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Parallel Subgradient-like Extragradient Approaches for Variational Inequality and Fixed Point Problems with Bregman Relatively Asymptotical Nonexpansivity

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Abstract: In a uniformly smooth and p-uniformly convexBanach space, let the pair of variational inequality and fixed point problems (VIFPPs) consist of two variational inequality problems (VIPs) involving two uniformly continuous and pseudomonotone mappings and two fixed point problems implicating two uniformly continuous and Bregman relatively asymptotically nonexpansive mappings. This article designs two parallel subgradient-like extragradient algorithms with inertial effect for solving this pair of VIFPPs, where each algorithm consists of two parts which are of symmetric structure mutually. With the help of suitable registrations, it is proven that the sequences generated by the suggested algorithms converge weakly and strongly to a solution of this pair of VIFPPs, respectively. Lastly, an illustrative instance is furnished to verify the implementability and applicability of the suggested approaches.

Keywords: parallel subgradient-like extragradient approach; variational inequality problem; inertial effect; bregman relatively asymptotically nonexpansive mapping; bregman distance; bregman projection

MSC: 47H05; 47H10; 65K15; 65Y05; 68W25

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1. Introduction

Let $\emptyset \neq C \subset H$ and P_C be the metric projection from H onto C with C being convex and closed in a real Hilbert space H. Suppose that the $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and induced norm in H, respectively. Given a nonlinear operator $S : C \to C$. We denote by $\text{Fix}(S)$ the fixed-point set of S. Also, the $\mathbf{R}$, $\rightharpoonup$ and $\to$ are used to stand for the real-number set, the weak convergence and the strong convergence, respectively. A self-mapping S on C is known as being of asymptotical nonexpansivity if $\exists$ (nonnegative real sequence) $\{\theta_n\}$ s.t.

$$(\theta_n + 1)\|v - u\| \geq \|S^n v - S^n u\| \quad \forall v, u \in C,\ n \geq 1, \tag{1.1}$$

with $\theta_n \to 0$. In particular, in case $\theta_n = 0\ \forall n \geq 1$, S reduces to a nonexpansive mapping. Let $A : H \to H$ be a mapping. Recall that so-called variational inequality problem (VIP) is to find $v \in C$ such that

$$\langle Av, x - v \rangle \geq 0 \quad \forall x \in C, \tag{1.2}$$

Here $\text{VI}(C, A)$ denotes the set of solutions of the VIP. In 1976, under weaker assumptions, Korpelevich [24] put forward the extragradient rule for approximating an element of $\text{VI}(C, A)$, i.e., for any starting $t_0 \in C$, $\{t_n\}$ is the sequence generated by

$$\begin{cases} s_n = P_C(t_n - \epsilon A t_n), \\ t_{n+1} = P_C(t_n - \epsilon A s_n) \quad \forall n \geq 0, \end{cases}$$

with $\epsilon \in (0, \frac{1}{L})$. If $\mathrm{VI}(C, A) \neq \emptyset$, then $\{t_n\}$ converges weakly to an element in $\mathrm{VI}(C, A)$. To the best of our understanding, the Korpelevich extragradient rule is one of the most effective approaches for solving the VIP till now. The literature on the VIP is vast and the Korpelevich extragradient rule has acquired the extensive attention paid by numerous scholars, who ameliorated it in various ways; see e.g., [1-6, 8-9, 13-16, 19, 21-23, 25-28, 31, 34].

Recently, Thong and Hieu [21] first put forth the inertial subgradient extragradient rule, i.e., for any starting $t_0, t_1 \in H$, $\{t_n\}$ is the sequence generated by

$$\begin{cases} y_n = t_n + \alpha_n(t_n - t_{n-1}), \\ s_n = P_C(y_n - \ell A y_n), \\ C_n = \{v \in H : \langle y_n - \ell A y_n - s_n, v - s_n \rangle \leq 0\}, \\ t_{n+1} = P_{C_n}(y_n - \ell A s_n) \quad \forall n \geq 1, \end{cases}$$

with constant $\ell \in (0, \frac{1}{L})$. Under suitable assumptions, they proved the weak convergence of $\{t_n\}$ to an element of $\mathrm{VI}(C, A)$. Subsequently, Thong and Hieu [15] introduced two inertial subgradient extragradient algorithms with linear-search process for solving the VIP with Lipschitz continuous monotone mapping A and the fixed-point problem (FPP) of a quasi-nonexpansive mapping S with demiclosedness property in H.

Algorithm 1.1 (see [15, Algorithm 1]). **Initialization:** Given $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$. Choose any initial $t_0, t_1 \in H$.

Iterations: Compute t_{n+1} below:

Step 1. Put $s_n = t_n + \alpha_n(t_n - t_{n-1})$ and calculate $y_n = P_C(s_n - \ell_n A s_n)$, wherein ℓ_n is picked as the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ s.t. $\ell\|As_n - Ay_n\| \leq \mu\|s_n - y_n\|$.

Step 2. Calculate $u_n = P_{C_n}(s_n - \ell_n A y_n)$, where $C_n := \{u \in H : \langle s_n - \ell_n A s_n - y_n, u - y_n \rangle \leq 0\}$.

Step 3. Calculate $t_{n+1} = (1 - \beta_n)s_n + \beta_n S u_n$. When $s_n = u_n = t_{n+1}$, one has $s_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Put $n := n + 1$ and return to Step 1.

Algorithm 1.2 (see [15, Algorithm 2]). **Initialization:** Given $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$. Choose any initial $t_0, t_1 \in H$.

Iterative steps: Compute t_{n+1} below:

Step 1. Putt $s_n = t_n + \alpha_n(t_n - t_{n-1})$ and calculate $y_n = P_C(s_n - \ell_n A s_n)$, wherein ℓ_n is picked as the largest $\ell \in \{\gamma, \gamma l, \gamma l^2, ...\}$ s.t. $\ell\|As_n - Ay_n\| \leq \mu\|s_n - y_n\|$.

Step 2. Calculate $u_n = P_{C_n}(s_n - \ell_n A y_n)$, where $C_n := \{u \in H : \langle s_n - \ell_n A s_n - y_n, u - y_n \rangle \leq 0\}$.

Step 3. Calculate $t_{n+1} = (1 - \beta_n)t_n + \beta_n S u_n$. When $s_n = u_n = t_n = t_{n+1}$, one has $t_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Put $n := n + 1$ and return to Step 1.

With the help of suitable assumptions, it was proved in [15] that the sequences generated by the suggested algorithms converge weakly to a point in $\mathrm{VI}(C, A) \cap \mathrm{Fix}(S)$. In addition, combining the subgradient extragradient method and the Halpern's iteration rule, Kraikaew and Saejung [22] proposed the Halpern subgradient extragradient rule for solving the VIP, and showed that the sequence generated by the proposed rule converges strongly to a point in $\mathrm{VI}(C, A)$. Recently, Reich et al. [27] put forward two gradient-projection algorithms for solving the VIP for uniformly continuous pseudomonotone mapping. In particular, they used a novel Armijo-type line search to acquire a hyperplane which strictly separates the current iterate from the solutions of the VIP under consideration. They proved that the sequences generated by two algorithms converge weakly and strongly to a point in $\mathrm{VI}(C, A)$, respectively.

On the other hand, let $\emptyset \neq C \subset E$ where C is convex and closed in a uniformly smooth and p-uniformly convex Banach space E for $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Let J_E^p be the duality mapping of E, and let E^* be the dual of E with the duality $J_{E^*}^q$. Suppose that the norm and the duality pairing between E and E^* are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $f_p(u) = \|u\|^p/p \; \forall u \in E$, D_{f_p} be the Bregman distance with respect to (w.r.t) f_p and Π_C be Bregman's projection w.r.t. f_p from E onto C, and presume that $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ s.t. $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Assume that $A : E \to E^*$ is uniformly continuous and pseudomonotone operator and S is Bregman relatively nonexpansive self-mapping on C. Very recently, inspired by the research works in [27], Eskandani et al. [31] proposed the hybrid projection approach with linesearch process for approximating a point in $\mathrm{VI}(C, A) \cap \mathrm{Fix}(S)$.

Algorithm 1.3 (see [31]). **Initialization:** Given $l \in (0,1)$, $\nu > 0$, $\lambda \in (0, \frac{1}{\nu})$ and choose $u, t_1 \in C$ randomly.
Iterations: Compute t_{n+1} $(n \ge 1)$ below:
Step 1. Calculate $y_n = \Pi_C(J_{E^*}^q(J_E^p t_n - \lambda A t_n))$ and $r_\lambda(t_n) := t_n - y_n$. If $r_\lambda(t_n) = 0$ and $St_n = t_n$, then stop; $t_n \in \blacksquare = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S)$. If this case does not occur, then,
Step 2. Calculate $s_n = t_n - \epsilon_n r_\lambda(t_n)$, with both $\epsilon_n := l^{k_n}$ and k_n being the smallest $k \ge 0$ s.t. $\langle At_n - A(t_n - l^k r_\lambda(t_n)), r_\lambda(t_n)\rangle \le \frac{\nu}{2} D_{f_p}(t_n, y_n)$.
Step 3. Calculate $u_n = J_{E^*}^q(\beta_n J_E^p t_n + (1-\beta_n) J_E^p(S\Pi_{C_n} t_n))$ and $t_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p u_n))$, with $C_n := \{y \in C : 0 \ge h_n(y)\}$ and $h_n(y) = \langle As_n, y - t_n\rangle + \frac{\epsilon_n}{2\lambda} D_{f_p}(t_n, y_n)$.
Again put $n := n+1$ and return to Step 1.

With the help of suitable conditions, it was proven in [31] that $\{t_n\}$ converges strongly to $\Pi_{\blacksquare} u$.

This article designs two parallel subgradient-like extragradient algorithms with inertial effect for resolving a pair of variational inequality and fixed point problems (VIFPPs) in uniformly smooth and p-uniformly convex Banach space E. Here two variational inequality problems (VIPs) involve two uniformly continuous pseudomonotone operators and two fixed point problems implicate two uniformly continuous Bregman relatively asymptotically nonexpansive mappings. Moreover, each algorithm consists of two parts which are of symmetric structure mutually. With the help of appropriate registrations, it is proven that the sequences generated by the suggested algorithms converge weakly and strongly to a solution of this pair of VIFPPs, respectively. Lastly, an illustrative instance is furnished to check the implementability and applicability of the proposed approaches.

The structure of the article is described as follows: Section 2 releases certain terminologies and preliminary results for later applications. Section 3 is focused on discussing the convergence of the suggested algorithms. In Section 4, the major outcomes are utilized to deal with the CFPP and VIPs in an illustrative instance. Our results improve and develop the revelent ones obtained previously in [15, 27, 31].

2. Preliminaries

Let $(E, \|\cdot\|)$ be a real Banach space, whose dual is denoted by E^*. We use the $y_n \to y$ and $y_n \rightharpoonup y$ to represent the strong and weak convergence of $\{y_n\}$ to $y \in E$, respectively. Moreover, the set of weak cluster points of $\{y_n\}$ is denoted by $\omega_w(y_n)$, i.e., $\omega_w(y_n) = \{y^\dagger \in E : y_{n_k} \rightharpoonup y^\dagger$ for some $\{y_{n_k}\} \subset \{y_n\}\}$. Let $U = \{y \in E : \|y\| = 1\}$ and $1 < q \le 2 \le p$ with $\frac{1}{p} + \frac{1}{q} = 1$. A Banach space E is referred to as being strictly convex if for each $y, x \in U$ with $y \ne x$, one has $\|y + x\|/2 < 1$. E is referred to as being uniformly convex if $\forall \varsigma \in (0,2]$, $\exists \delta > 0$ s.t. $\forall y, x \in U$ with $\|y - x\| \ge \varsigma$, one has $\|y + x\|/2 \le 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. The modulus of convexity of E is the function $\delta : [0,2] \to [0,1]$ defined by $\delta(\varsigma) = \inf\{1 - \|y + x\|/2 : y, x \in U \text{ with } \|y - x\| \ge \varsigma\}$. It is also known that E is uniformly convex if and only if $\delta(\varsigma) > 0$ $\forall \varsigma \in (0,2]$. Moreover, E is referred to as being p-uniformly convex if $\exists c > 0$ s.t. $\delta(\varsigma) \ge c\varsigma^p$ for $0 \le \varsigma \le 2$.

The nonnegative function $\rho_E(\cdot)$ on $[0,\infty)$ is called the modulus of smoothness of E if $\rho_E(\varsigma) := \sup\{(\|y + \varsigma x\| + \|y - \varsigma x\|)/2 - 1 : y, x \in U\}$ $\forall \varsigma \in [0,\infty)$. E is said to be uniformly smooth if $\lim_{\varsigma\to 0} \rho_E(\varsigma)/\varsigma = 0$, and q-uniformly smooth if $\exists C_q > 0$ s.t. $\rho_E(\varsigma) \le C_q \varsigma^q$ $\forall \varsigma > 0$. Recall that E is of p-uniform convexity iff E^* is of q-uniform smoothness; see e.g., [32] for more details. Putting $B(0,r) = \{y \in E : \|y\| \le r\}$ for each

$r > 0$, we say that $f : E \to \mathbf{R}$ is uniformly convex on bounded sets (see [31]) if $\rho_r(\varsigma) > 0\ \forall r, \varsigma > 0$, where $\rho_r(\varsigma) : [0, \infty) \to [0, \infty]$ is specified below

$$\begin{aligned}\rho_r(\varsigma) \quad &= \inf\{[\epsilon f(y) + (1-\epsilon)f(x) - f(\epsilon y + (1-\epsilon)x)]/\epsilon(1-\epsilon) : \\ &\quad \epsilon \in (0,1) \text{ and } y, x \in B(0,r) \text{ with } \|y - x\| = \varsigma\} \quad \forall \varsigma \geq 0.\end{aligned}$$

The ρ_r is known as the gauge function of f with uniform convexity. It is clear that the gauge ρ_r is nondecreasing.

Let $f : E \to \mathbf{R}$ be a convex function. If the limit $\lim_{\varsigma \to 0^+} \frac{f(y+\varsigma x)-f(y)}{\varsigma}$ exists for each $x \in E$, then f is referred to as being Gâteaux differentiable at y. In this case, the gradient $\nabla f(y)$ of f at y is of linearity, and is formulated as $\langle \nabla f(y), x\rangle := \lim_{\varsigma \to 0^+} \frac{f(y+\varsigma x)-f(y)}{\varsigma}\ \forall x \in E$. The f is referred to as being of Gâteaux differentiablility if it is of Gâteaux differentiablility at any $y \in E$. In case $\lim_{\varsigma \to 0^+} \frac{f(y+\varsigma x)-f(y)}{\varsigma}$ is achieved uniformly for any $x \in U$, we say that f is of Fréchet differentiablility at y. Besides, f is referred to as being of uniform Fréchet differentiablility on a subset $K \subset E$ if $\lim_{\varsigma \to 0^+} \frac{f(y+\varsigma x)-f(y)}{\varsigma}$ is achieved uniformly for $(y, x) \in K \times U$. When the norm of E is of Gâteaux differentiablility, E is said to be of smoothness.

