Communication

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Abstract: This research work aims to develop a mathematical modeling for production-inventory systems. Two mathematical models are present: An infinite dimensional, partial differential equation (PDE), production level modeling is present and a finite dimensional system for a coupled dynamic pricing, production rate and inventory level ordinary differential equations (ODE), integrating a proper Lyapunov stability analysis of the dynamical system and simulations.

Keywords: production-inventory system; Lyapunov stability; infinite dimensional system; finite dimensional system

MSC: 90B30

1. Introduction

Mathematical modeling plays an important role in industrial engineering, moreover in production-inventory systems, which is the case for high volume production systems that presents low variability. The application of PDE and ODE are important considering the greater amounts of data and business decisions, which are facing decision-makers. There is a growing interest in using real-time information to improve, support and validate business decision-making [1]. Today's enterprises work towards flexible, reliable and responsive business operations. Therefore, they need to implement systematic decision-making processes [2]. In production and logistics systems, supply chain management (SCM) and Industry 4.0 networks: Uncertainty, feedback cycles and system dynamics are mandatory goals [3]. This research work aims to explore the dynamic nature of a production-inventory system, via the development of infinite dimensional systems (PDEs) and finite dimensional systems, subject to demand fluctuations. Inventory management (IM) presents a crucial role in the operations and management sciences. SCM refers to the cooperation process management

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of materials and information flows between supply chains partners [4]. A supply chain is a network of facilities and distribution entities such as: Suppliers, manufacturers, distributors, retailers [5].

A supply chain is characterized by three flows: materials and cash (forward flow) and information (backward flow). By definition, IM is part of the SCM that plans, implements and controls the forward, and reverse flow of goods and services (as well as storage), in an efficient and effective way, between the point of origin and the point of consumption in order to meet customer requirements [6]. Inventories, in nature, are present all along the supply chain (SC) and inventory control is a crucial activity by a company's management [7]. To maintain appropriate inventory levels is crucial task for a company [8], considering that quick and positive response to customers are related to high inventory levels (which increase the cost), while low inventory levels might cause scarcity. In SCM systems there are disruption in: Supply, transportation, production and demand fluctuation [9]. In production-inventory systems some works analyzes supply disruption [10], with multi-echelon production-inventory system supply disruption [11]. In relation to production disruptions in [12] presents deteriorating items, while random disruptions are present in [13,14] and a system dynamics approach is developed for production process disruption in [15]. Supply chain dynamism builds the strategic orientation supply chain disruption orientation [16]. In a typical supply chain network efficient collaboration between all parts is vital, in times of uncertainty and unpredictable disruption [17]. Product demand variation, which is increased or decreased in a time horizon period corresponds to demand fluctuation. Table 1 summarizes a literature review in demand fluctuation with proper application sector.

Table 1. Demand Fluctuation Literature review.

Paper	Contribution	Application
Morikawa [18]	The aim of this paper presents evidence on the relationship between short-term demand fluctuations and the total factor productivity of service industries.	Service industries
Paul, et al [19]	This paper considers a supplier–retailer system, with an imperfect production process and a possibility of having demand fluctuation. Also, a dynamic planning process is analyzed to deal with short-term demand fluctuations.	Supply chain
Tian, et al [20]	This paper main contribution is the statistical physics method is applied to the demand fluctuation of two different bike sharing system.	Urban traffic
Yang, et al [21]	This paper presents a comparison for two types of flexibility investment, which are flexible technology and flexible capacity, under demand fluctuations.	Flexible manufacturing systems
Xiong and Helo [22]	This paper presents a fuzzy inventory model to counteract the demand fluctuation in supply demand networks, applying fuzzy logic controller.	Supply demand networks

Figure 1 presents a general taxonomy for production systems, from a mathematical modeling context, considering the demand profile and production level along this research work.

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This technical note is integrated in section 2 by the mathematical formulation and solution of the infinite dimensional system for production level modeling, in section 3, a finite dimensional production-inventory system is analyzed with a proper stability analysis. Finally, conclusions and future work are present in section 4.

