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Not peer-reviewed version

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Posted Date: 26 July 2023

doi: 10.20944/preprints202307.1764.v1

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Article

On a Model of Propagation of Two Component Nonlinear Optical Waves

Aleksandr O. Smirnov ^{1,*}  and Eugeni A. Frolov ²

¹ St.-Petersburg State University of Aerospace Instrumentation, Sankt-Petersburg, Bolshaya Morskaya str., 67A, 190000, Russia; alsmir@guap.ru

² St.-Petersburg State University of Aerospace Instrumentation, Sankt-Petersburg, Bolshaya Morskaya str., 67A, 190000, Russia; e.frolov@guap.ru

* Correspondence: alsmir@guap.ru

Abstract: Currently, two-component integrable nonlinear equations from the hierarchies of the vector nonlinear Schrödinger equation and the vector derivative nonlinear Schrödinger equation are being actively investigated. Usually, such two-component integrable nonlinear evolutionary equations are considered, the form of which does not depend on the replacement of one component with another. In this paper, we propose for consideration and usage a new hierarchy of two-component integrable nonlinear equations, which have an important difference from the already known equations. Among the hierarchy equations there are analogues of the two-component nonlinear Schrödinger equation (second equation from hierarchy) and the two-component modified Korteweg-de Vries equation (fourth equation from hierarchy). The third equation of the hierarchy is a combination of the nonlinear Schrödinger equation for one component and the modified Korteweg-de Vries equation for the second component. The equations for the individual components are very different from each other, even if they have the same order. Let us note that even hierarchy equations can be reduced to well-known variants of the scalar derivative nonlinear Schrödinger equations, and odd equations can't be reduced. To construct the hierarchy equations, we use the monodromy matrix method, first proposed by B.A. Dubrovin. Knowledge of the monodromy matrix makes it possible to construct spectral curves of multiphase solutions, as well as to find stationary equations that these solutions satisfy. The last section presents the simplest solutions in the form of solitons and periodic one-phase waves, as well as spectral curves corresponding to these solutions.

Keywords: spectral curve; derivative NLS equation; lax pair; vector NLS equation; monodromy matrix

Introduction

The transmission of information in the optical fiber is carried out by means of modulation of the reference laser signal. The fiber material is selected in such a way that the nonlinear effects resulting from the interaction of the wave with the medium compensate for the dispersion. The simplest model for the propagation of a polarized signal in an optical fiber is the focusing nonlinear Schrödinger equation

$$ip_z + p_{tt} + 2|p|^2 p = 0. \quad (1)$$

Here p is the slowly changing complex amplitude of the modulated signal superimposed on the laser reference wave, z is the coordinate along the direction of signal propagation, t is a linear combination of time and longitudinal coordinate (see, for example, [1] and references therein). It is not difficult to understand that equation (1) is an equation in dimensionless variables, i.e. it is obtained from the original equation by replacing variables and functions. This model is obtained from Maxwell's equations by discarding terms that have little effect on the behavior of a nonlinear wave, i.e. it describes the real process with some accuracy [1]. The advantage of equation (1) is that it refers to integrable nonlinear equations (see, for example, [1–6]), which have solutions in the form of solitary waves (solitons). Solitons are nonlinear waves that propagate indefinitely without loss of shape and speed.

Naturally, in real waveguides, solitons lose energy over time, but amplifiers and repeaters compensate for these losses. In the case of using lasers generating femtosecond pulses, the dispersion terms of the third, fourth and fifth order of the order must be taken into account in the models. These models correspond to the integrable Hirota equation [7–10]

$$ip_z + \alpha(p_{tt} + 2|p|^2 p) - i\beta(p_{ttt} + 6|p|^2 p_t) = 0.$$

and also integrable higher nonlinear Schrodinger equations [11–14]. There are also non-integrable models, which we will not discuss in this paper. To account for other types of interaction of waves with the waveguide medium, various variants of derivatives of nonlinear Schrödinger equations can be used [15–33], including the Kundu-Eckhaus equation [34–36]. Recently, models of waves with double polarization have been actively studied, since with the help of appropriate signals it is possible to transmit twice as much information [37–42].

Nonlinear signals are studied and filtered using a nonlinear Fourier transform [2,4,5,20,33,39,41–47]. In this case, spectral analysis is performed not of the nonlinear signal itself, but of the first operator from the Lax pair. Each simplest integrable nonlinear differential equation can be obtained as a condition for the compatibility of two linear differential equations, called a Lax pair [1–6]. In particular, the Lax pair for equation (1) has the form [1–6,48,49]

$$\begin{aligned}\partial_t \Psi &= U \Psi, \\ \partial_z \Psi &= V_1 \Psi,\end{aligned}$$

where $U = \lambda J + U^0$, $V_1 = \lambda U + V_1^0$,

$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & ip \\ -iq & 0 \end{pmatrix}, \quad V_1^0 = \begin{pmatrix} -ipq & -p_t \\ -q_t & ipq \end{pmatrix},$$

$$q = -p^*, i^2 = -1.$$

Note that each simplest integrable nonlinear equation is the first equation from an infinite sequence of equations called a hierarchy. Each equation from the hierarchy corresponds to its own second Lax pair operator. In particular, the nonlinear Schrodinger equation is the first equation from the Ablowitz-Kaup-Newell-Sigur hierarchy [48–50]. One of the useful features of hierarchies of integrable nonlinear equations is the fact that there are functions $\hat{p}(t, z_1, z_2, \dots)$ that satisfy all the equations of the hierarchy simultaneously. Hence, the functions $p(t, z) = \hat{p}(t, \alpha_1(z), \alpha_2(z), \dots)$ will be solutions of the so-called mixed equations [11–14,48,49]. Therefore, consideration of the mixed equations is one of the ways to increase the number of integrable models of wave propagation in nonlinear media. Another way to construct new integrable models corresponding to the new properties of the studied nonlinear signals is to consider new Lax pairs. In particular, the propagation of bi-polarized waves in nonlinear optical waveguides is described by the Manakov system, which is a compatibility condition of linear matrix differential equations with third-order matrices [37–42,51–56]. Also, the compatibility conditions of Lax pairs with third-order matrices lead to two-component derivative nonlinear Schrödinger equations that describe more complex models of bi-polarized waves [57–61].