Let $\frac{1}{p} + \frac{1}{q} = 1$ for $p, q > 1$. The $J_E^p : E \to E^*$ is specified below

$$J_E^p(y) = \{\varphi \in E^* : \langle \varphi, y\rangle = \|y\|^p \text{ and } \|\varphi\| = \|y\|^{p-1}\} \quad \forall y \in E.$$

It is known that E is of smoothness iff J_E^p is of single value from E into E^*. Also, E is of reflexivity iff J_E^p is of surjectivity, and E is strictly convex iff J_E^p is of one-to-one property. So it follows that, when the smooth Banach space E is of both strict convexity and reflexivity, J_E^p is the bijection and in this case, $J_{E^*}^q = (J_E^p)^{-1}$. Also, recall that E is of uniform smoothness iff the function $f_p(y) = \|y\|^p/p$ is of uniform Fréchet differentiablility on bounded sets iff J_E^p is of uniform continuity on bounded sets. Moreover, E is of uniform convexity iff the function f_p is of uniform convexity (see [32]).

Let the function $f : E \to \mathbf{R}$ be of both Gâteaux differentiablility and convexity. Bregman's distance w.r.t. f is specified below

$$D_f(t, s) := f(t) - f(s) - \langle \nabla f(s), t - s\rangle \quad \forall t, s \in E.$$

It is worth mentioning that the $D_f(\cdot,\cdot)$ is not a metric in the common sense of the terminology. Evidently, $D_f(t,t) = 0$ but $D_f(t,s) = 0$ can not lead to $t = s$. Generally, D_f is not of symmetry and fulfill no triangle inequality. However, D_f fulfills the three point equequality

$$D_f(t,s) + D_f(s,u) = D_f(t,u) - \langle \nabla f(s) - \nabla f(u), t - s\rangle.$$

See [20] for many details.

It is remarkable that the J_E^p on the smooth E is Gâteaux's derivative of f_p. Thus, Bregman's distance w.r.t. f_p is specified below

$$\begin{aligned}D_{f_p}(y,x) \quad &= \|y\|^p/p - \|x\|^p/p - \langle J_E^p(x), y - x\rangle \\ &= \|y\|^p/p + \|x\|^p/q - \langle J_E^p(x), y\rangle \\ &= (\|x\|^p - \|y\|^p)/q - \langle J_E^p(x) - J_E^p(y), y\rangle.\end{aligned}$$

In the p-uniformly convex and smooth Banach space E for $2 \leq p < \infty$, there holds the following relationship between the metric and Bregman distance:

$$\tau\|y - x\|^p \leq D_{f_p}(y,x) \leq \langle J_E^p(y) - J_E^p(x), y - x\rangle, \tag{2.1}$$

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where $\tau > 0$ is some fixed number (see [12]). Via (2.1) it can be easily seen that for each $\{y_n\} \subset E$ of boundedness, the relation is valid:

$$y_n \to y \quad \Leftrightarrow \quad D_{f_p}(y_n, y) \text{ converges to } 0 \quad \text{ as } n \to \infty.$$

Let $\emptyset \neq C \subset E$ with C being convex and closed in a strictly convex, smooth and reflexive Banach space E. Bregman's projection is formulated as minimizers of Bregman's distance. Bregman's projection of $y \in E$ onto C w.r.t. f_p indicates a unique point $\Pi_C y \in C$ s.t. $D_{f_p}(\Pi_C y, y) = \min_{x \in C} D_{f_p}(x, y)$. In the case of Hilbert space, Bregman's projection w.r.t. f_2 reduces to the metric projection. Using [30, Theorem 2.1] and [18, Corollary 4.4], in a uniformly convex Banach space, the characterization of Bregman's projection is formulated by:

$$\langle J_E^p(y) - J_E^p(\Pi_C y), x - \Pi_C y\rangle \leq 0 \quad \forall x \in C. \tag{2.2}$$

Meantime, (2.2) is equivalent to the descent property

$$D_{f_p}(x, \Pi_C y) + D_{f_p}(\Pi_C y, y) \leq D_{f_p}(x, y) \quad \forall x \in C. \tag{2.3}$$

When $p = 2$, J_E^p reduces to the normalized duality mapping and is written as J. The $\phi : E^2 \to \mathbf{R}$ is formulated below

$$\phi(t, s) = \|t\|^2 - 2\langle Js, t\rangle + \|s\|^2 \quad \forall t, s \in E,$$

and $\Pi_C(t) = \mathrm{argmin}_{s \in C} \phi(s, t)\ \forall t \in E$.

In terms of [31], the function $V_{f_p} : E \times E^* \to [0, \infty)$ associated with f_p is specified below

$$V_{f_p}(y, y^*) = \|y\|^p/p - \langle y^*, y\rangle + \|y^*\|^q/q \quad \forall (y, y^*) \in E \times E^*. \tag{2.4}$$

So, $V_{f_p}(y, y^*) = D_{f_p}(y, J_{E^*}^q(y^*))\ \forall (y, y^*) \in E \times E^*$. Moreover, by the subdifferential inequality, we obtain

$$V_{f_p}(y, y^*) + \langle x^*, J_{E^*}^q(y^*) - y\rangle \leq V_{f_p}(y, y^* + x^*) \quad \forall y \in E,\ y^*, x^* \in E^*. \tag{2.5}$$

In addition, V_{f_p} is convex in the second variable. Hence one has

$$D_{f_p}(z, J_{E^*}^q(\sum_{j=1}^n \varsigma_j J_E^p(y_j))) \leq \sum_{j=1}^n \varsigma_j D_{f_p}(z, y_j) \ \ \forall z \in E, \{y_j\}_{j=1}^n \subset E, \{\varsigma_j\}_{j=1}^n \subset [0,1] \text{ with } \sum_{j=1}^n \varsigma_j = 1. \tag{2.6}$$

Lemma 2.1 ([30]). Let E be a uniformly convex Banach space and $\{s_n\}, \{t_n\}$ be two sequences in E such that the first one is bounded. If $\lim_{n\to\infty} D_{f_p}(t_n, s_n) = 0$, then $\lim_{n\to\infty} \|t_n - s_n\| = 0$.

Assume that S is a self-mapping on C. Let the $\mathrm{Fix}(S)$ indicate the set of fixed points of S, that is, $\mathrm{Fix}(S) = \{y \in C : y = Sy\}$. A point $y^\dagger \in C$ is referred to as an asymptotic fixed point of S if $\exists\{y_n\} \subset C$ s.t. $y_n \rightharpoonup y^\dagger$ and $(I - S)y_n \to 0$. Let the $\widehat{\mathrm{Fix}}(S)$ denote the asymptotic fixed point set of S. The terminology of asymptotic fixed points was invented in [11]. A self-mapping S on C is known as being the one of Bregman's relatively asymptotical nonexpansivity w.r.t. f_p if $\mathrm{Fix}(S) = \widehat{\mathrm{Fix}}(S) \neq \emptyset$, and $\exists\{\theta_n\} \subset [0, \infty)$ with both $\theta_n \to 0$ $(n \to \infty)$ and

$$(\theta_n + 1)D_{f_p}(y, x) \geq D_{f_p}(y, S^n x) \quad \forall y \in \mathrm{Fix}(S), x \in C,\ n \geq 1.$$

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In particular, if $\theta_n = 0 \; \forall n \geq 1$, then S reduces to the one of Bregman's relatively nonexpansivity w.r.t. f_p, that is, $\mathrm{Fix}(S) = \widehat{\mathrm{Fix}}(S) \neq \emptyset$ and $D_{f_p}(y, Sx) \leq D_{f_p}(y, x) \; \forall y \in \mathrm{Fix}(S), x \in C$. In addition, a mapping $A : C \to E^*$ is known as being

(i) monotone on C if $\langle Av - Ay, v - y\rangle \geq 0 \; \forall v, y \in C$;

(ii) pseudo-monotone if $\langle Ay, v - y\rangle \geq 0 \Rightarrow \langle Av, v - y\rangle \geq 0 \; \forall v, y \in C$;

(iii)ℓ-Lipschitz continuous or ℓ-Lipschitzian if $\exists \ell > 0$ s.t. $\|At - Ay\| \leq \ell\|t - y\| \forall t, y \in C$;

(iv) of weakly sequential continuity if $\forall \{t_n\} \subset C$,the relation holds: $t_n \rightharpoonup t \Rightarrow At_n \rightharpoonup At$.

Lemma 2.2 ([31]). Let $r > 0$ be a constant and suppose that $f : E \to \mathbf{R}$ is a uniformly convex function on any bounded subset of a Banach space E. Then

$$f(\sum_{k=1}^{n} \alpha_k t_k) \leq \sum_{k=1}^{n} \alpha_k f(t_k) - \alpha_i \alpha_j \rho_r(\|t_i - t_j\|),$$

$\forall i, j \in \{1, 2, ..., n\}$, $\{t_k\}_{k=1}^n \subset B(0, r)$ and $\{\alpha_k\}_{k=1}^n \subset (0, 1)$ for $\sum_{k=1}^n \alpha_k = 1$, with ρ_r being the gauge of f with uniform convexity.

Proof. It is easy to show the conclusion.

Lemma 2.3 ([28]). Let E_i be a Banach space for $i = 1, 2$ and suppose that $A : E_1 \to E_2$ is of uniform continuity on any bounded subset of E_1 and $D \subset E_1$ is of boundedness. Then $A(D) \subset E_2$ is of boundedness.

Lemma 2.4 ([10]). Assume $\emptyset \neq C \subset E$ with C being convex and closed, and let $A : C \to E^*$ be of both pseudomonotonicity and continuity. Given $y^\dagger \in C$. Then $\langle Ay^\dagger, y - y^\dagger\rangle \geq 0 \; \forall y \in C \; \Leftrightarrow \; \langle Ay, y - y^\dagger\rangle \geq 0 \; \forall y \in C$.

Lemma 2.5. Suppose that E is a smooth and p-uniformly convex Banach space for $p \geq 2$, where J_E^p is of weakly sequential continuity. Assume $\{s_n\} \subset E$ and $\emptyset \neq \blacksquare \subset E$. If $\omega_w(s_n) \subset \blacksquare$, and $\{D_{f_p}(z, s_n)\}$ converges for each $z \in \blacksquare$. Then one has the weak convergence of $\{s_n\}$ to an element of $\blacksquare$.

Proof. First, one has $\tau\|z - s_n\|^p \leq D_{f_p}(z, s_n) \; \forall z \in \blacksquare$ by (2.1). Thus we obtain that $\{s_n\}$ is of boundedness. Since E is reflexive, we get $\omega_w(s_n) \neq \emptyset$. Also, one claims that $\{s_n\}$ converges weakly to an element of $\blacksquare$. Indeed, let $\bar{s}, \hat{s} \in \omega_w(s_n)$ with $\bar{s} \neq \hat{s}$. Then, $\exists\{s_{n_k}\} \subset \{s_n\}$ and $\exists\{s_{m_k}\} \subset \{s_n\}$ s.t. $s_{n_k} \rightharpoonup \bar{s}$ and $s_{m_k} \rightharpoonup \hat{s}$. Since J_E^p is weakly sequentially continuous, we obtain both $J_E^p(s_{n_k}) \rightharpoonup J_E^p\bar{s}$ and $J_E^p(s_{m_k}) \rightharpoonup J_E^p\hat{s}$. Note that $D_{f_p}(\bar{s}, \hat{s}) + D_{f_p}(\hat{s}, s_n) = D_{f_p}(\bar{s}, s_n) - \langle J_E^p\hat{s} - J_E^p s_n, \bar{s} - \hat{s}\rangle$. So, utilizing the convergence of the sequences $\{D_{f_p}(\bar{s}, s_n)\}$ and $\{D_{f_p}(\hat{s}, s_n)\}$, we conclude that

$$\begin{aligned}
-\langle J_E^p\hat{s} - J_E^p\bar{s}, \bar{s} - \hat{s}\rangle &= \lim_{k\to\infty}[-\langle J_E^p\hat{s} - J_E^p s_{n_k}, \bar{s} - \hat{s}\rangle] \\
&= \lim_{n\to\infty}[D_{f_p}(\bar{s}, \hat{s}) + D_{f_p}(\hat{s}, s_n) - D_{f_p}(\bar{s}, s_n)] \\
&= \lim_{k\to\infty}[-\langle J_E^p\hat{s} - J_E^p s_{m_k}, \bar{s} - \hat{s}\rangle] = -\langle J_E^p\hat{s} - J_E^p\hat{s}, \bar{s} - \hat{s}\rangle = 0,
\end{aligned}$$

which hence yields $\langle J_E^p\bar{s} - J_E^p\hat{s}, \bar{s} - \hat{s}\rangle = 0$. From (2.1) we get $0 < \tau\|\bar{s} - \hat{s}\|^p \leq D_{f_p}(\bar{s}, \hat{s}) \leq \langle J_E^p\bar{s} - J_E^p\hat{s}, \bar{s} - \hat{s}\rangle = 0$. This arrives at a contradiction. Thereupon, this means that $\{s_n\}$ converges weakly to an element of $\blacksquare$.

The lemma below was put forth in $\mathbf{R}^m$ by [29]. It is easy to verify that the proof of the lemma in Banach spaces is actually the same as in $\mathbf{R}^m$.

Lemma 2.6. Assume $\emptyset \neq C \subset E$ with C being convex and closed. Suppose that $K := \{y \in C : h(y) \leq 0\}$ where $h : E \to \mathbf{R}$ is defined on E. If $K \neq \emptyset$ and h is Lipschitz continuous on C with modulus $\theta > 0$, then $\theta \mathrm{dist}(x, K) \geq \max\{h(x), 0\} \; \forall x \in C$, where $\mathrm{dist}(x, K)$ stands for the distance of x to K.

Lemma 2.7 ([17]). Let $\{\mathbf{\Gamma}_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\mathbf{\Gamma}_{n_k}\} \subset \{\mathbf{\Gamma}_n\}$ s.t. $\mathbf{\Gamma}_{n_k} < \mathbf{\Gamma}_{n_k+1}$ for all k. Assume that $\{\varphi(n)\}_{n\geq n_0}$ is defined as $\varphi(n) = \max\{k \leq n : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}$, with integer $n_0 \geq 1$ satisfying $\{k \leq n_0 : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\} \neq \emptyset$. Then the following hold:

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(i) $\varphi(n_0) \le \varphi(n_0+1) \le \cdots$ and $\varphi(n) \to \infty$;
(ii) $\mathbf{\Gamma}_{\varphi(n)} \le \mathbf{\Gamma}_{\varphi(n)+1}$ and $\mathbf{\Gamma}_n \le \mathbf{\Gamma}_{\varphi(n)+1}$ $\forall n \ge n_0$.

