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Is ItWorth Testing Large Numbers for Collatz Conjecture?

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Article

Is It Worth Testing Large Numbers for Collatz Conjecture?

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Abstract: Starting with a positive odd integer n, apply a function $f(n)$ repetitively such that $f(n) = 3n+1$ if $n \not\equiv 0(\text{mod } 2)$ or $f(n) = n/2$ if $n \equiv 0(\text{mod } 2)$. Let the sequence of all the odd integers obtained form an orbit $n, f^1(n), f^2(n), \ldots, f^k(n), f^{k+1}(n)$. The k^{th} odd integer is written as $f^k(n) = (3^k + \mathbf{B(k)})/2^{\mathbf{z(k)}}$ where $\mathbf{B(k)}$ is the collection of terms without the coefficient n. We show that when $1 - \mathbf{B(k)}/(2^{\mathbf{z(k)}}n) \approx 1$, the condition for unbounded orbit is given by OEIS A022921. Further, if there exists at least one orbit for which $f^{k+1}(n) = n$ when $(k+1) < 3$, we show that there exists no orbit with $k+1 > 3$ for which $f^{k+1}(n) = n$. The final result is that the odd integers that can violate the Collatz conjecture are smaller than 5.

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1. Introduction

The Collatz problem [1–4], involves a function $f(n)$ defined on positive odd integers n. The function takes an odd integer, multiplies it by 3, and adds 1. If the resulting integer is even, it is divided by 2. The Collatz conjecture posits that the only integer for which $f^{k+1}(n) = n$ is 1. It is known as the trivial cycle.

If there exists an odd integer $n > 1$ for which the Collatz orbit converges back to n or diverges to infinity, the conjecture would be proven false. This article focuses on orbits where there is a repeating integer other than 1, resulting in a finite value for $k+1$. The possible value of $k+1$ for a repeating orbit is $\{1, 2, 3, \ldots\}$, and one known solution is $k+1 = 1$.

The $(k+1)^{th}$ odd integer in the Collatz orbit is given by

$$f^{k+1}(n) = 3\left\{\frac{\left\{3\frac{\left\{3\frac{\left\{\frac{3n+1}{2^{z_1}}+1\right\}}{2^{z_2}}+1\right\}}{2^{z_3}}+1\right\}}{\vdots}\right\}+1$$

$$f^{k+1}(n) = \frac{3^{k+1}n + \mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}}$$

Where the number of even steps between two consecutive odd integers is z_i, and

$$\begin{aligned}\mathbf{B(k+1)} &= 3^k + 3^{k-1}2^{z_1} + \cdots + 3^1 2^{z_1+z_2+\cdots+z_{k-2}} + 2^{z_1+z_2+\cdots+z_{k-1}+z_k} \\ \mathbf{z(k+1)} &= z_1 + z_2 + \cdots + z_k + z_{k+1}\end{aligned}$$

$\mathbf{B(k)}$ and $\mathbf{z(k)}$ are defined similarly for the k^{th} odd integer.

Numerous experiments with large numbers have been carried out to test the hypothesis [5]. It is anticipated that, at some point, a sufficiently large number will disprove the Collatz conjecture. Motivated by this idea, we consider an integer n as a large value and attempt to establish an upper limit for its magnitude.

The next section develops a methodology to show that the integers capable of violating the Collatz conjecture are smaller than 5.

2. Methodology

A Collatz orbit collapses to the trivial cycle if it has a finite stopping time [6] or the orbit has a bounded value [7] (we define a bounded orbit as an orbit that falls below the value of the starting integer). Therefore, the following two condition are essential for a repeating orbit:

- The Collatz orbit has to be unbounded till the k^{th} odd term to prevent convergence to the trivial cycle.
- The k^{th} odd integer should give an even integer of the form $2^{z_{k+1}}n$.

The relation between the indices of 3 and 2 are evaluated for the above two conditions to occur. Finally, the value of $\mathbf{B(k+1)}/2^{\mathbf{z(k+1)}}$ is estimated which gives an estimate of the value of the integer n.

3. Condition for unbounded orbits

Convergence to the trivial cycle is avoided if $f^k(n) > n$

$$\frac{3^k n + \mathbf{B(k)}}{2^{\mathbf{z(k)}}} > n$$
$$\frac{3^k}{2^{\mathbf{z(k)}}} > 1 - \frac{\mathbf{B(k)}}{2^{\mathbf{z(k)}} n}$$

Let the integer n be so huge that $1 - \mathbf{B(k)}/(2^{\mathbf{z(k)}} n) \approx 1$.

$$\frac{3^k}{2^{\mathbf{z(k)}}} > 1 \tag{1}$$

The value of z_i for which the Equation (1) holds is given by OEIS-A022921 [8].