Naturally, all the models are regularly tested in practice when experimenters try to detect certain forms of signals obtained theoretically [62–65]. Therefore, one of the goals of theorists is to create new integrable models that could be used to describe nonlinear phenomena. In this paper, we propose for research a new integrable model describing the propagation of two interacting nonlinear waves

$$\begin{aligned}ip_{z_2} &= p_{tt} + ip^2 p_t^* + \frac{1}{2} \left(|p|^4 p + 8|u|^2 p + 4u^2 p^* \right), \\ iu_{z_2} &= u_{tt} - ip^2 u_t^* - 2i(pu^* + up^*)p_t + \frac{1}{2} (3|p|^4 u + 2|p|^2 p^2 u^* + 4|u|^2 u).\end{aligned}$$

In the absence of one of the waves ($u \equiv 0$), the model is reduced to the Gerdjikov-Ivanov equation [25–30,32,33]. The presented article consists of an introduction, five sections and concluding remarks. In the first section, we consider various possible variants of the Lax operator in the case of a quadratic spectral bundle. Based on the results of this section, we decided to investigate a model with a more general than usual Lax operator. The Section 2 of the paper is devoted to finding the structure of the monodromy matrix and the recurrent relations between its elements.

In Section 3, the stationary equations are derived and equations of spectral curves are considered. As in the case of the scalar derivative NLS equations [33], the stationary equations form two groups. But, if in the scalar case it was possible to use equations from only one group, then in the case of this model it is necessary to use both pairs of equations. Note also that in the case of usually vector NLS equations [42,66], both components p_1 and p_2 satisfy similar stationary equations. In this paper, the components p and u satisfy stationary equations with different structures.

The Section 4 defines the sequence of the second equations from the Lax pair and the evolutionary integrable nonlinear equations from the corresponding hierarchy. Note that even hierarchy equations differ from odd ones. In particular, for $n = 3$ we have

$$\begin{aligned} ip_{z_3} &= 2u_{tt} - 2i(p^*u_t - up_t^*)p - i(6pu^* + 4p^*u)p_t + (|p|^4 - 4|u|^2)u, \\ u_{z_3} &= -p_{ttt} - i(p^*p_{tt} + pp_{tt}^*)p - 6(u^*p_t + p^*u_t)u - (4u^*u_t + 2uu_t^*)p \\ &\quad - 2ip|p_t|^2 - \frac{3}{2}|p|^4p_t - 2i|pu|^2p - \frac{i}{2}|p|^6p. \end{aligned}$$

If we put $u = 0$ in an odd equation, then it ceases to be an evolutionary integrable nonlinear equation. In Section 5 we present the simplest solutions of the second equation from the hierarchy we have constructed. In particular, we found a solution in the form of a solitary wave, the components of which are described by different formulas. This is due to the fact that each component satisfies its own nonlinear differential equation.

1. Structure of the Lax Operator for a Quadratic Spectral Bundle

Let the Lax pair be given by the equations

$$i\Psi_t + U\Psi = 0, \quad (2)$$

$$i\Psi_z + V\Psi = 0, \quad (3)$$

where

$$U = \lambda^2 J + \lambda Q_1 + Q_2, \quad V = \lambda^2 K + \lambda S_1 + S_2, \quad (4)$$

J, K are constant matrices, λ is a spectral parameter.

The condition of compatibility of the equations (2) and (3) has the form

$$iV_t - iU_z + UV - VU = 0$$

or

$$\begin{aligned} \lambda^4(JK - KJ) + \lambda^3(JS_1 - S_1J + Q_1K - KQ_1) \\ + \lambda^2(JS_2 - S_2J + Q_1S_1 - S_1Q_1 + Q_2K - KQ_2) + \dots = 0. \end{aligned} \quad (5)$$

The condition (5), along with integrable nonlinear equations, gives algebraic constraints on the elements of the matrices U and V . These restrictions have the following form

$$JK = KJ, \quad (6)$$

$$[J, S_1] = [K, Q_1], \quad (7)$$

$$[J, S_2] + Q_1 S_1 = [K, Q_2] + S_1 Q_1. \quad (8)$$

Usually solutions of the equation (7) is written using the relations

$$S_1 = [K, R], \quad Q_1 = [J, R], \quad (9)$$

where R is some matrix of the same size.

Substituting (9) into (8) and simplifying, we get

$$\begin{aligned} & J(S_2 + RKR - R^2K) - (S_2 + RKR - R^2K)J \\ &= K(Q_2 + RJR - R^2J) - (Q_2 + RJR - R^2J)K. \end{aligned}$$

or

$$[J, S_2 + [RK, R]] = [K, Q_2 + [RJ, R]].$$

Therefore, the following representation of the matrices S_2 and Q_2 can be used

$$\begin{aligned} S_2 + [RK, R] &= [K, R_2], \\ Q_2 + [RJ, R] &= [J, R_2] \end{aligned}$$

or

$$\begin{aligned} S_2 &= [K, R_2] + [R, RK], \\ Q_2 &= [J, R_2] + [R, RJ], \end{aligned} \quad (10)$$

where R_2 is a new matrix of the same size. Note that in the right-hand sides of the equalities (9) and (10) it is possible to add linear combinations of matrices commuting with J and K .