Lemma 2.8 ([7]). Let $\{\sigma_n\}$ be a sequence in $[0,\infty)$ satisfying $\sigma_{n+1} \le (1-\mu_n)\sigma_n + \mu_n c_n$ $\forall n \ge 1$, with $\{\mu_n\}$ and $\{c_n\}$ being real sequences satisfying the conditions: (i) $\{\mu_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \mu_n = \infty$; and (ii) $\limsup_{n\to\infty} c_n \le 0$ or $\sum_{n=1}^{\infty} |\mu_n c_n| < \infty$. Then $\sigma_n \to 0$ as $n \to \infty$.

Lemma 2.9 ([33]). Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le (1+\delta_n)a_n + b_n$ $\forall n \ge 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, let $\emptyset \ne C \subset E$ with C being convex and closed in uniformly smooth and p-uniformly convex Banach space E for $p \ge 2$. We are now in a position to present and analyze our iterative algorithms for approximating a common solution of a pair of VIFPPs, where each algorithm consists of two parts which are of symmetric structure mutually. Assume always that the following conditions hold:

(C1) $S_1, S_2 : C \to C$ are the mappings of both uniform continuity and Bregman's relatively asymptotical nonexpansivity with sequences $\{\theta_n\}_{n=1}^{\infty}$ and $\{\bar{\theta}_n\}_{n=1}^{\infty}$, respectively.

(C2) For $i = 1,2$, $A_i : E \to E^*$ is of both uniform continuity and pseudomonotonicity on C, s.t. $\|A_i y^\dagger\| \le \liminf_{n\to\infty} \|A_i y_n\|$ $\forall \{y_n\} \subset C$ with $y_n \rightharpoonup y^\dagger$.

(C3) $\blacksquare = \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S_i) \ne \emptyset$.

Algorithm 3.1. Initialization: Given $x_0, x_1 \in C$ arbitrarily and let $\epsilon > 0$, $\mu_i > 0$, $\lambda_i \in (0, \frac{1}{\mu_i})$, $l_i \in (0,1)$ for $i = 1,2$. Choose $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\ell_n\} \subset (0,\infty)$ s.t. $0 < \liminf_{n\to\infty} \alpha_n(1-\alpha_n)$, $0 < \liminf_{n\to\infty} \beta_n(1-\beta_n)$ and $\sum_{n=1}^{\infty} \ell_n < \infty$. Moreover, assume $\sum_{n=1}^{\infty} \theta_n < \infty$, and given the iterates x_{n-1} and x_n $(n \ge 1)$, choose ϵ_n s.t. $0 \le \epsilon_n \le \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\|}\} & \text{if } x_n \ne x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterations: Compute x_{n+1} below:

Step 1. Put $g_n = J_{E^*}^q((1-\epsilon_n)J_E^p S_1^n x_n + \epsilon_n J_E^p(S_1^n x_n + x_n - x_{n-1}))$ and calculate $s_n = J_{E^*}^q(\beta_n J_E^p x_n + (1-\beta_n)J_E^p g_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p s_n - \lambda_1 A_1 s_n))$, $e_{\lambda_1}(s_n) := s_n - y_n$ and $t_n = s_n - \tau_n e_{\lambda_1}(s_n)$, with $\tau_n := l_1^{k_n}$ and k_n being the smallest $k \ge 0$ s.t.

$$\frac{\mu_1}{2} D_{f_p}(s_n, y_n) \ge \langle A_1 s_n - A_1(s_n - l_1^k e_{\lambda_1}(s_n)), s_n - y_n\rangle. \tag{3.1}$$

Step 2. Calculate $w_n = \Pi_{C_n}(s_n)$, with $C_n := \{y \in C : h_n(y) \le 0\}$ and

$$h_n(y) = \langle A_1 t_n, y - s_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(s_n, y_n). \tag{3.2}$$

Step 3. Calculate $\bar{y}_n = \Pi_C(J_{E^*}^q(J_E^p w_n - \lambda_2 A_2 w_n))$, $e_{\lambda_2}(w_n) := w_n - \bar{y}_n$ and $\bar{t}_n = w_n - \bar{\tau}_n e_{\lambda_2}(w_n)$, with $\bar{\tau}_n := l_2^{j_n}$ and j_n being the smallest $j \ge 0$ s.t.

$$\frac{\mu_2}{2} D_{f_p}(w_n, \bar{y}_n) \ge \langle A_2 w_n - A_2(w_n - l_2^j e_{\lambda_2}(w_n)), w_n - \bar{y}_n\rangle. \tag{3.3}$$

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Step 4. Calculate $v_n = J^q_{E^*}(\alpha_n J^p_E w_n + (1-\alpha_n)J^p_E(S^n_2 w_n))$ and $x_{n+1} = \Pi_{\bar{C}_n \cap Q_n}(w_n)$, with $Q_n := \{y \in C : (1+\bar{\theta}_n)D_{f_p}(y, w_n) \geq D_{f_p}(y, v_n)\}$, $\bar{C}_n := \{y \in C : \bar{h}_n(y) \leq 0\}$ and

$$\bar{h}_n(y) = \langle A_2 \bar{t}_n, y - w_n \rangle + \frac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \bar{y}_n). \tag{3.4}$$

Again set $n := n + 1$ and go to Step 1.

The following lemmas are used in the proofs of our main results in the sequel.

Lemma 3.1. Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.1. Then the following hold: $\frac{1}{\lambda_1} D_{f_p}(s_n, y_n) \leq \langle A_1 s_n, e_{\lambda_1}(s_n) \rangle$ and $\frac{1}{\lambda_2} D_{f_p}(w_n, \bar{y}_n) \leq \langle A_2 w_n, e_{\lambda_2}(w_n) \rangle$.

Proof. Note that the former inequality is analogous to the latter. So it suffices to show that the latter holds. Indeed, using the definition of $\bar{y}_n$ and properties of Π_C, one has

$$0 \geq \langle J^p_E w_n - \lambda_2 A_2 w_n - J^p_E \bar{y}_n, y - \bar{y}_n \rangle \quad \forall y \in C.$$

Setting $y = w_n$ in the last inequality, from (2.1) we get

$$\lambda_2 \langle A_2 w_n, w_n - \bar{y}_n \rangle \geq \langle J^p_E w_n - J^p_E \bar{y}_n, w_n - \bar{y}_n \rangle \geq D_{f_p}(w_n, \bar{y}_n),$$

which completes the proof.

Lemma 3.2. Linesearch rules (3.1), (3.3) of Armijo-type and sequence $\{x_n\}$ constructed in Algorithm 3.1 are well defined.

Proof. Observe that the rule (3.1) is analogous to the one (3.3). So it suffices to show that the latter is valid. Using the uniform continuity of A_2 on C, from $l_2 \in (0,1)$ one gets $\lim_{j\to\infty} \langle A_2 w_n - A_2(w_n - l^j_2 e_{\lambda_2}(w_n)), e_{\lambda_2}(w_n) \rangle = 0$. In the case of $e_{\lambda_2}(w_n) = 0$, it is explicit that $j_n = 0$. In the case of $e_{\lambda_2}(w_n) \neq 0$, we obtain that $\exists j_n \geq 0$ s.t. (3.3) holds.

It is not hard to check that Q_n and $\bar{C}_n$ are convex and closed for all n. Let us show that $Q_n \cap \bar{C}_n \supset$ ■. Choose a $z \in$ ■ arbitrarily. Since S_2 is Bregman's relatively asymptotically nonexpansive mapping, by Lemma 2.2 one gets

$$\begin{aligned}
&D_{f_p}(z, v_n) \leq \alpha_n D_{f_p}(z, w_n) + (1-\alpha_n) D_{f_p}(z, S^n_2 w_n) - \alpha_n(1-\alpha_n)\rho^*_{b_{w_n}} \| J^P_E w_n - J^P_E S^n_2 w_n \| \\
&\leq \alpha_n D_{f_p}(z, w_n) + (1-\alpha_n)(1+\bar{\theta}_n) D_{f_p}(z, w_n) - \alpha_n(1-\alpha_n)\rho^*_{b_{w_n}} \| J^P_E w_n - J^P_E S^n_2 w_n \| \\
&\leq (1+\bar{\theta}_n) D_{f_p}(z, w_n) - \alpha_n(1-\alpha_n)\rho^*_{b_{w_n}} \| J^P_E w_n - J^P_E S^n_2 w_n \| \\
&\leq (1+\bar{\theta}_n) D_{f_p}(z, w_n),
\end{aligned}$$

which hence leads to $z \in Q_n$. Meanwhile, from Lemma 2.4, we get $\langle A_2 \bar{t}_n, \bar{t}_n - z \rangle \geq 0$. Thus,

$$\begin{aligned}
\bar{h}_n(z) &= \langle A_2 \bar{t}_n, z - w_n \rangle + \tfrac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \bar{y}_n) \\
&= -\langle A_2 \bar{t}_n, w_n - \bar{t}_n \rangle - \langle A_2 \bar{t}_n, \bar{t}_n - z \rangle + \tfrac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \bar{y}_n) \\
&\leq -\bar{\tau}_n \langle A_2 \bar{t}_n, e_{\lambda_2}(w_n) \rangle + \tfrac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \bar{y}_n).
\end{aligned} \tag{3.5}$$

So it follows from (3.3) that

$$\frac{\mu_2}{2} D_{f_p}(w_n, \bar{y}_n) \geq \langle A_2 w_n - A_2 \bar{t}_n, e_{\lambda_2}(w_n) \rangle.$$

By Lemma 3.1 we have

$$(\frac{1}{\lambda_2}-\frac{\mu_2}{2})D_{f_p}(w_n,\bar{y}_n)\le\langle A_2w_n,e_{\lambda_2}(w_n)\rangle-\frac{\mu_2}{2}D_{f_p}(w_n,\bar{y}_n)\le\langle A_2\bar{t}_n,e_{\lambda_2}(w_n)\rangle,$$

which together with (3.5), attains

$$\bar{h}_n(z)\le-\frac{\bar{\tau}_n}{2}(\frac{1}{\lambda_2}-\mu_2)D_{f_p}(w_n,\bar{y}_n)\le 0.$$

Therefore, $\blacksquare\subset\bar{C}_n\cap Q_n$. As a result, the sequence $\{x_n\}$ is well defined.

Lemma 3.3. Suppose that $\{y_n\}$ and $\{\bar{y}_n\}$ are the sequences generated by Algorithm 3.1. If $\lim_{n\to\infty}\|s_n-y_n\|=0$ and $\lim_{n\to\infty}\|w_n-\bar{y}_n\|=0$, then $\omega_w(s_n)\subset\mathrm{VI}(C,A_1)$ and $\omega_w(w_n)\subset\mathrm{VI}(C,A_2)$.

Proof. Note that the former inclusion is analogous to the latter. So it suffices to show that the latter is valid. Indeed, taking a $z\in\omega_w(w_n)$ arbitrarily, we know that $\exists\{w_{n_k}\}\subset\{w_n\}$, s.t. $w_{n_k}-\bar{y}_{n_k}\to 0$ and $w_{n_k}\rightharpoonup z$. So, we have $\bar{y}_{n_k}\rightharpoonup z$. Noticing the convexity and closedness of C, according to $\bar{y}_{n_k}\rightharpoonup z$ and $\{\bar{y}_n\}\subset C$, one gets $z\in C$. In what follows, ones consider two aspects. If $A_2z=0$, then $z\in\mathrm{VI}(C,A_2)$ (due to $\langle A_2z,x-z\rangle\ge 0$ for all $x\in C$). If $A_2z\ne 0$, by the condition on A_2, one gets $0<\|A_2z\|\le\liminf_{k\to\infty}\|A_2w_{n_k}\|$. So, we might assume that $\|A_2w_{n_k}\|\ne 0\ \forall k\ge 1$. From (2.2), we get

$$\langle J_E^p w_{n_k}-\lambda_2A_2w_{n_k}-J_E^p\bar{y}_{n_k},y-\bar{y}_{n_k}\rangle\le 0\quad\forall y\in C,$$

and hence

$$\frac{1}{\lambda_2}\langle J_E^p w_{n_k}-J_E^p\bar{y}_{n_k},y-\bar{y}_{n_k}\rangle+\langle A_2w_{n_k},\bar{y}_{n_k}-w_{n_k}\rangle\le\langle A_2w_{n_k},y-w_{n_k}\rangle\quad\forall y\in C.\tag{3.6}$$

Since A_2 is uniformly continuous, using Lemma 2.3 we deduce that $\{A_2w_{n_k}\}$ is of boundedness. Observe that $\{\bar{y}_{n_k}\}$ is also of boundedness. So, using the uniform continuity of J_E^p on any bounded subset of E, from (3.6) we have

$$\liminf_{k\to\infty}\langle A_2w_{n_k},y-w_{n_k}\rangle\ge 0\quad\forall y\in C.\tag{3.7}$$

To prove that z lies in $\mathrm{VI}(C,A_2)$, one picks $\{\kappa_k\}$ in $(0,1)$ s.t. $\kappa_k\downarrow 0$. For any k, we choose the smallest $m_k>0$ s.t. for all $j\ge m_k$,

$$\langle A_2w_{n_j},y-w_{n_j}\rangle+\kappa_k\ge 0.\tag{3.8}$$

Because $\{\kappa_k\}$ is decreasing, we get the increasing property of $\{m_k\}$. For the sake of simplicity, $\{A_2w_{n_{m_k}}\}$ is still written as $\{A_2w_{m_k}\}$. It is known that $A_2w_{m_k}\ne 0$ for all k (due to $\{A_2w_{m_k}\}\subset\{A_2w_{n_k}\}$). Then, putting $\bar{g}_{m_k}=\frac{A_2w_{m_k}}{\|A_2w_{m_k}\|^{\frac{q}{q-1}}}$, one gets $\langle A_2w_{m_k},J_{E^*}^q\bar{g}_{m_k}\rangle=1\ \forall k\ge 1$. Indeed, it is evident that $\langle A_2w_{m_k},J_{E^*}^q\bar{g}_{m_k}\rangle=\langle A_2w_{m_k},(\frac{1}{\|A_2w_{m_k}\|^{\frac{q}{q-1}}})^{q-1}J_{E^*}^q A_2w_{m_k}\rangle=(\frac{1}{\|A_2w_{m_k}\|^{\frac{q}{q-1}}})^{q-1}\|A_2w_{m_k}\|^q=1\ \forall k\ge 1$. So, by (3.8) one has $\langle A_2w_{m_k},y+\kappa_kJ_{E^*}^q\bar{g}_{m_k}-w_{m_k}\rangle\ge 0\ \forall k\ge 1$. Again from the pseudomonotonicity of A_2 one has

$$\langle A_2(y+\kappa_kJ_{E^*}^q\bar{g}_{m_k}),y+\kappa_kJ_{E^*}^q\bar{g}_{m_k}-w_{m_k}\rangle\ge 0\quad\forall y\in C.\tag{3.9}$$

Let us show that $\lim_{k\to\infty}\kappa_kJ_{E^*}^q\bar{g}_{m_k}=0$. Indeed, noticing $\kappa_k\downarrow 0$ and $\{w_{m_k}\}\subset\{w_{n_k}\}$, we obtain that

$$0\le\limsup_{k\to\infty}\|\kappa_kJ_{E^*}^q\bar{g}_{m_k}\|=\limsup_{k\to\infty}\frac{\kappa_k}{\|A_2w_{m_k}\|}\le\frac{\limsup_{k\to\infty}\kappa_k}{\liminf_{k\to\infty}\|A_2w_{n_k}\|}=0.$$

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Hence one gets $\kappa_k J^q_{E^*}\bar{g}_{m_k} \to 0$ as $k \to \infty$. Thus, taking the limit as $k \to \infty$ in (3.9), from (C3) one has $\langle A_2 y, y - z\rangle \geq 0$ for all $y \in C$. In terms of Lemma 2.4 we conclude that z lies in VI(C, A_2).

