Article

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Abstract: We assume that the current mathematical knowledge $\mathcal{K}$ is a finite set of statements from both formal and constructive mathematics, which is time-dependent and publicly available. Any formal theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We assume that mathematical sets are atemporal entities. In this article, they exist formally in ZFC theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal. Algorithms always terminate. We explain the distinction between algorithms whose existence is provable in ZFC and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$.

Keywords: constructive mathematics; constructively defined algorithms; current mathematical knowledge; informal mathematics; known algorithms

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1. Introduction

We assume that the current mathematical knowledge $\mathcal{K}$ is a finite set of statements from both formal and constructive mathematics, which is time-dependent and publicly available. Any formal theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. We assume that mathematical sets are atemporal entities. In this article, they exist formally in ZFC theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal.

Let $\mathcal{T}$ denote the set of twin primes. The true statement "*There is no known constructively defined integer n such that* $\text{card}(\mathcal{T}) < \omega \Rightarrow \mathcal{T} \subseteq (-\infty, n]$" is not formal, belongs to $\mathcal{K}$, and may not belong to $\mathcal{K}$ in the future. Paul Cohen proved in 1963 that the equality $2^{\aleph_0} = \aleph_1$ is independent of ZFC, see [3]. Before 1963, the false statement "*There is a known constructively defined integer $n \geqslant 1$ such that $2^{\aleph_0} = \aleph_n$*" was outside $\mathcal{K}$. Since 1963, this statement is outside $\mathcal{K}$ forever. The true statement "*There exists a set $\mathcal{X} \subseteq \{1, \ldots, 49\}$ such that* $\text{card}(\mathcal{X}) = 6$ *and $\mathcal{X}$ never occurred as the winning six numbers in the Polish Lotto lottery*" refers to the current non-mathematical knowledge and is outside $\mathcal{K}$.

Algorithms always terminate. We explain the distinction between algorithms whose existence is provable in ZFC and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$. These results are mainly from [18].

Observation 1 is known from the beginning of computability theory and shows that the predicate of the current mathematical knowledge slightly increases the intuitive mathematics. In Observation 2, the predicate of the current mathematical knowledge trivially increases the constructive mathematics.

Observation 1. *Church's thesis is based on the fact that the currently known computable functions are recursive, where the notion of a computable function is informal.*

Observation 2. *There exists a prime number p greater than the largest known prime number.*

In the traditional Brouwerian intuitionism, the truth of a mathematical statement means that we know a constructive proof, see [10, p. 135]. Below is the English summary of [17] available at the internet address of [17].

The basic philosophical idea of intuitionism is that mathematical entities exist only as mental constructions and that the notion of truth of a proposition should be equated with its verification or the existence of proof. However different intuitionists explained the existence of a proof in fundamentally different ways. There seem to be two main alternatives: the actual and potential existence of a proof. The second proposal is also understood in two alternative ways: as knowledge of a method of construction of a proof or as knowledge-independent and tenseless existence of a proof. This paper is a presentation and analysis of these alternatives.

2. Basic Definitions and Examples

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in *ZFC*) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [2], [5], [13, p. 9]. A definition of an integer n is called *constructive*, if it provides a known algorithm with no input that returns n. Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. *We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs: 1 (one), $+$ (addition), $-$ (subtraction), $\cdot$ (multiplication), $\hat{\ }$ (exponentiation with exponent in $\mathbb{N}$), $!$ (factorial of a non-negative integer), $($ (left parenthesis), $)$ (right parenthesis).*

The set of known elements of $\mathcal{X}$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let t denote the largest twin prime that is smaller than $(((((((((9!)!)!)!)!)!)!)!)!$. The number t is an unknown element of $\mathcal{T}$.

Definition 2. *Conditions (1)-(5) concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
(1) A known algorithm with no input returns an integer n satisfying $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
(2) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.
(3) There is no known algorithm with no input that returns the logical value of the statement $\mathrm{card}(\mathcal{X}) = \omega$.
(4) There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.
(5) $\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.