Taking into account the remaining constraints from the condition (5) leads to the following matrix of operators (2) and (3)

$$\begin{aligned} S_1 &= [K, R], \quad Q_1 = [J, R], \\ S_2 &= [K, R_2] + [R, RK] + s_{20}K, \\ Q_2 &= [J, R_2] + [R, RJ] + q_{20}J, \end{aligned} \quad (11)$$

where s_{20} and q_{20} are some functions. It is easy to see that the matrix J completely defines the structure of the operator (2). For example, if $J = \text{diag}(1, -1)$, then

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & 2r_{12} \\ -2r_{21} & 0 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} 0 & 2\tilde{r}_{12} \\ -2\tilde{r}_{21} & 0 \end{pmatrix} + \begin{pmatrix} 2r_{12}r_{21} & -2r_{11}r_{12} \\ 2r_{21}r_{22} & -2r_{12}r_{21} \end{pmatrix} + \begin{pmatrix} q_{20} & 0 \\ 0 & -q_{20} \end{pmatrix}. \end{aligned}$$

In the case of the main variants of the derivative nonlinear Schrödinger equations, we have ($q = -p^*$)

- in the case of the Kaup-Newell equation [15–17,20,25,26,29,30,32,33]

$$r_{12} = ip/2, \quad r_{21} = -iq/2, \quad \tilde{r}_{12} = r_{11}r_{12}, \quad \tilde{r}_{21} = r_{21}r_{22}, \quad q_{20} = -2r_{12}r_{21};$$

- in the case of the Chen-Lee-Liu equation [22,25,26,29,30,32,33]

$$r_{12} = ip/2, \quad r_{21} = -iq/2, \quad \tilde{r}_{12} = r_{11}r_{12}, \quad \tilde{r}_{21} = r_{21}r_{22}, \quad q_{20} = -pq/4;$$

- in the case of the Gerdjikov-Ivanov equation [25–30,32,33]

$$r_{12} = ip/2, \quad r_{21} = -iq/2, \quad \tilde{r}_{12} = r_{11}r_{12}, \quad \tilde{r}_{21} = r_{21}r_{22}, \quad q_{20} = 0.$$

By choosing other values of q_{20} , other special cases of the generalized derivative nonlinear Schrödinger equation can be obtained [31,33,67–69]. Note that the solutions of these equations are connected by a gauge transformation preserving the amplitude (see for ex. [26,32,33,70–72]).

It is easy to notice that in these models, the matrices Q_1 and Q_2 depend only on two functions p and q , and the matrix Q_2 is diagonal. Therefore, the addition of new functions u and v to the non-diagonal terms of the matrix Q_2 allows us to explore a new nonlinear integrable model:

$$r_{12} = p/2, \quad r_{21} = q/2, \quad \tilde{r}_{12} = r_{11}r_{12} + u/2, \quad \tilde{r}_{21} = r_{21}r_{22} + v/2, \quad q_{20} = 0.$$

This model at $u = v = 0$ is transformed into one of the special cases of the generalized derivative nonlinear Schrödinger equation [33]. The matrices J , Q_1 and Q_2 in the case of the new model are equal

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & p \\ -q & 0 \end{pmatrix}, \quad Q_2 = \frac{1}{2} \begin{pmatrix} pq & 2u \\ -2v & -pq \end{pmatrix}. \quad (12)$$

2. The Monodromy Matrix

The monodromy matrix is a key object of spectral analysis of periodic solutions of the integrable nonlinear models. Spectral data in the case of periodic nonlinear signals is the spectral curve, its genus and its parameters. The equation of the spectral curve is the characteristic equation of the monodromy matrix M , which is a polynomial of the spectral parameter λ

$$M = \sum_{j=0}^N m_j(t) \lambda^j,$$

and which satisfies the equation [73]

$$iM_x + UM - MU = 0. \quad (13)$$

The monodromy matrix also exists in the limiting cases when the solution periods become infinite. In particular, solitary waves are the limiting cases of periodic waves when the period of a nonlinear wave becomes infinitely large.

From equation (13), where the matrix U is determined by the equalities (4), (12), the following structure of the matrix M follows

$$M = W_n + \sum_{k=1}^{n-1} c_k W_{n-k} + c_n U + c_{n+1} W_{-1} + c_{n+2} J, \quad (14)$$

where $W_{-1} = \lambda J + Q_1$, $U = \lambda W_{-1} + Q_2$, $W_1 = \lambda U + W_1^0$,

$$W_{k+1} = \lambda V_k + W_{k+1}^0, \quad W_k^0 = \begin{pmatrix} F_k & H_k \\ -G_k & -F_k \end{pmatrix}, \quad k \geq 1,$$

c_k are some real constants. From equation (6) it also follows the recurrence relations on the elements of the matrix W_k^0 :

$$\begin{aligned} H_1 &= -\frac{i}{2}p_t, & G_1 &= \frac{i}{2}q_t, \\ H_2 &= pF_1 - \frac{i}{2}u_t, & G_2 &= qF_1 + \frac{i}{2}v_t, \\ H_{k+2} &= pF_{k+1} + uF_k - \frac{1}{2}pqH_k - \frac{i}{2}\partial_t H_k, \\ G_{k+2} &= qF_{k+1} + vF_k - \frac{1}{2}pqG_k + \frac{i}{2}\partial_t G_k, \\ \partial_t F_k &= i(vH_k - uG_k + qH_{k+1} - pG_{k+1}). \end{aligned} \quad (15)$$

In particular,

$$\begin{aligned} F_1 &= \frac{1}{2}(pv + qu), \\ H_2 &= \frac{1}{2}(pqu + p^2v - iu_t), \\ G_2 &= \frac{1}{2}(pqv + q^2u + iv_t), \\ F_2 &= \frac{1}{2}uv - \frac{1}{8}p^2q^2 + \frac{i}{4}(pq_t - qp_t), \\ H_3 &= uv p + \frac{1}{2}u^2q - \frac{1}{8}p^3q^2 + \frac{i}{4}p^2q_t - \frac{1}{4}p_{tt}, \\ G_3 &= uv q + \frac{1}{2}v^2p - \frac{1}{8}p^2q^3 - \frac{i}{4}q^2p_t - \frac{1}{4}q_{tt}, \\ F_3 &= \frac{1}{4}(uq + pv)pq + \frac{i}{4}(pv_t - vp_t + uq_t - qu_t), \\ H_4 &= \frac{1}{2}u^2v - \frac{1}{8}p^2q^2u - \frac{i}{4}(2qu + 3pv)p_t + \frac{i}{4}(uq_t - qu_t)p - \frac{1}{4}u_{tt}, \\ G_4 &= \frac{1}{2}uv^2 - \frac{1}{8}p^2q^2v + \frac{i}{4}(3qu + 2pv)q_t + \frac{i}{4}(pv_t - vp_t)q - \frac{1}{4}v_{tt}, \\ F_4 &= pquv - \frac{1}{16}p^3q^3 + \frac{3}{8}(p^2v^2 + q^2u^2) + \frac{1}{8}p_tq_t \\ &\quad + \frac{i}{4}(uv_t - vu_t) - \frac{1}{8}(pq_{tt} + qp_{tt}). \end{aligned}$$