Lemma 3.4. Suppose that $\{y_n\}$ and $\{\bar{y}_n\}$ are the sequences generated by Algorithm 3.1. Then the following hold:

(i) $\lim_{n\to\infty} \tau_n D_{f_p}(s_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} D_{f_p}(s_n, y_n) = 0$;

(ii) $\lim_{n\to\infty} \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) = 0 \Rightarrow \lim_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = 0$.

Proof. Note that the claim (i) is analogous to the one (ii). So it suffices to show that the second is valid. To verify the second, we discuss two cases. In case $\liminf_{n\to\infty} \bar{\tau}_n > 0$, one may presume that $\exists \bar{\tau} > 0$ satisfying $\bar{\tau}_n \geq \bar{\tau} > 0$ for all n, which immediately leads to

$$D_{f_p}(w_n, \bar{y}_n) = \frac{1}{\bar{\tau}_n}\bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) \leq \frac{1}{\bar{\tau}} \cdot \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n). \tag{3.10}$$

This together with $\lim_{n\to\infty} \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) = 0$, arrives at $\lim_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = 0$.

In case $\liminf_{n\to\infty} \bar{\tau}_n = 0$, we assume that $\limsup_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = \hat{a} > 0$. This ensures that $\exists\{m_j\} \subset \{n\}$ satisfying

$$\lim_{j\to\infty} \bar{\tau}_{m_j} = 0 \quad \text{and} \quad \lim_{j\to\infty} D_{f_p}(w_{m_j}, \bar{y}_{m_j}) = \hat{a} > 0.$$

We define $\widehat{t_{m_j}} = \frac{1}{l_2}\bar{\tau}_{m_j}\bar{y}_{m_j} + (1 - \frac{1}{l_2}\bar{\tau}_{m_j})w_{m_j}\ \forall j \geq 1$. Noticing $\lim_{j\to\infty} \bar{\tau}_{m_j} D_{f_p}(w_{m_j}, \bar{y}_{m_j}) = 0$, From (2.1) we get $\lim_{j\to\infty} \bar{\tau}_{m_j}\|w_{m_j} - \bar{y}_{m_j}\|^p = 0$ and hence

$$\lim_{j\to\infty} \|\widehat{t_{m_j}} - w_{m_j}\|^p = \lim_{j\to\infty} \frac{\bar{\tau}^{p-1}_{m_j}}{l_2^p} \cdot \bar{\tau}_{m_j}\|w_{m_j} - \bar{y}_{m_j}\|^p = 0. \tag{3.11}$$

Because A_2 is uniformly continuous on bounded subsets of C, we obtain

$$\lim_{j\to\infty} \|A_2 w_{m_j} - A_2\widehat{t_{m_j}}\| = 0. \tag{3.12}$$

From the step size rule (3.3) and the definition of $\widehat{t_{m_j}}$, it follows that

$$\langle A_2 w_{m_j} - A_2\widehat{t_{m_j}}, w_{m_j} - \bar{y}_{m_j}\rangle > \frac{\mu_2}{2} D_{f_p}(w_{m_j}, \bar{y}_{m_j}). \tag{3.13}$$

Now, taking the limit as $j \to \infty$, from (3.12) we have $\lim_{j\to\infty} D_{f_p}(w_{m_j}, \bar{y}_{m_j}) = 0$. This, however, yields a contradiction. As a result, $D_{f_p}(w_n, \bar{y}_n) \to 0$ as $n \to \infty$.

In what follows, we show the first main result.

Theorem 3.1. Suppose that E is uniformly smooth and p-uniformly convex, where J^p_E is of weakly sequential continuity. If under Algorithm 3.1, $S^{n+1}_1 x_n - S^n_1 x_n \to 0$ and $S^{n+1}_2 w_n - S^n_2 w_n \to 0$, then $x_n \rightharpoonup z \in \blacksquare \Leftrightarrow \sup_{n\geq 0}\|x_n\| < \infty$.

Proof. Note that that the necessity is valid. So we need to only show the statement of sufficiency. Presume $\sup_{n\geq 0}\|x_n\| < \infty$. Choose a $z \in \blacksquare$ arbitrarily. Clearly, $x_{n-1} \neq x_n \Leftrightarrow J^p_E S^n_1 x_n \neq J^p_E(S^n_1 x_n - x_{n-1} + x_n)$. Using the definition of ϵ_n, we get $\epsilon_n\|J^p_E S^n_1 x_n - J^p_E(S^n_1 x_n + x_n - x_{n-1})\| \leq \ell_n\ \forall n \geq 1$. From (2.1), (2.6) and the three point identity of D_{f_p} we get

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$$
\begin{aligned}
&D_{f_p}(z,g_n) \le (1-\epsilon_n)D_{f_p}(z,S_1^n x_n) + \epsilon_n D_{f_p}(z,S_1^n x_n + x_n - x_{n-1})\\
&= D_{f_p}(z,S_1^n x_n) + \epsilon_n[D_{f_p}(z,S_1^n x_n + x_n - x_{n-1}) - D_{f_p}(z,S_1^n x_n)]\\
&= D_{f_p}(z,S_1^n x_n) + \epsilon_n[D_{f_p}(S_1^n x_n, S_1^n x_n + x_n - x_{n-1})\\
&\quad + \langle J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1}), z - S_1^n x_n\rangle]\\
&\le (1+\theta_n)D_{f_p}(z,x_n) + \epsilon_n\langle J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1}), z + x_{n-1} - x_n - S_1^n x_n\rangle\\
&\le (1+\theta_n)D_{f_p}(z,x_n) + \epsilon_n\|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\|\|z + x_{n-1} - x_n - S_1^n x_n\|\\
&\le (1+\theta_n)D_{f_p}(z,x_n) + \ell_n M,
\end{aligned}
$$

where $\sup_{n\ge1}\|z + x_{n-1} - x_n - S_1^n x_n\| \le M$ for some $M > 0$. By Lemma 2.2 we get

$$
\begin{aligned}
&D_{f_p}(z,s_n) = V_{f_p}(z, \beta_n J_E^p x_n + (1-\beta_n)J_E^p g_n)\\
&\le \frac{1}{p}\|z\|^p - \beta_n\langle J_E^p x_n, z\rangle - (1-\beta_n)\langle J_E^p g_n, z\rangle + \frac{\beta_n}{q}\|J_E^p x_n\|^q\\
&\quad + \frac{(1-\beta_n)}{q}\|J_E^p g_n\|^q - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\
&= \frac{1}{p}\|z\|^p - \beta_n\langle J_E^p x_n, z\rangle - (1-\beta_n)\langle J_E^p g_n, z\rangle + \frac{\beta_n}{q}\|x_n\|^p\\
&\quad + \frac{(1-\beta_n)}{q}\|g_n\|^p - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\
&= \beta_n D_{f_p}(z,x_n) + (1-\beta_n)D_{f_p}(z,g_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\
&\le \beta_n D_{f_p}(z,x_n) + (1-\beta_n)[(1+\theta_n)D_{f_p}(z,x_n) + \ell_n M] - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\
&\le (1+\theta_n)D_{f_p}(z,x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|.
\end{aligned}
$$

Noticing $w_n = \Pi_{C_n}s_n$, by (2.1) and (2.3) we get

$$
\begin{aligned}
D_{f_p}(z,w_n) &\le D_{f_p}(z,s_n) - D_{f_p}(w_n,s_n)\\
&= D_{f_p}(z,s_n) - D_{f_p}(\Pi_{C_n}s_n, s_n)\\
&\le D_{f_p}(z,s_n) - \tau\|\Pi_{C_n}s_n - s_n\|^p\\
&\le D_{f_p}(z,s_n) - \tau\|P_{C_n}s_n - s_n\|^p\\
&= D_{f_p}(z,s_n) - \tau[\mathrm{dist}(C_n,s_n)]^p.
\end{aligned}
$$

Because $x_{n+1} = \Pi_{\bar{C}_n\cap Q_n}w_n$, by (2.1) and (2.3) we have

$$
\begin{aligned}
D_{f_p}(z,x_{n+1}) &\le D_{f_p}(z,w_n) - D_{f_p}(\Pi_{\bar{C}_n\cap Q_n}w_n, w_n)\\
&\le D_{f_p}(z,w_n) - D_{f_p}(\Pi_{\bar{C}_n}w_n, w_n)\\
&\le D_{f_p}(z,w_n) - \tau\|\Pi_{\bar{C}_n}w_n - w_n\|^p\\
&\le D_{f_p}(z,w_n) - \tau\|P_{\bar{C}_n}w_n - w_n\|^p\\
&= D_{f_p}(z,w_n) - \tau[\mathrm{dist}(\bar{C}_n,w_n)]^p.
\end{aligned}
$$

Combining these inequalities and (3.13), leads to

$$
\begin{aligned}
&D_{f_p}(z,x_{n+1}) \le D_{f_p}(z,w_n) - D_{f_p}(\Pi_{\bar{C}_n\cap Q_n}w_n, w_n)\\
&\le D_{f_p}(z,s_n) - D_{f_p}(w_n,s_n) - D_{f_p}(x_{n+1},w_n)\\
&\le D_{f_p}(z,s_n) - \tau[\mathrm{dist}(C_n,s_n)]^p - \tau[\mathrm{dist}(\bar{C}_n,w_n)]^p\\
&\le (1+\theta_n)D_{f_p}(z,x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\
&\quad - \tau[\mathrm{dist}(C_n,s_n)]^p - \tau[\mathrm{dist}(\bar{C}_n,w_n)]^p,
\end{aligned}
\tag{3.14}
$$

which hence leads to

$$
D_{f_p}(z,x_{n+1}) \le (1+\theta_n)D_{f_p}(z,x_n) + \ell_n M.
$$

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Since $\sum_{n=1}^{\infty} \ell_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$, by Lemma 2.9 we deduce that $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists. In addition, by the boundedness of $\{x_n\}$, we conclude that $\{g_n\}, \{s_n\}, \{v_n\}, \{w_n\}, \{y_n\}, \{\bar{y}_n\}, \{t_n\}, \{\bar{t}_n\}, \{S_1^n x_n\}$ and $\{S_2^n w_n\}$ are also bounded. From (3.14) we obtain

$$\begin{aligned}&D_{f_p}(w_n, s_n) + D_{f_p}(x_{n+1}, w_n) \le D_{f_p}(z, s_n) - D_{f_p}(z, x_{n+1})\\ &\le (1+\theta_n)D_{f_p}(z, x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^* \|J_E^p x_n - J_E^p g_n\| - D_{f_p}(z, x_{n+1}),\end{aligned}$$

which immediately yields

$$\begin{aligned}&D_{f_p}(w_n, s_n) + D_{f_p}(x_{n+1}, w_n) + \beta_n(1-\beta_n)\rho_b^* \|J_E^p x_n - J_E^p g_n\|\\ &\le (1+\theta_n)D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M.\end{aligned}$$

Since $\lim_{n\to\infty} \ell_n = 0$, $\lim_{n\to\infty} \theta_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists, it follows that $\lim_{n\to\infty} D_{f_p}(w_n, s_n) = 0$, $\lim_{n\to\infty} D_{f_p}(x_{n+1}, w_n) = 0$, and $\lim_{n\to\infty} \rho_b^* \|J_E^p x_n - J_E^p g_n\| = 0$, which hence yields $\lim_{n\to\infty} \|J_E^p x_n - J_E^p g_n\| = 0$. From $s_n = J_{E^*}^q(\beta_n J_E^p x_n + (1-\beta_n) J_E^p g_n)$, it is readily known that $\lim_{n\to\infty} \|J_E^p s_n - J_E^p x_n\| = 0$. Noticing $g_n = J_{E^*}^q((1-\epsilon_n) J_E^p S_1^n x_n + \epsilon_n J_E^p(S_1^n x_n + x_n - x_{n-1}))$, we obtain from $\lim_{n\to\infty} \ell_n = 0$ and the definition of ϵ_n that

$$\|J_E^p g_n - J_E^p S_1^n x_n\| = \epsilon_n \|J_E^p(S_1^n x_n + x_n - x_{n-1}) - J_E^p S_1^n x_n\| \le \ell_n \to 0 \quad (n\to\infty).$$

Hence, using (2.1) and uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we conclude that $\lim_{n\to\infty} \|g_n - S_1^n x_n\| = 0$ and

$$\lim_{n\to\infty} \|w_n - s_n\| = \lim_{n\to\infty} \|x_{n+1} - w_n\| = \lim_{n\to\infty} \|x_n - S_1^n x_n\| = \lim_{n\to\infty} \|s_n - x_n\| = 0. \tag{3.15}$$

