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On the Simplest Entangled Mixed States

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Abstract: Quantum entanglement is a fascinating topic with both theoretical and technological impacts. Yet, even in the simplest scenarios, there are still various open questions involving pure state and mixed state entanglement. Here we present the *simplest* entangled mixed states. We consider both qubits (quantum bits) and qudits (quantum digits). We first provide our simplest yet most important result for rank-2 bi-partite states, and we then generalize our results to go beyond rank-2 and to go beyond bi-partite states.

Keywords: entanglement; quantum information; multipartite quantum systems

1. Introduction

Quantum entanglement is a fascinating topic, both as a fundamental concept in physics [1–3], as well as in quantum information, communication, cryptography and computing [4–7]. In particular, entangled mixed states are still providing many challenges with both theoretical [4,8–14] and technological [15] impacts.

Here we present the *simplest* entangled mixed states. We consider both qubits (quantum bits) and qudits (quantum digits), and we clarify the challenge already in the introduction. We provide a fascinating result for rank-2 bi-partite states, in Section 2: The simplest entangled mixed state is simply *any* mixture of a single bipartite pure entangled state and a single bipartite pure product state. We provide extensions beyond rank-2, in the next two sections. Finally in Section 5 we generalize our main result from the bipartite to the multi-partite case. Some conclusions and open questions are mentioned in the Discussion.

1.1. A few simple examples/challenges

To clarify the problem, we present here a challenge:

Let two subsystems be of any dimension (quantum digits — qudits), and let $|\Psi^+\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$, and consider the following mixed states:

$$\rho = p|00\rangle\langle 00| + (1-p)|\Psi^+\rangle\langle\Psi^+|\,, \tag{1}$$

$$\rho = p|01\rangle\langle 01| + (1-p)|\Psi^+\rangle\langle\Psi^+|\,, \tag{2}$$

$$\rho = p_0|00\rangle\langle 00| + p_1|11\rangle\langle 11| + p_2|\Psi^+\rangle\langle\Psi^+|\,, \tag{3}$$

$$\rho = p_0|00\rangle\langle 00| + p_1|01\rangle\langle 01| + p_2|\Psi^+\rangle\langle\Psi^+|\,, \tag{4}$$

and lastly

$$\rho = p_0|22\rangle\langle 22| + p_1|23\rangle\langle 23| + p_2|\Psi^+\rangle\langle\Psi^+|\,. \tag{5}$$

Is it easy to see if each is entangled or not, for probabilities p, $1-p$ and p_i not equal zero? The answer is that one of the above states, and just one, is not entangled, with some choices of its parameters.

In this paper we will solve for these states and various generalizations.

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1.2. Near the completely mixed state

First some known examples. If ρ^A and ρ^B are the completely mixed state then $\rho^A \otimes \rho^B$ is also the completely mixed state and for p close enough to 1, the state $p\rho^A \otimes \rho^B + (1-p)\rho_1$ will be in a ball of separable states [16] around the completely mixed state for any state ρ_1, mixed or pure. A similar result was found even earlier [17] for n-qubits.

1.3. Werner state and a simple rank-4 separability-entanglement boundary

Another well known example, of two qubits and rank-4 mixed states, is the Werner states; consider the following states [18] of two qubits — an equal mixture of the four Bell states (i.e., a state identical to the completely mixed state of two qubits), plus some extra amount of one of them, e.g., WLG, an extra amount of the singlet state $|\Psi_-\rangle$. A Werner state can also be written as

$$\rho_W = \lambda|\Psi_-\rangle\langle\Psi_-| + \frac{1-\lambda}{3}\Big[|\Psi_+\rangle\langle\Psi_+| + |\Phi_+\rangle\langle\Phi_+| + |\Phi_-\rangle\langle\Phi_-|\Big] , \tag{6}$$

where $|\Psi^\pm\rangle = \dfrac{|01\rangle \pm |10\rangle}{\sqrt{2}}$ and $|\Phi^\pm\rangle = \dfrac{|00\rangle \pm |11\rangle}{\sqrt{2}}$, and $1/4 \leq \lambda < 1$; it is then a rank 4 state. For $\lambda = 1$ the formula gives the pure singlet state, and for $\lambda = 1/4$ the state is the completely mixed state. The interesting range, as Werner found, is $1/4 < \lambda \leq 1/2$, because at this range it is a state that might seem entangled but is separable. Exactly at $\lambda = 1/2$, the separability is easily proven by writing the state as an equal mixture of the singlet state with each of the other Bell states, and noticing that $|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-| = |01\rangle\langle 01| + |10\rangle\langle 10|$, and similarly with the other two Bell states when calculating in the x direction and y direction. Namely the state is shown to be separable with $N = 6$ product terms:

$$\rho_W = (1/6)\Big[|\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|\Big] + (1/6)\Big[|\Phi_+\rangle\langle\Phi_+| + |\Psi_-\rangle\langle\Psi_-|\Big] + (1/6)\Big[|\Phi_-\rangle\langle\Phi_-| + |\Psi_-\rangle\langle\Psi_-|\Big] . \tag{7}$$

1.4. Void state and a simple rank-3 counter example

A more complicated rank-3 mixed state example relies on the Werner states, in some sense, and on the notion of void states: Extending the Werner states definition to $\lambda < 1/4$ is also of interest; in particular, for $\lambda = 0$ it is a rank 3 state, which is an example of a state named a "void state" [13,19], in which a diagonalized density matrix has exactly one zero term on its diagonal.

