + m^{n-1}A_{n-1}(\varrho,\xi,z) \in \mathfrak{P}$$

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$\forall\, \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

In the context of Lemma 1 and torsion restriction, we get

$$[g(\varrho), \mathcal{G}(\xi, \ldots, \xi, z)] \pm [\varrho, \mathcal{G}(\xi, \ldots, \xi, z) \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Replacing z by ξ and using the given condition, we find that $[g(\varrho), \xi] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. In particular, $[g(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus by, Theorem 5, $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$. □

Corollary 9. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $[\mathscr{d}(\varrho), \mathscr{d}(\xi)] \pm [\mathscr{d}(\varrho), \xi] \pm [\varrho, \mathscr{d}(\xi)] \in \mathfrak{P}\ \forall\, \varrho, \xi \in \mathfrak{S}$, then either $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Theorem 10. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying the condition $[g(\varrho), g(\xi)] \pm g([\varrho, \xi]) \pm [\mathscr{d}(\varrho), \xi] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$, then one of the following assertions hold:*

1. $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$
2. *$\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. Assume that

$$[g(\varrho), g(\xi)] \pm g([\varrho, \xi]) \pm [\mathscr{d}(\varrho), \xi] \in \mathfrak{P}\ \forall\, \varrho, \xi \in \mathfrak{S}.$$

Replace ξ by $\xi + mz$, for $1 \leq m \leq n-1$ to get

$$[g(\varrho), g(\xi + mz)] \pm g([\varrho, \xi + mz]) \pm [\mathscr{d}(\varrho), \xi + mz] \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

After simplifying, it becomes

$$[g(\varrho), g(\xi)] + [g(\varrho), g(mz)] + [g(\varrho), \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})]$$
$$\pm g([\varrho, \xi]) \pm g([\varrho, mz]) \pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{[\varrho, \xi], \ldots, [\varrho, \xi]}_{(n-t)-\text{times}}, \underbrace{[\varrho, mz], \ldots, [\varrho, mz]}_{t-\text{times}}) \pm$$
$$[\mathscr{d}(\varrho), \xi] \pm [\mathscr{d}(\varrho), mz] \in \mathfrak{P}$$

$\forall\, \varrho, \xi, z \in \mathfrak{S}$. By employing the provided condition, we obtain

$$[g(\varrho), \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})] \pm$$
$$\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{[\varrho, \xi], \ldots, [\varrho, \xi]}_{(n-t)-\text{times}}, \underbrace{[\varrho, mz], \ldots, [\varrho, mz]}_{t-\text{times}}) \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

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$\forall\, \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

In light of Lemma 1 and torsion restriction, we get

$$[g(\varrho), \mathcal{G}(\xi, \ldots, \xi, z)] \pm \mathcal{G}([\varrho, \xi], \ldots, [\varrho, \xi], [\varrho, z]) \in \mathfrak{P}\; \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Replacing z by ξ and using the given condition, we find that $[d(\varrho), \xi] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. In particular, $[d(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$. Thus by, [2, Theorem 1.4], $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain or $d(\mathfrak{S}) \subseteq \mathfrak{P}$. □

Theorem 11. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $(n-1)!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying $g(\varrho^2) \pm \varrho^2 \in \mathfrak{P}\; \forall\, \varrho \in \mathfrak{S}$, then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. On replacing ϱ by $\varrho + m\xi$, $\xi \in \mathfrak{S}$ for $1 \leq m \leq n-1$ in the given condition, we get

$$g(\varrho + m\xi)^2 \pm (\varrho + m\xi)^2 \in \mathfrak{P}\; \forall\, \varrho, \xi \in \mathfrak{S}.$$

Further solving, we have

$$g(\varrho^2) + g(m(\varrho\xi + \xi\varrho)) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho^2, \ldots, \varrho^2}_{(n-t)-\text{times}}, \underbrace{m(\varrho\xi + \xi\varrho), \ldots, m(\varrho\xi + \xi\varrho)}_{t-\text{times}}) +$$
$$g((m\xi)^2) + \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho^2 + m(\varrho\xi + \xi\varrho), \ldots, \varrho^2 + m(\varrho\xi + \xi\varrho)}_{(n-t)-\text{times}}, \underbrace{(m\xi)^2, \ldots, (m\xi)^2}_{t-\text{times}})$$
$$\pm \varrho^2 \pm (m\xi)^2 \pm m(\varrho\xi + \xi\varrho) \in \mathfrak{P}\; \forall\, \varrho, \xi \in \mathfrak{S}.$$