It follows from recurrent relations (15) that when the reality conditions

$$q = \sigma p^*, \quad v = \sigma u^*, \quad \sigma = \pm 1, \quad (16)$$

are met, the elements of the monodromy matrix satisfy the following relations

$$G_k = \sigma H_k^*, \quad F_k^* = F_k. \quad (17)$$

3. Conservation Laws

Conservation laws are described by the following stationary nonlinear differential equations

$$\begin{aligned} H_{n+1} + \sum_{j=1}^n c_j H_{n+1-j} + c_{n+1}u + c_{n+2}p &= 0, \\ G_{n+1} + \sum_{j=1}^n c_j G_{n+1-j} + c_{n+1}v + c_{n+2}q &= 0, \end{aligned}$$

and

$$\begin{aligned} H_{n+2} - pF_{n+1} + \sum_{j=1}^{n-1} c_j (H_{n+2-j} - pF_{n+1-j}) \\ - \frac{ic_n}{2} u_x - \frac{c_{n+1}}{2} (p^2 q + ip_x) + c_{n+2} u = 0, \\ G_{n+2} - qF_{n+1} + \sum_{j=1}^{n-1} c_j (G_{n+2-j} - qF_{n+1-j}) \\ + \frac{ic_n}{2} v_x - \frac{c_{n+1}}{2} (pq^2 - iq_x) + c_{n+2} v = 0. \end{aligned}$$

These equations also follow from equation (13).

Any m -phase solution for $m \leq n$ and for all values of t and z satisfies these stationary equations. The parameters of the corresponding multiphase solution depend on the constants c_k and coefficients f_k of the spectral curve (see below). It follows from the realness conditions (17) that stationary equations admit reduction (16).

In particular, for $n = 0$, the stationary equations have the form

$$\begin{aligned} ip_t - 2c_1 u - 2c_2 p &= 0, \\ iq_t + 2c_1 v + 2c_2 q &= 0, \\ iu_t + c_1(p^2 q + ip_x) - 2c_2 u &= 0, \\ iv_t - c_1(pq^2 - iq_x) + 2c_2 v &= 0. \end{aligned} \quad (18)$$

It is easy to see that for $c_1 \neq 0$, the components u and v are connected to the components p and q using the following relations

$$u = -\frac{1}{2c_1}(2c_2 p - ip_x), \quad v = -\frac{1}{2c_1}(2c_2 q + iq_x).$$

As will be shown below, in this case all components are expressed in terms of elliptic functions or their degenerations. Note that for $c_1 = 0$, the system of stationary equations (18) splits into two separate identical systems, the solutions of which are plane waves. Accordingly, when $c_1 = 0$, additional components can be removed from the model by putting $u = v = 0$.

Recall that the characteristic equation of the monodromy matrix M is the equation of the corresponding spectral curve

$$\Gamma: \mathcal{R}(\mu, \lambda) = \det(M - \mu I) = 0. \quad (19)$$

Here I is identity matrix. It follows from the equations (14) and (19) that the equation of the spectral curve has the form

$$\mu^2 = \lambda^{2n+4} + \sum_{k=1}^{2n+4} f_k \lambda^{2n+4-k}, \quad (20)$$

where f_k are constants (integrals). Therefore, the spectral curve Γ is a hyperelliptic curve of genus $g = n + 1$. Naturally, this statement is true only for equations corresponding to non-degenerate connected curves.

For $n=0$ ($g=1$), the equation of the spectral curve has the form

$$\mu^2 = \lambda^4 + 2c_1 \lambda^3 + (c_1^2 + 2c_2) \lambda^2 + f_3 \lambda + f_4,$$

where the integrals f_3 and f_4 are equal

$$f_3 = 2c_1 c_2 - c_1 p q - u q - p v,$$

$$f_4 = c_2^2 + (c_2 - c_1^2)pq + \frac{1}{4}p^2q^2 - uv - c_1(pv + uq).$$

I.e., for $n = 0$, the nondegenerate spectral curve is elliptic.

For $n = 1$ ($g = 2$), the stationary equations have a more complex form

$$\begin{aligned} iu_t - (pv + qu)p + ic_1p_t - 2c_2u - 2c_3p &= 0, \\ iv_t + (pv + qu)q + ic_1q_t + 2c_2v + 2c_3q &= 0, \\ p_{tt} - 2(pv + qu)u - ipqp_t + 2ic_1u_t + 2c_2(p^2q + ip_t) - 4c_3u &= 0, \\ q_{tt} - 2(qu + pv)v + ipqq_t - 2ic_1v_t + 2c_2(pq^2 - iq_t) - 4c_3v &= 0. \end{aligned} \quad (21)$$

It follows from the first two equations of system (21) that in the case $g = 2$, the dependence of the components u and v on p and q can be found from the solution of the following linear matrix differential equation

$$i\partial_t \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} pq + 2c_2 & p^2 \\ -q^2 & -pq - 2c_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -ic_1\partial_t \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 2c_3 & 0 \\ 0 & -2c_3 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

It is easy to see that for $c_1 = c_3 = 0$, this equation admits solutions of the form $u = v = 0$. In this case, the stationary equations for the components p and q will take a simpler form

$$\begin{aligned} p_{tt} - ipqp_t + 2c_2(p^2q + ip_t) &= 0, \\ q_{tt} + ipqq_t + 2c_2(pq^2 - iq_t) &= 0. \end{aligned} \quad (22)$$

At the same time, it is not difficult to see that for $c_1 = c_3 = 0$ there are solutions of stationary equations (21) that satisfy the condition $u^2 + v^2 \neq 0$.