Using a similar logic as in the Werner-state example, the void variant of the Werner state with $\lambda = 0$, namely

$$\rho_v = (1/3)\Big[|\Psi_+\rangle\langle\Psi_+| + |\Phi_-\rangle\langle\Phi_-| + |\Phi_+\rangle\langle\Phi_+|\Big] , \tag{8}$$

is separable: each of the three Bell states can be equaly mixed with another of them,

$$\rho_v = (1/6)\Big[|\Psi_+\rangle\langle\Psi_+| + |\Phi_-\rangle\langle\Phi_-|\Big] + (1/6)\Big[|\Phi_+\rangle\langle\Phi_+| + |\Psi_+\rangle\langle\Psi_+|\Big] + (1/6)\Big[|\Phi_-\rangle\langle\Phi_-| + |\Phi_+\rangle\langle\Phi_+|\Big] . \tag{9}$$

Since this state also equal $(1/3)|\Psi_+\rangle\langle\Psi_+| + \Big[(1/3)|00\rangle\langle 00| + |11\rangle\langle 11|\Big]$, this provides the promised counter example — the state is separable with $N = 6$ product terms, solving the challenge presented in the introduction. In the rest of the paper we will clarify why all other examples provided in the challenge are mixed entangled states.

2. The bi-partite rank 2 case

In this section we present our main result — the simplest entangled mixed state. To the best of our knowledge, our result here was never mentioned or noticed or proven.

Theorem 1. *Given a product state $|\varphi_0^A\varphi_0^B\rangle$ and a pure entangled state $|\Psi\rangle$ of a 2 partite A, B system, the state*

$$\rho = p|\varphi_0^A\varphi_0^B\rangle\langle\varphi_0^A\varphi_0^B| + (1-p)|\Psi\rangle\langle\Psi| \tag{10}$$

is entangled for all p such that $0 < p < 1$ (the case $p = 0$ is trivial).

Proof. If ρ were separable then ρ could be written as

$$\rho = \sum_{i=1}^{N} p_i|\varphi_i^A\varphi_i^B\rangle\langle\varphi_i^A\varphi_i^B| \tag{11}$$

with $p_i > 0$, $\sum_i p_i = 1$ for some product states. We now show stepwise that eq. (10) and (11) cannot both hold without leading to a contradiction.

First, let $|\Phi_i\rangle$ denote the state $|\varphi_i^A\varphi_i^B\rangle$ for $1 \leq i \leq N$ as well as for $i = 0$ so that equating (10) and (11) gives, with $p > 0, 1-p > 0, p_i > 0$

$$p|\Phi_0\rangle\langle\Phi_0| + (1-p)|\Psi\rangle\langle\Psi| = \rho = \sum_{i=1}^{N} p_i|\Phi_i\rangle\langle\Phi_i| \tag{12}$$

From Lemma A1 it follows that

$$\mathrm{Span}\{|\Phi_0\rangle, |\Psi\rangle\} = \mathrm{Span}\{|\Phi_i\rangle \mid 1 \leq i \leq N\} \tag{13}$$

Let $\mathcal{H} = \mathrm{Span}\{|\Phi_0\rangle, |\Psi\rangle\}$; $\dim \mathcal{H} = 2$ since $|\Phi_0\rangle$ is separable and $|\Psi\rangle$ is not. The span of the states $\{|\Phi_i\rangle \mid 0 \leq i \leq N\}$ (this time including $i = 0$) being equal to $\mathcal{H}$, the set $\{|\Phi_0\rangle\}$ can thus be completed to a basis of $\mathcal{H}$ using one of the states $|\Phi_i\rangle$ with $1 \leq i \leq N$, which can be assumed without loss of generality to be with $i = 1$, so as to get

$$\mathcal{H} = \mathrm{Span}\{|\Phi_0\rangle, |\Phi_1\rangle\} \tag{14}$$

Notice $|\Phi_0\rangle, |\Phi_1\rangle$ is only an algebraic basis of $\mathcal{H}$. Those two states span $\mathcal{H}$ but are unlikely orthogonal; we will thus need to rely on properties of linear independent sets in vector spaces.

