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Article

A new Derivation of Extended Watson Summation Theorem due to Kim et al. with an Application

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Abstract: In applied mathematics, statistics, operation research, physics, and engineering mathematics, confluent representations of hypergeometric functions in one and two variables are known to exist, and their occurrence in a variety of applications is also well recognised. In this article, we intend to present a new derivation of the extended Watson summation theorem for the Kim et al. given series ${}_4F_3$. We assessed four attractive integrals involving generalized hypergeometric function as an application. With a few particular cases, this note will come to an end. In the results given above, symmetry appears on its own.

Keywords: Generalized hypergeometic function; extended Watson theorem; Gauss theorem; special cases

0. Introduction

Let z be a complex variable, and let $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of real and complex numbers, respectively. The generalized binomial coefficient for α and β for real or complex parameters is defined as, [9]

$$\binom{\lambda}{\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} = \binom{\lambda}{\lambda-\mu} \qquad (\lambda, \mu \in \mathbb{C}),$$

in which:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

indicates that the well-known gamma function for the special situation $Re(z) > 0$ can be reduced:

$$\binom{\lambda}{n} = \frac{(-1)^n(-\lambda)_n}{n!},$$

where $(\lambda)_r$ stands for the Pochhammer symbol [2], which is represented by:

$$(\lambda)_r = \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)} = \begin{cases} 1; & (r=0,\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+r-1); & (r \in \mathbb{N}, \lambda \in \mathbb{C}) \end{cases} \tag{0.1}$$

The generalized hypergeometric functions [2] are defined as follows using Pochhammer's symbol (1.1):

$${}_pF_q\left[\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ & & \\ \beta_1, & \ldots, & \beta_q \end{matrix} ;\ z\right] = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r \ldots (\alpha_p)_r}{(\beta_1)_r \ldots (\beta_q)_r} \frac{z^r}{r!} \tag{0.2}$$

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Its substantial to note here that symmetry occurs in the numerator $\alpha_1, \ldots, \alpha_p$ and denominator $\beta_1, \ldots, \beta_q$ parameters of the generalized hypergeometric function which means that by changing the order of each of the numerator parameters provides the same function and is the same case for denominator parameters. The convergent of the series (1.2) depend on the relation between the parameters of both numerator and denominator and by using elementary ratio test, we get the following cases, [2]:

i. If $p \leq q$, then series converge for all $|z| < \infty$.
ii. If $p = q+1$, the series converges for $|z| < 1$, otherwise it is diverges.
iii. If $p = q+1$, on the circle of converge $|z| = 1$, the series is:
   - Absolutely convergent if $\mathrm{Re}\left(\sum_{n=1}^{q}\beta_n - \sum_{n=1}^{p}\alpha_n\right) > 0$.
   - Conditionally convergent if $-1 < \mathrm{Re}\left(\sum_{n=1}^{q}\beta_n - \sum_{n=1}^{p}\alpha_n\right) \leq 0$.
   - Divergent if $\mathrm{Re}\left(\sum_{n=1}^{q}\beta_n - \sum_{n=1}^{p}\alpha_n\right) \leq -1$.
iv. $\forall\, z \neq 0$, the series generally diverges if $p > q+1$. However, the series comes to an end (terminates) and the generalized hypergeometric function becomes a polynomial in z when one or more of the numerator parameters $alpha_p$ are negative integers.

The function ${}_pF_q$ is implemented as HypergeometricPFQ in MATHEMATICA and can be used to calculate both symbolic and numerical data.

The classical summation theorems of Gauss, Gauss second, Kummer, and Bailey for the series ${}_2F_1$ and Watson, Dixon, Whipple, and Saalschütz for the serie ${}_3F_2$, among others, are crucial in the theory of generalized hypergeometric series.

Many scholars have been motivated and inspired to create and study hypergeometric functions of two or more variables as a result of the hypergeometric function's enormous popularity, huge applicability, and generalized hypergeometric functions of one variable. During 1992-1996, in a series of three research papers, the generalizations of the aforementioned classical summation theorems were established by Lavoie et al. [6,16,17] who also attained numerous special cases and limiting cases for their conclusions. Later on, Lewanowicz [18] and Vidunas [19] obtained further generalizations of Watson's and Kummer's summation theorems, respectively.