In accordance of the given condition and Lemma 1, we get

$$n\mathcal{G}(\varrho^2, \ldots, \varrho^2, \varrho\xi + \xi\varrho) \pm (\varrho\xi + \xi\varrho) \in \mathfrak{P}\; \forall\, \varrho, \xi \in \mathfrak{S}.$$

Replacing ξ by ϱ, we find that

$$2ng(\varrho^2) \pm 2\varrho^2 \in \mathfrak{P},$$

or

$$(2n-2)\varrho^2 \in \mathfrak{P}.$$

The application of the torsion restriction gives that $\varrho^2 \in \mathfrak{P}$. This implies that $\varrho\xi + \xi\varrho \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. Replacing ξ by ξz, we get $[\varrho, \xi]z \in \mathfrak{P}$. Again replacing z by $r[\varrho, \xi]$, we get $[\varrho, \xi]r[\varrho, \xi] \in \mathfrak{P}$ for all $\varrho, \xi, r \in \mathfrak{S}$. Using the primeness of $\mathfrak{P}$, we get $[\varrho, \xi] \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. Hence $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain by Lemma 2. □

Theorem 12. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

1. $[g(\varrho), g(\xi)] - [\varrho, \xi] \in \mathfrak{P}\; \forall\, \varrho, \xi \in \mathfrak{S}$
2. $[g(\varrho), g(\xi)] - [\xi, \varrho] \in \mathfrak{P}\; \forall\, \varrho, \xi \in \mathfrak{S}$.

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

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Proof. (i) Given that

$$[g(\varrho), g(\xi)] - [\varrho, \xi] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}. \tag{11}$$

Consider a positive integer m; $1 \leq m \leq n-1$. Replacing ξ by $\xi + mz$, where $z \in \mathfrak{S}$ in (11), we get

$$[g(\varrho), g(\xi + mz)] - [\varrho, \xi + mz] \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}.$$

On further solving, we get

$$[g(\varrho), g(\xi)] + [g(\varrho), g(mz)] + [g(\varrho), \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})] - [\varrho, \xi] - [\varrho, mz] \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}.$$

On taking account of hypothesis, we see that

$$[g(\varrho), \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})] \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}$$

which results in

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

Using Lemma 1 and torsion restriction, we have

$$[g(\varrho), \mathcal{G}(\xi, \ldots, \xi, z)] \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}.$$

In particular, for $z = \xi$, we get

$$[g(\varrho), g(\xi)] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}.$$

Now using the given condition, we find that

$$[\varrho, \xi] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}.$$

From Lemma 2, $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

(ii) Follows from the first implication with a slight modification. □

Following are the very interesting observations derived from Theorem 12.

Corollary 13. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:

1. $g(\varrho)g(\xi) \pm \varrho\xi \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$
2. $g(\varrho)g(\xi) \pm \xi\varrho \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$.

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Corollary 14. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:

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1. $[g(\varrho), g(\xi)] = [\varrho, \xi] \;\forall\; \varrho, \xi \in \mathfrak{S}$
2. $[g(\varrho), g(\xi)] = [\xi, \varrho] \;\forall\; \varrho, \xi \in \mathfrak{S}$.

   Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Proof. (i) Let us assume that

$$[g(\varrho), g(\xi)] + [\varrho, \xi] = 0 \;\forall\; \varrho, \xi \in \mathfrak{S}.$$

According to semiprimeness, there exists a family Γ of prime ideals $\mathfrak{P}$ such that $\bigcap_{\mathfrak{P}\in\Gamma} \mathfrak{P} = (0)$ thereby obtaining $[g(\varrho), g(\xi)] + [\varrho, \xi] \in \mathfrak{P}$ for all $\mathfrak{P} \in \Gamma$. Invoking the previous theorem, we conclude that $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain. Therefore, for all $\varrho, \xi \in \mathfrak{S}$, we have $[\varrho, \xi] \in \mathfrak{P}$ and since $\bigcap_{\mathfrak{P}\in\Gamma} \mathfrak{P} = (0)$. This implies that $[\varrho, \xi] = 0$. Hence, $\mathfrak{S}$ is commutative.