From $|\Phi_0\rangle = |\varphi_0^A\varphi_0^B\rangle$, $|\Phi_1\rangle = |\varphi_1^A\varphi_1^B\rangle$, $|\Psi\rangle \in \mathrm{Span}\{|\varphi_0^A\varphi_0^B\rangle, |\varphi_1^A\varphi_1^B\rangle\}$ and $|\Psi\rangle$ entangled, it follows that $\{|\varphi_0^A\rangle, |\varphi_1^A\rangle\}$ and $\{|\varphi_0^B\rangle, |\varphi_1^B\rangle\}$ are linearly independent sets. Indeed let $|\Psi\rangle = a_0|\varphi_0^A\varphi_0^B\rangle + a_1|\varphi_1^A\varphi_1^B\rangle$; if it held that $|\varphi_1^A\rangle = c|\varphi_0^A\rangle$ it would follow that $|\Psi\rangle = |\varphi_0^A\rangle \otimes (a_0|\varphi_0^B\rangle + a_1 c|\varphi_1^B\rangle)$ and $|\Psi\rangle$ would not be entangled. The same argument prevents $\{|\varphi_0^B\rangle, |\varphi_1^B\rangle\}$ from being linearly dependent. It then follows that the states

$$|\varphi_0^A\varphi_0^B\rangle, |\varphi_0^A\varphi_1^B\rangle, |\varphi_1^A\varphi_0^B\rangle, |\varphi_1^A\varphi_1^B\rangle \tag{15}$$

are linearly independent.

If a linear combination of $|\Phi_0\rangle$ and $|\Phi_1\rangle$ is a product state, it must take the form

$$(a_{10}|\varphi_0^A\rangle + a_{11}|\varphi_1^A\rangle) \otimes (a_{20}|\varphi_0^B\rangle + a_{21}|\varphi_1^B\rangle) \tag{16}$$

However, since the states in (15) are linearly independent, for (16) to be in the span of $|\Phi_0\rangle$ and $|\Phi_1\rangle$, it is required that $a_{10}a_{21} = 0$ and $a_{11}a_{20} = 0$. If $a_{10}a_{20} \neq 0$ then $a_{10} \neq 0$ and $a_{20} \neq 0$ and thus $a_{21} = 0$ and

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$a_{11} = 0$ so that (16) is equal to $a_{10}a_{20}|\Phi_0\rangle$; else $a_{11}a_{21} \neq 0$ and then (16) is equal to $a_{11}a_{21}|\Phi_1\rangle$. That means that all the product states $|\Phi_i\rangle$ are multiples of either $|\Phi_0\rangle$ or $|\Phi_1\rangle$. It then follows that

$$\rho = \sum_{i=1}^{N} p_i|\Phi_i\rangle\langle\Phi_i| = p'_0|\Phi_0\rangle\langle\Phi_0| + p'_1|\Phi_1\rangle\langle\Phi_1| \tag{17}$$

with two product states, $p'_0 > 0$, $p'_1 > 0$.

We are thus left with

$$p|\Phi_0\rangle\langle\Phi_0| + (1-p)|\Psi\rangle\langle\Psi| = p'_0|\Phi_0\rangle\langle\Phi_0| + p'_1|\Phi_1\rangle\langle\Phi_1| \tag{18}$$

with $|\Psi\rangle = a_0|\Phi_0\rangle + a_1|\Phi_1\rangle$ for some a_0, a_1 so that

$$|\Psi\rangle\langle\Psi| = |a_0|^2|\Phi_0\rangle\langle\Phi_0| + a_0\overline{a_1}|\Phi_0\rangle\langle\Phi_1| + \overline{a_0}a_1|\Phi_1\rangle\langle\Phi_0| + |a_1|^2|\Phi_1\rangle\langle\Phi_1|$$

However, by Lemma A4, the operators $|\Phi_0\rangle\langle\Phi_0|$, $|\Phi_0\rangle\langle\Phi_1|$, $|\Phi_1\rangle\langle\Phi_0|$ and $|\Phi_1\rangle\langle\Phi_1|$ are linearly independent; for the left hand side of (18) to be equal to its right hand side, the coefficients of $|\Phi_0\rangle\langle\Phi_1|$ and $|\Phi_1\rangle\langle\Phi_0|$ must consequently be 0, implying that $a_0\overline{a_1} = 0$ and thus either a_0 or a_1 is 0, so that $|\Psi\rangle$ must then be a product state, giving the desired contradiction. □

3. Beyond rank 2 mixed states — the two-qubit case

One might conjecture that a simple extension from Theorem 1, i.e., extending from rank 2 states into rank 3 states (or higher ranks), is possible, and think that:

"Mixing two (or more) pure product states with a single entangled pure state yields a mixed entangled state."

However, the void-state example in the introduction already provided a counter example. The following two propositions provide interesting generalizations of the above trivial counter example, via two steps, for both rank-3 states and rank-4 states.

