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An Algebraic Proof of the Jacobian Conjecture

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Abstract: In this paper, a short survey of the existed results concerning the Jacobian Conjecture is first given. Then the 3-fold linear type polynomial map will be analyzed in detail. The expansion of the Jacobian condition is deduced to obtain its equivalent algebraic equations, and the Jacobian condition will be analyzed to derive two coordinate transformations that can maintain the invariance of the Jacobian condition. Finally, it is proved by mathematical induction method that one general chain expression presented in this paper is just the inverse polynomial map of 3-fold linear type polynomial map for all $n \geq 2$, i.e. $LJC(n,[3])$ holds such that the Jacobian Conjecture holds.

Keywords: Jacobian Conjecture; 3-fold linear type map; Jacobian condition; coordinate transformation; equivalent algebraic equations; general chain expression

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1. Introduction

The last forty years the interest in the study of polynomial automorphisms is growing rapidly. The main motivation behind this interest is the existence of several very appealing open problems such as the Jacobian Conjecture [1–3].

The aim of this paper is to give a purely algebraic method to prove that the Jacobian Conjecture holds for all $n \geq 2$.

This paper is divided into five parts. In the first part, a short survey of the existed results concerning the Jacobian Conjecture is given. In the second part, the 3-fold linear type polynomial map will be analyzed in detail. Then we deduce the expansion of the Jacobian condition for 3-fold linear type polynomial map to obtain its equivalent algebraic equations, and analyze the Jacobian condition to derive two coordinate transformations that can maintain the invariance of the Jacobian condition. In the fourth part, we can prove by mathematical induction method that one general chain expression presented in this paper is just the inverse polynomial map of 3-fold linear type polynomial map for all $n \geq 2$. Finally, we can obtain several further results such as one result about the injectivity problem of the polynomial map.

2. The Jacobian Conjecture: a short survey

Let K be an algebraically closed field of characteristic zero and $F = (F_1, F_2, \cdots, F_n): K^n \to K^n$ be a polynomial map, i.e. a map of the form

$$(x_1, x_2, \cdots, x_n) \mapsto (F_1(x_1, x_2, \cdots, x_n), F_2(x_1, x_2, \cdots, x_n), \cdots, F_n(x_1, x_2, \cdots, x_n))$$

where each $F_i \in K[x] := K[x_1, x_2, \cdots, x_n]$.

Furthermore, for $x \in K^n$, put $F'(x) := \det(JF(x)) = |JF(x)|$, where

$$JF = \left(\frac{\partial F_i}{\partial x_j} \right)_{1 \leq i, j \leq n} \tag{1}$$

is the Jacobian matrix [2].

Put

$$\deg(F) := \max_{1\le i\le n} \deg(F_i) \tag{2}$$

where $\deg(F)$ means the total degree of F.

Now the famous Jacobian Conjecture can be shown below [2].

Conjecture 2.1. (Jacobian Conjecture ($JC(n)$)) Let $F: K^n \to K^n$ be a polynomial map such that $F'(x) \neq 0$ for all $x \in K^n$ (or equivalently $\det(JF) \in K^*$, which is defined as Jacobian condition), then F is invertible (i.e. F has an inverse which is also a polynomial map).

The Jacobian Conjecture is equivalent to the injectivity for F from the following beautiful result due to Bialynicki-Birula and Rosenlicht [2,4].

Theorem 2.2. Let K be an algebraically closed field of characteristic zero and $F: K^n \to K^n$ be a polynomial map. If F is injective, then F is surjective and its inverse is a polynomial map, i.e. F is a polynomial automorphism.

So the Jacobian Conjecture is equivalent to: if $F'(x) \neq 0$ for all $x \in K^n$, then F is injective or equivalently $F(a) \neq F(b)$ for all $a \neq b$, $a, b \in K^n$. Therefore, one proof that F is injective for all $n \geq 2$ is sufficient to prove that the Jacobian Conjecture holds.

From linear algebra we know that the Jacobian Conjecture is true if $\deg(F) = 1$. So the next case is $\deg(F) = 2$. It was only in 1980 that Stuart Wang proved that in this case the Jacobian Conjecture is true [2,5]:

Theorem 2.3. If $\deg(F) \leq 2$, then the Jacobian Conjecture is true.

Definition 2.4. We say that a polynomial map $F = (F_1, F_2, \cdots, F_n): K^n \to K^n$ is of homogeneous type if it has the form $F_i = x_i + h_i$ with h_i homogeneous of the same degree $d \geq 2$ [6].

Definition 2.5. We say that a polynomial map $F = (F_1, F_2, \cdots, F_n): K^n \to K^n$ is of d-fold linear type (denoted as L^d type), for $d \geq 2$, if there are linear forms $L_1, L_2, \cdots, L_n \in K[x_1, x_2, \cdots, x_n]$ such that F has the form $F_i = x_i + h_i$ with $h_i = L_i^d$ (which is defined as d-fold linear polynomial) [6].

Definition 2.6. We say that a polynomial map $F = (F_1, F_2, \cdots, F_n): K^n \to K^n$ is of general d-fold linear type (denoted as GL^d type), for $d \geq 2$, if F has the form $F_i = x_i + h_i$ that h_i is the linear combination of several d-fold linear polynomials or equivalently h_i is the sum of several d-fold linear polynomials (which is defined as general d-fold linear polynomial).

Definition 2.7. We say that a polynomial map $F = (F_1, F_2, \cdots, F_n): K^n \to K^n$ is of symmetric type if it is of special type and the Jacobian matrix JF is a symmetric matrix [6].

Definition 2.8. We say that a polynomial map $F = (F_1, F_2, \cdots, F_n): K^n \to K^n$ is of symmetric homogeneous type if it is both of symmetric type and of homogeneous type [6].

Conjecture 2.9. ($JC(n,[d])$) Let $F = x + h(x)$ be a polynomial map of homogeneous type having degree d with $F'(x) = 1$ for all $x \in K^n$, then F is invertible [6].

Conjecture 2.10. ($LJC(n,[d])$) Let $F = x + h(x)$ be a polynomial map of d-fold linear type having degree d with $F'(x) = 1$ for all $x \in K^n$, then F is invertible [6].

Conjecture 2.11. ($SJC(n,[d])$) Let $F = x + h(x)$ be a polynomial map of symmetric homogeneous type having degree d with $F'(x) = 1$ for all $x \in K^n$, then F is invertible [6].