The equation of the spectral curve for $n = 1$ has the form

$$\mu^2 = \lambda^6 + 2c_1\lambda^5 + (c_1^2 + 2c_2)\lambda^4 + 2(c_1c_2 + c_3)\lambda^3 + f_4\lambda^2 + f_5\lambda + f_6,$$

where the integrals f_4, f_5 and f_6 are equal

$$\begin{aligned} f_4 &= c_2^2 + 2c_1c_3 - c_1(pv + qu) - c_2pq + \frac{1}{4}p^2q^2 - uv - \frac{i}{2}(pq_t - qp_t), \\ f_5 &= 2c_2c_3 + (c_3 - c_1c_2)pq - c_1^2(pv + qu) + \frac{1}{2}c_1p^2q^2 - 2c_1uv + \frac{1}{2}pq(pv + qu) \\ &\quad - \frac{i}{2}c_1(pq_t - qp_t) - \frac{i}{2}(uq_t - vp_t), \\ f_6 &= c_3^2 + (c_1c_3 - c_2^2)pq + \frac{1}{4}c_1^2p^2q^2 - c_1^2uv + (c_3 - c_1c_2)(pv + qu) \\ &\quad + \frac{1}{2}c_1pq(pv + qu) + \frac{1}{4}(p^2v^2 + q^2u^2) + \frac{1}{2}pquv \\ &\quad - \frac{i}{2}c_2(pq_t - qp_t) - \frac{i}{2}c_1(uq_t - vp_t) - \frac{1}{4}p_tq_t. \end{aligned}$$

For $n = 2$ ($g = 3$), the stationary equations become even more complicated

$$\begin{aligned} u_{tt} + ip^2v_t + 2i(pv + qu)p_t - 2u^2v + p^3qv + \frac{3}{2}p^2q^2u \\ + c_1(p_{tt} - ipqp_t - 2(pv + qu)u) + 2ic_2u_t + 2c_3(p^2q + ip_t) - 4c_4u &= 0, \\ v_{tt} - iq^2u_t - 2i(pv + qu)q_t - 2uv^2 + pq^3u + \frac{3}{2}p^2q^2v \\ + c_1(q_{tt} + ipqq_t - 2(pv + qu)v) - 2ic_2v_t + 2c_3(pq^2 - iq_t) - 4c_4v &= 0, \end{aligned}$$

$$\begin{aligned}
& p_{tt} - ip^2q_t - 4uvp - 2qu^2 + \frac{1}{2}p^3q^2 + 2c_1(iu_t - (pv + qu)p) \\
& + 2ic_2p_t - 4c_3u - 4c_4p = 0, \\
& q_{tt} + iq^2p_t - 4uvq - 2pv^2 + \frac{1}{2}p^2q^3 - 2c_1(iv_t + (pv + qu)q) \\
& - 2ic_2q_t - 4c_3v - 4c_4q = 0.
\end{aligned}$$

It is easy to see that for $n = 2$, the dependence between the components (u, v) and (p, q) is described by nonlinear equations. At the same time, these equations also allow solutions of the form $u = v = 0$ for $c_1 = c_3 = 0$. The equations for the components p and q in this case have the form

$$\begin{aligned}
p_{tt} - ip^2q_t + \frac{1}{2}p^3q^2 + 2ic_2p_t - 4c_4p &= 0, \\
q_{tt} + iq^2p_t + \frac{1}{2}p^2q^3 - 2ic_2q_t - 4c_4q &= 0.
\end{aligned} \tag{23}$$

We will omit the values of the coefficients in the equation of the spectral curve for $n = 2$, since they are very cumbersome and are not interesting at the moment.

Let us note that the nonlinear stationary equations (22) and (23) are very different, despite the fact that they have the same order.

4. Integrable Nonlinear Evolutionary Equations

Let the second equation of the Lax pair have the form

$$i\Psi_{t_k} + V_k\Psi = 0, \tag{24}$$

where $V_k = 2^m W_k$, $k = 2m$ or $k = 2m - 1$.

Then from the compatibility condition of the equations (2) and (24) the following evolutionary equations

$$\begin{aligned}
ip_{z_k} + 2^{m+1}H_{k+1} &= 0, \\
-iq_{z_k} + 2^{m+1}G_{k+1} &= 0, \\
iu_{z_k} + 2^{m+1}(H_{k+2} - pF_{k+1}) &= 0, \\
-iv_{z_k} + 2^{m+1}(G_{k+2} - qF_{k+1}) &= 0
\end{aligned} \tag{25}$$

follow.

It is not difficult to see that the evolutionary equations (25), as well as the stationary equations, admit reduction (16). Accordingly, after replacing (16), an integrable nonlinear two-component equation with a different dependence of the components on the variables z_k will be obtained from the system of equations (25). Note that the well-known two-component integrable nonlinear equations [37,38,40–42,51–61] describe models with identical dependencies of components on variables z_k .

We present the first evolutionary integrable equations from the corresponding hierarchy. For $k = 1$, the structure of the first two equations of the system differs from the structure of the last two:

$$\begin{aligned}
ip_{z_1} &= 2iu_t - 2(pv + qu)p, \\
-iq_{z_1} &= -2iv_t - 2(pv + qu)q, \\
iu_{z_1} &= p_{tt} - ipqp_t - 2(pv + qu)u, \\
-iv_{z_1} &= q_{tt} + ipqq_t - 2(pv + qu)v.
\end{aligned} \tag{26}$$

Note that the last two equations of the system (26) when performing reduction (16) are analogous to the derivative nonlinear Schrödinger equation. At the same time, the first two equations describe a fairly simple relationship between the components.