3.1. Extending the basic rank 3 void-state counter example

Proposition 1. *The state*

$$\rho = p_0|00\rangle\langle 00| + p_1|11\rangle\langle 11| + p_2|\Psi^+\rangle\langle\Psi^+| \tag{19}$$

is entangled if and only if

$$p_2 > 2 \cdot \sqrt{p_0 p_1} \tag{20}$$

Proof. The Peres-Horodecki [20,21] criterion states that, in a 2×2 system, a state is separable iff its partial transpose is positive semi-definite. The partial transpose of $|\Psi^+\rangle\langle\Psi^+| = \big[|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\big]/2$ is

$$\frac{1}{2}\big[|01\rangle\langle 01| + |00\rangle\langle 11| + |11\rangle\langle 00| + |10\rangle\langle 10|\big]$$

that of $|00\rangle\langle 00|$ is $|00\rangle\langle 00|$ and that of $|11\rangle\langle 11|$ is $|11\rangle\langle 11|$. It follows that the partial transpose of ρ in matrix form (with rows and columns in the order 00, 11, 01, 10) is

$$\begin{bmatrix} p_0 & \frac{p_2}{2} & 0 & 0 \\ \frac{p_2}{2} & p_1 & 0 & 0 \\ 0 & 0 & \frac{p_2}{2} & 0 \\ 0 & 0 & 0 & \frac{p_2}{2} \end{bmatrix}$$

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That matrix is block diagonal and it is positive semi definite if and only if the block

$$\begin{bmatrix} p_0 & p_2/2 \\ p_2/2 & p_1 \end{bmatrix}$$

has no negative eigenvalue. Those are solution of the characteristic equation
$(\lambda - p_0)(\lambda - p_1) - (p_2/2)^2 = 0 \quad$ i.e.

$$\lambda^2 - (p_0 + p_1)\lambda + p_0 p_1 - (p_2/2)^2 = 0$$

The roots are

$$\frac{(p_0 + p_1) \pm \sqrt{(p_0 + p_1)^2 - 4(p_0 p_1 - (p_2/2)^2)}}{2}$$

and they are non negative iff

$$(p_0 + p_1)^2 \geq (p_0 + p_1)^2 - 4(p_0 p_1 - (p_2/2)^2)$$

i.e. $p_0 p_1 \geq (p_2/2)^2$ so that ρ is separable iff $p_0 p_1 \geq (p_2/2)^2$ or equivaletly ρ is entangled iff $(p_2/2)^2 > p_0 p_1$. □

The void state example in the introduction is a special case of this one, with $p_0 = p_1 = p_2 = 1/3$, hence $(1/6)^2 < 1/9$ proving separability.

3.2. A more general case - beyond void states

Proposition 2. *The state*

$$\rho = \sum_{i=0}^{1} \sum_{j=0}^{1} p_{ij} |ij\rangle\langle ij| + p' |\Psi^+\rangle\langle\Psi^+|$$

is entangled if and only if

$$p' > 2 \cdot \sqrt{p_{00} p_{11}} \tag{21}$$

Proof. The partial transpose of $p_{ij}|ij\rangle\langle ij|$ is $p_{ij}|ij\rangle\langle ij|$. The partial transpose of $p'|\Psi^+\rangle\langle\Psi^+|$ is

$$\frac{p'}{2}\left[|01\rangle\langle 01| + |00\rangle\langle 11| + |11\rangle\langle 00| + |10\rangle\langle 10|\right]$$

In matrix form, the partial transpose of ρ with rows and columns in the 00, 11, 01 and 10 order is

$$\begin{bmatrix} p_{00} & \frac{p'}{2} & 0 & 0 \\ \frac{p'}{2} & p_{11} & 0 & 0 \\ 0 & 0 & p_{01} + \frac{p'}{2} & 0 \\ 0 & 0 & 0 & p_{10} + \frac{p'}{2} \end{bmatrix}$$

That matrix is block diagonal and it is positive semi definite if and only if the block

$$\begin{bmatrix} p_{00} & p'/2 \\ p'/2 & p_{11} \end{bmatrix}$$

has no negative eigenvalue. Those are solution of the characteristic equation
$(\lambda - p_{00})(\lambda - p_{11}) - (p'/2)^2 = 0$ i.e.

$$\lambda^2 - (p_{00} + p_{11})\lambda + p_{00} p_{11} - (p'/2)^2 = 0$$

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The roots are

$$\frac{(p_{00}+p_{11}) \pm \sqrt{(p_{00}+p_{11})^2 - 4(p_{00}p_{11} - (p'/2)^2)}}{2}$$

and they are non negative iff

$$(p_{00}+p_{11})^2 \geq (p_{00}+p_{11})^2 - 4(p_{00}p_{11} - (p'/2)^2)$$

i.e. $p_{00}p_{11} \geq (p'/2)^2$ so that, since ρ is in a 2×2 dimensional system, ρ is separable iff $p_{00}p_{11} \geq (p'/2)^2$ or equivalently ρ is entangled iff $(p'/2)^2 > p_{00}p_{11}$.

□

Corollary 1. *ρ in the preceding proposition is entangled as soon as the coefficient of $|00\rangle\langle 00|$ or $|11\rangle\langle 11|$ is 0 and $p' > 0$.*

4. The bi-partite case, higher ranks

While we just saw that it is not trivial to find simple mixed entangled state in a bi-partite system when the rank of the mixed state is 3 or 4, here we show a way to find simple mixed entangled states for states of any rank larger than 2. For simplicity we focus on extending the rank of subsystem B only.