Since $\{x_n\}$ is of boundedness and E is of reflexivity, we obtain that $\omega_w(x_n)$ is nonempty. Next, let us show that $\blacksquare \supset \omega_w(x_n)$. Choose a z in $\omega_w(x_n)$ arbitrarily. It is known that $\exists \{x_{n_k}\} \subset \{x_n\}$ satisfying $x_{n_k} \rightharpoonup z$. By (3.15) one gets $w_{n_k} \rightharpoonup z$. Since $\{A_1 t_n\}$ is of boundedness, one knows that $\exists L_1 > 0$ satisfying $\|A_1 t_n\| \le L_1$. So it follows that for all $u, v \in C_n$,

$$|h_n(u) - h_n(v)| = |\langle A_1 t_n, u - v\rangle| \le \|A_1 t_n\| \|u - v\| \le L_1 \|u - v\|,$$

which implies that $h_n(y)$ is L_1-Lipschitz continuous on C_n. Using Lemma 2.6, we get

$$\mathrm{dist}(C_n, s_n) \ge \frac{1}{L_1} h_n(s_n) = \frac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n). \tag{3.16}$$

Since x_{n+1} lies in Q_n, by (3.14) one has

$$\begin{aligned}D_{f_p}(x_{n+1}, v_n) &\le (1+\bar{\theta}_n)[D_{f_p}(z, w_n) - D_{f_p}(z, x_{n+1})]\\ &\le (1+\bar{\theta}_n)[D_{f_p}(z, s_n) - D_{f_p}(z, x_{n+1})]\\ &\le (1+\bar{\theta}_n)[(1+\theta_n)D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M].\end{aligned}$$

Since $\lim_{n\to\infty} \theta_n = 0$, $\lim_{n\to\infty} \bar{\theta}_n = 0$, $\lim_{n\to\infty} \ell_n = 0$ and $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists, we have $D_{f_p}(x_{n+1}, v_n) \to 0$ and thus $x_{n+1} - v_n \to 0$. By (3.15) we get

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.17}$$

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Furthermore, by Lemma 2.2, we have

$$\begin{aligned}
&D_{f_p}(z,v_n) = V_{f_p}(z,\alpha_n J_E^p w_n + (1-\alpha_n)J_E^p S_2^n w_n) \\
&\le \tfrac{1}{p}\|z\|^p - \alpha_n\langle J_E^p w_n, z\rangle - (1-\alpha_n)\langle J_E^p S_2^n w_n, z\rangle + \tfrac{\alpha_n}{q}\|J_E^p w_n\|^q \\
&\quad + \tfrac{(1-\alpha_n)}{q}\|J_E^p S_2^n w_n\|^q - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_2^n w_n\| \\
&= \tfrac{1}{p}\|z\|^p - \alpha_n\langle J_E^p w_n, z\rangle - (1-\alpha_n)\langle J_E^p S_2^n w_n, z\rangle + \tfrac{\alpha_n}{q}\|w_n\|^p \\
&\quad + \tfrac{(1-\alpha_n)}{q}\|S_2^n w_n\|^p - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_2^n w_n\| \\
&= \alpha_n D_{f_p}(z,w_n) + (1-\alpha_n)D_{f_p}(z,S_2^n w_n) - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_2^n w_n\| \\
&\le (1+\bar{\theta}_n)D_{f_p}(z,w_n) - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_2^n w_n\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_2^n w_n\| \le (1+\bar{\theta}_n)D_{f_p}(z,w_n) - D_{f_p}(z,v_n) \\
&\le D_{f_p}(z,w_n) - D_{f_p}(z,v_n) + D_{f_p}(w_n,v_n) + \bar{\theta}_n D_{f_p}(z,w_n) \\
&= \langle J_E^p v_n - J_E^p w_n, z - w_n\rangle + \bar{\theta}_n D_{f_p}(z,w_n).
\end{aligned}$$

Taking the limit in the last inequality as $n\to\infty$, and using uniform continuity of J_E^p on bounded subsets of E, (3.17) and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we get $\lim_{n\to\infty}\rho_b^*\|J_E^p w_n - J_E^p S_2^n w_n\| = 0$ and hence $\lim_{n\to\infty}\|J_E^p w_n - J_E^p S_2^n w_n\| = 0$. Since $J_{E^*}^q$ is uniformly continuous on any bounded subset of E^*, we deduce that

$$\lim_{n\to\infty}\|w_n - S_2^n w_n\| = 0. \tag{3.18}$$

Now let us show $z\in\bigcap_{i=1}^2 \mathrm{VI}(C,A_i)$. Since $\{A_2\bar{t}_n\}$ is of boundedness, it follows that $\exists L_2>0$ satisfying $\|A_2\bar{t}_n\|\le L_2$. Thus we obtain that for all $u,v\in\bar{C}_n$,

$$|\bar{h}_n(u)-\bar{h}_n(v)| = |\langle A_2\bar{t}_n, u-v\rangle| \le \|A_2\bar{t}_n\|\|u-v\| \le L_2\|u-v\|,$$

which guarantees that $\bar{h}_n(y)$ is L_2-Lipschitz continuous on $\bar{C}_n$. By Lemma 2.6, we get

$$\mathrm{dist}(\bar{C}_n,w_n) \ge \frac{1}{L_2}\bar{h}_n(w_n) = \frac{\bar{\tau}_n}{2\lambda_2 L_2}D_{f_p}(w_n,\bar{y}_n). \tag{3.19}$$

Combining (3.14), (3.16) and (3.19), we have

$$\begin{aligned}
&(1+\theta_n)D_{f_p}(z,x_n) - D_{f_p}(z,x_{n+1}) + \ell_n M \\
&\ge D_{f_p}(z,s_n) - D_{f_p}(z,x_{n+1}) \\
&\ge \tau[\tfrac{\tau_n}{2\lambda_1 L_1}D_{f_p}(s_n,y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2}D_{f_p}(w_n,\bar{y}_n)]^p.
\end{aligned} \tag{3.20}$$

Thus,

$$\lim_{n\to\infty}\tau_n D_{f_p}(s_n,y_n) = \lim_{n\to\infty}\bar{\tau}_n D_{f_p}(w_n,\bar{y}_n) = 0.$$

According to Lemma 3.4, we have

$$\lim_{n\to\infty}\|y_n - s_n\| = \lim_{n\to\infty}\|\bar{y}_n - w_n\| = 0.$$

In addition, from (3.15) and $x_{n_k} \rightharpoonup z$ we infer that $s_{n_k} \rightharpoonup z$ and $w_{n_k} \rightharpoonup z$. By Lemma 3.3 we obtain that $z \in \omega_w(s_n) \subset \mathrm{VI}(C, A_1)$ and $z \in \omega_w(w_n) \subset \mathrm{VI}(C, A_2)$. Consequently,

$$z \in \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i).$$

Next we claim that $z \in \bigcap_{i=1}^{2} \mathrm{Fix}(S_i)$. Indeed, by (3.15) we immediately get

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - w_n\| + \|w_n - s_n\| + \|s_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.21}$$

We first claim that $\lim_{n\to\infty} \|x_n - S_1x_n\| = 0$ and $\lim_{n\to\infty} \|w_n - S_2w_n\| = 0$. Actually, using (3.15), (3.18) and uniform continuity of S_i on C for $i = 1, 2$, we obtain that $S_1x_n - S_1^{n+1}x_n \to 0$ and $S_2w_n - S_2^{n+1}w_n \to 0$. Thus, from $S_1^{n+1}x_n - S_1^n x_n \to 0$ and $S_2^{n+1}w_n - S_2^n w_n \to 0$ (due to the assumptions) we deduce that

$$\|x_n - S_1x_n\| \le \|x_n - S_1^n x_n\| + \|S_1^n x_n - S_1^{n+1}x_n\| + \|S_1^{n+1}x_n - S_1x_n\| \to 0 \quad (n \to \infty)$$

and

$$\|w_n - S_2w_n\| \le \|w_n - S_2^n w_n\| + \|S_2^n w_n - S_2^{n+1}w_n\| + \|S_2^{n+1}w_n - S_2w_n\| \to 0 \quad (n \to \infty).$$

These together with $x_{n_k} \rightharpoonup z$ and $w_{n_k} \rightharpoonup z$, ensure that $z \in \bigcap_{i=1}^{2} \widehat{\mathrm{Fix}}(S_i) = \bigcap_{i=1}^{2} \mathrm{Fix}(S_i)$. Therefore, $z \in \blacksquare$. This means that $\omega_w(x_n) \subset \blacksquare$. As a result, by Lemma 2.5 one gets the desired conclusion.

In what follows, we prove the second main outcome for finding a solution of a pair of VIFPPs for two operators of both uniform continuity and pseudomonotonicity, and two mappings of both uniform continuity and Bregman's relatively asymptotical nonexpansivity in E.

Algorithm 3.2. Initialization: Given $x_0, x_1 \in C$ arbitrarily and let $\epsilon > 0$, $\mu_\iota > 0$, $l_\iota \in (0,1)$ and $\lambda_\iota \in (0, \frac{1}{\mu_\iota})$ for $\iota = 1, 2$. Choose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\ell_n\} \subset (0,\infty)$ s.t. $\lim_{n\to\infty} \ell_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\theta_n + \bar{\theta}_n}{\alpha_n} = 0$, $0 < \liminf_{n\to\infty} \beta_n(1-\beta_n)$ and $0 < \liminf_{n\to\infty} \gamma_n(1-\gamma_n)$. Moreover, given the iterates x_{n-1} and x_n $(n \ge 1)$, choose ϵ_n s.t. $0 \le \epsilon_n \le \overline{\epsilon_n}$, where $\sup_{n\ge1} \frac{\epsilon_n}{\alpha_n} < \infty$ and

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_1^n x_n - J_E^p(S_1^n x_n - x_{n-1} + x_n)\|}\} & \text{if } x_{n-1} \ne x_n, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterations: Compute x_{n+1} below:

Step 1. Put $g_n = J_{E^*}^q((1-\epsilon_n)J_E^p S_1^n x_n + \epsilon_n J_E^p(S_1^n x_n + x_n - x_{n-1}))$, and calculate $s_n = J_{E^*}^q(\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p g_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p s_n - \lambda_1 A_1 s_n))$, $e_{\lambda_1}(s_n) := s_n - y_n$ and $t_n = s_n - \tau_n e_{\lambda_1}(s_n)$, where $\tau_n := l_1^{k_n}$ and k_n is the smallest $k \ge 0$ s.t.

$$\frac{\mu_1}{2} D_{f_p}(s_n, y_n) \ge \langle A_1 s_n - A_1(s_n - l_1^k e_{\lambda_1}(s_n)), s_n - y_n \rangle.$$

Step 2. Calculate $w_n = \Pi_{C_n}(s_n)$, with $C_n := \{y \in C : h_n(y) \le 0\}$ and

$$h_n(y) = \langle A_1 t_n, y - s_n \rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(s_n, y_n).$$

Step 3. Calculate $\bar{y}_n = \Pi_C(J_{E^*}^q(J_E^p w_n - \lambda_2 A_2 w_n))$, $e_{\lambda_2}(w_n) := w_n - \bar{y}_n$ and $\bar{t}_n = w_n - \bar{\tau}_n e_{\lambda_2}(w_n)$, where $\bar{\tau}_n := l_2^{j_n}$ and j_n is the smallest $j \ge 0$ s.t.

$$\frac{\mu_2}{2} D_{f_p}(w_n, \bar{y}_n) \ge \langle A_2 w_n - A_2(w_n - l_2^j e_{\lambda_2}(w_n)), w_n - \bar{y}_n \rangle.$$

Step 4. Set $z_n = \Pi_{\bar{C}_n}(w_n)$, and calculate $v_n = J_{E^*}^q(\beta_n J_E^p z_n + (1-\beta_n)J_E^p(S_2^n z_n))$ and $x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)$, where $\bar{C}_n := \{y \in C : \bar{h}_n(y) \le 0\}$ and

$$\bar{h}_n(y) = \langle A_2 \bar{t}_n, y - w_n \rangle + \frac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \bar{y}_n).$$

Again put $n := n + 1$ and return to Step 1.

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Theorem 3.2. Suppose that the conditions (C1)-(C3) hold. If under Algorithm 3.2, $S_1^{n+1}x_n - S_1^n x_n \to 0$ and $S_2^{n+1}z_n - S_2^n z_n \to 0$, then $x_n \to \Pi_{\blacksquare} u \Leftrightarrow \sup_{n\geq 0}\|x_n\| < \infty$.

Proof. It is explicit that the necessity of Theorem 3.2 holds. Hence we need to only prove the statement of sufficiency. Assume that $\sup_{n\geq 0}\|x_n\| < \infty$. In what follows, we divide our proof into four claims.

Claim 1. One shows that

$$\begin{aligned}&(1-\alpha_n)(1+\bar{\theta}_n)\gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (\theta_n+1)(\bar{\theta}_n+1)D_{f_p}(\hat{u},x_n) - D_{f_p}(\hat{u},x_{n+1}) + (\bar{\theta}_n+1)\ell_n M,\end{aligned}$$

for some $M > 0$. In fact, put $\hat{u} = \Pi_{\blacksquare} u$. Noticing $w_n = \Pi_{C_n} s_n$ and $z_n = \Pi_{\bar{C}_n} w_n$, we obtain from (2.1) and (2.3) that

$$\begin{aligned}D_{f_p}(\hat{u},w_n) &\leq D_{f_p}(\hat{u},s_n) - D_{f_p}(w_n,s_n)\\ &\leq D_{f_p}(\hat{u},s_n) - \tau[\mathrm{dist}(C_n,s_n)]^p,\end{aligned}$$

and

$$\begin{aligned}D_{f_p}(\hat{u},z_n) &\leq D_{f_p}(\hat{u},w_n) - D_{f_p}(z_n,w_n)\\ &\leq D_{f_p}(\hat{u},w_n) - \tau[\mathrm{dist}(\bar{C}_n,w_n)]^p.\end{aligned}$$

By the similar reasonings to those in the proof of the above theorem, we obtain

$$\begin{aligned}D_{f_p}(\hat{u},g_n) &\leq D_{f_p}(\hat{u},S_1^n x_n) + \epsilon_n\|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\|\\ &\times \|\hat{u} + x_{n-1} - x_n - S_1^n x_n\| \leq (1+\theta_n)D_{f_p}(\hat{u},x_n) + \ell_n M,\end{aligned}$$

where $\sup_{n\geq 1}\|\hat{u} + x_{n-1} - x_n - S_1^n x_n\| \leq M$ for some $M > 0$. This ensures that $\{g_n\}$ is bounded.

Using (2.6) and the last two inequalities, from $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ we obtain

$$\begin{aligned}&D_{f_p}(\hat{u},x_{n+1}) \leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)D_{f_p}(\hat{u},v_n)\\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)(1+\bar{\theta}_n)D_{f_p}(\hat{u},z_n)\\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)(1+\bar{\theta}_n)[\gamma_n D_{f_p}(\hat{u},x_n) + (1-\gamma_n)D_{f_p}(\hat{u},g_n)\\ &\quad - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|]\\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1+\bar{\theta}_n)[(1+\theta_n)D_{f_p}(\hat{u},x_n) + \ell_n M]\\ &\quad - (1-\alpha_n)(1+\bar{\theta}_n)\gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|,\end{aligned}$$

which immediately arrives at the desired claim. In addition, it is easily known that $\{s_n\}, \{v_n\}, \{w_n\}, \{y_n\}, \{\bar{y}_n\}, \{z_n\}, \{t_n\}, \{\bar{t}_n\}$ and $\{S_2^n z_n\}$ are of boundedness.