For $k = 2$, the evolutionary equations have the form

$$\begin{aligned} ip_{z_2} &= p_{tt} - ip^2 q_t + \frac{1}{2}(p^3 q^2 - 8uvp - 4u^2 q), \\ -iq_{z_2} &= q_{tt} + iq^2 p_t + \frac{1}{2}(p^2 q^3 - 8uvq - 4v^2 p), \\ iu_{z_2} &= u_{tt} + ip^2 v_t + 2i(pv + qu)p_t + \frac{1}{2}(3p^2 q^2 u + 2p^3 qv - 4u^2 v), \\ -iv_{z_2} &= v_{tt} - i2q^2 u_t - 2i(pv + qu)q_t + \frac{1}{2}(3p^2 q^2 v + 2pq^3 u - 4uv^2). \end{aligned} \quad (27)$$

It is easy to see that equations (27) are analogs of the two-component coupled derivative nonlinear Schrödinger equation. Assuming $q = -p^*$ and $v = -u^*$, we obtain from equation (27) a new two-component derivative nonlinear Schrödinger equation

$$\begin{aligned} ip_{z_2} &= p_{tt} + ip^2 p_t^* + \frac{1}{2}(|p|^4 p + 8|u|^2 p + 4u^2 p^*), \\ iu_{z_2} &= u_{tt} - ip^2 u_t^* - 2i(pu^* + up^*)p_t + \frac{1}{2}(3|p|^4 u + 2|p|^2 p^2 u^* + 4|u|^2 u). \end{aligned} \quad (28)$$

Unlike the usual vector derivative nonlinear Schrödinger equation [57–61], in this case the evolution of the components p and u is described by different equations.

For $k = 3$, the evolutionary equations again divided into two groups. The first two equations are analogs of the nonlinear Schrödinger equation, and the second are analogs of the modified Korteweg–de Vries equation

$$\begin{aligned} ip_{z_3} &= 2u_{tt} + 2i(qu_t - uq_t)p + i(6pv + 4qu)p_t + (p^2 q^2 - 4uv)u, \\ -iq_{z_3} &= 2v_{tt} - 2i(pv_t - vp_t)q - i(6qu + 4pv)q_t + (p^2 q^2 - 4uv)v, \\ u_{z_3} &= -p_{ttt} + i(qp_{tt} + pq_{tt})p + 6(vp_t + qu_t)u + (4vu_t + 2uv_t)p \\ &\quad + 2ip p_t q_t - \frac{3}{2}p^2 q^2 p_t - 2ip^2 quv + \frac{i}{2}p^4 q^3, \\ v_{z_3} &= -q_{ttt} - i4(qp_{tt} + pq_{tt})q + 6(pv_t + uq_t)v + (2vu_t + 4uv_t)q \\ &\quad - 2iq p_t q_t - \frac{3}{2}p^2 q^2 q_t + 2ipq^2 uv - \frac{i}{2}p^3 q^4. \end{aligned} \quad (29)$$

Assuming $q = -p^*$ and $v = -u^*$, we obtain from equation (29) a two-component mixed equation

$$\begin{aligned} ip_{z_3} &= 2u_{tt} - 2i(p^* u_t - up_t^*)p - i(6pu^* + 4p^* u)p_t + (|p|^4 - 4|u|^2)u, \\ u_{z_3} &= -p_{ttt} - i(p^* p_{tt} + pp_{tt}^*)p - 6(u^* p_t + p^* u_t)u - (4u^* u_t + 2uu_t^*)p \\ &\quad - 2ip|p_t|^2 - \frac{3}{2}|p|^4 p_t - 2i|pu|^2 p - \frac{i}{2}|p|^6 p. \end{aligned}$$

First equation is an analogue of the derivative NLS equation, and second equation is an analogue of the modified Korteweg–de Vries equation.

Note that further the structure of the evolutionary equations (25) will depend on the parity of the number of the equation. For odd k , the order of derivatives with respect to t will be different, for even k , the same. In particular, for $k = 4$, the system of evolutionary equations will be an analogue of the vector modified Korteweg–de Vries equation. Note also that even equations admit reduction $u = v = 0$, whereas odd ones do not. In this case, equations (27) under this reduction and under condition (16) pass into the Gerdjikov-Ivanov equation [25–30,32,33]. I.e., the model considered in this paper generalizes the already known integrable models of nonlinear wave propagation.

5. One-Phase Solutions

To show differences in components behaviors, we consider examples with $n = 0$ and $c_1 \neq 0$. To find solutions to system (26), we express u and v from its first two equations and substitute them into the rest. After simplification we have

$$\begin{aligned} p_{tt} - 2i(c_1^2 - 2c_2)p_t - 2c_1^2 p^2 q - 4c_2^2 p &= 0, \\ q_{tt} + 2i(c_1^2 - 2c_2)q_t - 2c_1^2 p q^2 - 4c_2^2 q &= 0. \end{aligned} \quad (30)$$

Let us note that equation (30) differs from equations (22) and (23).

Following [74], we will make a replacement in these equations

$$p = \sqrt{r} \exp \left\{ - \int \frac{w}{2r} dt \right\}, \quad q = \sqrt{r} \exp \left\{ \int \frac{w}{2r} dt \right\}, \quad (31)$$

where

$$r = pq, \quad w = pq_x - qp_x.$$

After simplification we have

$$w = -2i(c_1^2 - 2c_2)r + ic_3, \quad (32)$$

$$2rr_{tt} - (r_t)^2 - 8c_1^2 r^3 + 4c_1^2 (c_1^2 - 4c_2)r^2 - c_3^2 = 0. \quad (33)$$

Here c_3 is an integration constant.

Additional relations follow from the equation of the spectral curve. Converting expressions for constants f_3 and f_4 using substitutions (31) and (32), we obtain

$$\begin{aligned} f_3 &= 2c_1 c_2 - \frac{c_3}{2c_1}, \\ (r_t)^2 &= 4c_1^2 r^3 - 4c_1^2 (c_1^2 - 4c_2)r^2 + 4c_1^2 (4c_2^2 - c_3 - 4f_4)r - c_3^2. \end{aligned} \quad (34)$$

It is not difficult to check the compatibility of the equations (33) and (34). It follows from these equations that the function $r(t)$ is an elliptic function or one of its degenerations.