4.1. A bipartite case — with any rank on subsystem B

We focus here on extending the previous theorem such that, while subsystem A is still of rank 2, there is no limit on the rank of subsystem B.

Theorem 2. *Given a product state $|\varphi^A\rangle\langle\varphi^A| \otimes \rho^B$ with ρ^B a mixed state (of rank $K > 1$) on the B system and a pure entangled state $|\Psi\rangle$ of a bipartite A, B system, the state*

$$\rho = p|\varphi^A\rangle\langle\varphi^A| \otimes \rho^B + (1-p)|\Psi\rangle\langle\Psi| \tag{22}$$

is entangled for all p such that $0 < p < 1$ (the case $p = 0$ is trivial).

Proof. Let $\rho^B = \sum_{k=1}^{K} \lambda_k |\varphi_k^B\rangle\langle\varphi_k^B|$ be an eigen-decomposition of ρ^B with $\lambda_k > 0$ and the $|\varphi_k^B\rangle$ normalized and orthogonal. The state ρ can then be written

$$\rho = \sum_{k=1}^{K} p\lambda_k |\varphi^A\varphi_k^B\rangle\langle\varphi^A\varphi_k^B| + (1-p)|\Psi\rangle\langle\Psi| \tag{23}$$

where the states $|\varphi^A\varphi_k^B\rangle$ are pairwise orthogonal and consequently linearly independent. Also $|\Psi\rangle$ cannot be in their span else it would take the form $|\varphi^A\rangle \otimes (\sum_j \alpha_j |\varphi_j^B\rangle)$ and thus be separable. It follows that

$$\dim \mathrm{Span}\{|\varphi^A\varphi_1^B\rangle, \ldots, |\varphi^A\varphi_K^B\rangle, |\Psi\rangle\} = K + 1\,. \tag{24}$$

We now assume that ρ is separable, for proving it cannot be. If ρ was separable then ρ could be written as

$$\rho = \sum_{i=1}^{N} p_i |\psi_i^A\psi_i^B\rangle\langle\psi_i^A\psi_i^B| \tag{25}$$

with $p_i > 0$, $\sum_i p_i = 1$ for some product states. Note that these product states are not expected to be orthogonal. We now show that if both eq. (23) and (25) hold with $|\Psi\rangle$ being entangled we reach a contradiction.

Since ρ is assumed to be both equal to (23) and (25), Lemma A1 gives

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$$\text{Span}\{|\varphi^A\varphi_1^B\rangle,\ldots,|\varphi^A\varphi_k^B\rangle,|\Psi\rangle\} = \text{Span}\{|\psi_i^A\psi_i^B\rangle \mid 1 \le i \le N\} \tag{26}$$

Let $\mathcal{H}$ be that common span; $\dim\mathcal{H} = K+1$; $\mathcal{H}$ is also the span of the set

$$\{|\varphi^A\varphi_j^B\rangle \mid 1 \le j \le K\} \cup \{|\psi_i^A\psi_i^B\rangle \mid 1 \le i \le N\}$$

where the set on the left side is composed of K linearly independent vectors in $\mathcal{H}$; those K vectors can be completed to a basis of $\mathcal{H}$ using one of the $|\psi_i^A\psi_i^B\rangle$ which we may assume, WLG, to be with $i=1$ so that

$$\{|\psi_1^A\psi_1^B\rangle,|\varphi^A\varphi_j^B\rangle \mid 1 \le j \le K\} \tag{27}$$

is a basis of $\mathcal{H}$. We now proceed in two different ways to reach the contradiction:

On the one hand, if the B system is traced out, all remaining pure states are in the span of $\{|\varphi^A\rangle,|\psi_1^A\rangle\}$. It thus follows that for all $1 < i \le N$,

$$|\psi_i^A\rangle = a_i|\varphi^A\rangle + b_i|\psi_1^A\rangle$$

for some $a_i, b_i \in \mathbb{C}$ so that

$$|\psi_i^A\psi_i^B\rangle = a_i|\varphi^A\psi_i^B\rangle + b_i|\psi_1^A\psi_i^B\rangle \tag{28}$$

On the other hand, $|\psi_i^A\psi_i^B\rangle \in \mathcal{H}$ and is thus in the span of (27), i.e. it must hold that

$$|\psi_i^A\psi_i^B\rangle = a_i'|\varphi^A\eta_i^B\rangle + b_i'|\psi_1^A\psi_1^B\rangle \tag{29}$$

for some $a_i', b_i' \in \mathbb{C}$ and $|\eta_i^B\rangle \in \text{Span}(|\varphi_1^B\rangle,\ldots,|\varphi_K^B\rangle\}$.