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The following reduction, now standard knowledge, is proved in [3,6]:

Theorem 2.12. (Cubic Reduction) For any fixed integer $d \geq 3$, we have

$$JC(n) \Leftrightarrow JC(n,[d]) \quad \text{for all} \quad n \geq 2 \tag{3}$$

In particular, proving the Jacobian Conjecture is reduced to proving $JC(n,[3])$ for all $n \geq 2$. An even stronger reduction was proved by L. M. Druzkowski in [6,7]:

Theorem 2.13. (Cubic Linear Reduction) For any fixed integer $d \geq 3$, we have

$$JC(n) \Leftrightarrow LJC(n,[d]) \quad \text{for all} \quad n \geq 2 \tag{4}$$

In particular, proving the Jacobian Conjecture is then reduced to proving $LJC(n,[3])$ for all $n \geq 2$. In a recent breakthrough, M. de Bondt and A. van den Essen proved the following intriguing reduction [6,8]:

Theorem 2.14. (Symmetric Reduction) For any fixed integer $d \geq 3$, we have

$$JC(n) \Leftrightarrow SJC(n,[d]) \quad \text{for all} \quad n \geq 2 \tag{5}$$

In particular, proving the Jacobian Conjecture is reduced to proving $SJC(n,[3])$ for all $n \geq 2$.

In this paper, $LJC(n,[3])$ for all $n \geq 2$ will be used to prove the Jacobian Conjecture by means of mathematical induction method for the rank of the Jacobian matrix $Jh(x)$.

3.The 3-fold linear type polynomial map

The 3-fold linear type (i.e. L^3 type) polynomial map is rewritten as follows.

$$f := F(x) = (x_1 + (\sum a_{1j}x_j)^3, x_2 + (\sum a_{2j}x_j)^3, \cdots, x_n + (\sum a_{nj}x_j)^3) \tag{6}$$

$$h := h(x) = ((\sum a_{1j}x_j)^3, (\sum a_{2j}x_j)^3, \cdots, (\sum a_{nj}x_j)^3) \tag{7}$$

Let A be the system matrix of L^3 type polynomial map:

$$A = (a_{ij}) \in K^{n \times n} \tag{8}$$

Theorem 3.1. In the n-dimensional case where $F = x + h$ with h homogeneous, we have these equivalences [2]:

$$\det(JF) = 1 \Leftrightarrow Jh \text{ is nilpotent} \Leftrightarrow (Jh)^{r_0} = 0,\ (Jh)^i \neq 0,\ 0 \leq i \leq r_0 - 1 \tag{9}$$

where r_0 is the nilpotent index of the nilpotent matrix Jh (which is the maximum order of all Jordan submatrices in the Jordan canonical form of the nilpotent matrix Jh, and may depend on the specific details of the matrix Jh).

Theorem 3.2. In the n-dimensional case where $F = x + h$ with h homogeneous, r_0 is the nilpotent index of the nilpotent matrix Jh, then

$$(Jh)^i = 0,\ i \geq r_0 \tag{10}$$

Theorem 3.3. Let $r := rank(Jh)$ be the rank of the nilpotent matrix Jh for all $n \geq 2$ where $F = x + h$ with h homogeneous, then

$$(Jh)^i = 0,\ i \geq r + 1 \tag{11}$$

It is obvious that $r_0 \le r+1$.

In particular, it is worth noted that the parameter r_0 may depend on the specific details of the matrix Jh, but the parameter r is only related to the overall property of the matrix Jh and not its details. In the following algebraic proof, $LJC(n,[3])$ for all $n \ge 2$ will be proved by mathematical induction method for the parameter r.

The L^3 or GL^3 type polynomial map $F(x) = x + h(x)$ for all $n \ge 2$ satisfies the following mathematical conditions:

$$H := Jh/3,\ r = rank(H) \tag{12}$$

$$h = H \cdot x = (Jh)x/3 \tag{13}$$

$$f = F(x) = x + h \tag{14}$$

$$\det(JF) = \det(I + 3H) = 1,\ \text{for all } x \in K^n \Leftrightarrow H \text{ is nilpotent} \Leftrightarrow (H)^{r+1} = 0 \tag{15}$$

where r is the rank of the nilpotent matrix H (such that Jh). There must be $r \le n-1$.

In fact, the homogeneous type polynomial map with the degree three (i.e. $d = 3$) also satisfies the conditions (12) ~ (15).

Definition 3.4. The general chain expression $\varphi[f,h,i]$ is defined as follows.

$$\begin{cases} \varphi[f,h,0] = f \\ \varphi[f,h,i+1] = f - h(\varphi[f,h,i]),\ i \ge 0 \end{cases} \tag{16}$$

or

$$\varphi[f,h,i] := f - \underbrace{h(f - h(f - \cdots h(f)))}_{i \text{ copies for } h} \tag{17}$$

where i is defined as the index of the general chain expression $\varphi[f,h,i]$. In particular, $\varphi[f,h,0] = \varphi[f,0,0] = f$.

The following results are now standard knowledge [9].

Theorem 3.5. Let $f = F(x) = x + h(x)$ be a polynomial map of homogeneous type having degree d with $F'(x) = 1$ for all $x \in K^n$,

(i) if $h = 0$, $r = 0$, then $G(f) := x = f = \varphi[f,h,0]$ is the inverse polynomial map of $F(x)$;

(ii) if $(Jh)^2 = 0$, then $r_0 = 2$, and $x = G(f) = f - h(f) = \varphi[f,h,1]$ is the inverse polynomial map of $F(x)$.

Therefore, the following proposition holds.

Proposition 3.6. Let $f = F(x) = x + h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x) = 1$ for all $x \in K^n$,

(i) if $h = 0$, $r = 0$, then $x = G(f) = \varphi[f,h,0] = f$ is the inverse polynomial map of $F(x)$;

(ii) if $r = 1$, then $r_0 = 2$, $(Jh)^2 = 0$, and $x = G(f) = \varphi[f,h,1] = f - h(f)$ is the inverse polynomial map of $F(x)$.

For L^3 or GL^3 type map with $F'(x) = 1$ for all $x \in K^n$, if $r \ge 2$, is $F(x)$ invertible? If $F(x)$ is invertible, what is its inverse polynomial map? Through the later proof in this paper, we can be sure that the general chain expression $\varphi[f,h,r]$ is just its inverse polynomial map.