In 2010-2011, the above-mentioned classical summation theorems were most broadly developed and extended by Rakha and Rathie [20] and Kim et al. [5]. Computer programmes in MATHEMATICA and MAPLE have also been used to obtain and verify those results.

In the theory of hypergeometric and generalized hypergeometric series, the ${}_3F_2$ hypergeometric function has a particularly noteworthy role. Despite this, the ${}_3F_2$ hypergeometric function has several uses in mathematics, see [10], for further information on these applications. Also it has a lot of applications in physics and statistics such as: Random Walks: further details regarding this application may be found at[11]. $3j$, $6j$ and $9j$ Symbols, see [12,13]. For more applications, see [14,15]. In the Generalized hypergeometric function (1.2), with $p = 3$, $q = 2$, $\alpha_1 = a$, $\alpha_2 = b$, $\alpha_3 = c$, $\beta_1 = \frac{1}{2}(a+b+1)$ and $\beta_2 = 2c$ with argument $z = 1$, we will get the well-known Watson summation theorem [2] viz.

$$
{}_3F_2\left[\begin{matrix} \alpha, & \beta, & \gamma \\ \frac{1}{2}(\alpha+\beta+1), & 2\gamma & \end{matrix} ; 1\right]
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)}, \tag{0.3}
$$

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provided $Re(2\gamma-\alpha-\beta)>-1$.

Bailey [1], in his paper mentioned several interesting applications by using the aforementioned classical summation theorems. In 2010, these classical summation theorems have been extended by Kim et al. [5]. However, here we would like to mention some of the extended summation theorems that will be required in our present investigations.

- Extension of Gauss second summation theorem:

$$
{}_3F_2\left[\begin{matrix}\alpha, & \beta, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & \delta & \end{matrix};\frac{1}{2}\right]=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{3}{2}\right)}
$$
$$
\times\left\{\frac{\left(\frac{1}{2}(\alpha+\beta-1)-\frac{\alpha\beta}{\delta}\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}+\frac{\left(\frac{\alpha+\beta+1}{\delta}-2\right)}{\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\beta\right)}\right\}, \tag{0.4}
$$

for $\delta=\frac{1}{2}(\alpha+\beta+1)$, the result (1.4) reduces to the following well-known Gauss second summation theorem [2,8] viz.

$$
{}_2F_1\left[\begin{matrix}\alpha, & \beta \\ \frac{1}{2}(\alpha+\beta+1) & \end{matrix};\frac{1}{2}\right]=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}. \tag{0.5}
$$

- Extension of Watson summation theorem [5]:

$$
{}_4F_3\left[\begin{matrix}\alpha, & \beta, & \gamma, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & 2\gamma, & \delta & \end{matrix};\ 1\right]
$$
$$
=\frac{2^{\alpha+\beta-2}\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}\right)}{(\alpha-\beta-1)(\alpha-\beta+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)\Gamma(\beta)}
$$
$$
\times\left\{\gamma_1\frac{\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\gamma-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)}+\gamma_2\frac{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\gamma-\frac{1}{2}\alpha\right)\Gamma\left(\gamma-\frac{1}{2}\beta\right)}\right\}, \tag{0.6}
$$

provided $Re(2\gamma-\alpha-\beta)>1$. Also, the constant γ_1 and γ_2 are given by $\gamma_1=\alpha(2\gamma-\alpha)+\beta(2\gamma-\beta)-2\gamma+1-\frac{\alpha\beta}{\delta}(4\gamma-\alpha-\beta-1)$ and $\gamma_2=\frac{4}{\delta}(\alpha+\beta+1)-8$. For $\delta=\frac{1}{2}(\alpha+\beta+1)$.
For $\delta=\frac{1}{2}(\alpha+\beta+1)$, the result (1.6) reduces to the classical Watson summation theorem (1.3).

- Gauss summation theorem [2,8]:

$$
{}_2F_1\left[\begin{matrix}\alpha, & \beta \\ \gamma & \end{matrix};1\right]=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{0.7}
$$

provided $Re(\gamma-\alpha-\beta)>0$.