4. Condition for repeating integers

The orbit has remained unbounded till the k^{th} term. Now we need $f^{k+1}(n) = n$, i.e.,

$$\frac{3^{k+1} n + \mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}} = n$$
$$n\left(1 - \frac{3^{k+1}}{2^{\mathbf{z(k+1)}}}\right) = \frac{\mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}} \tag{2}$$

Since the RHS and n are positive integers,

$$\frac{3^{k+1}}{2^{\mathbf{z(k+1)}}} < 1 \tag{3}$$

The indices of 3 and 2 should satisfy Equation (3) if the orbit has repeating integers.

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5. Estimates on the value of $\frac{\mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}}$

From equation (2), we have

$$\frac{\mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}} = \frac{3^k}{2^{\mathbf{z(k+1)}}} + \frac{3^{k-1}2^{z_1}}{2^{\mathbf{z(k+1)}}} + \ldots + \frac{3^1 2^{z_1+z_2+\cdots+z_{k-2}}}{2^{\mathbf{z(k+1)}}} + \frac{2^{z_1+z_2+\cdots+z_{k-1}+z_k}}{2^{\mathbf{z(k+1)}}}$$

Using using Equation (1) and OEIS-A022921, it can be shown each term is greater than $\frac{1}{2^{z_{k+1}}3}$. Similarly, using Equation (3) and OEIS-A022921, it can be shown that each term is less than $\frac{1}{3}$. There are $(k+1)$ terms, hence

$$\frac{k+1}{3} > \frac{\mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}} > \frac{k+1}{2^{z_{k+1}}3} \tag{4}$$

6. Estimates on the value of $n\left(1 - \frac{3^{k+1}}{2^{\mathbf{z(k+1)}}}\right)$

The value of RHS of Equation (2) has been estimated in the previous section. Let the LHS of Equation (2) be greater than 1 when the integer n repeats.

$$\begin{aligned} n\left(1 - \frac{3^{k+1}}{2^{\mathbf{z(k+1)}}}\right) &> 1 \\ \frac{\mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}} &> 1 && \text{(From Equation (2))} \\ \frac{k+1}{3} &> 1 && \text{(From Equation (4))} \\ k+1 &> 3 \end{aligned}$$

Based on the above result, it is established that the value of $k+1$ in a repeating orbit exceeds 3, which automatically excludes the trivial cycle where $k+1=1$. Thus, the assumption that the left-hand side of Equation (2) is greater than 1 does not encompass the entire solution set for $k+1$. Consequently, it can be deduced that the left-hand side of Equation (2) is greater than some fraction $\mathbf{Q}$, where $\mathbf{Q} < 1$, such that the value of $k+1$ still includes the trivial cycle.

7. Value of $k+1$ for repeating orbits

Based on the discussion in the previous section, it is established that

$$\begin{aligned} n\left(1 - \frac{3^{k+1}}{2^{\mathbf{z(k+1)}}}\right) &< 1 \\ \frac{\mathbf{B(k+1)}}{2^{\mathbf{z(k+1)}}} &< 1 && \text{(From Equation (2))} \\ \frac{k+1}{2^{z_{k+1}}3} &< 1 && \text{(From Equation (4))} \\ k+1 &< 2^{z_{k+1}}3 \end{aligned}$$

It can be seen that the trivial cycle is included in the solution as $\lim_{z_{k+1}\to\infty} k+1 \in \{1,2,3,\ldots\}$. Furthermore, once an integer in the form of $2^{z_{k+1}}n$ is reached, subsequent steps are exclusively even until arriving at the integer n, resulting in a constant value of $k+1$ from that point on-wards. Hence, if the number of odd steps required to reach $2^{z_{k+1}}n$ is $k+1$, the same applies to reaching $2^{z_k}n$, $2^{z_{k-1}}n$, and ultimately the integer n, each necessitating $k+1$ odd steps. However, the integer n reoccurs in

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the orbit when $2^{z_{k+1}}$ reduces to 1 through division by 2, i.e., $2^{z_{k+1}} = 1$. By inserting $2^{z_{k+1}} = 1$ into the aforementioned equation, we can deduce that the number of odd steps in any repeating cycle within the Collatz sequence is limited by $k+1 < 3$.

7.1. Result and Conclusion

We made the assumption that a large odd number n violates the Collatz conjecture by becoming part of a repeating cycle in the orbit. We established the fundamental conditions for an orbit to be unbounded and for an integer to repeat in the sequence. Notably, we discovered that the value of $k+1$ must be below 3 for an orbit to repeat. This finding, in conjunction with OEIS-A022921, sets the limit for the value of n to be less than 5.
Extending this observation to Collatz-like sequences, we can make the following observations:

- For the $n+1$ sequence, every orbit is bounded.
- In the $5n+1$ sequence, the number of odd steps in a repeating orbit is limited to less than 5.
- Similarly, for the $7n+1$ sequence, the number of odd steps in a repeating orbit is limited to less than 7.

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