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Proposition 3.7. Let $f = F(x) = x + h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x) = 1$ for all $x \in K^n$, if $x = G(f) = \varphi[f, h, r]$ is its inverse polynomial map, then

$$x = G(f) = \varphi[f, h, i] = \varphi[f, h, r],\ i \ge r \tag{18}$$

where the formula (18) is called an extension of the index r.

Let $r_{\min} (\le r)$ be the minimum index that satisfies $x = G(f) = \varphi[f, h, i]$, then

$$x = G(f) = \varphi[f, h, i_1] = \varphi[f, h, i_2] = \varphi[f, h, r_{\min}],\ i_1 > i_2 \ge r_{\min} \tag{19}$$

The formula (19) is also an extension of the index $r_{\min}$. Whether $r_{\min}$ is related to the nilpotent index r_0 of the nilpotent matrix H (such that Jh) is still unknown.

It is worth noted that L^3 type polynomial map is a special case of GL^3 type polynomial map, the following result is easily obtained.

Theorem 3.8. For all $n \ge 2$, $h \in L^3$, $Jh = 3H$, $rank(H) = N \le n-1$; $\bar{h} \in GL^3$, $J\bar{h} = 3\bar{H}$, $rank(\bar{H}) = N$, h and $\bar{h}$ satisfy the conditions (12) ~ (15) at the same time, if $\bar{f} = \bar{F}(x)$ is invertible and its inverse polynomial map is $x = \bar{G}(\bar{f}) = \varphi[\bar{f}, \bar{h}, N]$, then $f = F(x)$ must be invertible and its inverse polynomial map is $x = G(f) = \varphi[f, h, N]$.

There is one important issue to be dealt with before further analysis can be advanced: is the reverse for Theorem 3.8 true? Since the general chain expression $\varphi[f, h, r]$ is only related to f, h and r, and not to the details of h, this is really true. It will be proved below.

Lemma 3.9. Let $f = F(x) = x + h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x) = 1$ for all $x \in K^n$, $Jh = 3H$, $r = rank(H)$, if $f = F(x)$ is invertible and its inverse polynomial map is $x = G(f)$, then

$$F(G(f)) = id_f,\ G(F(x)) = id_x \Rightarrow JF(x)JG(f) = I \tag{20}$$

$$JF(x)JG(f) = I \Leftrightarrow JG(f) = (I + 3H(x))^{-1} = \sum_{i=0}^{r} (-3H(x))^i,\ \ (H(x))^j = 0, j \ge r+1 \tag{21}$$

Lemma 3.10. Let $f = F(x) = x + h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x) = 1$ for all $x \in K^n$, $Jh = 3H$, $r = rank(H)$, if there exists a polynomial map $G(f) \in K^n[f]$ with zero constant term which satisfies the following condition, then $G(f)$ is unique.

$$JG(f) = \sum_{i=0}^{r} (-3H(x))^i \tag{22}$$

Proof. Let's assume that there exists another polynomial map $\tilde{G}(f) \in K^n[f]$ with zero constant term which also satisfies the condition (22), and $\tilde{G}(f) \ne G(f)$.

$$J\tilde{G}(f) = \sum_{i=0}^{r} (-3H(x))^i \tag{23}$$

$$\Delta G := G(f) - \tilde{G}(f) \ne 0 \tag{24}$$

$$J(\Delta G) = JG(f) - J\tilde{G}(f) = 0 \tag{25}$$

$$\frac{\partial(\Delta G_i)}{\partial f_j} = 0 \quad i, j = 1, 2, \cdots, n \tag{26}$$

We note that the constant terms of the polynomials $G(f)$ and $\tilde{G}(f)$ are zero, so

$$\Delta G_i = 0 \quad i = 1, 2, \cdots, n \tag{27}$$

$$\Delta G = 0 \tag{28}$$

The formula (28) is contradictory to the formula (24).
Therefore, the polynomial map $G(f)$ is unique.□

Combining Lemma 3.9 and Lemma 3.10, the following conclusion can be obtained.

Lemma 3.11. Let $f = F(x) = x + h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x) = 1$ for all $x \in K^n$, $Jh = 3H$, $r = rank(H)$, if there exists a polynomial map $G(f) \in K^n[f]$ with zero constant term which satisfies the condition (22), then $G(f)$ is unique and the following formula (29) holds, i.e. $f = F(x)$ is invertible and its inverse polynomial map is $x = G(f)$.

$$F(G(f)) = id_f, G(F(x)) = id_x \Leftrightarrow JF(x)JG(f) = I \tag{29}$$

Theorem 3.12. For all $n \geq 2$, $h \in L^3$, $Jh = 3H$, $rank(H) = N \leq n-1$; $\bar{h} \in GL^3$, $J\bar{h} = 3\bar{H}$, $rank(\bar{H}) = N$, h and $\bar{h}$ satisfy the conditions (12) ~ (15) at the same time, if $f = F(x)$ is invertible and its inverse polynomial map is $x = G(f) = \varphi[f, h, N]$, then $\bar{f} = \bar{F}(x)$ must also be invertible and its inverse polynomial map is $x = \bar{G}(\bar{f}) = \varphi[\bar{f}, \bar{h}, N]$.

Proof. It is noted that the polynomial map $x = G(f) = \varphi[f, h, N]$ is the inverse polynomial map of $f = F(x)$ and satisfies the conditions (12) ~ (15). From the formula (18), there are the following derivations.

$$x = G(f) = \varphi[f, h, N+1] = f - h(\varphi[f, h, N]) = f - h(G(f)) \tag{30}$$

$$JG(f) = I - J(h(G(f))) = I - Jh(x)JG(f) = I - 3H(x)JG(f) \tag{31}$$

$$(I + 3H(x))JG(f) = I \tag{32}$$

Then the formula (22) will be derived from (32).

On the other hand, the formula (22) can also be considered to be derived by substituting the conditions (12) ~ (15) into the expression $JG(f) = J(\varphi[f, h, N])$, although its derivation process is very complex and tedious.