Condition (3) implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Let $[\cdot]$ denote the integer part function.

Example 1. *The set*
$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } \left[\frac{(((((((((9!)!)!)!)!)!)!)!)!}{\pi}\right] \text{ is odd} \\ \varnothing, & \text{otherwise} \end{cases}$$

does not satisfy Condition (3) because we know an algorithm with no input that computes $\left[\frac{(((((((((9!)!)!)!)!)!)!)!)!}{\pi}\right]$. The set of known elements of $\mathcal{X}$ is empty. Hence, Condition (5) fails for $\mathcal{X}$.

Example 2 ([2], [5],([13]p. 9)). *The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = (\{0, \dots, k\} \times \{1\}) \cup (\{k+1, \ k+2, \ k+3, \ \dots\} \times \{0\})$$

No known algorithm computes the function h.

Example 3. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds} \\ \varnothing, & \text{otherwise} \end{cases}$$

is decidable. This $\mathcal{X}$ satisfies Conditions (1) and (3) and does not satisfy Conditions (2), (4), and (5). These facts will hold forever.

3. Number-Theoretic Results

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [15], [16], [20].

Statement 1. *Condition (1) remains unproven for $\mathcal{X} = \mathcal{P}_{n^2+1}$.*

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\texttt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition (1), but the algorithm $\texttt{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. $\square$

Statement 2. *The statement*

$$\exists n \in \mathbb{N} \ (\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n+3])$$

remains unproven in ZFC and classical logic without the law of excluded middle.

Statement 3. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : \ \text{card}(\mathcal{P}_{n^2+1} \cap [-1, k]) > 10^{10^{10}}\} \cup \{k \in \mathcal{P}_{n^2+1} : \ \text{card}(\mathcal{P}_{n^2+1} \cap [-1, k]) \leqslant 10^{10^{10}}\}$$

satisfies Conditions (2)-(4). Condition (1) fails for $\mathcal{X}$, $\text{card}(\mathcal{X}) < \omega \Rightarrow \text{card}(\mathcal{X}) \leqslant 10^{10^{10}}$.

Proof. Since $\text{card}(\mathcal{P}_{n^2+1} \cap [2, 10^{28})) = 2199894223892$ ([16]) and the inequality $\text{card}(P_{n^2+1}) \geqslant 10^{10^{10}}$ remains unproven, Conditions (3) and (4) hold. $\square$

Conjecture 1. *([1, p. 443], [6]). The are infinitely many primes of the form $k! + 1$.*

For a non-negative integer n, let $\rho(n)$ denote $29.5 + \dfrac{11!}{3n+1} \cdot \sin(n)$.

Statement 4. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } \rho(n) \text{ primes of the form } k! + 1\}$$

satisfies Conditions (1)-(5) except the requirement that $\mathcal{X}$ is naturally defined. $501893 \in \mathcal{X}$. Condition (1) holds with $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0, 501893]) = 159827$. $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} : \text{the interval } [-1, n]$ contains at least 30 primes of the form $k! + 1\}$.

Proof. For every integer $n \geqslant 11!$, 30 is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!, \infty)$, then $n + 1, \ n + 2, \ n + 3, \ \ldots \ \in \mathcal{X}$. Hence, Condition (1) holds with $n = 11! - 1$. We explicitly know 24 positive integers k such that $k! + 1$ is prime, see [4]. The inequality

$$\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k! + 1 \text{ is prime}\}) > 24$$

remains unproven. Since $24 < 30$, Condition (3) holds. The interval $[-1, 11! - 1]$ contains exactly three primes of the form $k! + 1$: $1! + 1, \ 2! + 1, \ 3! + 1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line $[501893.0, \ 159827]$, which proves Condition (1) with $n = 501893$ and Condition (4) with $\mathrm{card}(\mathcal{X}) \geqslant 159827$. $\square$