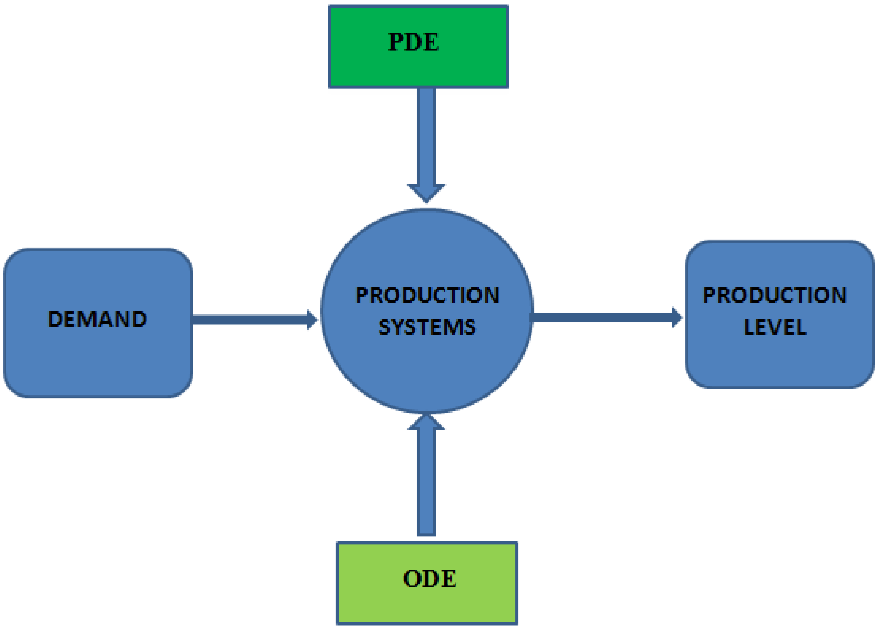

Figure 1. General taxonomy for production systems

2. Infinite dimensional production level modeling

In general, a production system combines humans, machinery and equipment that are next to common material and information flow [23]. In a previous work [24], a PDE for production level subject to dynamic pricing and time, was presented, with the form:

$$v_p \frac{\partial^2 \varphi(p,t)}{\partial p \partial t} + (\beta + a_p) \frac{\partial \varphi(p,t)}{\partial p} + \alpha\beta \frac{\partial \varphi(p,t)}{\partial t} + K\varphi(p,t) = D(p,t) \quad (1)$$

Where $\varphi(p,t)$ is the production level subject to price and time, $D(p,t)$ is the demand level, v_p is the price velocity, a_p is the price acceleration, β is the damping coefficient, α is the scaling production factor and K is the production system resilience, with

$$\beta + a_p > 0.$$

Our aim is to solve equation (1) analytically, subject to similar conditions for which a production system approaches to demand levels near to zero.

Theorem 1. Considering the PDE in (1) assumming that: $D(p,t) = wP^{'}$, $\beta + a_p = 1$ *and* $\alpha\beta = 1$, which gives the following solution:

$$\varphi(p,t) = e^{-\alpha p}\left(e^{-\left(\frac{K-\alpha}{1-\alpha v_p}\right)t} + \frac{\alpha w}{\alpha - K}\right)$$

Proof.

$$v_p \frac{\partial^2 \varphi(p,t)}{\partial p \partial t} + \frac{\partial \varphi(p,t)}{\partial p} + \frac{\partial \varphi(p,t)}{\partial t} + K\varphi(p,t) = wP' \quad (2)$$

Applying the separation of variables method, we have the following:

$$\varphi(p,t) = P(p)T(t). \quad (3)$$

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Substituting equation (3) in the PDE (2):

$$v_p P'T' + P'T - wP' + PT' + KPT = 0. \quad (4)$$

Where: $P' = \frac{dP}{dp}$ and $T' = \frac{dT}{dt}$

Grouping terms from equation (4):

$$P'(v_p T' + T - w) = -P(T' + KT). \quad (5)$$

Separating terms in equation (5):

$$-\frac{P'}{P} = \frac{(T' + KT)}{v_p T' + T - w} = \alpha. \quad (6)$$

For separate ordinary differential equation (ODE), in time domain:

$$\frac{P'}{P} = -\alpha. \quad (7)$$

Therefore:

$$\frac{dP}{P} = -\alpha dp. \quad (8)$$

Solving equation (8):

$$P(p) = e^{-\alpha p}. \quad (9)$$

For the ODE, in time domain:

$$T' + KT = \alpha(v_p T' + T - w). \quad (10)$$

After some algebraic manipulation, equation (10) becomes:

$$T' + \left(\frac{K-\alpha}{1-\alpha v_p}\right) T = -\left(\frac{\alpha w}{1-\alpha v_p}\right). \quad (11)$$