Since the general chain expression $\varphi[f, h, r]$ is only related to f, h and r, and not to the details of h, for all $n \geq 2$, $\bar{h} \in GL^3$, $J\bar{h} = 3\bar{H}$, $rank(\bar{H}) = N$, $\bar{h}$ also satisfies the conditions (12) ~ (15), then the following formula (33) will also be derived by substituting the conditions (12) ~ (15) into the expression $J\bar{G}(\bar{f}) = J(\varphi[\bar{f}, \bar{h}, N])$.

$$J\bar{G}(\bar{f}) = \sum_{i=0}^{N} (-3\bar{H}(x))^i \tag{33}$$

From Lemma 3.11, then $\bar{G}(\bar{f})$ is unique, $\bar{f} = \bar{F}(x)$ must also be invertible and its inverse polynomial map is $x = \bar{G}(\bar{f}) = \varphi[\bar{f}, \bar{h}, N]$. □

Theorem 3.13. For all $n \geq 2$, $h \in L^3$, $Jh = 3H$, $rank(H) = N \leq n-1$; $\bar{h} \in GL^3$, $J\bar{h} = 3\bar{H}$, $rank(\bar{H}) = N$, h and $\bar{h}$ satisfy the conditions (12) ~ (15) at the same time, then $f = F(x)$ is invertible and its inverse

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polynomial map is $x = G(f) = \varphi[f, h, N]$ if and only if $\overline{f} = \overline{F}(x)$ is invertible and its inverse polynomial map is $x = \overline{G}(\overline{f}) = \varphi[\overline{f}, \overline{h}, N]$.

After the above important issue has been solved satisfactorily, the following algebraic proof in this paper only needs to be done for L^3 type polynomial map.

4. The Jacobian condition and coordinate transformations

First, we deduce the expansion of the Jacobian condition for L^3 type (or GL^3 type) polynomial map to obtain its equivalent algebraic equations, and then analyze the Jacobian condition to derive two coordinate transformations that can maintain the invariance of the Jacobian condition.

Definition 4.1. For $B := (B_{ij}) \in K^{n\times n}[x]$, $1 \le i_1 < i_2 < \cdots < i_k \le n$, $B[i_1, i_2, \cdots, i_k]$ is defined as a k-order principal submatrix of the matrix B, which is the k-order submatrix of the matrix B composed of the elements that are located at the intersections of rows $i_1, i_2, \cdots, i_k$ and columns $i_1, i_2, \cdots, i_k$ in B. $|B[i_1, i_2, \cdots, i_k]|$ or $\det(B[i_1, i_2, \cdots, i_k])$ is a k-order principal minor of the matrix B. In particular, $B[i_1, i_2, \cdots, i_n] = B[1, 2, \cdots, n] = B$.

Lemma 4.2. For $B := (B_{ij}) \in K^{n\times n}[x]$, let

$$I_{n,n} := I = diag[1, 1, \cdots, 1]$$

$$I_{n,n-1} := \begin{bmatrix} I^{(n-1)\times(n-1)} & 0^{(n-1)\times 1} \\ 0^{1\times(n-1)} & 0^{1\times 1} \end{bmatrix}, \quad I_{n,n-k} := \begin{bmatrix} I^{(n-k)\times(n-k)} & 0^{(n-k)\times k} \\ 0^{k\times(n-k)} & 0^{k\times k} \end{bmatrix}, \ 0 \le k \le n$$

Then there are the following formulas.

$$|I_{n,0} + B| = |B[1, 2, \cdots, n]| = |B|$$

$$|I_{n,1} + B| = |B[2, 3, \cdots, n]| + |B|$$

$$\cdots$$

$$|I_{n,n-2} + B| = |B[n-1, n]| + \sum_{1 \le i_1 \le n-2} |B[i_1, n-1, n]| + \sum_{1 \le i_1 < i_2 \le n-2} |B[i_1, i_2, n-1, n]| + \cdots + \sum_{1 \le i_1 < i_2 < \cdots < i_{n-2} \le n-2} |B[i_1, i_2, \cdots, i_{n-2}, n-1, n]|$$

where $B[i_1, i_2, \cdots, i_{n-2}, n-1, n] = B$.

$$|I_{n,n-1} + B| = |B[n]| + \sum_{1 \le i_1 \le n-1} |B[i_1, n]| + \sum_{1 \le i_1 < i_2 \le n-1} |B[i_1, i_2, n]| + \cdots + \sum_{1 \le i_1 < i_2 < \cdots < i_{n-1} \le n-1} |B[i_1, i_2, \cdots, i_{n-1}, n]|$$

where $B[i_1, i_2, \cdots, i_{n-1}, n] = B$.

Proof. (i) If $n = 1$, then $I_{1,0} + B = B$, $|I_{1,0} + B| = |B| = |B[1]|$, the conclusion holds.

(ii) If $n = 2$, then

$$|I_{2,0} + B| = |B| = |B[1, 2]|$$

$$|I_{2,1}+B|=\begin{vmatrix}1+B_{11} & B_{12}\\ B_{21} & B_{22}\end{vmatrix}=\begin{vmatrix}1 & B_{12}\\ 0 & B_{22}\end{vmatrix}+\begin{vmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{vmatrix}=|B[2]|+|B[1,2]|=|B[2]|+|B|$$

The conclusions also hold.

(iii) Let's assume that the conclusions are true if $n=N$, then while $n=N+1$, there are the following derivations.

$$|I_{N+1,0}+B|=|B|=|B[1,2,\cdots,N,N+1]|$$

$$\begin{aligned}|I_{N+1,1}+B|&=|I_{N,0}+B[2,3,\cdots,N,N+1]|+|B[1,2,\cdots,N,N+1]|\\&=|B[2,3,\cdots,N,N+1]|+|B|\end{aligned}$$

$$\cdots$$

$$\begin{aligned}|I_{N+1,N}+B|&=|I_{N,N-1}+B[1,2,\cdots,N-1,N+1]|+|I_{N+1,N-1}+B[1,2,\cdots,N,N+1]|\\&=\left\{\begin{matrix}|B[N+1]|+\sum\limits_{1\le i_1\le N-1}|B[i_1,N+1]|+\sum\limits_{1\le i_1<i_2\le N-1}|B[i_1,i_2,N+1]|\\+\cdots+\sum\limits_{1\le i_1<i_2<\cdots<i_{N-1}\le N-1}|B[i_1,i_2,\cdots,i_{N-1},N+1]|\end{matrix}\right\}+\\&\quad\left\{\begin{matrix}|B[N,N+1]|+\sum\limits_{1\le i_1\le N-1}|B[i_1,N,N+1]|+\sum\limits_{1\le i_1<i_2\le N-1}|B[i_1,i_2,N,N+1]|\\+\cdots+\sum\limits_{1\le i_1<i_2<\cdots<i_{N-1}\le N-1}|B[i_1,i_2,\cdots,i_{N-1},N,N+1]|\end{matrix}\right\}\\&=|B[N+1]|+\sum_{1\le i_1\le N}|B[i_1,N+1]|+\sum_{1\le i_1<i_2\le N}|B[i_1,i_2,N+1]|\\&\quad+\cdots+\sum_{1\le i_1<i_2<\cdots<i_N\le N}|B[i_1,i_2,\cdots,i_N,N+1]|\end{aligned}$$

Therefore, these conclusions are also true.□

Lemma 4.3. For $B=(B_{ij})\in K^{n\times n}[x]$, then

$$|I+B|=1+\sum_{1\le i_1\le n}|B[i_1]|+\sum_{1\le i_1<i_2\le n}|B[i_1,i_2]|+\cdots+\sum_{1\le i_1<i_2<\cdots<i_n\le n}|B[i_1,i_2,\cdots,i_n]| \quad (34)$$

where $B[i_1,i_2,\cdots,i_n]=B$.