The equality of (28) and (29) means that

$$\begin{aligned}&|\varphi^A\rangle \otimes a_i|\psi_i^B\rangle + |\psi_1^A\rangle \otimes b_i|\psi_i^B\rangle = \\ &|\varphi^A\rangle \otimes a_i'|\eta_i^B\rangle + |\psi_1^A\rangle \otimes b_i'|\psi_1^B\rangle\end{aligned}$$

Since $|\varphi^A\rangle$ and $|\psi_1^A\rangle$ are linearly independent, by Lemma A2, for the equality to hold, it must hold that $a_i|\psi_i^B\rangle = a_i'|\eta_i^B\rangle$ but more importantly that $b_i|\psi_i^B\rangle = b_i'|\psi_1^B\rangle$. Since both $|\psi_i^B\rangle$ and $|\psi_1^B\rangle$ are normalized states, this implies that they are equal (up to a phase factor) and, since i was chosen arbitrarily that implies

$$|\psi_i^B\rangle\langle\psi_i^B| = |\psi_1^B\rangle\langle\psi_1^B| \qquad\qquad 1 < i \le N$$

From this result along with 25, the mixed state is not a complicated separable state but a product state $\rho = [\sum_{i=1}^N p_i|\psi_i^A\rangle\langle\psi_i^A|] \otimes |\psi_1^B\rangle\langle\psi_1^B|$. But now, if we trace-out the A system, we get a state of rank 1 for subsystem B. On the other hand, if we trace out the A subsystem in (23) we get a state of rank at least 2 which gives the desired contradiction. □

We conclude that the state discussed in Theorem 2, see eq.22, is entangled.

4.2. A bipartite case — with any rank — a question for thought

Observing the cases presented in the introduction, we can design similar examples not yet solved by the methods we presented here. E.g., this case, for subsystems of arbitrary dimension

$$\rho = p_0|22\rangle\langle 22| + p_1|33\rangle\langle 33| + p_2|\Psi^+\rangle\langle\Psi^+| \tag{30}$$

(and its extensions) is left as an open problem for future research.

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5. The multi-partite rank 2 case

The basic result can be extended to an n partite system with a proof that follows the same lines.

Theorem 3. *Given a product state $|\varphi_0^{A_1}\varphi_0^{A_2}\dots\varphi_0^{A_n}\rangle$ and a pure entangled state $|\Psi\rangle$ of a n partite system $(A_1, A_2, \dots, A_n)$, the state*

$$\rho = p|\varphi_0^{A_1}\varphi_0^{A_2}\dots\varphi_0^{A_n}\rangle\langle\varphi_0^{A_1}\varphi_0^{A_2}\dots\varphi_0^{A_n}| + (1-p)|\Psi\rangle\langle\Psi| \tag{31}$$

is entangled for all p such that $0 < p < 1$ (the case $p = 0$ is trivial).

Proof. If ρ were separable then it could be written as $\sum_{i=1}^N p_i|\Phi_i\rangle\langle\Phi_i|$ with $p_i > 0$, $|\Phi_i\rangle = |\varphi_i^{A_1}\varphi_i^{A_2}\dots\varphi_i^{A_n}\rangle$ so that, again, and for the same reasons

$$p|\Phi_0\rangle\langle\Phi_0| + (1-p)|\Psi\rangle\langle\Psi| = \rho = \sum_{i=1}^N p_i|\Phi_i\rangle\langle\Phi_i| \tag{32}$$

$$\mathcal{H} = \mathrm{Span}\{|\Phi_0\rangle, |\Psi\rangle\} = \mathrm{Span}\{|\Phi_i\rangle \mid 1 \le i \le N\}\,, \tag{33}$$

and we can still assume

$$\mathcal{H} = \mathrm{Span}\{|\Phi_0\rangle, |\Phi_1\rangle\}\,, \tag{34}$$

with $|\Phi_0\rangle = |\varphi_0^{A_1}\varphi_0^{A_2}\dots\varphi_0^{A_n}\rangle$ and $|\Phi_1\rangle = |\varphi_1^{A_1}\varphi_1^{A_2}\dots\varphi_1^{A_n}\rangle$. Some of the sets $\{|\varphi_0^{A_j}\rangle, |\varphi_1^{A_j}\rangle\}$ for $1 \le j \le n$ must be linearly independent else $|\Phi_0\rangle$ and $|\Phi_1\rangle$ would be linearly dependent. We may assume that this holds for $1 \le j \le k$ and $|\varphi_1^{A_j}\rangle = |\varphi_0^{A_j}\rangle$ for $j > k$ (constants can always be moved from one system to another). The states

$$|\varphi_{i_1}^{A_1}\varphi_{i_2}^{A_2}\dots\varphi_{i_k}^{A_k}\varphi_0^{A_{k+1}}\dots\varphi_0^{A_n}\rangle \qquad i_j \in \{0,1\}, 1 \le j \le k \tag{35}$$

are then linearly independent. We first show that $k > 1$. Otherwise $|\Psi\rangle$ would be in the span of $|\varphi_0^{A_1}\varphi_0^{A_2}\dots\varphi_0^{A_n}\rangle$ and $|\varphi_1^{A_1}\varphi_0^{A_2}\dots\varphi_0^{A_n}\rangle$ and would be equal to $(a_0|\varphi_0^{A_1}\rangle + a_1|\varphi_1^{A_1}\rangle) \otimes |\varphi_0^{A_2}\dots\varphi_0^{A_n}\rangle$ for some $a_0, a_1 \in \mathbb{C}$ and thus be a product state contrary to the hypothesis.