- Special case of (1.7) [[8],p.49]:

$$
{}_2F_1\left[\begin{matrix}-\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2} \\ \gamma+\frac{1}{2} & \end{matrix};1\right]=\frac{2^n(\gamma)_n}{(2\gamma)_n}. \tag{0.8}
$$

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- A definite integral due to MacRobert [7]:

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}[1+ax+b(1-x)]^{-\lambda-\mu}dx = \frac{1}{(1+a)^{\lambda}(1+b)^{\mu}}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}, \tag{0.9}$$

provided $Re(\lambda) > 0, Re(\mu) > 0$ and the constant a and b are such that none of the expressions $1+a, 1+b$ and $[1+ax+b(1-x)], 0 \le x \le 1$ is not zero.
- Relation between Pochhammer symbol and Gamma function:

$$(d)_n = \frac{\Gamma(d+n)}{\Gamma(d)}. \tag{0.10}$$

- Elementary identities:

$$(-n)_{2m} = 2^{2m}\left(-\frac{1}{2}n\right)_m\left(-\frac{1}{2}n+\frac{1}{2}\right)_m = \frac{n!}{(n-2m)!}. \tag{0.11}$$

$$(\beta)_{n+2m} = (\beta)_{2m}(\beta+2m)_n. \tag{0.12}$$

$$(\alpha)_{2n} = 2^{2n}\left(\frac{1}{2}\alpha\right)_n\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n. \tag{0.13}$$

- A result recorded in Rainville [[8], Equ. 8, p.57]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n+2m). \tag{0.14}$$

The paper is organised as follows. In section 1, we shall give a new derivation of the extended Watson summation theorem . As an applications, in section 2, we shall evaluate integrals involving generalized hypergeometric function by employing extended Watson summation theorem (1.6), while section 3, deals with some of the interesting special cases of our main findings.

The following is how the paper is set up. We will present a new derivation of the extended Watson summation theorem (ref. 1.3) in section 1. The extended Watson summation theorem (1.6) will be used in section 2 as an application for obtaining integrals involving generalized hypergeometric function ${}_3F_2$, and section 3 will discuss some of the remarkable particular cases of our major results. endnote.

1. A new derivation of the result (1.6)

In this section, we shall give a new derivation of the extended Watson summation theorem (1.6). For this, In order to derive the result (1.6), we proceed as follows.
Consider the integral, For $Re(d) > 0$:

$$I = \int_0^{\infty} e^{-t}t^{d-1}\,{}_4F_4\left[\begin{matrix}\alpha, & \beta, & \gamma, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & 2\gamma, & \delta, & d\end{matrix}\;;\; t\right]dt.$$

By describing ${}_4F_4$ as a series and changing the integration and series order, which is simply supported by the series' uniform convergence, we have

$$I = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\gamma)_n(\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n(2\gamma)_n(\delta)_n(d)_n n!}\int_0^{\infty} e^{-t}t^{d+n-1}dt.$$

Evaluating the well-known gamma integral and making use of the result (1.10), we have after some simplification:

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n (\gamma)_n (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (2\gamma)_n (\delta)_n n!}. \tag{1.1}$$

Summing up the series, we have

$$I = \Gamma(d)\, {}_4F_3 \left[\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta+1 \\ & & & ; \; 1 \\ \frac{1}{2}(\alpha+\beta+3), & 2\gamma, & \delta, & \end{array} \right]. \tag{1.2}$$

On the other hand, writing (2.1) in the following form:

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^n n!} \left\{ \frac{2^n (\gamma)_n}{(2\gamma)_n} \right\}.$$

Now, making use of the result (1.8), we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^n n!}\, {}_2F_1 \left[\begin{array}{cc} -\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2} \\ & ; 1 \\ \gamma+\frac{1}{2} & \end{array} \right].$$

Further, expressing ${}_2F_1$ as a series, we have after some simplification,

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{(\alpha)_n (\beta)_n (\delta+1)_n \left(-\frac{1}{2}n\right)_m \left(-\frac{1}{2}n+\frac{1}{2}\right)_m}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^n \left(\gamma+\frac{1}{2}\right)_m m! n!}.$$