Claim 2. One shows that

$$\begin{aligned}&(\bar{\theta}_n+1)[D_{f_p}(w_n,s_n) + D_{f_p}(z_n,w_n)]\\ &\leq \alpha_n\langle J_E^p u - J_E^p\hat{u}, \zeta_n - \hat{u}\rangle - D_{f_p}(\hat{u},x_{n+1}) + (\bar{\theta}_n+1)D_{f_p}(\hat{u},s_n).\end{aligned}$$

Indeed, set $b = \sup_{n\geq 1}\{\|x_n\|^{p-1}, \|g_n\|^{p-1}, \|z_n\|^{p-1}, \|S_2^n z_n\|^{p-1}\}$. By Lemma 2.2 we get

$$\begin{aligned}&D_{f_p}(\hat{u},s_n) = V_{f_p}(\hat{u},\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p g_n)\\ &\leq \tfrac{1}{p}\|\hat{u}\|^p - \gamma_n\langle J_E^p x_n,\hat{u}\rangle - (1-\gamma_n)\langle J_E^p g_n,\hat{u}\rangle + \tfrac{\gamma_n}{q}\|J_E^p x_n\|^q\\ &\quad + \tfrac{(1-\gamma_n)}{q}\|J_E^p g_n\|^q - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\ &= \tfrac{1}{p}\|\hat{u}\|^p - \gamma_n\langle J_E^p x_n,\hat{u}\rangle - (1-\gamma_n)\langle J_E^p g_n,\hat{u}\rangle + \tfrac{\gamma_n}{q}\|x_n\|^p\\ &\quad + \tfrac{(1-\gamma_n)}{q}\|g_n\|^p - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\ &= \gamma_n D_{f_p}(\hat{u},x_n) + (1-\gamma_n)D_{f_p}(\hat{u},g_n) - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\ &\leq \gamma_n D_{f_p}(\hat{u},x_n) + (1-\gamma_n)[(1+\theta_n)D_{f_p}(z,x_n) + \ell_n M]\\ &\quad - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|\\ &\leq (1+\theta_n)D_{f_p}(\hat{u},x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|,\end{aligned} \tag{3.22}$$

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and

$$\begin{aligned}
& D_{f_p}(\hat{u}, v_n) = V_{f_p}(\hat{u}, \beta_n J_E^p z_n + (1-\beta_n) J_E^p S_2^n z_n) \\
& \le \beta_n D_{f_p}(\hat{u}, z_n) + (1-\beta_n) D_{f_p}(\hat{u}, S_2^n z_n) - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \| \\
& \le \beta_n D_{f_p}(\hat{u}, z_n) + (1-\beta_n)(1+\bar{\theta}_n) D_{f_p}(\hat{u}, z_n) - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \| \\
& \le (1+\bar{\theta}_n) D_{f_p}(\hat{u}, z_n) - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \| \\
& \le (1+\bar{\theta}_n) D_{f_p}(\hat{u}, w_n) - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \|.
\end{aligned} \tag{3.23}$$

Set $\zeta_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)$. From (2.5), we have

$$\begin{aligned}
& D_{f_p}(\hat{u}, x_{n+1}) \le V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n) \\
& \le V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n - \alpha_n(J_E^p u - J_E^p \hat{u})) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle \\
& \le (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle \\
& \le (1+\bar{\theta}_n) D_{f_p}(\hat{u}, w_n) - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \| + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle \\
& \le (1+\bar{\theta}_n) D_{f_p}(\hat{u}, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle.
\end{aligned} \tag{3.24}$$

Furthermore, from (3.23) one has

$$\begin{aligned}
D_{f_p}(\hat{u}, v_n) \ & \le (1+\bar{\theta}_n) D_{f_p}(\hat{u}, z_n) - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \| \\
& \le (1+\bar{\theta}_n)[D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n)] - \beta_n(1-\beta_n)\rho_b^* \| J_E^p z_n - J_E^p S_2^n z_n \| \\
& \le (1+\bar{\theta}_n)[D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n)].
\end{aligned}$$

This together with (3.24), arrives at

$$\begin{aligned}
& D_{f_p}(\hat{u}, x_{n+1}) \le (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle \\
& \le (1+\bar{\theta}_n)[D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n)] + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle \\
& \le (1+\bar{\theta}_n)[D_{f_p}(\hat{u}, s_n) - D_{f_p}(w_n, s_n) - D_{f_p}(z_n, w_n)] + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
& (1+\bar{\theta}_n)[D_{f_p}(w_n, s_n) + D_{f_p}(z_n, w_n)] \\
& \le \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u} \rangle - D_{f_p}(\hat{u}, x_{n+1}) + (1+\bar{\theta}_n) D_{f_p}(\hat{u}, s_n).
\end{aligned} \tag{3.25}$$

Claim 3. One shows that

$$\begin{aligned}
& (\bar{\theta}_n + 1)(1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p\} \\
& \le \alpha_n D_{f_p}(\hat{u}, u) - D_{f_p}(\hat{u}, x_{n+1}) + (\theta_n + 1)(\bar{\theta}_n + 1) D_{f_p}(\hat{u}, x_n) + (\bar{\theta}_n + 1)\ell_n M.
\end{aligned}$$

Indeed, by the analogous reasonings to those of (3.20), one gets

$$\begin{aligned}
D_{f_p}(\hat{u}, z_n) \ & \le D_{f_p}(\hat{u}, w_n) - \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p \\
& \le D_{f_p}(\hat{u}, s_n) - \tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p - \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p.
\end{aligned} \tag{3.26}$$

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Applying (3.26), (3.23) and (3.22), we have

$$
\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le D_{f_p}(\hat{u}, J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n)J^p_E v_n)) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[(\bar{\theta}_n + 1)D_{f_p}(\hat{u}, z_n) - \beta_n(1-\beta_n)\rho_b^*\|z_n - S_2^n z_n\|] \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)(\bar{\theta}_n + 1)\{D_{f_p}(\hat{u}, s_n) - \tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p \\
&\quad - \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p\} \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (\bar{\theta}_n + 1)D_{f_p}(\hat{u}, s_n) - (1-\alpha_n)(\bar{\theta}_n + 1)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p \\
&\quad + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p\} \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (\bar{\theta}_n + 1)[(\theta_n + 1)D_{f_p}(\hat{u}, x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J^p_E x_n - J^p_E g_n\|] \\
&\quad - (1-\alpha_n)(\bar{\theta}_n + 1)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p\} \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (\bar{\theta}_n + 1)[(\theta_n + 1)D_{f_p}(\hat{u}, x_n) + \ell_n M] \\
&\quad - (1-\alpha_n)(\bar{\theta}_n + 1)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p\}.
\end{aligned}
\tag{3.27}
$$

Claim 4. One shows that $\lim_{n\to\infty}\|x_n - \hat{u}\| = 0$. Indeed, since E is reflexive and $\{x_n\}$ is bounded, one has $\omega_w(x_n) \neq \emptyset$. Choose a z in $\omega_w(x_n)$ arbitrarily. It is known that $\exists\{x_{n_k}\} \subset \{x_n\}$ satisfying $x_{n_k} \rightharpoonup z$. One writes $\mathbf{\Gamma}_n = D_{f_p}(\hat{u}, x_n)$ for all n. In what follows, let us prove $\{\mathbf{\Gamma}_n\} \to 0$ $(n \to \infty)$ in the two possible aspects below.

Aspect 1. Assume that $\exists n_0 \ge 1$ s.t. $\{\mathbf{\Gamma}_n\}_{n=n_0}^\infty$ is non-increasing. It is known that $\mathbf{\Gamma}_n \to d < +\infty$ and hence $\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} \to 0$. From (3.25) and (3.22) we get

$$
\begin{aligned}
&(\bar{\theta}_n + 1)[D_{f_p}(w_n, s_n) + D_{f_p}(z_n, w_n)] \\
&\le \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle - D_{f_p}(\hat{u}, x_{n+1}) + (\bar{\theta}_n + 1)D_{f_p}(\hat{u}, s_n) \\
&\le (\bar{\theta}_n + 1)[(\theta_n + 1)D_{f_p}(\hat{u}, x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J^p_E x_n - J^p_E g_n\|] \\
&\quad - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle,
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
&(\bar{\theta}_n + 1)[D_{f_p}(w_n, s_n) + D_{f_p}(z_n, w_n) + \gamma_n(1-\gamma_n)\rho_b^*\|J^p_E x_n - J^p_E g_n\|] \\
&\le (\theta_n + 1)(\bar{\theta}_n + 1)D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + (\bar{\theta}_n + 1)\ell_n M + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle \\
&= (\theta_n + 1)(\bar{\theta}_n + 1)\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + (\bar{\theta}_n + 1)\ell_n M + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle.
\end{aligned}
$$

Since $\ell_n \to 0$, $\bar{\theta}_n \to 0$, $\theta_n \to 0$, $\alpha_n \to 0$, $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$, $\mathbf{\Gamma}_n \to d$ and $\{\zeta_n\}$ is of boundedness, one obtains $\lim_{n\to\infty} D_{f_p}(w_n, s_n) = 0$, $\lim_{n\to\infty} D_{f_p}(z_n, w_n) = 0$, and $\lim_{n\to\infty}\rho_b^*\|J^p_E x_n - J^p_E g_n\| = 0$, which hence yields $\lim_{n\to\infty}\|J^p_E x_n - J^p_E g_n\| = 0$. From $u_n = J^q_{E^*}(\gamma_n J^p_E x_n + (1-\gamma_n)J^p_E g_n)$, it is easily known that $\lim_{n\to\infty}\|J^p_E s_n - J^p_E x_n\| = 0$. Noticing $g_n = J^q_{E^*}((1-\epsilon_n)J^p_E S_1^n x_n + \epsilon_n J^p_E(S_1^n x_n + x_n - x_{n-1}))$, we infer from $\lim_{n\to\infty}\ell_n = 0$ and the definition of ϵ_n that

$$
\|J^p_E g_n - J^p_E S_1^n x_n\| = \epsilon_n\|J^p_E(S_1^n x_n + x_n - x_{n-1}) - J^p_E S_1^n x_n\| \le \ell_n \to 0 \quad (n \to \infty).
$$

Hence, using (2.1) and uniform continuity of $J^q_{E^*}$ on any bounded subset of E^*, we conclude that $\lim_{n\to\infty}\|g_n - S_1^n x_n\| = 0$ and

$$
\lim_{n\to\infty}\|w_n - s_n\| = \lim_{n\to\infty}\|z_n - w_n\| = \lim_{n\to\infty}\|x_n - S_1^n x_n\| = \lim_{n\to\infty}\|s_n - x_n\| = 0. \tag{3.28}
$$

Furthermore, from (3.24) and (3.22) we have

$$\begin{aligned}
&(1-\alpha_n)\beta_n(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_2^n z_n\| \\
&\le (1+\bar{\theta}_n)D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1+\bar{\theta}_n)(1+\theta_n)D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + (1+\bar{\theta}_n)\ell_n M + \alpha_n\langle J_E^p u - J_E^p\hat{u}, \zeta_n - \hat{u}\rangle.
\end{aligned}$$

By the similar reasonings, we deduce that $\lim_{n\to\infty}\|J_E^p z_n - J_E^p S_2^n z_n\| = 0$, which hence leads to $\lim_{n\to\infty}\|J_E^p v_n - J_E^p z_n\| = 0$ (due to $v_n = J_{E^*}^q(\beta_n J_E^p z_n + (1-\beta_n)J_E^p S_2^n z_n)$). Using uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we get

$$\lim_{n\to\infty}\|z_n - S_2^n z_n\| = \lim_{n\to\infty}\|v_n - z_n\| = 0. \tag{3.29}$$

This together with (3.28) implies that

$$\|v_n - x_n\| \le \|v_n - z_n\| + \|z_n - w_n\| + \|w_n - s_n\| + \|s_n - x_n\| \to 0 \quad (n\to\infty).$$

It is clear that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.30}$$

Let us show that $z \in \bigcap_{i=1}^2 \mathrm{Fix}(S_i)$. Indeed, since $\zeta_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)$, it can be readily seen that

$$\lim_{n\to\infty}\|\zeta_n - x_n\| = 0. \tag{3.31}$$

In addition, using (2.3), (3.22) and (3.23), we have

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (\bar{\theta}_n + 1)D_{f_p}(\hat{u}, w_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (\bar{\theta}_n + 1)D_{f_p}(\hat{u}, s_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (\bar{\theta}_n + 1)[(\theta_n + 1)D_{f_p}(\hat{u}, x_n) + \ell_n M] - D_{f_p}(x_{n+1}, \zeta_n),
\end{aligned}$$

which hence yields

$$\begin{aligned}
D_{f_p}(x_{n+1}, \zeta_n) &\le \alpha_n D_{f_p}(\hat{u}, u) + (1+\bar{\theta}_n)(1+\theta_n)D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + (1+\bar{\theta}_n)\ell_n M \\
&= \alpha_n D_{f_p}(\hat{u}, u) - \mathbf{\Gamma}_{n+1} + (\theta_n + 1)(\bar{\theta}_n + 1)\mathbf{\Gamma}_n + (\bar{\theta}_n + 1)\ell_n M.
\end{aligned}$$

So it follows that $D_{f_p}(x_{n+1}, \zeta_n) \to 0$ and hence $\|x_{n+1} - \zeta_n\| \to 0$. Thus, from (3.31) we get

$$\|x_n - x_{n+1}\| \le \|x_n - \zeta_n\| + \|\zeta_n - x_{n+1}\| \to 0 \quad (n\to\infty). \tag{3.32}$$

We now claim that $\lim_{n\to\infty}\|x_n - S_1 x_n\| = 0$ and $\lim_{n\to\infty}\|w_n - S_2 w_n\| = 0$. Indeed, using (3.28), (3.29) and uniform continuity of S_i on C for $i = 1, 2$, we obtain that $S_1 x_n - S_1^{n+1}x_n \to 0$ and $S_2 z_n - S_2^{n+1}z_n \to 0$. Thus, from $S_1^{n+1}x_n - S_1^n x_n \to 0$ and $S_2^{n+1}z_n - S_2^n z_n \to 0$ (due to the assumptions) we deduce that

$$\|x_n - S_1 x_n\| \le \|x_n - S_1^n x_n\| + \|S_1^n x_n - S_1^{n+1}x_n\| + \|S_1^{n+1}x_n - S_1 x_n\| \to 0 \quad (n\to\infty)$$

and

$$\|z_n - S_2 z_n\| \le \|z_n - S_2^n z_n\| + \|S_2^n z_n - S_2^{n+1}z_n\| + \|S_2^{n+1}z_n - S_2 z_n\| \to 0 \quad (n\to\infty).$$

These together with $x_{n_k} \rightharpoonup z$ and $z_{n_k} \rightharpoonup z$ (due to (3.30)), ensure that $z \in \bigcap_{i=1}^2 \widehat{\mathrm{Fix}}(S_i) = \bigcap_{i=1}^2 \mathrm{Fix}(S_i)$.