Solving the equation (11), we have:

$$T(t) = Ce^{-\left(\frac{K-\alpha}{1-\alpha v_p}\right)t} + \frac{\alpha w}{K-\alpha}. \quad (12)$$

Finally, the solution for the PDE production level subject to price and time is:

$$\varphi(p,t) = e^{-\alpha p}\left(e^{-\left(\frac{K-\alpha}{1-\alpha v_p}\right)t} + \frac{\alpha w}{\alpha - K}\right). \quad (13)$$

In Figure 2, a PDE production level normalized solution in function of price and time is present, considering a tendency to zero production level conforming price and time tends to infinity.

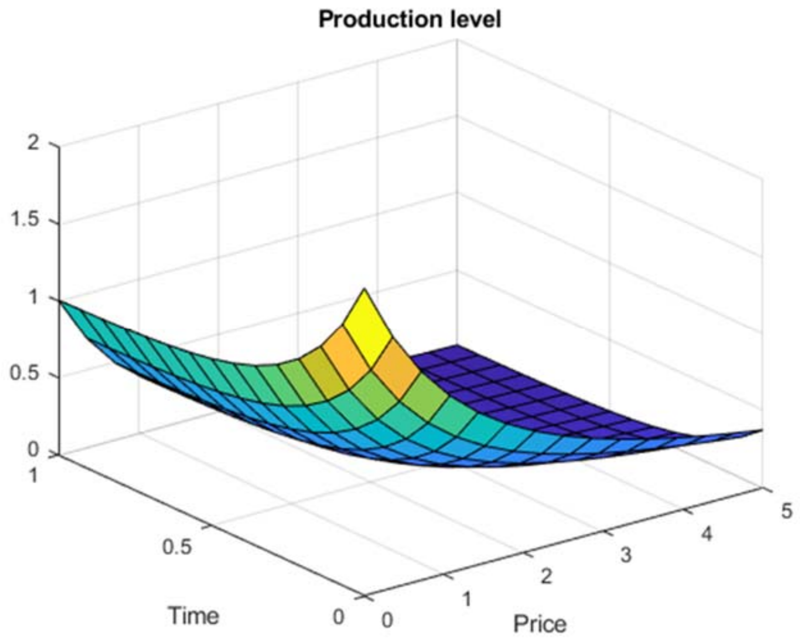

Figure 2. PDE production level solution in function of price and time

3. Finite dimensional production-inventory modeling

In [25] a dynamic production-inventory system is presented via optimal control theory. For mathematical modeling purposes an extension, of this previous work is present here, considering dynamic pricing and time delayed effects.

3.1. Mathematical modeling

A time-delayed production inventory system is proposed via a set of coupled ordinary differential equations which are described as follows: In order to incorporate a dynamic pricing perspective in the production-inventory system, we propose:

$$\frac{dp}{dt} = u_1(t) - p(t) + k\frac{I(t)}{C} \tag{14}$$

Where from equation (14), u_1= denotes the purchase price level; p = denote the sales price level; I= inventory level; k= associated cost for inventory and C=Capacity level.

To develop the production-inventory system dynamics for production level and production rates, the following equation is proposed:

$$C\frac{d^2Q}{dt^2} + I(t)\frac{dQ}{dt} + \gamma Q(t-\theta) = d(t) \tag{15}$$

From equation (15), Q= production level; dQ/dt= production rate; d= demand level, θ= lead time and γ= production resilience factor.

The following ODE describes the inventory level of the system:

$$\frac{dI}{dt} = \mu I(t-\theta) + \gamma Q(t) - d(t) \tag{16}$$

In order to provide a linear approximation, via Taylor series, for the time delay terms as in [26], and neglecting high order terms:

$$x(t-\tau) \approx x(t) - \tau\dot{x}(t) \tag{17}$$

Developing a state space formulation for the time delayed production inventory system from equations (14)-(16) and applying equation (17) our new state space system is:

$$\dot{x_1}(t) = -x_1(t) + k_1 x_4(t) \tag{18}$$

$$\dot{x_2}(t) = x_3(t) \tag{19}$$

$$\dot{x_3}(t) = -k_2 x_3(t) x_4(t) - k_3 x_2(t) + k_4 x_3(t) \tag{20}$$

$$\dot{x_4}(t) = k_5 x_2(t) + k_6 x_4(t) \tag{21}$$

Where $x_1(t) = p(t)$, $x_2(t)=Q(t)$, $x_3(t)=dQ/dt$, $x_4(t) = I(t)$ and the constants $k_1 = k/C$, $k_2 = 1/C$, $k_3 = \gamma/C$, $k_4 = \gamma\theta/C$, $k_5 = \gamma/(1+\mu\theta)$, $k_6 = \mu/(1+\mu\theta)$.