Now, if a linear combination of $|\Phi_0\rangle$ and $|\Phi_1\rangle$ is a product state, it must take the form

$$(a_{10}|\varphi_0^{A_1}\rangle + a_{11}|\varphi_1^{A_1}\rangle) \otimes \dots \otimes (a_{k0}|\varphi_0^{A_k}\rangle + a_{k1}|\varphi_1^{A_k}\rangle) \otimes |\varphi_0^{A_{k+1}}\dots\varphi_0^{A_n}\rangle \tag{36}$$

with $k > 1$. The states in (35) being linearly independent, for (36) to be in the span of $|\Phi_0\rangle$ and $|\Phi_1\rangle$, the products $a_{1i_1}a_{2i_2}\dots a_{ki_k}$ must be 0 when the i_j for $1 \le j \le k$ are not all equal. Also $a_{10}a_{20}\cdots a_{k0} \neq 0$ or $a_{11}a_{21}\cdots a_{k1} \neq 0$ else eq. (36) is equal to 0. If

$$a_{10}a_{20}\dots a_{k0} \neq 0 \qquad\qquad k \geq 2$$

then $a_{10}\dots\widehat{a_{i0}}\dots a_{k0} \neq 0$ where the hat represents removing that given term from the product; since $(a_{10}\dots\widehat{a_{i0}}\dots a_{k0})a_{i1}$ (obtained by replacing a_{i0} by a_{i1}) must be 0, it follows that $a_{i1} = 0$ for $1 \le i \le k$. That means that (36) is equal to

$$a_{10}\dots a_{k0}|\Phi_0\rangle$$

If $a_{11}a_{21}\dots a_{k1} \neq 0$ then (36) must be equal to $a_{11}\dots a_{k1}|\Phi_1\rangle$. It then follows that all the states $|\Phi_i\rangle$ are multiples of either $|\Phi_0\rangle$ or $|\Phi_1\rangle$ so that the right-hand side of eq. 32 can be rewritten as

$$\rho = p_0'|\Phi_0\rangle\langle\Phi_0| + p_1'|\Phi_1\rangle\langle\Phi_1|$$

which is nothing but eq. (17). The rest of the proof is identical to what follows eq. (17) at the end of the proof of Theorem 1. □

6. Discussion

We presented various cases in which is is easy to theoretically build an entangled mixed state. Our results shed some light also on the opposite question which is still open in many cases — given a mixed state, is it entangled or separable? Our result can also be found useful in analysis of experiments and technologies where it is potentially important to know what level of noise and interaction can still leave a state somewhat entangled.

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Abbreviations

Abbreviations

The following abbreviation is used in this manuscript:

WLG Without loss of generality

Appendix A

Appendix A.1. Lemmas

Lemma A1. *If in some Hilbert space*

$$\sum_{i=1}^{n} p_i|\Phi_i\rangle\langle\Phi_i| = \sum_{j=1}^{m} q_j|\Psi_j\rangle\langle\Psi_j| \qquad p_i > 0, q_j > 0 \tag{A1}$$

then

$$\mathrm{Span}\{|\Phi_i\rangle \mid 1 \le i \le n\} = \mathrm{Span}\{|\Psi_j\rangle \mid 1 \le j \le m\}. \tag{A2}$$

Proof. Let $\mathcal{H}$ be the span of the states $|\Phi_i\rangle$ and $\mathcal{H}'$ be that of the $|\Psi_j\rangle$. For all $|\varphi\rangle \in \mathcal{H}^\perp$ it holds that $\langle\varphi|\Phi_i\rangle\langle\Phi_i|\varphi\rangle = 0$ for $1 \le i \le n$ which implies by eq. (A1) that

$$0 = \sum_{j=1}^{m} q_j\langle\varphi|\Psi_j\rangle\langle\Psi_j|\varphi\rangle = \sum_{j=1}^{m} q_j|\langle\varphi|\Psi_j\rangle|^2$$

and, since $q_j > 0$ this implies $\langle\varphi|\Psi_j\rangle = 0$ for $1 \le j \le m$ i.e. $|\varphi\rangle \in \mathcal{H}'^\perp$ which implies that $\mathcal{H}^\perp \subseteq \mathcal{H}'^\perp$ and thus $\mathcal{H}' \subseteq \mathcal{H}$. By symmetry $\mathcal{H} \subseteq \mathcal{H}'$ so that $\mathcal{H} = \mathcal{H}'$. □