Proof. (i) If $n=1$, then $|I+B|=|1+B_{11}|=1+B_{11}=1+|B[1]|=1+|B|$, the formula (34) holds.

(ii) If $n=2$, then

$$\begin{aligned}|I+B|&=\begin{vmatrix}1+B_{11} & B_{12}\\ B_{21} & 1+B_{22}\end{vmatrix}=\begin{vmatrix}1 & B_{12}\\ 0 & 1+B_{22}\end{vmatrix}+\begin{vmatrix}B_{11} & B_{12}\\ B_{21} & 1+B_{22}\end{vmatrix}\\&=\{1+|B[2]|\}+\{|B[1]|+|B[1,2]|\}=1+|B[1]|+|B[2]|+|B[1,2]|\\&=1+|B[1]|+|B[2]|+|B|\end{aligned}$$

The formula (34) also holds.

(iii) Let's assume that the formula (34) holds if $n=N$, then while $n=N+1$, there are the following derivations.

$$
\begin{aligned}
|I+B| &= |I_{N+1,N}+B| + |I_{N,N}+B[1,2,\cdots,N]| \\
&= \left\{ \begin{aligned} & |B[N+1]| + \sum_{1\le i_1\le N} |B[i_1,N+1]| + \sum_{1\le i_1<i_2\le N} |B[i_1,i_2,N+1]| \\ & +\cdots+ \sum_{1\le i_1<i_2<\cdots<i_N\le N} |B[i_1,i_2,\cdots,i_N,N+1]| \end{aligned} \right\} \\
&\quad + \left\{ 1+\sum_{1\le i_1\le N} |B[i_1]| + \sum_{1\le i_1<i_2\le N} |B[i_1,i_2]| +\cdots+ \sum_{1\le i_1<i_2<\cdots<i_N\le N} |B[i_1,i_2,\cdots,i_N]| \right\} \\
&= 1+\sum_{1\le i_1\le N+1} |B[i_1]| + \sum_{1\le i_1<i_2\le N+1} |B[i_1,i_2]| +\cdots+ \sum_{1\le i_1<i_2<\cdots<i_{N+1}\le N+1} |B[i_1,i_2,\cdots,i_{N+1}]|
\end{aligned} \quad)
$$

where $B[i_1,i_2,\cdots,i_{N+1}]=B$.

Therefore, the formula (34) also holds.□

Theorem 4.4. Let $f=F(x)=x+h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x)=1$ for all $x\in K^n$, $Jh=3H$, $r=rank(H)$, then

$$
1=|JF|=1+\sum_{1\le i_1\le n} 3^1 |H[i_1]| + \sum_{1\le i_1<i_2\le n} 3^2 |H[i_1,i_2]| +\cdots+ \sum_{1\le i_1<i_2<\cdots<i_n\le n} 3^n |H[i_1,i_2,\cdots,i_n]| \quad)
$$

$$
\begin{cases} \sum_{1\le i_1\le n} |H[i_1]| = 0 \\ \sum_{1\le i_1<i_2\le n} |H[i_1,i_2]| = 0 \\ \cdots \\ \sum_{1\le i_1<i_2<\cdots<i_n\le n} |H[i_1,i_2,\cdots,i_n]| = |H| = 0 \end{cases} \tag{35}
$$

When the matrix H has one column (such as column n) with all zero entries, (35) will be reduced to one $(n-1)$ dimensional case, and at the same time, the invertibility problem of the n dimensional polynomial map is equivalent to the invertibility problem of the $(n-1)$ dimensional polynomial map. This property can be developed into one kind of coordinate transformation to reduce the dimension and the rank of the matrix H for L^3 type map, while the L^3 type map will be transformed into one GL^3 type map and the invariance of the Jacobian condition can be maintained.

The formula (35) are first equivalent algebraic equations of the Jacobian condition for L^3 or GL^3 type map. It is worth noted that the last equation in (35) means $r=rank(H)\le n-1$, such that (35) can be further simplified as follows.

$$
\begin{cases} \sum_{1\le i_1\le n} |H[i_1]| = 0 \\ \sum_{1\le i_1<i_2\le n} |H[i_1,i_2]| = 0 \\ \cdots \\ \sum_{1\le i_1<i_2<\cdots<i_r\le n} |H[i_1,i_2,\cdots,i_r]| = 0 \end{cases} \tag{36}
$$

Definition 4.5. For all $x\in K^n$, the degree υ of $x_1^{\sigma_1}x_2^{\sigma_2}\cdots x_n^{\sigma_n}$ is defined as

$$
\upsilon=\sum_{i=1}^{n}\sigma_i \tag{37}
$$

Then (36) can be further decomposed into a series of equations about polynomials' coefficients of L^3 or GL^3 type map in the following.

$$
\alpha(\sigma_1,\sigma_2,\cdots,\sigma_n)=0,\ \ \upsilon=2q, 1\le q\le r \tag{38}
$$

where $\alpha(\sigma_1,\sigma_2,\cdots,\sigma_n)$ is the coefficient of the term $x_1^{\sigma_1}x_2^{\sigma_2}\cdots x_n^{\sigma_n}$ on the left-hand side of the equations as shown in (36).

The formula (38) are second equivalent algebraic equations of the Jacobian condition for L^3 or GL^3 type polynomial map.

In the following, we only analyze the Jacobian condition for L^3 type polynomial map to derive two coordinate transformations that can maintain the invariance of the Jacobian condition.

Lemma 4.6. For a nilpotent matrix $H \in K^{n\times n}[x]$, its similar matrix PHP^{-1} is also nilpotent.