3.2. Stability analysis

Theorem 2. In order to have a stability analysis, for the system of equation 18-21, the following conditions must be satisfied, $\gamma\vartheta > 0$, $\mu > 0$ and $C > 1 + \mu\theta$.

Proof.

In order to proof theorem 2, the following Lyapunov candidate function is proposed:

$$V(x_1, x_2, x_3, x_4) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}(x_3 - x_4)^2 \tag{22}$$

Applying the derivative with respect to time, to the Lyapunov candidate function:

$$\dot{V}(x_1, x_2, x_3, x_4) = x_1\dot{x}_1 + x_2\dot{x}_2 + (x_3 - x_4)(\dot{x}_3 - \dot{x}_4) \tag{23}$$

After some algebraic manipulation, and applying the LaSalle Theorem in the equilibrium point.

$$\dot{V}(x_1, x_2, x_3, x_4) < -x_1^2 + k_4 x_3^2 + k_6 x_4^2 - (k_3 - k_5 - 1)|Q|^2 - (k_4 + k_6)|O|^2 - (k_5 - k_3)|P|^2 \tag{24}$$

Equation 24, can be simplified to:

$$\dot{V}(x_1, x_2, x_3, x_4) < -x_1^2 - (k_4 + k_6)|O|^2 - (k_5 - k_3)|P|^2 \tag{25}$$

In order to achieve stability, the following condition must be satisfied:

$$k_4 + k_6 > 0$$

$$k_5 - k_3 > 0$$

Therefore: $k_4 > 0$ and $k_6 > 0$

Considering that: $k_4 = \frac{\gamma\theta}{C}$ and $k_6 = \frac{\mu}{1+\mu\theta}$

Also,

$$k_5 > k_3$$

From which we can conclude that:

$$\gamma\vartheta > 0 \quad , \quad \mu > 0 \quad \textit{and} \quad C > 1 + \mu\theta$$

3.3. Simulations

Simulations were developed applying MATLAB, for the equations (18)-(21), in which in Figure 3 presents a decaying performance for price level of the finite dimensional production system. This is considering the Equation 14, which presents the dynamic pricing analysis for production-inventory system. Conforme time increases the pricel level tends to zero.

In Figure 4 a production level achieves a maximum for the finite dimensional system which is based on the context that this is the level the production rate. From Figure 5, unproduce is related with a negative production rate performance. This effect is present aroung time unit two, and conforms tends to higher values the production rate decreases.

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Figure 6 presents a decaying inventory level conforms production rate tends to a negative value, which is an effect of the relation of maximum production level towards lower levels for inventory. This is present from Equation 16, which presents a dynamic inventory level with time delaying.