In particular, if the spectral curve is given by equation

$$\mu^2 = \left((\lambda - a)^2 + b^2 \right)^2,$$

then the constants c_k and f_4 have the following values

$$c_1 = -2a, \quad c_2 = a^2 + b^2, \quad c_3 = 0, \quad f_4 = (a^2 + b^2)^2.$$

For these values of constants, the function $r(t)$ satisfies the equation

$$(r_t)^2 = 16a^2 r^3 + 64a^2 b^2 r^2.$$

Solving this equation, we have

$$r(t) = -4b^2 \operatorname{sech}^2(4abt + \alpha), \quad \alpha = \text{const.}$$

In this case

$$w(t) = 16ib^2(a^2 - b^2) \operatorname{sech}^2(4abt + \alpha).$$

Substituting this value of the functions $r(t)$ and $w(t)$ into formulas (31) and (27), we get the solution to the equations (27)

$$\begin{aligned} p &= 2ib \operatorname{sech}(\phi_1(t, z_2)) e^{i\phi_2(t, z_2)}, \\ q &= 2ib \operatorname{sech}(\phi_1(t, z_2)) e^{-i\phi_2(t, z_2)}, \\ u &= 2b \operatorname{sech}(\phi_1(t, z_2)) (ia - b \tanh(\phi_1(t, z_2))) e^{i\phi_2(t, z_2)}, \\ v &= 2b \operatorname{sech}(\phi_1(t, z_2)) (ia + b \tanh(\phi_1(t, z_2))) e^{-i\phi_2(t, z_2)}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \phi_1(t, z_2) &= 4abt - 16ab(a^2 - b^2)z_2, \\ \phi_2(t, z_2) &= 2(a^2 - b^2)t + 4(a^4 - 6a^2b^2 + b^4)z_2. \end{aligned}$$

It is easy to see that the components of solution (35) satisfy the reduction $q = -p^*$ and $v = -u^*$. The amplitudes of the components p and u for $a = 2, b = 3$ are shown in Figure 1.

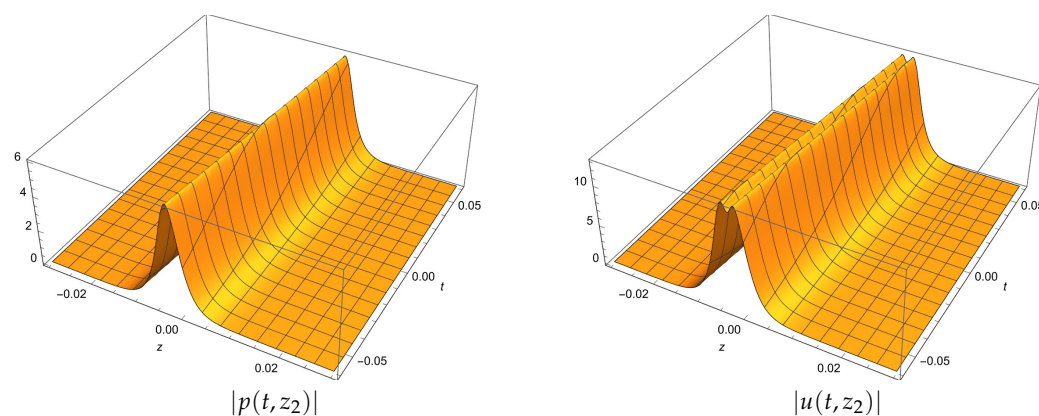


Figure 1. The amplitudes of the solution (35) for $a = 2, b = 3$.

It is not difficult to see that the shape of the component u is quite different from the shape of the component p . The component p is a classical soliton. At the same time, the component u is defined using a completely new expression.

Assuming $c_3 = 0, 0 < k < 1$, it is possible to construct three different solutions in elliptic Jacobi functions [75,76].

If $c_2 = (c_1^4 - 1 - k^2)/(4c_1^2)$, then the function $r(t)$ is expressed in terms of the $\operatorname{sn}(t; k)$:

$$r(t) = \frac{k^2}{c_1^2} \operatorname{sn}^2(t; k).$$

The spectral curve of this solution is determined by the equation

$$\mu^2 = \prod_{j=1}^4 (\lambda - \lambda_j), \quad (36)$$

where

$$\lambda_{1,2} = -\frac{1 + c_1^2 \pm k}{2c_1}, \quad \lambda_{3,4} = \frac{1 - c_1^2 \pm k}{2c_1}.$$

In this case

$$\begin{aligned} p(t, z_2) &= \frac{k}{c_1} \operatorname{sn}(\phi_1(t, z_2); k) e^{i\phi_2(t, z_2)}, \\ u(t, z_2) &= -\frac{k}{2c_1^2} \left(c_1^2 \operatorname{sn}(\phi_1(t, z_2); k) - i \operatorname{sn}'(\phi_1(t, z_2); k) \right) e^{i\phi_2(t, z_2)}, \\ q &= p^*, \quad v = u^*, \end{aligned} \quad (37)$$

where

$$\begin{aligned} \phi_1(t, z_2) &= t + \frac{c_1^4 + 1 + k^2}{c_1^2} z_2, \\ \phi_2(t, z_2) &= \frac{c_1^4 + 1 + k^2}{2c_1^2} t + \frac{c_1^8 + 6(k^2 + 1)c_1^4 + 1 + 4k^2 + k^4}{4c_1^4} z_2. \end{aligned}$$

Since the solution (37) to the equations (27) satisfies the conditions $q = p^*$, $v = u^*$, it is not suitable for describing the propagation of nonlinear optical signals.