T. Nagell proved in [11] (cf. [14, p. 104]) that the equation $x^2 - 17 = y^3$ has exactly 16 integer solutions, namely $(\pm 3, -2), \ (\pm 4, -1), \ (\pm 5, 2), \ (\pm 9, 4), \ (\pm 23, 8), \ (\pm 282, 43), \ (\pm 375, 52), \ (\pm 378661, 5234)$. The set

$$\bigcup_{\substack{(x, y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) = 0}} \{(x + 8)^8\}$$

has exactly 23 elements. Among them, there are 14 integers from the interval $[1, 2199894223892]$. Let $\mathcal{W}$ denote the set

$$\bigcup_{\substack{(x, y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) = 0}} \{k \in \mathbb{N} : \ k \text{ is the } (x + 8)^8 - \text{th element of } \mathcal{P}_{n^2 + 1}\}$$

From [16], it is known that $\mathrm{card}(\mathcal{P}_{n^2 + 1} \cap [2, 10^{28}]) = 2199894223892$. Hence, $\mathrm{card}(\mathcal{W} \cap [2, 10^{28}]) = 14$ and 14 elements of $\mathcal{W}$ can be practically computed. The inequality $\mathrm{card}(\mathcal{P}(n^2 + 1)) \geqslant (378661 + 8)^8$ remains unproven. The last two sentences and Statement 4 imply the following corollary.

Corollary 1. *If we add $\mathcal{W}$ to $\mathcal{X}$, then the following statements hold:*

Condition (1) fails for $\mathcal{X}$,

$159827 + 14 \leqslant \mathrm{card}(\mathcal{X})$,

the above lower bound is currently the best known,

$\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathrm{card}(\mathcal{X}) \leqslant 159827 + 23$,

the above upper bound is currently the best known,

$\mathcal{X}$ satisfies Conditions (2)-(5) except the requirement that $\mathcal{X}$ is naturally defined.
Analogical statements hold, if we add to $\mathcal{X}$ the set

$$\bigcup_{\substack{x \in \mathbb{N} \\ x \text{ divides } 99!}} \{k \in \mathbb{N} : \ k - 501894 \text{ is the } x - th \text{ element of } \mathcal{P}_{n^2+1}\}$$

Definition 3. *Conditions (1a)-(5a) concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
(1a) A known algorithm with no input returns a positive integer n satisfying $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
(2a) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.
(3a) There is no known algorithm with no input that returns the logical value of the statement $\mathrm{card}(\mathcal{X}) < \omega$.
(4a) There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is finite.
(5a) $\mathcal{X}$ is naturally defined. The finiteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.

Statement 5. *The set*

$$\mathcal{X} = \Big\{ n \in \mathbb{N} : \text{ the interval } [-1, n] \text{ contains more than}$$

$$6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \text{ squares of the form } k!+1 \Big\}$$

satisfies Conditions (1a)-(5a) except the requirement that $\mathcal{X}$ is naturally defined. $95151 \in \mathcal{X}$. Condition (1a) holds with $n = 95151$. $\mathrm{card}(\mathcal{X} \cap [0, 95151]) = 30311$. $\mathcal{X} \cap [95152, \infty) = \{n \in \mathbb{N} : \text{ the interval } [-1, n] \text{ contains at least 7 squares of the form } k!+1\}$.

Proof. For every integer $n > 10^6$, 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$. By this, if $n \in \mathcal{X} \cap (10^6, \infty)$, then $n+1$, $n+2$, $n+3$, ... $\in \mathcal{X}$. Hence, Condition (1a) holds with $n = 10^6$. It is conjectured that $k!+1$ is a square only for $k \in \{4, 5, 7\}$, see [19, p. 297]. Hence, the inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \text{ is a square}\}) > 3$ remains unproven. Since $3 < 7$, Condition (3a) holds. The interval $[-1, 10^6]$ contains exactly three squares of the form $k!+1$: $4!+1$, $5!+1$, $7!+1$. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line $[95151.0, \ 30311]$, which proves Condition (1a) with $n = 95151$ and Condition (4a) with $\mathrm{card}(\mathcal{X}) \geqslant 30311$. $\square$