Lemma A2. *Let $|\varphi_0^A\rangle$ and $|\varphi_1^A\rangle$ be linearly independent normalized states of subsystem A, and consider two subsystems A and B.*
If

$$|\varphi_0^A\rangle \otimes |\psi_0^B\rangle + |\varphi_1^A\rangle \otimes |\psi_1^B\rangle = |\varphi_0^A\rangle \otimes |\psi_0'^B\rangle + |\varphi_1^A\rangle \otimes |\psi_1'^B\rangle \tag{A3}$$

then $|\psi_0^B\rangle = |\psi_0'^B\rangle$ and $|\psi_1^B\rangle = |\psi_1'^B\rangle$

Proof. Let us write, WLG, $|\varphi_0^A\rangle$ as $|0\rangle$, and, with nonzero coefficient b (note that a can be zero or nonzero), $|\varphi_1^A\rangle = a|0\rangle + b|1\rangle$, such that $|0\rangle$ and $|1\rangle$ normalized and orthogonal, and $|a|^2 + |b|^2 = 1$. Eq.(A3) can then be rewritten as

$$|0\rangle \otimes |\psi_0^B\rangle + \Big[a|0\rangle + b|1\rangle\Big] \otimes |\psi_1^B\rangle = |0\rangle \otimes |\psi_0'^B\rangle + \Big[a|0\rangle + b|1\rangle\Big] \otimes |\psi_1'^B\rangle$$

and, grouping terms

$$|0\rangle \otimes \left[|\psi_0^B\rangle + a|\psi_1^B\rangle\right] + |1\rangle \otimes b|\psi_1^B\rangle = |0\rangle \otimes \left[|\psi_0'^B\rangle + a|\psi'^B_1\rangle\right] + |1\rangle \otimes b|\psi_1'^B\rangle\ .$$

It follows from lemma A3 that $b|\psi_1^B\rangle = b|\psi_1'^B\rangle$, and therefore $|\psi_1^B\rangle = |\psi_1'^B\rangle$ since $b \neq 0$. It also follows from lemma A3 that $|\psi_0^B\rangle + a|\psi_1^B\rangle = |\psi_0'^B\rangle + a|\psi_1'^B\rangle$; Using $|\psi_1^B\rangle = |\psi_1'^B\rangle$ it now follows that $|\psi_0^B\rangle + a|\psi_1^B\rangle = |\psi_0'^B\rangle + a|\psi_1^B\rangle$, hence $|\psi_0^B\rangle = |\psi_0'^B\rangle$. □

Lemma A3. *Let there be two subsystems with $|0\rangle$ and $|1\rangle$ the computation basis for the first (the left) subsystem. For clarity and consistency we call the right subsystem B.*
If $|0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle = |0\rangle|\psi_0'\rangle + |1\rangle|\psi_1'\rangle$ then $|\psi_0\rangle = |\psi_0'\rangle$ and $|\psi_1\rangle = |\psi_1'\rangle$.

Proof. Let $|j\rangle$ be an orthonormal basis of a subsystem B, and let $|\psi_i^B\rangle = \sum_j a_{ij}|j^B\rangle$ and $|\psi'^B_i\rangle = \sum_j a'_{ij}|j^B\rangle$, for $i = \{0,1\}$. Then the equality can be rewritten

$$\sum_j a_{0j}|0j\rangle + \sum_j a_{1j}|1j\rangle = \sum_j a'_{0j}|0j\rangle + \sum_j a'_{1j}|1j\rangle$$

which, due to orthonormaility, implies $a_{0j} = a'_{0j}$ for all j i.e. $|\psi_0^B\rangle = |\psi'^B_0\rangle$ also $a_{1j} = a'_{1j}$ for all j i.e. $|\psi_1^B\rangle = |\psi'^B_1\rangle$. □

Lemma A4. *If $(|\Phi_i\rangle)_{1\le i\le r}$ are linearly independent states of some Hilbert space $\mathcal{H}$ of dimension N, then the operators $\left(|\Phi_i\rangle\langle\Phi_j|\right)_{1\le i,j\le r}$ are linearly independent.*

Proof. If $\sum_{ij} a_{ij}|\Phi_i\rangle\langle\Phi_j| = 0$ then $\sum_{ij} a_{ij}|\Phi_i\rangle\langle\Phi_j|\Psi\rangle = \sum_i \left(\sum_j a_{ij}\langle\Phi_j|\Psi\rangle\right)|\Phi_i\rangle = 0$ for all $|\Psi\rangle \in \mathcal{H}$ and linear independence of the $|\Phi_i\rangle$ implies that the coefficient of $|\Phi_i\rangle$ is 0 for all i and $|\Psi\rangle$ i.e.

$$\sum_j a_{ij}\langle\Phi_j|\Psi\rangle = 0 = \overline{\langle\Psi|\sum_j \overline{a_{ij}}\Phi_j\rangle}$$

so that, for all i, $\sum_j \overline{a_{ij}}|\Phi_j\rangle = 0$; linear independence of the $|\Phi_j\rangle$ implies $\overline{a_{ij}} = 0 = a_{ij}$ for all i, j. □

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