(ii) Similarly, if $[g(\varrho), g(\xi)] - [\xi, \varrho] = 0$ for all $\varrho, \xi \in \mathfrak{S}$, then the same reasoning proves the required result. □

Corollary 15. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a n-derivation of $\mathfrak{S}$ with trace $\ell : \mathfrak{S} \to \mathfrak{S}$ satisfying any one of the following conditions:

1. $[\ell(\varrho), \ell(\xi)] = [\varrho, \xi] \;\forall\; \varrho, \xi \in \mathfrak{S}$
2. $[\ell(\varrho), \ell(\xi)] = [\xi, \varrho] \;\forall\; \varrho, \xi \in \mathfrak{S}$

   Then $\mathfrak{S}$ is commutative.

Theorem 16. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and P be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\ell : \mathfrak{S} \to \mathfrak{S}$ satisfying $g(\varrho \circ \xi) \pm \varrho \circ \xi \in \mathfrak{P}$ $\forall\; \varrho, \xi \in \mathfrak{S}$ then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. The given condition is that

$$g(\varrho \circ \xi) \pm \varrho \circ \xi \in \mathfrak{P} \;\forall\; \varrho, \xi \in \mathfrak{S}.$$

Put $\xi + mz$ in place of ξ, where $1 \leq m \leq n-1$ to get

$$g(\varrho \circ (\xi + mz)) \pm \varrho \circ (\xi + mz) \in \mathfrak{P} \;\forall\; \varrho, \xi, z \in \mathfrak{S}.$$

Upon simplifying, we arrive at

$$g(\varrho \circ \xi) + g(\varrho \circ mz) + \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\varrho \circ \xi, \ldots, \varrho \circ \xi}_{(n-t)-\text{times}}, \underbrace{\varrho \circ mz, \ldots, \varrho \circ mz}_{t-\text{times}}) \pm \varrho \circ \xi \pm \varrho \circ mz \in \mathfrak{P}$$

$\forall\; \varrho, \xi, z \in \mathfrak{S}$. In light of the given condition, we obtain

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\varrho \circ \xi, \ldots, \varrho \circ \xi}_{(n-t)-\text{times}}, \underbrace{\varrho \circ mz, \ldots, \varrho \circ mz}_{t-\text{times}}) \in \mathfrak{P} \;\forall\; \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\; \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

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Because of Lemma 1 and torsion restriction, we get

$$\mathcal{G}(\varrho \circ \xi, \ldots, \varrho \circ \xi, \varrho \circ z) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Replacing z by ξ and using the given condition, we find that $\varrho \circ \xi \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}$ i.e., $\mathfrak{S} \circ \mathfrak{S} \subseteq \mathfrak{P}$. Using the Lemma 2, we get $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain. □

Corollary 17. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$- torsion free semiprime ring. If $\mathfrak{S}$ admits a nonzero permuting n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying $d(\varrho \circ \xi) \pm \varrho \circ \xi = 0, \forall$ $\varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}$ is commutative.

Theorem 18. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying any one of the following conditions:*

1. $g([\varrho, \xi]) \pm \varrho \circ \xi \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}$
2. $g(\varrho \circ \xi) \pm [\varrho, \xi] \in \mathfrak{P} \,\forall\, \varrho, \xi \in \mathfrak{S}$,

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Proof. (i) Replacing ξ by $\xi + mz$ for $1 \leq m \leq n-1$, $z \in \mathfrak{S}$ in the given condition, we get

$$g([\varrho, \xi + mz]) \pm \varrho \circ (\xi + mz) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}.$$