Lemma 4.7. Let $f = F(x) = x + h(x)$ be one L^3 type polynomial map with $F'(x) = 1$ for all $x \in K^n$, $Jh = 3H$, $r = rank(H)$, its system matrix A satisfies the following formula.

$$rank(A) = rank(H) = r \le n-1 \tag{39}$$

For simplicity of analysis, we want to place the r mutually linearly independent columns of the matrix A in the front r columns of this matrix, which can be achieved by exchanging the subscripts of the variables (such as f and x) with multiple elementary matrix transformations.

Definition 4.8. The elementary transformation matrix $D(i;j)$ is defined as the matrix obtained by exchanging row i and row j of the identity matrix I. And the matrix $D(i;j)$ satisfies

$$\det(D(i;j)) = |D(i;j)| = -1 \tag{40}$$

$$(D(i;j))^{-1} = D(i;j) \tag{41}$$

It is supposed that the front r columns of the matrix A are mutually linearly independent after a series of elementary transformation matrices (whose product is denoted as D) have been used for a series of row and column exchange transformations of the matrix A in turn. This is one kind of coordinate transformation for L^3 type polynomial map, which is equivalent to applying a series of row exchange transformations to f and x at the same time and a similar transformation to the nilpotent matrix H, as shown below.

$$Df = Dx + DHx = Dx + (DHD^{-1})Dx \tag{42}$$

$$\begin{cases} \hat{f} := Df \\ \hat{x} := Dx \\ \hat{H} := DHD^{-1} \end{cases} \tag{43}$$

$$\hat{F}(\hat{x}) := \hat{f} = \hat{x} + \hat{H}\hat{x} \tag{44}$$

$$1 = |J\hat{F}| = |I + 3\hat{H}| = |I + 3DHD^{-1}| = |I + 3H| = |JF| \tag{45}$$

Therefore, the matrix $\hat{H}$ is the similar matrix of H such that it is nilpotent. The polynomial map $\hat{f} = \hat{F}(\hat{x}) = \hat{x} + \hat{H}\hat{x}$ is also of L^3 type, and the invariance of the Jacobian condition will be maintained after the coordinate transformation (as shown in (43)) has exchanged the subscripts of the variables.

It is no problem to suppose that the front r columns of the matrix A have been mutually linearly independent so that this will not be mentioned in the latter part of this paper.

In order to apply the mathematical induction method to proof the inverse map of L^3 type polynomial map, we also need another coordinate transformation to reduce the dimension (from n to $r(<n)$) and the rank (from r to $\bar{r}(<r)$) of the matrix H for L^3 type map, while the L^3 type map will be transformed into one GL^3 type polynomial map and the invariance of the Jacobian condition will also be maintained after the coordinate transformation (as shown in the description of (35)).

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Definition 4.9. For the system matrix A for L^3 type map with $r = rank(A) = rank(H)$, whose front r columns have been mutually linearly independent, then we define

$$\begin{cases} A := [A_1^T, A_2^T, \cdots, A_n^T]^T \\ A := [\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_n] \end{cases}$$

$$\overline{A}_j = \sum_{i=1}^{r} \beta_{ji} \overline{A}_i\,,\quad j = r+1, r+2, \cdots, n \tag{46}$$

$$Ax = \sum_{i=1}^{r} \overline{A}_i (x_i + \sum_{j=r+1}^{n} \beta_{ji} x_j) \tag{47}$$

The L^3 type map will be performed with the following coordinate transformation to reduce the rank of the matrix H from r to $\overline{r} (\le r-1)$.

$$\begin{cases} \hat{x}_i := (x_i + \sum_{j=r+1}^{n} \beta_{ji} x_j),\ i = 1, 2, \cdots, r \\ \hat{x}_j := x_j,\ \ j = r+1, r+2, \cdots, n \end{cases} \tag{48}$$

The formula (48) is rewritten in matrix form as follows.

$$\hat{x} := [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n]^T = \hat{D}x\,,\ \ \det(\hat{D}) = 1,\ \ \hat{D} \in K^{n\times n} \tag{49}$$

So there are the following derivations.

$$\begin{cases} \hat{f} := \hat{D}f \\ \hat{x} := \hat{D}x \\ \hat{H}(\hat{x}) := \hat{D}H(x)\hat{D}^{-1} \end{cases} \tag{50}$$

$$\begin{cases} \hat{A} := [\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_r, 0, 0, \cdots, 0] \\ \hat{A} := [\hat{A}_1^T, \hat{A}_2^T, \cdots, \hat{A}_n^T]^T \end{cases}$$

$$\hat{A} = A\hat{D}^{-1}$$

$$\sum_{i=1}^{r} \overline{A}_i \hat{x}_i = \sum_{i=1}^{r} \overline{A}_i (x_i + \sum_{j=r+1}^{n} \beta_{ji} x_j) = \sum_{i=1}^{r} \overline{A}_i x_i + \sum_{j=r+1}^{n} (\sum_{i=1}^{r} \beta_{ji} \overline{A}_i) x_j = \sum_{i=1}^{r} \overline{A}_i x_i + \sum_{j=r+1}^{n} \overline{A}_j x_j = \sum_{i=}^{n}$$

$$\sum_{i=1}^{r} \overline{A}_i \hat{x}_i = \hat{A}\hat{x} = \left(A\hat{D}^{-1}\right)\left(\hat{D}x\right) = Ax$$

$$A_i x = \hat{A}_i \hat{x},\ (A_i x)^2 = \left(\hat{A}_i \hat{x}\right)^2,\ i = 1, 2, \cdots, n$$

$$H(x)\hat{D}^{-1} = \left(diag[(A_1 x)^2, (A_2 x)^2, \cdots, (A_n x)^2]\right) A\hat{D}^{-1} = \left(diag[\left(\hat{A}_1\hat{x}\right)^2, \left(\hat{A}_2\hat{x}\right)^2, \cdots, \left(\hat{A}_n\hat{x}\right.\right.$$

$$\hat{H}(\hat{x}) = \hat{D}H(x)\hat{D}^{-1} = \hat{D}\left(diag[\left(\hat{A}_1\hat{x}\right)^2, \left(\hat{A}_2\hat{x}\right)^2, \cdots, \left(\hat{A}_n\hat{x}\right)^2]\right)\hat{A}$$

$$\hat{F}(\hat{x}) := \hat{f} = \hat{x} + \hat{H}(\hat{x})\hat{x} \tag{51}$$

$$1 = \left|J\hat{F}\right| = \left|I + 3\hat{H}\right| = \left|I + 3\hat{D}H\hat{D}^{-1}\right| = \left|I + 3H\right| = \left|JF\right| \tag{52}$$

$$\overline{x} := [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_r]^T \tag{53}$$

$$\overline{f} := [\hat{f}_1, \hat{f}_2, \cdots, \hat{f}_r]^T \tag{54}$$

In (50), since the last $(n-r)$ columns in the matrix $\hat{H}(\hat{x})$ are all zero columns, the front r columns will be constructed as a single matrix $\tilde{H}(\overline{x})$.