When $c_2 = (c_1^4 + 2 - k^2)/(4c_1^2)$ the function $r(t)$ has the form

$$r(t) = -\frac{1}{c_1^2} \operatorname{dn}^2(t; k).$$

The spectral curve is again determined by equation (36). Only the branching points in this case are not real, but complex conjugate

$$\lambda_{1,2} = -\frac{c_1}{2} \pm i \frac{1 + \sqrt{1 - k^2}}{2c_1}, \quad \lambda_{3,4} = -\frac{c_1}{2} \pm i \frac{1 - \sqrt{1 - k^2}}{2c_1}.$$

The following periodic solution to the equations (27) corresponds to this curve:

$$\begin{aligned} p(t, z_2) &= \frac{i}{c_1} \operatorname{dn}(\phi_1(t, z_2); k) e^{i\phi_2(t, z_2)}, \\ u(t, z_2) &= -\frac{i}{2c_1^2} \left(c_1^2 \operatorname{dn}(\phi_1(t, z_2); k) - i \operatorname{dn}'(\phi_1(t, z_2); k) \right) e^{i\phi_2(t, z_2)}, \\ q &= -p^*, \quad v = -u^*, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \phi_1(t, z_2) &= t + \frac{c_1^4 - 2 + k^2}{c_1^2} z_2, \\ \phi_2(t, z_2) &= \frac{c_1^4 - 2 + k^2}{2c_1^2} t + \frac{c_1^8 + 6(k^2 - 2)c_1^4 + 6 - 6k^2 + k^4}{4c_1^4} z_2. \end{aligned}$$

Since reductions $q = -p^*$, $v = -u^*$ are performed, solution (38) can be used to describe the propagation of a periodic nonlinear two-component wave. The amplitudes of the components p and u for $c_1 = 0.5$, $k = 0.7$ are shown in Figure 2. It is easy to see that the component p is an ordinary “dnoidal” wave, while component u has a non-standard shape.

The third elliptic solution to the equations (27) for $c_2 = (c_1^4 - 1 + 2k^2)/(4c_1^4)$ is expressed in terms of the $\operatorname{cn}(t; k)$ [75,76]:

$$r(t) = -\frac{k^2}{c_1^2} \operatorname{cn}^2(t; k),$$

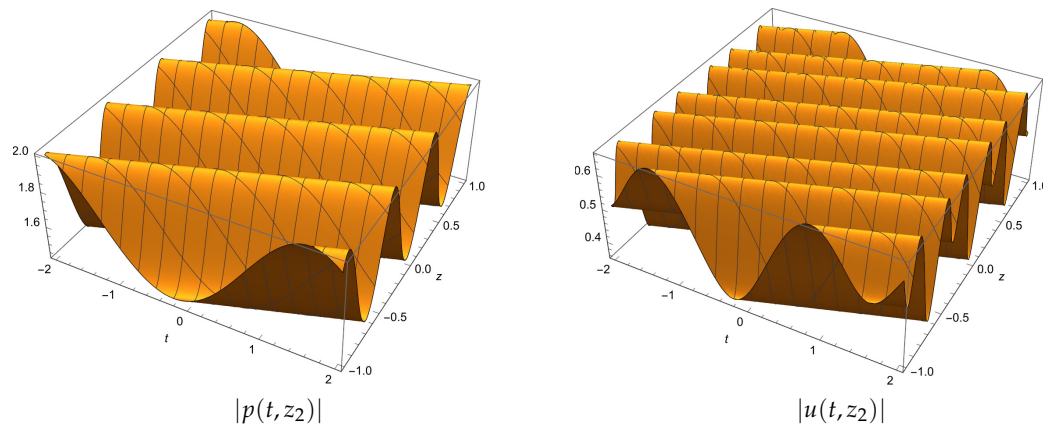


Figure 2. The amplitudes of the solution (38) for $c_1 = 0.5, k = 0.7$.

and

$$\begin{aligned} p(t, z_2) &= \frac{ik}{c_1} \operatorname{cn}(\phi_1(t, z_2); k) e^{i\phi_2(t, z_2)}, \\ u(t, z_2) &= -\frac{ik}{2c_1^2} \left(c_1^2 \operatorname{cn}(\phi_1(t, z_2); k) - i \operatorname{cn}'(\phi_1(t, z_2); k) \right) e^{i\phi_2(t, z_2)}, \\ q &= -p^*, \quad v = -u^*, \end{aligned} \quad (39)$$

where

$$\begin{aligned} \phi_1(t, z_2) &= t + \frac{c_1^4 + 1 - 2k^2}{c_1^2} z_2, \\ \phi_2(t, z_2) &= \frac{c_1^4 + 1 - 2k^2}{2c_1^2} t + \frac{c_1^8 + 6(1 - 2k^2)c_1^4 + 1 - 6k^2 + 6k^4}{4c_1^4} z_2. \end{aligned}$$

The spectral curve of the solution (39) is given by equation (36), where

$$\lambda_{1,2} = -\frac{c_1}{2} + \frac{\sqrt{1 - k^2} \pm ik}{2c_1}, \quad \lambda_{3,4} = -\frac{c_1}{2} - \frac{\sqrt{1 - k^2} \pm ik}{2c_1}.$$

The examples we have considered have shown that, as in the case of the standard nonlinear Schrödinger equation, the location of the branching points of the spectral curve corresponds to the sign of reduction. If $q = p^*$, then the branching points are on the real axis. If $q = -p^*$, then the branching points form complex conjugate pairs.

Concluding Remarks

From (15) and (25) the following equality

$$\partial_t F_{k+1} = 2^{-m} \partial_{z_k} F_1 \quad (40)$$

follows. Therefore, there exists the function Φ such that

$$F_1 = \partial_t \Phi, \quad F_{k+1} = 2^{-m} \partial_{z_k} \Phi.$$

Note that the same statement is also true for other hierarchies (see, for example, [42,66,77,78]). As it was shown in [66], using equation (40), it is possible to construct a new vector analogue of the Kundu-Eckhaus equation.

Funding: The research was supported by the Russian Science Foundation (grant agreement No 22-11-00196)

Conflicts of Interest: The author declares no conflict of interest.

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