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In what follows, we show that $z \in \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)$. From (3.27), we have

$$\begin{aligned}
&(\bar{\theta}_n + 1)(1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n)]^p\} \\
&\le \alpha_n D_{f_p}(\hat{u}, u) - D_{f_p}(\hat{u}, x_{n+1}) + (\theta_n + 1)(\bar{\theta}_n + 1) D_{f_p}(\hat{u}, x_n) + (\bar{\theta}_n + 1)\ell_n M \\
&= \alpha_n D_{f_p}(\hat{u}, u) - \mathbf{\Gamma}_{n+1} + (\theta_n + 1)(\bar{\theta}_n + 1)\mathbf{\Gamma}_n + (\bar{\theta}_n + 1)\ell_n M.
\end{aligned}$$

So it follows that $\lim_{n\to\infty} \frac{\tau_n}{2\lambda_1 L_1} D_{f_p}(s_n, y_n) = \lim_{n\to\infty} \frac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \bar{y}_n) = 0$, and hence

$$\lim_{n\to\infty} \tau_n D_{f_p}(s_n, y_n) = \lim_{n\to\infty} \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) = 0. \tag{3.33}$$

By Lemma 3.4, we obtain

$$\lim_{n\to\infty} \|y_n - s_n\| = \lim_{n\to\infty} \|\bar{y}_n - w_n\| = 0. \tag{3.34}$$

Applying (3.34) and Lemma 3.3, one gets $z \in \bigcap_{j=1}^{2} \mathrm{VI}(C, A_j)$. Thus one has $\omega_w(x_n) \subset \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)$. Consequently, $\blacksquare \supset \omega_w(x_n)$. Finally, let us prove $\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \le 0$. One can pick $\{x_{n_j}\} \subset \{x_n\}$ s.t.

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle.$$

Because E is reflexive and $\{x_n\}$ is bounded, we might assume $x_{n_j} \rightharpoonup \bar{z}$. Using (2.2) and $\bar{z} \in \blacksquare$ we infer that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle = \langle J_E^p u - J_E^p \hat{u}, \bar{z} - \hat{u}\rangle \le 0, \tag{3.35}$$

which along with (3.31), arrives at

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \le 0.$$

From (3.24) and (3.22), we get

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le (1-\alpha_n)(1+\bar{\theta}_n) D_{f_p}(\hat{u}, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n)(1+\bar{\theta}_n) D_{f_p}(\hat{u}, s_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n) D_{f_p}(\hat{u}, s_n) + \bar{\theta}_n D_{f_p}(\hat{u}, s_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n)[(1+\theta_n) D_{f_p}(\hat{u}, x_n) + \epsilon_n \|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\| \\
&\quad \times \|\hat{u} + x_{n-1} - x_n - S_1^n x_n\|] + \bar{\theta}_n D_{f_p}(\hat{u}, s_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n) D_{f_p}(\hat{u}, x_n) + \epsilon_n \|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\| \\
&\quad \times \|\hat{u} + x_{n-1} - x_n - S_1^n x_n\| + \theta_n D_{f_p}(\hat{u}, x_n) + \bar{\theta}_n D_{f_p}(\hat{u}, s_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&= (1-\alpha_n) D_{f_p}(\hat{u}, x_n) + \alpha_n \{\tfrac{\epsilon_n}{\alpha_n} \|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\| \\
&\quad \times \|\hat{u} + x_{n-1} - x_n - S_1^n x_n\| + \tfrac{\theta_n}{\alpha_n} D_{f_p}(\hat{u}, x_n) + \tfrac{\bar{\theta}_n}{\alpha_n} D_{f_p}(\hat{u}, s_n) + \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle\}.
\end{aligned} \tag{3.36}$$

Using uniform continuity of J_E^p on any bounded subset of E, from (3.32) and the boundedness of $\{x_n\}$ we get

$$\lim_{n\to\infty} \|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\| \|\hat{u} + x_{n-1} - x_n - S_1^n x_n\| = 0.$$

Noticing $\sup_{n\geq 1}\frac{\epsilon_n}{\alpha_n}<\infty$, $\lim_{n\to\infty}\frac{\theta_n+\bar{\theta}_n}{\alpha_n}=0$ and $\limsup_{n\to\infty}\langle J_E^p u-J_E^p\hat{u},\zeta_n-\hat{u}\rangle\leq 0$, we deduce that

$$\begin{aligned}\limsup_{n\to\infty}\{&\tfrac{\epsilon_n}{\alpha_n}\|J_E^p S_1^n x_n-J_E^p(S_1^n x_n+x_n-x_{n-1})\|\|\hat{u}+x_{n-1}-x_n-S_1^n x_n\|\\&+\tfrac{\theta_n}{\alpha_n}D_{f_p}(\hat{u},x_n)+\tfrac{\bar{\theta}_n}{\alpha_n}D_{f_p}(\hat{u},s_n)+\langle J_E^p u-J_E^p\hat{u},\zeta_n-\hat{u}\rangle\}\leq 0.\end{aligned}$$

Thanks to $\{\alpha_n\}\subset(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$, utilizing Lemma 2.8 to (3.36) one gets $D_{f_p}(\hat{u},x_n)\to 0$ and hence $\|x_n-\hat{u}\|\to 0$.

Aspect 2. Assume that $\exists\{\mathbf{\Gamma}_{n_k}\}\subset\{\mathbf{\Gamma}_n\}$ satisfying $\mathbf{\Gamma}_{n_k}<\mathbf{\Gamma}_{n_k+1}$ for all k, with $\mathcal{N}$ being the natural-number set. Let $\varphi:\mathcal{N}\to\mathcal{N}$ be formulated below

$$\varphi(n):=\max\{j\leq n:\mathbf{\Gamma}_j<\mathbf{\Gamma}_{j+1}\}.$$

Using Lemma 2.7, one has

$$\max\{\mathbf{\Gamma}_{\varphi(n)},\mathbf{\Gamma}_n\}\leq\mathbf{\Gamma}_{\varphi(n)+1}. \tag{3.37}$$

From (3.25) and (3.22) it follows that

$$\begin{aligned}&(1+\bar{\theta}_{\varphi(n)})[D_{f_p}(w_{\varphi(n)},s_{\varphi(n)})+D_{f_p}(z_{\varphi(n)},w_{\varphi(n)})\\&\quad+\gamma_{\varphi(n)}(1-\gamma_{\varphi(n)})\rho_b^*\|J_E^p x_{\varphi(n)}-J_E^p g_{\varphi(n)}\|]\\&\leq(1+\bar{\theta}_{\varphi(n)})(1+\theta_{\varphi(n)})\mathbf{\Gamma}_{\varphi(n)}-\mathbf{\Gamma}_{\varphi(n)+1}+(1+\bar{\theta}_{\varphi(n)})\ell_{\varphi(n)}M\\&\quad+\alpha_{\varphi(n)}\langle J_E^p u-J_E^p\hat{u},\zeta_{\varphi(n)}-\hat{u}\rangle.\end{aligned}$$

Noticing $g_{\varphi(n)}=J_{E^*}^q((1-\epsilon_{\varphi(n)})J_E^p S_1^{\varphi(n)}x_{\varphi(n)}+\epsilon_{\varphi(n)}J_E^p(S_1^{\varphi(n)}x_{\varphi(n)}+x_{\varphi(n)}-x_{\varphi(n)-1}))$ and $s_{\varphi(n)}=J_{E^*}^q(\gamma_{\varphi(n)}J_E^p x_{\varphi(n)}+(1-\gamma_{\varphi(n)})J_E^p g_{\varphi(n)}))$, we obtain that $\lim_{n\to\infty}\|g_{\varphi(n)}-S_1^{\varphi(n)}x_{\varphi(n)}\|=0$ and

$$\lim_{n\to\infty}\|w_{\varphi(n)}-s_{\varphi(n)}\|=\lim_{n\to\infty}\|z_{\varphi(n)}-w_{\varphi(n)}\|=\lim_{n\to\infty}\|x_{\varphi(n)}-S_1^{\varphi(n)}x_{\varphi(n)}\|=\lim_{n\to\infty}\|s_{\varphi(n)}-x_{\varphi(n)}\|=0. \tag{3.38}$$

Also, from (3.24) and (3.22) we have

$$\begin{aligned}&(1-\alpha_{\varphi(n)})\beta_{\varphi(n)}(1-\beta_{\varphi(n)})\rho_b^*\|J_E^p z_{\varphi(n)}-J_E^p S_2^{\varphi(n)}z_{\varphi(n)}\|\\&\leq(1+\bar{\theta}_{\varphi(n)})(1+\theta_{\varphi(n)})\mathbf{\Gamma}_{\varphi(n)}-\mathbf{\Gamma}_{\varphi(n)+1}+(1+\bar{\theta}_{\varphi(n)})\ell_{\varphi(n)}M+\alpha_{\varphi(n)}\langle J_E^p u-J_E^p\hat{u},\zeta_{\varphi(n)}-\hat{u}\rangle.\end{aligned}$$

Noticing $v_{\varphi(n)}=J_{E^*}^q(\beta_{\varphi(n)}J_E^p z_{\varphi(n)}+(1-\beta_{\varphi(n)})J_E^p S^{\varphi(n)}z_{\varphi(n)})$ and using the similar reasonings to those in Case 1, we get

$$\lim_{n\to\infty}\|z_{\varphi(n)}-S_2^{\varphi(n)}z_{\varphi(n)}\|=\lim_{n\to\infty}\|v_{\varphi(n)}-z_{\varphi(n)}\|=0.$$

This together with (3.38) implies that

$$\lim_{n\to\infty}\|v_{\varphi(n)}-x_{\varphi(n)}\|=\lim_{n\to\infty}\|z_{\varphi(n)}-x_{\varphi(n)}\|=0. \tag{3.39}$$

Noticing $\zeta_{\varphi(n)}=J_{E^*}^q(\alpha_{\varphi(n)}J_E^p u+(1-\alpha_{\varphi(n)})J_E^p v_{\varphi(n)})$, by (3.39) one gets

$$\lim_{n\to\infty}\|x_{\varphi(n)}-\zeta_{\varphi(n)}\|=0. \tag{3.40}$$

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Applying the same reasonings as in Case 1, one has that $\lim_{n\to\infty}\|x_{\varphi(n)} - x_{\varphi(n)+1}\| = 0$,

$$\lim_{n\to\infty}\|s_{\varphi(n)} - y_{\varphi(n)}\| = \lim_{n\to\infty}\|w_{\varphi(n)} - \bar{y}_{\varphi(n)}\| = 0, \tag{3.41}$$

and

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, \zeta_{\varphi(n)} - \hat{u}\rangle \le 0. \tag{3.42}$$

Using (3.36), we get

$$\begin{aligned} &D_{f_p}(\hat{u}, x_{\varphi(n)+1}) \\ &\le (1-\alpha_{\varphi(n)})D_{f_p}(\hat{u}, x_{\varphi(n)}) + \alpha_{\varphi(n)}\{\tfrac{\epsilon_{\varphi(n)}}{\alpha_{\varphi(n)}}\|J_E^p S_1^{\varphi(n)}x_{\varphi(n)} - J_E^p(S_1^{\varphi(n)}x_{\varphi(n)} + x_{\varphi(n)} - x_{\varphi(n)-1})\| \\ &\quad \times \|\hat{u} + x_{\varphi(n)-1} - x_{\varphi(n)} - S_1^{\varphi(n)}x_{\varphi(n)}\| + \tfrac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}}D_{f_p}(\hat{u}, x_{\varphi(n)}) + \tfrac{\bar{\theta}_{\varphi(n)}}{\alpha_{\varphi(n)}}D_{f_p}(\hat{u}, s_{\varphi(n)}) \\ &\quad + \langle J_E^p u - J_E^p \hat{u}, \zeta_{\varphi(n)} - \hat{u}\rangle\}, \end{aligned} \tag{3.43}$$

which together with (3.37), hence yields

$$\begin{aligned} &\mathbf{\Gamma}_{\varphi(n)} \\ &\le \tfrac{\epsilon_{\varphi(n)}}{\alpha_{\varphi(n)}}\|J_E^p S_1^{\varphi(n)}x_{\varphi(n)} - J_E^p(S_1^{\varphi(n)}x_{\varphi(n)} + x_{\varphi(n)} - x_{\varphi(n)-1})\|\|\hat{u} + x_{\varphi(n)-1} - x_{\varphi(n)} - S_1^{\varphi(n)}x_{\varphi(n)}\| \\ &\quad + \tfrac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}}D_{f_p}(\hat{u}, x_{\varphi(n)}) + \tfrac{\bar{\theta}_{\varphi(n)}}{\alpha_{\varphi(n)}}D_{f_p}(\hat{u}, s_{\varphi(n)}) + \langle J_E^p u - J_E^p \hat{u}, \zeta_{\varphi(n)} - \hat{u}\rangle. \end{aligned}$$

As a result, from (3.42) we deduce that

$$\lim_{n\to\infty}\mathbf{\Gamma}_{\varphi(n)} = 0. \tag{3.44}$$

From (3.42), (3.43) and (3.44), one concludes that

$$\mathbf{\Gamma}_{\varphi(n)+1} \to 0 \quad (n\to\infty). \tag{3.45}$$

Again using (3.37), one gets $\mathbf{\Gamma}_n \to 0$. Therefore, $x_n - \hat{u} \to 0$. This completes the proof.

Remark 3.1. It can be easily seen from the proof of Theorem 3.2 that if the assumption that $\lim_{n\to\infty}\frac{\ell_n}{\alpha_n} = 0$, is used in place of the one that $\lim_{n\to\infty}\ell_n = 0$ and $\sup_{n\ge1}\frac{\epsilon_n}{\alpha_n} < \infty$, then Theorem 3.2 is still valid.

Under Algorithm 3.1, setting $A_2 = 0$ one obtains the algorithm below for approximating a point in $\blacksquare = \mathrm{VI}(C, A_1) \cap (\bigcap_{i=1}^2 \mathrm{Fix}(S_i))$.