$$\tilde{H}(\overline{x}) = \hat{D}\left(diag[\left(\hat{A}_1\hat{x}\right)^2, \left(\hat{A}_2\hat{x}\right)^2, \cdots, \left(\hat{A}_n\hat{x}\right)^2]\right)[\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_r] \tag{55}$$

$$\hat{h}(\overline{x}) := \hat{H}(\hat{x})\hat{x} = \tilde{H}(\overline{x})\overline{x} = \hat{D}\left(diag[\left(\hat{A}_1\hat{x}\right)^2, \left(\hat{A}_2\hat{x}\right)^2, \cdots, \left(\hat{A}_n\hat{x}\right)^2]\right)[\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_r]\overline{x} \tag{56}$$

$$\hat{f} = \hat{x} + \hat{H}(\hat{x})\hat{x} = \hat{x} + \tilde{H}(\overline{x})\overline{x} = \hat{x} + \hat{h}(\overline{x}) \tag{57}$$

Then, the front r rows in the matrix $\hat{h}(\overline{x})$ can be constructed as a new matrix $\overline{h}(\overline{x})$.

$$\begin{cases} \overline{f} = \overline{x} + \overline{h}(\overline{x}) \\ \hat{f}_j = \hat{x}_j + \hat{h}_j(\overline{x}), \quad j = r+1, r+2, \cdots, n \end{cases} \tag{58}$$

It is obvious that the polynomial map $\overline{f} = \overline{x} + \overline{h}(\overline{x})$ in the r dimensional case is of GL^3 type.

Definition 4.10. For the r dimensional GL^3 type polynomial map $\overline{f} = \overline{x} + \overline{h}(\overline{x})$ as shown in (58), $\overline{H}(\overline{x}) := J\overline{h}(\overline{x})/3$, $\overline{r} := rank(\overline{H}(\overline{x}))$, $\overline{F}(\overline{x}) := \overline{f} = \overline{x} + \overline{h}(\overline{x}) = \overline{x} + \overline{H}(\overline{x})\overline{x}$.

The conclusions that $\det(J\overline{F}(\overline{x})) = 1$ and $\overline{H}(\overline{x})$ is nilpotent can be easily deduced from (35) and (46) ~ (58). This means that $\det(J\overline{F}(\overline{x})) = 1$ and matrix $\overline{H}(\overline{x})$ is nilpotent for the r dimensional GL^3 type polynomial map $\overline{f} = \overline{x} + \overline{h}(\overline{x})$ as shown in (58) if and only if $\det(JF(x)) = 1$ and matrix $H(x)$ is nilpotent for the n dimensional L^3 type polynomial map $f = x + h(x)$.

Theorem 4.11. For the r dimensional GL^3 type polynomial map $\overline{f} = \overline{x} + \overline{h}(\overline{x})$ as shown in (58), $\det(J\overline{F}(\overline{x})) = 1$, $\overline{H}(\overline{x})$ is nilpotent, and $\overline{r} = rank(\overline{H}(\overline{x})) \le r-1$.

5. The inverse polynomial map of the 3-fold linear type polynomial map

Now, we can prove by mathematical induction method that the general chain expression $\varphi[f, h, r]$ is just the inverse polynomial map of the L^3 or GL^3 type polynomial map $f = x + h(x)$ with $F'(x) = 1$ for all $x \in K^n$.

It is obvious that $LJC(n,[3])$ holds for all $n \ge 2$ and $r = 0,1$ from Proposition 3.6, we just need to prove that if $LJC(n,[3])$ holds when $r \le N$, then $LJC(n,[3])$ also holds when $r = N+1 (\le n-1)$.

We already know from Proposition 3.6 that if $r = 0$ and $r = 1$, the general chain expression $\varphi[f, h, r]$ is the inverse map of the L^3 or GL^3 type polynomial map $f = x + h(x)$ with $F'(x) = 1$ for all $x \in K^n$. Let's assume that the expression $\varphi[f, h, r]$ is the inverse map of the L^3 or GL^3 type polynomial map $f = x + h(x)$ with $F'(x) = 1$ for all $x \in K^n$ if $r \le N$, then when $r = N+1$, there is the following derivation process for the L^3 type polynomial map $f = x + h(x)$ with $F'(x) = 1$ for all $x \in K^n$.

By means of the coordinate transformation as shown in (46) ~ (58), the n dimensional L^3 type polynomial map $f = x + h(x)$ will be transformed into one r dimensional GL^3 type polynomial map $\overline{f} = \overline{x} + \overline{h}(\overline{x})$ as shown in (58). Since $\overline{H}(\overline{x})$ is nilpotent such that $\overline{r} = rank(\overline{H}(\overline{x})) \le r-1 = N$ (as shown in the description of (35)), the expression $\varphi[\overline{f}, \overline{h}, \overline{r}]$ is the inverse map of the GL^3 type polynomial map $\overline{f} = \overline{x} + \overline{h}(\overline{x})$.