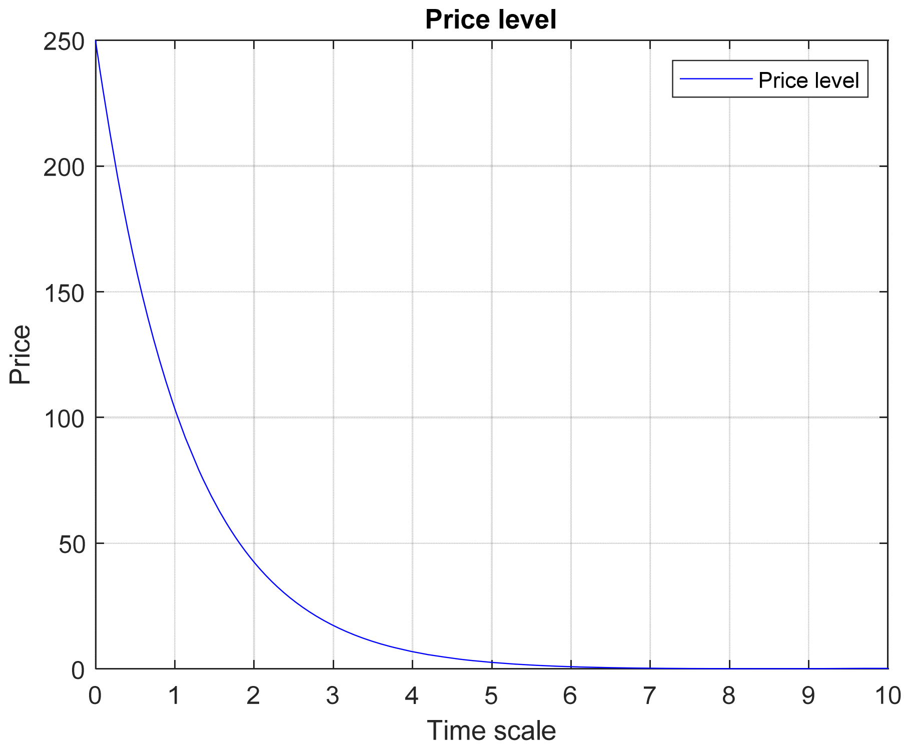

Figure 3. Price level for the finite dimensional production system

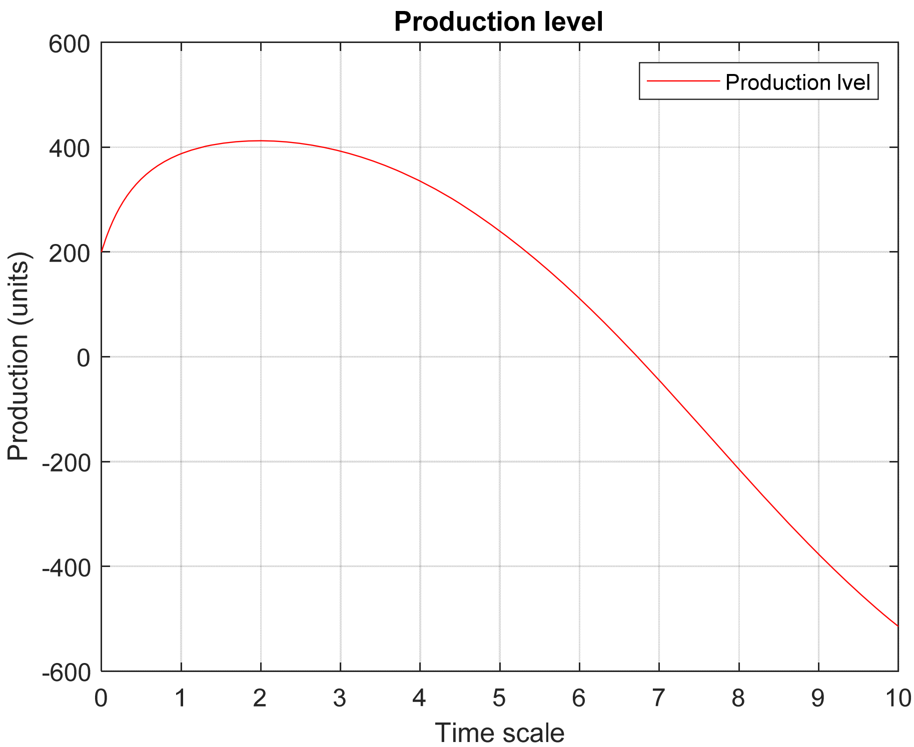

Figure 4. Production level for the finite dimensional production system

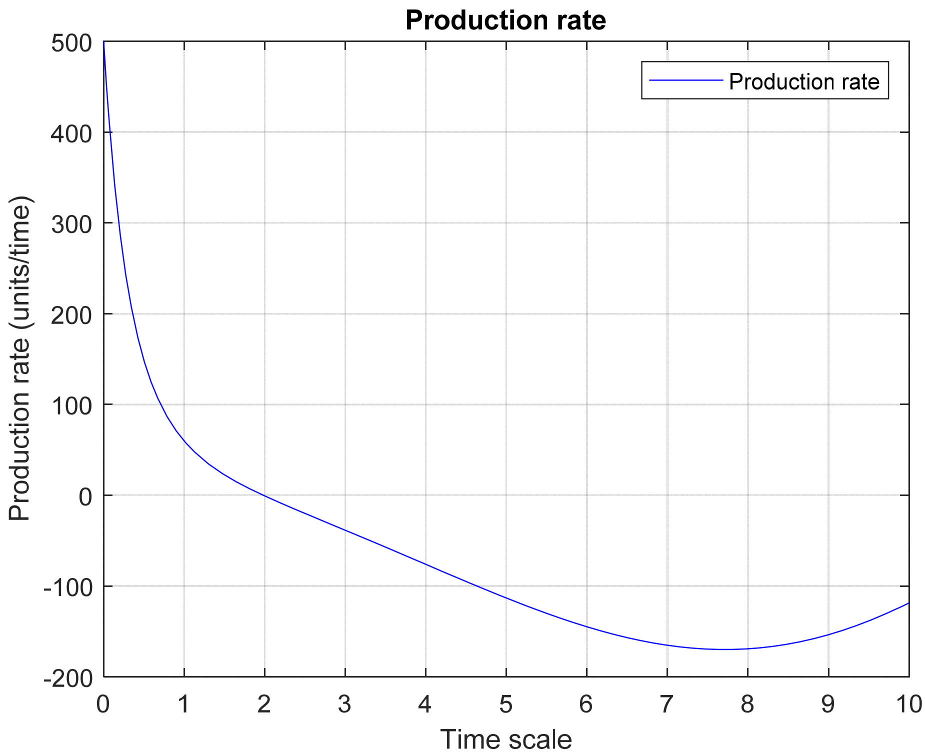

Figure 5. Production rate for the finite dimensional production system

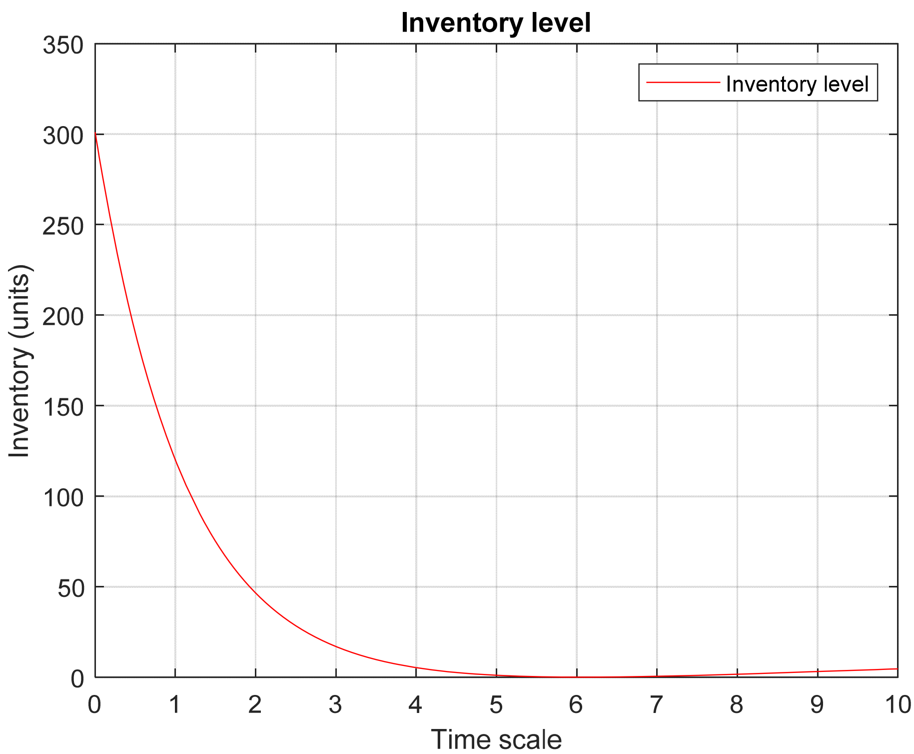

Figure 6. Inventory level for finite dimensional production system

4. Conclusions

This communication presents that mathematical modeling in high volume production systems, is important via the application of infinite and finite dimensional approaches, which corresponds to the use of partial differential equations (PDE) and ordinary differential equations (ODE), respectively.

This mathematical modeling considers low variability and cause-effect relation for deterministic approach in high volume production systems. This research work presents two novel approaches via PDE and ODE to model high volume production systems, with emphasis on demand fluctuation. In general, demand fluctuation corresponds to product demand variation, which is increased or decreased in a time horizon period.

In future work, our interest is to apply optimal control theory in order to extend the results based on mathematical modeling for both dynamical systems.

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