After simplification, it becomes

$$g([\varrho, \xi]) + g([\varrho, mz]) + \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{[\varrho, \xi], \ldots, [\varrho, \xi]}_{(n-t)-\text{times}}, \underbrace{[\varrho, mz], \ldots, [\varrho, mz]}_{t-\text{times}})$$
$$\pm \varrho \circ \xi \pm \varrho \circ mz \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Using the specified condition, we get

$$\sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{[\varrho, \xi], \ldots, [\varrho, \xi]}_{(n-t)-\text{times}}, \underbrace{[\varrho, mz], \ldots, [\varrho, mz]}_{t-\text{times}}) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\, \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times. Using Lemma 1 and using the fact that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we get

$$\mathcal{G}([\varrho, \xi], \ldots, [\varrho, \xi], [\varrho, z]) \in \mathfrak{P} \,\forall\, \varrho, \xi, z \in \mathfrak{S}. \tag{12}$$

For $z = \xi$, we get $g([\varrho, \xi]) \in \mathfrak{P}$ then our hypothesis reduces to $\varrho \circ \xi \in \mathfrak{P}$. Using the Lemma 2, we get $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

(ii) Proceeding in the same way as in (i), we conclude. □

Corollary 19. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation with trace $d : \mathfrak{S} \to \mathfrak{S}$. If d satisfying any one of the following conditions:

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1. $\mathscr{d}([\varrho,\xi]) \pm \varrho \circ \xi \in \mathfrak{P} \, \forall \, \varrho, \xi \in \mathfrak{S}$
2. $\mathscr{d}(\varrho \circ \xi) \pm [\varrho,\xi] \in \mathfrak{P} \, \forall \, \varrho, \xi \in \mathfrak{S}$

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Corollary 20. [9, Theorem 2.10] For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$- torsion free semiprime ring. If $\mathfrak{S}$ admits a nonzero permuting n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying $\mathscr{d}([\varrho,\xi]) = \varrho \circ \xi, \forall \, \varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}$ is commutative.

Theorem 21. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

1. $g([\varrho,\xi]) \pm g(\varrho) \pm [\varrho,\xi] \in \mathfrak{P} \, \forall \, \varrho, \xi \in \mathfrak{S}$
2. $g([\varrho,\xi]) \pm g(\xi) \pm [\varrho,\xi] \in \mathfrak{P} \, \forall \, \varrho, \xi \in \mathfrak{S}$

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Proof. (i) Given that

$$g([\varrho,\xi]) \pm g(\varrho) \pm [\varrho,\xi] \in \mathfrak{P} \, \forall \, \varrho, \xi \in \mathfrak{S}.$$

Replacing ϱ by $\varrho + mz$, where $z \in \mathfrak{S}$ and $1 \leq m \leq n-1$ in the given condition, we get

$$g([\varrho + mz,\xi]) \pm g(\varrho + mz) \pm [\varrho + mz,\xi] \in \mathfrak{P} \, \forall \, \varrho, \xi \in \mathfrak{S}$$

which on solving

$$g([\varrho,\xi]) + g([mz,\xi]) \pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{[\varrho,\xi],\ldots,[\varrho,\xi]}_{(n-t)-\text{times}},\underbrace{[mz,\xi],\ldots,[mz,\xi]}_{t-\text{times}})$$
$$\pm g(\varrho) \pm g(mz) \pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}}) \pm [\varrho,\xi] \pm [mz,\xi] \in \mathfrak{P}$$

$\forall \, \varrho, \xi, z \in \mathfrak{S}$. By using hypothesis, we get

$$\sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{[\varrho,\xi],\ldots,[\varrho,\xi]}_{(n-t)-\text{times}},\underbrace{[mz,\xi],\ldots,[mz,\xi]}_{t-\text{times}})$$
$$\pm \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{mz,\ldots,mz}_{t-\text{times}}) \in \mathfrak{P} \, \forall \, \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho,\xi,z) + m^2 A_2(\varrho,\xi,z) + \cdots + m^{n-1} A_{n-1}(\varrho,\xi,z) \in \mathfrak{P}$$

$\forall \, \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho,\xi,z)$ represents the term in which z appears t-times.