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$$\begin{cases} \bar{G}(\bar{f}) := \bar{x} = \varphi[\bar{f}, \bar{h}, \bar{r}], \ \bar{r} \le r-1 = N \\ \hat{x}_j = \hat{f}_j - \hat{h}_j(\bar{x}) = \hat{f}_j - \hat{h}_j(\bar{G}(\bar{f})), \ j = N+2, N+3, \cdots, n \end{cases} \tag{59}$$

$$\begin{cases} \bar{x}_i = \varphi_i[\bar{f}, \bar{h}, \bar{r}] = \bar{f}_i - \bar{h}_i(\bar{x}) = \bar{f}_i - \bar{h}_i(\varphi[\bar{f}, \bar{h}, \bar{r}]), \ i = 1, 2, \cdots, N+1 \\ \hat{x}_j = \hat{f}_j - \hat{h}_j(\bar{G}(\bar{f})) = \hat{f}_j - \hat{h}_j(\varphi[\bar{f}, \bar{h}, \bar{r}]), \ j = N+2, N+3, \cdots, n \end{cases} \tag{60}$$

$$\hat{x} = \hat{f} - \hat{h}(\bar{x}) = \hat{f} - \hat{h}(\varphi[\bar{f}, \bar{h}, \bar{r}]) \tag{61}$$

$$\hat{D}^{-1}\hat{x} = \hat{D}^{-1}\hat{f} - \hat{D}^{-1}\hat{h}(\bar{x}) = \hat{D}^{-1}\hat{f} - \hat{D}^{-1}\hat{h}(\varphi[\bar{f}, \bar{h}, \bar{r}]) \tag{62}$$

$$\begin{cases} \tilde{h}(\bar{x}) := \hat{D}^{-1}\hat{h}(\bar{x}) = \hat{D}^{-1}\hat{h}(\varphi[\bar{f}, \bar{h}, \bar{r}]) \\ x = f - \tilde{h}(\bar{x}) = f - \tilde{h}(\varphi[\bar{f}, \bar{h}, \bar{r}]) \end{cases} \tag{63}$$

$$\begin{aligned} \tilde{h}(\bar{x}) &= \hat{D}^{-1}\hat{h}(\bar{x}) = \hat{D}^{-1}\tilde{H}(\bar{x})\bar{x} = \hat{D}^{-1}\hat{H}(\hat{x})\hat{x} = diag[\left(\hat{A}_1\hat{x}\right)^2, \left(\hat{A}_2\hat{x}\right)^2, \cdots, \left(\hat{A}_n\hat{x}\right)^2][\bar{A}_1, \bar{A}_2, \cdot \\ &= diag[\left(\hat{A}_1\hat{x}\right)^2, \left(\hat{A}_2\hat{x}\right)^2, \cdots, \left(\hat{A}_n\hat{x}\right)^2]\hat{A}\hat{x} \\ &= diag[\left(A_1x\right)^2, \left(A_2x\right)^2, \cdots, \left(A_nx\right)^2]Ax = h(x) \end{aligned} \tag{64}$$

$$\tilde{h}(\bar{x}) = h(x) \tag{65}$$

From the formula (65), the following formula as shown in (66) can be obtained naturally.

$$\begin{cases} \tilde{h}(\varphi[\bar{f}, \bar{h}, 0]) = \tilde{h}(\bar{f}) = h(f) \\ \tilde{h}(\varphi[\bar{f}, \bar{h}, k+1]) = \tilde{h}(\bar{f} - \bar{h}(\varphi[\bar{f}, \bar{h}, k])) = h(f - \tilde{h}(\varphi[\bar{f}, \bar{h}, k])), \ k \ge 0 \end{cases} \tag{66}$$

The following results will be obtained by substituting (66) from left to right into the general chain expression as shown in (63).

$$\begin{aligned} x &= f - \tilde{h}(\varphi[\bar{f}, \bar{h}, \bar{r}]) = f - \tilde{h}(\bar{f} - \bar{h}(\varphi[\bar{f}, \bar{h}, \bar{r}-1])) = f - h(f - \tilde{h}(\varphi[\bar{f}, \bar{h}, \bar{r}-1])) \\ &= f - h(f - h(f - \tilde{h}(\varphi[\bar{f}, \bar{h}, \bar{r}-2]))) \\ &= \cdots \\ &= f - h(\varphi[f, h, \bar{r}]) = \varphi[f, h, \bar{r}+1], \ \bar{r}+1 \le r = N+1 \end{aligned} \tag{67}$$

$$G(f) := x = \varphi[f, h, \bar{r}+1], \ \bar{r}+1 \le r = N+1 \tag{68}$$

If $\bar{r}+1 < r$, then the index of the general chain expression $\varphi[f, h, \bar{r}+1]$ will be extended from $\bar{r}+1$ to r by means of (19), while the invariance of the inverse map will be maintained.

$$x = G(f) = \varphi[f, h, \bar{r}+1] = \varphi[f, h, r], \ \bar{r}+1 \le r = N+1 \tag{69}$$

Therefore, when $r = N+1$, the general chain expression $\varphi[f, h, r]$ is just the inverse map of the L^3 type (such that GL^3 type) polynomial map $f = x + h(x)$ with $F'(x) = 1$ for all $x \in K^n$. We have several main conclusions in the following.

Theorem 5.1. Let $f = F(x) = x + h(x)$ be a polynomial map of L^3 or GL^3 type with $F'(x) = 1$ for all $x \in K^n$, then the general chain expression $\varphi[f, h, r]$ is just its inverse polynomial map for all $n \ge 2$ and $0 \le r \le n-1$.

Theorem 5.2. $LJC(n,[3])$ holds for all $n \ge 2$.

Theorem 5.3. $JC(n)$ holds for all $n \ge 2$.

6. Several further results

It is worth noted that Theorem 5.1 holds not only for the case of the L^3 or GL^3 type polynomial map, but also for the case of the polynomial maps of L^d type (such that GL^d type) or homogeneous type with the degree $d \geq 2$. As the same derivations as above, we can obtain several further results as follows.

Theorem 6.1. Let $f = F(x) = x + h(x)$ be a polynomial map of L^d or GL^d type with the degree $d \geq 2$, $F'(x) = 1$ for all $x \in K^n$, then the general chain expression $\varphi[f, h, r]$ is its inverse polynomial map for all $n \geq 2$ and $0 \leq r \leq n-1$.

Theorem 6.2. Let $f = F(x) = x + h(x)$ be a polynomial map of homogeneous type with the degree $d \geq 2$, $F'(x) = 1$ for all $x \in K^n$, then the general chain expression $\varphi[f, h, r]$ is its inverse polynomial map for all $n \geq 2$ and $0 \leq r \leq n-1$.

Since $JC(n)$ holds for all $n \geq 2$, there is another result about the injectivity problem of the polynomial map $F(x)$ as shown below.

Theorem 6.3. Let K be an algebraically closed field of characteristic zero and $F: K^n \to K^n$ be a polynomial map. if $F'(x) \neq 0$ for all $x \in K^n$, then F is injective (i.e. $F(a) \neq F(b)$ for all $a \neq b$, $a, b \in K^n$). On the other hand, if $F(a) = F(b)$ for some $a \neq b$, $a, b \in K^n$, then $F'(x) = 0$ for some $x \in K^n$.

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