Statement 6. *The set*
$$\mathcal{X} = \{k \in \mathbb{N} : \mathrm{card}([-1, k] \cap \mathcal{P}_{n^2+1}) < 10^{10000}\}$$

satisfies the conjunction

$$\neg(\text{Condition 1a}) \wedge (\text{Condition 2a}) \wedge (\text{Condition 3a}) \wedge (\text{Condition 4a}) \wedge (\text{Condition 5a})$$

For a non-negative integer n, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n. Let $\kappa : \mathbb{N} \to \mathbb{N}$ be defined by setting $\kappa(n)$ to be the exponent of 2 in the prime factorization of $n + 1$.

Statement 7. *([18, p. 250]). The set*

$$\mathcal{X} = \{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \ is \ prime\}$$

satisfies Conditions (1)-(5) except the requirement that $\mathcal{X}$ is naturally defined. Condition (1) holds with $n = 10^{10^{10}}$.

Statement 8. *There exists a naturally defined set $\mathcal{C} \subseteq \mathbb{N}$ which satisfies the following Conditions (6)-(11).*
(6) A known and simple algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{C}$.
(7) There is no known algorithm with no input that returns the logical value of the statement $\mathrm{card}(\mathcal{C}) = \omega$.
(8) There is no known algorithm with no input that returns the logical value of the statement $\mathrm{card}(\mathbb{N} \setminus \mathcal{C}) = \omega$.
(9) It is conjectured, though so far unproven, that $\mathcal{C}$ is infinite.
(10) There is no known algorithm with no input that returns an integer n satisfying $\mathrm{card}(\mathcal{C}) < \omega \Rightarrow \mathcal{C} \subseteq (-\infty, n]$.
(11) There is no known algorithm with no input that returns an integer m satisfying $\mathrm{card}(\mathbb{N} \setminus \mathcal{C}) < \omega \Rightarrow \mathbb{N} \setminus \mathcal{C} \subseteq (-\infty, m]$.

Proof. Conditions (6)-(11) hold for $\mathcal{C} = \{k \in \mathbb{N} : 2^{2^k} + 1 \ is \ composite\}$. It follows from the following three observations. It is an open problem whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [8, p. 159] and [12, p. 74]. It is an open problem whether or not there are infinitely many prime numbers of the form $2^{2^k} + 1$, see [8, p. 158] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [7, p. 23]. □

4. A Consequence of the Physical Limits of Computation

Statement 9. *No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition (1). Since Conditions (1)-(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X}, \ n + 2 \notin \mathcal{X}, \ n + 3 \notin \mathcal{X}, \ \ldots \tag{T}$$
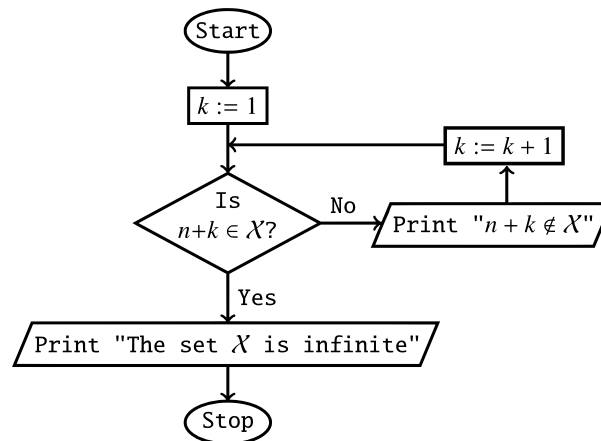
Figure 1. Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite.

The sentences from the sequence (T) and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition (4). $\square$

The physical limits of computation ([9]) disprove the assumption of Statement 9.