Algorithm 3.3. Initialization: Given $x_0, x_1 \in C$ arbitrarily and let $\epsilon > 0$, $\mu_1 > 0$, $\lambda_1 \in (0, \frac{1}{\mu_1})$, $l_1 \in (0,1)$. Choose $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\ell_n\} \subset (0,\infty)$ s.t. $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$ and $\sum_{n=1}^\infty \ell_n < \infty$. Moreover, assume $\sum_{n=1}^\infty \theta_n < \infty$, and given the iterates x_{n-1} and x_n $(n \ge 1)$, choose ϵ_n s.t. $0 \le \epsilon_n \le \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_1^n x_n - J_E^p(S_1^n x_n + x_n - x_{n-1})\|}\} & \text{if } x_n \ne x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterations: Compute x_{n+1} below:

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Step 1. Put $g_n = J^q_{E^*}((1-\epsilon_n)J^p_E S^n_1 x_n + \epsilon_n J^p_E(S^n_1 x_n + x_n - x_{n-1}))$, and calculate $s_n = J^q_{E^*}(\beta_n J^p_E x_n + (1-\beta_n)J^p_E g_n)$, $y_n = \Pi_C(J^q_{E^*}(J^p_E s_n - \lambda_1 A_1 s_n))$, $e_{\lambda_1}(s_n) := s_n - y_n$ and $t_n = s_n - \tau_n e_{\lambda_1}(s_n)$, with $\tau_n := l_1^{k_n}$ and k_n being the smallest $k \geq 0$ s.t.

$$\frac{\mu_1}{2} D_{f_p}(s_n, y_n) \geq \langle A_1 s_n - A_1(s_n - l_1^k e_{\lambda_1}(s_n)), s_n - y_n\rangle.$$

Step 2. Calculate $w_n = \Pi_{C_n}(s_n)$, with $C_n := \{y \in C : h_n(y) \leq 0\}$ and

$$h_n(y) = \langle A_1 t_n, y - s_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(s_n, y_n).$$

Step 3. Calculate $v_n = J^q_{E^*}(\alpha_n J^p_E w_n + (1-\alpha_n)J^p_E(S^n_2 w_n))$ and $x_{n+1} = \Pi_{Q_n}(w_n)$, with $Q_n := \{y \in C : D_{f_p}(y, v_n) \leq (\bar{\theta}_n + 1)D_{f_p}(y, w_n)\}$.

Again put $n := n+1$ and return to Step 1.

Corollary 3.1. Let the terms (C1)-(C2) with $A_2 = 0$, be valid, and assume $\blacksquare = \mathrm{VI}(C, A_1) \cap (\bigcap_{i=1}^2 \mathrm{Fix}(S_i)) \neq \emptyset$. If under Algorithm 3.3, $S_1^{n+1}x_n - S_1^n x_n \to 0$ and $S_2^{n+1}w_n - S_2^n w_n \to 0$, then $x_n \rightharpoonup z \in \blacksquare \Leftrightarrow \sup_{n\geq 0}\|x_n\| < \infty$.

Next, put $S_2 = I$ the identity mapping of E. Then we get $\blacksquare = \mathrm{Fix}(S_1) \cap (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i))$. In this case, Algorithm 3.2 can be rewritten as the iterative scheme below for settling a pair of VIPs and the FPP of S_1. By Theorem 3.2 one derives the strong convergence outcome below.

Corollary 3.2. Suppose that the condition (C2) holds, and let $\blacksquare = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap \mathrm{Fix}(S_1) \neq \emptyset$. For initial $x_0, x_1 \in C$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J^p_E S^n_1 x_n - J^p_E(S^n_1 x_n + x_n - x_{n-1})\|}\} & \text{if } x_n \neq x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Suppose that $\{x_n\}$ is the sequence constructed by

$$\begin{cases} g_n = J^q_{E^*}((1-\epsilon_n)J^p_E S^n_1 x_n + \epsilon_n J^p_E(S^n_1 x_n + x_n - x_{n-1})), \\ s_n = J^q_{E^*}(\gamma_n J^p_E x_n + (1-\gamma_n)J^p_E g_n), \\ y_n = \Pi_C(J^q_{E^*}(J^p_E s_n - \lambda_1 A_1 s_n)), \\ t_n = (1-\tau_n)s_n + \tau_n y_n, \\ w_n = \Pi_{C_n} s_n, \\ \bar{y}_n = \Pi_C(J^q_{E^*}(J^p_E w_n - \lambda_2 A_2 w_n)), \\ \bar{t}_n = (1-\bar{\tau}_n)w_n + \bar{\tau}_n \bar{y}_n, \\ z_n = \Pi_{\bar{C}_n} w_n, \\ x_{n+1} = \Pi_C(J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n)J^p_E z_n) \quad \forall n \geq 1, \end{cases}$$

where $\tau_n := l_1^{k_n}$, $\bar{\tau}_n := l_2^{j_n}$ and k_n, j_n are the smallest nonnegative integers k and j satisfying, respectively,

$$\langle A_1 s_n - A_1(s_n - l_1^k(s_n - y_n)), s_n - y_n\rangle \leq \frac{\mu_1}{2} D_{f_p}(s_n, y_n),$$

$$\langle A_2 w_n - A_2(w_n - l_2^j(w_n - \bar{y}_n)), w_n - \bar{y}_n\rangle \leq \frac{\mu_2}{2} D_{f_p}(w_n, \bar{y}_n),$$

and the sets $C_n, \bar{C}_n$, are constructed below

(i) $C_n := \{y \in C : h_n(y) \leq 0\}$ and $h_n(y) = \langle A_1 t_n, y - s_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(s_n, y_n)$;

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(ii) $\bar{C}_n := \{y \in C : \bar{h}_n(y) \leq 0\}$ and $\bar{h}_n(y) = \langle A_2\bar{t}_n, y - w_n\rangle + \frac{\bar{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \bar{y}_n)$.

Then, $x_n \to \Pi_{\blacksquare} u \Leftrightarrow \sup_{n\geq 0} \|x_n\| < \infty$ provided $S_1^{n+1}x_n - S_1^n x_n \to 0$.

4. Implementability and Applicability

In this section, we provide an illustrative example to demonstrate the applicability and implementability of our suggested approaches. For $i = 1, 2$, we take $\epsilon = \frac{1}{3}$, $\mu_i = 1$ and $l_i = \lambda_i = \frac{1}{3}$. First, we present an instance involving the mappings $A_1, A_2 : E \to E^*$ of both uniform continuity and pseudomonotonicity, and the mappings $S_1, S_2 : C \to C$ of both uniform continuity and Bregman's relatively asymptotical nonexpansivity satisfying $\blacksquare \neq \emptyset$. Put $C = [-2, 2]$ and $E = H = \mathbf{R}$ with the inner product and induced norm being written as $\langle a, b\rangle = ab$ and $\|\cdot\| = |\cdot|$, respectively. The starting x_0, x_1 are randomly chosen in C. For $i = 1, 2$, let $A_i : H \to H$ be defined as $A_1 y := \frac{1}{1+|\sin y|} - \frac{1}{1+|y|}$ and $A_2 y := y + \sin y$ for all $y \in H$. Next, let us prove that A_1 is the mapping of both Lipschitz continuity and pseudomonotonicity. In fact, for each $v, w \in H$ one has

$$\begin{aligned}\|A_1 v - A_1 w\| &= |\tfrac{1}{1+\|\sin v\|} - \tfrac{1}{1+\|v\|} - \tfrac{1}{1+\|\sin w\|} + \tfrac{1}{1+\|w\|}| \\ &\leq |\tfrac{\|y\|-\|w\|}{(1+\|v\|)(1+\|w\|)}| + |\tfrac{\|\sin y\|-\|\sin w\|}{(1+\|\sin v\|)(1+\|\sin w\|)}| \\ &\leq \|v - w\| + \|\sin v - \sin w\| \leq 2\|v - w\|.\end{aligned}$$

Thus, A_1 is of Lipschitz continuity. Also, one shows that A_1 is of pseudomonotonicity. For any $v, w \in H$, it is easily known that

$$\begin{aligned}&\langle A_1 v, w - v\rangle = (\tfrac{1}{1+|\sin v|} - \tfrac{1}{1+|v|})(w - v) \geq 0 \\ &\Rightarrow \langle A_1 w, w - v\rangle = (\tfrac{1}{1+|\sin w|} - \tfrac{1}{1+|w|})(w - v) \geq 0.\end{aligned}$$

It is easy to see that A_2 is of both Lipschitz continuity and monotonicity. Indeed, we deduce that $\|A_2 v - A_2 y\| \leq \|v - y\| + \|\sin v - \sin y\| \leq 2\|v - y\|$ and

$$\langle A_2 v - A_2 y, v - y\rangle = \|v - y\|^2 + \langle \sin v - \sin y, v - y\rangle \geq \|v - y\|^2 - \|v - y\|^2 = 0.$$

Now, let $S_1 : C \to C$ and $S_2 : C \to C$ be defined as $S_1 y = S_2 y := Sy = \frac{4}{5}\sin y$. It is clear that $\mathrm{Fix}(S_i) = \mathrm{Fix}(S) = \{0\}$ for $i = 1, 2$.

Also, $S : C \to C$ is the mapping of Bregman's relatively asymptotical nonexpansivity with $\theta_n = (\frac{4}{5})^n$, and $\forall\{\varrho_n\} \subset C$ we get $\|S^{n+1}\varrho_n - S^n\varrho_n\| \to 0$. In fact, note that

$$\|S^n v - S^n w\|^2 \leq (\frac{4}{5})^2\|S^{n-1}v - S^{n-1}w\|^2 \leq \cdots \leq (\frac{4}{5})^{2n}\|v - w\|^2 \leq (1 + \theta_n)\|v - w\|^2,$$

$$\|S^{n+1}\varrho_n - S^n\varrho_n\| \leq (\frac{4}{5})^{n-1}\|S^2\varrho_n - S\varrho_n\| = (\frac{4}{5})^{n-1}\|\frac{4}{5}\sin(S\varrho_n) - \frac{4}{5}\sin\varrho_n\| \leq 2(\frac{4}{5})^n \to 0 \ \ (n \to \infty),$$

and

$$\lim_{n\to\infty} \frac{\theta_n}{1/2(n+1)} = \lim_{n\to\infty} \frac{(4/5)^n}{1/2(n+1)} = 0.$$

Consequently,

$$\blacksquare = \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap \mathrm{Fix}(S_i) = \{0\} \neq \emptyset.$$

In addition, putting $\beta_n = \frac{n+2}{2(n+1)}$ $\forall n \geq 1$, we obtain

$$\lim_{n\to\infty} \beta_n(1 - \beta_n) = \lim_{n\to\infty} \frac{n+2}{2(n+1)}(1 - \frac{n+2}{2(n+1)}) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4} > 0$$

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In this case, the conditions (C1)-(C3) are satisfied.

Example 4.1. Let $\ell_n = \frac{1}{2(n+1)^2}$ and $\alpha_n = \beta_n = \frac{n+2}{2(n+1)}$ $\forall n \geq 1$. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Algorithm 3.1 is rewritten as follows:

$$\begin{cases} g_n = S^n x_n + \epsilon_n (x_n - x_{n-1}), \\ s_n = \frac{n+2}{2(n+1)} x_n + \frac{n}{2(n+1)} g_n, \\ y_n = P_C(s_n - \frac{1}{3} A_1 s_n), \\ t_n = (1 - \tau_n) s_n + \tau_n y_n, \\ w_n = P_{C_n} s_n, \\ \bar{y}_n = P_C(w_n - \frac{1}{3} A_2 w_n), \\ \bar{t}_n = (1 - \bar{\tau}_n) w_n + \bar{\tau}_n \bar{y}_n, \\ v_n = \frac{n}{2(n+1)} S^n w_n + \frac{n+2}{2(n+1)} w_n, \\ Q_n = \{y \in C : \|v_n - y\|^2 \leq (1 + (\frac{4}{5})^n) \|w_n - y\|^2\}, \\ x_{n+1} = P_{\bar{C}_n \cap Q_n} w_n, \end{cases} \tag{4.1}$$

with the sets $C_n, \bar{C}_n$ and the step-sizes $\tau_n, \bar{\tau}_n$ being picked as in Algorithm 3.1. By Theorem 3.1, one obtain $x_n \to 0 \in \blacksquare = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap \mathrm{Fix}(S))$.

Example 4.2. Let $\ell_n = \frac{1}{2(n+1)^2}$, $\alpha_n = \frac{1}{2(n+1)}$ and $\beta_n = \gamma_n = \frac{n+2}{2(n+1)}$ $\forall n \geq 1$. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Algorithm 3.2 is rewritten as follows:

$$\begin{cases} g_n = S^n x_n + \epsilon_n (x_n - x_{n-1}), \\ u_n = \frac{n+2}{2(n+1)} x_n + \frac{n}{2(n+1)} g_n, \\ y_n = P_C(s_n - \frac{1}{3} A_1 s_n), \\ s_n = (1 - \tau_n) s_n + \tau_n y_n, \\ w_n = P_{C_n} s_n, \\ \bar{y}_n = P_C(w_n - \frac{1}{3} A_2 w_n), \\ \bar{t}_n = (1 - \bar{\tau}_n) w_n + \bar{\tau}_n \bar{y}_n, \\ z_n = P_{\bar{C}_n} w_n, \\ v_n = \frac{n}{2(n+1)} S^n z_n + \frac{n+2}{2(n+1)} z_n, \\ x_{n+1} = P_C(\frac{2n+1}{2(n+1)} v_n + \frac{1}{2(n+1)} u) \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

with the sets $C_n, \bar{C}_n$ and the step-sizes $\tau_n, \bar{\tau}_n$ being picked as in Algorithm 3.2. By Theorem 3.2, we deduce that $x_n \to 0 \in \blacksquare = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap \mathrm{Fix}(S)$.

5. Conclusions

This article designs iterative algorithms for resolving a pair of VIFPPs in uniformly smooth and p-uniformly convex Banach spaces. With the help of parallel subgradient-like extragradient methods with both inertial effect and linesearch process, we fabricate two algorithms for approximating a common solution

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of the two pseudomonotone VIPs and the CFPP of two mappings of Bregman's relatively asymptotical nonexpansivity. We are focused on discussing the weak and strong convergence of the proposed algorithms by using standard terms and novel manoeuvres. Besides, an illustrative example is furnished to bear up the applicability and implementability of our proposed approaches. Finally, it is worth mentioning that part of our future research is aimed at achieving the weak and strong convergence results for the modifications of our proposed approaches with Nesterov double inertial extrapolation steps (see [34]) and adaptive stepsizes.

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