Making use of Lemma 1 and torsion restriction, we see that

$$\mathcal{G}([\varrho,\xi],\ldots,[\varrho,\xi],[z,\xi]) \pm \mathcal{G}(\varrho,\ldots,\varrho,z) \in \mathfrak{P} \, \forall \, \varrho, \xi, z \in \mathfrak{S}.$$

Replace z by ϱ to get

$$g([\varrho,\xi]) \pm g(\varrho) \in \mathfrak{P} \, \forall \, \varrho, \xi, z \in \mathfrak{S}.$$

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Hence, by using the given condition, we find that $[\varrho, \xi] \in \mathfrak{P}$. On taking account of Lemma 2, we get $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

(ii) Follows from the first implication with a slight modification. □

Corollary 22. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:

1. $d([\varrho, \xi]) \pm d(\varrho) \pm [\varrho, \xi] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$
2. $d([\varrho, \xi]) \pm d(\xi) \pm [\varrho, \xi] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Corollary 23. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:

1. $d([\varrho, \xi]) \pm d(\varrho) \pm [\varrho, \xi] = 0 \; \forall \; \varrho, \xi \in \mathfrak{S}$
2. $d([\varrho, \xi]) \pm d(\xi) \pm [\varrho, \xi] = 0 \; \forall \; \varrho, \xi \in \mathfrak{S}$

Then $\mathfrak{S}$ is commutative.

Theorem 24. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

1. $g(\varrho) \circ g(\xi) \pm \varrho \circ \xi \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$
2. $[g(\varrho), g(\xi)] \pm \varrho \circ \xi \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$
3. $g(\varrho) \circ g(\xi) \pm [\varrho, \xi] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$

Then $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Proof. (i) We assume that
$$g(\varrho) \circ g(\xi) \pm \varrho \circ \xi \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}.$$
Replacing ξ by $\xi + mz$, where $z \in \mathfrak{S}$ and $1 \leq m \leq n-1$ in the given condition, we get

$$g(\varrho) \circ g(\xi + mz) \pm \varrho \circ (\xi + mz) \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}$$

which on solving

$$g(\varrho) \circ g(\xi) + g(\varrho) \circ g(mz) + g(\varrho) \circ \sum_{t=1}^{n-1} {}^nC_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})$$
$$\pm \varrho \circ \xi \pm \varrho \circ mz \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}.$$

By using hypothesis, we get

$$g(\varrho) \circ \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}}) \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

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$\forall\, \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

Making use of Lemma 1, we see that

$$n(g(\varrho) \circ \mathcal{G}(\xi, \ldots, \xi, z)) \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we get

$$g(\varrho) \circ \mathcal{G}(\xi, \ldots, \xi, z) \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

In particular, $z = \xi$, we get

$$g(\varrho) \circ g(\xi) \in \mathfrak{P}\ \forall\, \varrho, \xi \in \mathfrak{S}.$$

Hence, by using the given condition, we find that $\varrho \circ \xi \in \mathfrak{P}$. On taking account of Lemma 2, we get $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

(ii) Given that

$$[g(\varrho), g(\xi)] \pm \varrho \circ \xi \in \mathfrak{P}\ \forall\, \varrho, \xi \in \mathfrak{S}.$$

Replace ξ by $\xi + mz$, where $z \in \mathfrak{S}$ and $1 \leq m \leq n-1$, we get

$$[g(\varrho), g(\xi + mz)] \pm \varrho \circ (\xi + mz) \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

After simplification, we obtain

$$[g(\varrho), g(\xi)] + [g(\varrho), g(mz)] + [g(\varrho), \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})]$$
$$\pm\, \varrho \circ \xi \pm \varrho \circ mz \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

By using hypothesis, we get

$$[g(\varrho), \sum_{t=1}^{n-1} {}^{n}C_t \mathcal{G}(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \ldots, mz}_{t-\text{times}})] \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall\, \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times. Application of Lemma 1 gives that

$$n[g(\varrho), \mathcal{G}(\xi, \ldots, \xi, z)] \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we get

$$[g(\varrho), \mathcal{G}(\xi, \ldots, \xi, z)] \in \mathfrak{P}\ \forall\, \varrho, \xi, z \in \mathfrak{S}.$$

In particular, $z = \xi$, we get

$$[g(\varrho), g(\xi)] \in \mathfrak{P}\ \forall\, \varrho, \xi \in \mathfrak{S}.$$

Hence, by using the given condition, we find that $\varrho \circ \xi \in \mathfrak{P}$ and using Lemma 2, we get $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

(iii) Proceeding in the same way as in (i), we conclude. $\square$

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Corollary 25. For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be n-derivation of $\mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$. If $\mathscr{d}$ satisfy any one of the following:

1. $\mathscr{d}(\varrho) \circ \mathscr{d}(\xi) \pm \varrho \circ \xi = 0 \; \forall \; \varrho, \xi \in \mathfrak{S}$
2. $[\mathscr{d}(\varrho), \mathscr{d}(\xi)] \pm \varrho \circ \xi = 0 \; \forall \; \varrho, \xi \in \mathfrak{S}$
3. $\mathscr{d}(\varrho) \circ \mathscr{d}(\xi) \pm [\varrho, \xi] = 0 \; \forall \; \varrho, \xi \in \mathfrak{S}$

Then $\mathfrak{S}$ is commutative.

Theorem 26. *For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \longrightarrow \mathfrak{S}$ satisfying $\mathscr{d}(\varrho)\mathscr{d}(\xi) - \varrho \circ \xi \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S}$ then either $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.*

Proof. Replacing ξ by $\xi + mz$ for $1 \leq m \leq n-1, z \in \mathfrak{S}$ in the hypothesis, we get

$$\mathscr{d}(\varrho)\mathscr{d}(\xi + mz) - \varrho \circ (\xi + mz) \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}.$$

After simplification, it becomes

$$\mathscr{d}(\varrho) \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\xi, \dots, \xi}_{(n-t)-\text{times}}, \underbrace{mz, \dots, mz}_{t-\text{times}}) \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}$$

which implies that

$$mA_1(\varrho, \xi, z) + m^2 A_2(\varrho, \xi, z) + \cdots + m^{n-1} A_{n-1}(\varrho, \xi, z) \in \mathfrak{P}$$

$\forall \; \varrho, \xi, z \in \mathfrak{S}$ where $A_t(\varrho, \xi, z)$ represents the term in which z appears t-times.

Using Lemma 1 and the fact that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free , we have

$$\mathscr{d}(\varrho)\mathfrak{D}(\xi, \dots, \xi, z) \in \mathfrak{P} \; \forall \; \varrho, \xi, z \in \mathfrak{S}. \tag{13}$$

Replace z by zr in above relation and using the above relation, we have

$$\mathscr{d}(\varrho) z \mathfrak{D}(\xi, \dots, \xi, r) \in \mathfrak{P} \; \forall \; \varrho, \xi, z, r \in \mathfrak{S}. \tag{14}$$

Also,

$$z\mathscr{d}(\varrho)\mathfrak{D}(\xi, \dots, \xi, r) \in \mathfrak{P} \; \forall \; \varrho, \xi, z, r \in \mathfrak{S}. \tag{15}$$

Using (14) and (15), we get

$$[\mathscr{d}(\varrho), z]\mathfrak{D}(\xi, \dots, \xi, r) \in \mathfrak{P} \; \forall \; \varrho, \xi, r \in \mathfrak{S}.$$

Writing $r\xi$ instead of r, we get

$$[\mathscr{d}(\varrho), z] r \mathscr{d}(\xi) \in \mathfrak{P}.$$

In particular for $z = \varrho$,

$$[\mathscr{d}(\varrho), \varrho] r \mathscr{d}(\xi) \in \mathfrak{P}.$$

Since $\mathfrak{P}$ is prime, it follows that either $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$ or $\mathscr{d}(\xi) \in \mathfrak{P}$ for all $\varrho, \xi \in \mathfrak{S}$. Using the Lemma 3, we get $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain. □

Corollary 27. [9, Theorem 2.11] For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$- torsion free semiprime ring. If $\mathfrak{S}$ admits a nonzero permuting n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying $\mathscr{d}(\varrho)\mathscr{d}(\xi) = \varrho \circ \xi, \forall \; \varrho, \xi \in \mathfrak{S}$, then $\mathscr{d}$ is commuting on $\mathfrak{S}$.

3. Conclusions

In summary, this research has achieved the successful establishment of a substantial link between the structural properties of quotient rings and the behaviour of traces of symmetric generalized n-derivations fulfilling certain algebraic identities involving prime ideals of an arbitrary ring $\mathfrak{S}$. Through a comprehensive analysis and rigorous investigation, the study has provided persuasive and substantial evidence of the interplay between these fundamental algebraic concepts.

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