5. Satisfiable Conjunctions which Consist of Conditions (1)-(5) and Their Negations

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions (1)-(5)?*

Open Problem 1 asks about the existence of a year $t \geqslant 2024$ in which the conjunction

$$(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge (\text{Condition } 3) \wedge (\text{Condition } 4) \wedge (\text{Condition } 5)$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geqslant 2024$ and for every $i \in \{1, 2, 3\}$, a positive solution to Open Problem i in the year t may change in the future. Currently, the answers to Open Problems 1–7 are negative.

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge (\text{Condition } 3) \wedge (\text{Condition } 4) \wedge (\text{Condition } 5)$$

The set $\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge (\text{Condition } 3) \wedge (\text{Condition } 4) \wedge \neg(\text{Condition } 5)$$

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer n. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\}, \textit{ otherwise} \end{cases}$$

satisfies the conjunction

$$(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge \neg(\text{Condition } 3) \wedge (\text{Condition } 4) \wedge \neg(\text{Condition } 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge \neg(\text{Condition } 3) \wedge (\text{Condition } 4) \wedge (\text{Condition } 5)?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [8, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [8, p. 159] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [7, p. 23]. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \dots, 10^6\} \cup \\ \{n \in \mathbb{N} : n \; is \; the \; sixth \; prime \; number \; of \; the \; form \; 2^{2^k} + 1\}, \; otherwise \end{cases}$$

satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \; is \; composite\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(Condition\ 1) \land (Condition\ 2) \land \#(Condition\ 3) \land (Condition\ 4) \land \#(Condition\ 5)$$

where # denotes the negation $\neg$ or the absence of any symbol.

Table 1. Five satisfiable conjunctions.

	(Cond. 2) $\land$ (Cond. 3) $\land$ (Cond. 4)	(Cond. 2) $\land$ ¬(Cond. 3) $\land$ (Cond. 4)
(Cond. 1) $\land$ (Cond. 5)	Open Problem 1	Open Problem 2
(Cond. 1) $\land$ ¬(Cond. 5)	$\mathcal{X} = \{n \in \mathbb{N} : \; the \; interval \; [-1, n] \; contains \; more \; than \; 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \; primes \; of \; the \; form \; k! + 1\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \dots, 10^6\}, \; otherwise \end{cases}$
¬(Cond. 1) $\land$ (Cond. 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
¬(Cond. 1) $\land$ ¬(Cond. 5)	$\mathcal{X} = \{0, \dots, 10^6\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \; if \; 2^{2^{f(9^9)}} + 1 \; is \; composite \\ \{0, \dots, 10^6\} \cup \{n \in \mathbb{N} : n \; is \\ the \; sixth \; prime \; number \; of \\ the \; form \; 2^{2^k} + 1\}, \; otherwise \end{cases}$

6. Subsets of $\mathbb{N}$ and Their Threshold Numbers

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\operatorname{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Definition 5. *We say that a non-negative integer n is a weak threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if* $\operatorname{card}(\mathcal{X}) < \omega \Rightarrow \operatorname{card}(\mathcal{X}) \leqslant n$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is infinite, then any non-negative integer n is a weak threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is finite, then the all weak threshold numbers of $\mathcal{X}$ form the set $[\operatorname{card}(\mathcal{X}), \infty) \cap \mathbb{N}$.

Example 4. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : \text{ any proof in ZFC of length } k \text{ or less does not show that } \varnothing \neq \varnothing\}$$

conjecturally equals $\mathbb{N}$. No effectively computable integer n is a threshold number of $\mathcal{X}$. No effectively computable non-negative integer n is a weak threshold number of $\mathcal{X}$.

Open Problem 4. *Is there a known threshold number of $\mathcal{P}_{n^2+1}$?*

Open Problem 5. *Is there a known weak threshold number of $\mathcal{P}_{n^2+1}$?*

Open Problem 6. *Is there a known threshold number of $\mathcal{T}$?*

Open Problem 7. *Is there a known weak threshold number of $\mathcal{T}$?*

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