Now, making use of the identity (1.11), we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{[\frac{n}{2}]} \frac{(\alpha)_n (\beta)_n (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n \left(\gamma+\frac{1}{2}\right)_m (\delta)_n 2^{2m+n} m! (n-2m)!}.$$

Next, replacing n by $n+2m$ and making use of the result (1.14), we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{n+2m} (\beta)_{n+2m} (\delta+1)_{n+2m}}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_{n+2m} (\delta)_{n+2m} \left(\gamma+\frac{1}{2}\right)_m 2^{n+4m} m! n!}.$$

Now, making use of the identity (1.12) and after some simplification

$$I = \Gamma(d) \sum_{m=0}^{\infty} \frac{(\alpha)_{2m} (\beta)_{2m} (\delta+1)_{2m}}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_{2m} \left(\gamma+\frac{1}{2}\right)_m (\delta)_{2m} 2^{4m} m!} \sum_{n=0}^{\infty} \frac{(\alpha+2m)_n (\beta+2m)_n (\delta+1+2m)_n}{\left(\frac{1}{2}(\alpha+\beta+3)+2m\right)_n (\delta+2m)_n 2^n n!}.$$

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Summing up the inner series, we have

$$I = \Gamma(d) \sum_{m=0}^{\infty} \frac{(\alpha)_{2m}(\beta)_{2m}(\delta+1)_{2m}}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_{2m}\left(\gamma+\frac{1}{2}\right)_m (\delta)_{2m} 2^{4m} m!}$$
$$\times {}_3F_2\left[\begin{matrix} \alpha+2m, & \beta+2m, & \delta+1+2m \\ \frac{1}{2}(\alpha+\beta+3)+2m, & \delta+2m & \end{matrix} ; \frac{1}{2}\right].$$

We now observe that the ${}_3F_2$ appearing can be expressed with the help of the result (1.4) and once it has been simplified and using the result (1.13) separating into four parts then summarizing the series and employing the result(1.7), we obtain

$$I = \Gamma(d) \frac{2^{\alpha+\beta-2}\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}\right)}{(\alpha-\beta-1)(\alpha-\beta+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)\Gamma(\beta)}$$
$$\times \left\{\gamma_1 \frac{\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\gamma-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)} + \gamma_2 \frac{\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\gamma-\frac{1}{2}\alpha\right)\Gamma\left(\gamma-\frac{1}{2}\beta\right)}\right\}, \quad (1.3)$$

where the constant γ_1 and γ_2 are the same as given in equation (1.6). Finally, equating the results (2.2) and (2.3), we arrive at our desired result (1.6). This completes the derivation of the result (1.6).

2. Application

In this section, by employing extended Watson summation theorem (1.6), the following integrals employing generalized hypergeometric function ${}_3F_2$ will be evaluated. With the constants a and b are such that none of the expressions $1+a$, $1+b$ and $[1+ax+b(1-x)]$, $0 \le x \le 1$ is not a zero., these integrals are

$$\int_0^1 x^{\gamma-1}(1-x)^{\gamma-1}[1+ax+b(1-x)]^{-2\gamma}$$
$$\times {}_3F_2\left[\begin{matrix} \alpha, & \beta, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & \delta & \end{matrix} ; \frac{(1+a)x}{1+ax+b(1-x)}\right] dx$$
$$= \frac{1}{(1+a)^{\gamma}(1+b)^{\gamma}} \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \Omega, \quad (2.1)$$

provided $Re(\gamma) > 0$, $Re(2\gamma-\alpha-\beta) > 1$. Also Ω is the same as given in (1.6). Then

$$\int_0^1 x^{\beta-1}(1-x)^{2\gamma-\beta-1}[1+ax+b(1-x)]^{-2\gamma}$$
$$\times {}_3F_2\left[\begin{matrix} \alpha, & \gamma, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & \delta & \end{matrix} ; \frac{(1+a)x}{1+ax+b(1-x)}\right] dx$$
$$= \frac{1}{(1+a)^{\beta}(1+b)^{2\gamma-\beta}} \frac{\Gamma(\beta)\Gamma(2\gamma-\beta)}{\Gamma(2\gamma)} \Omega, \quad (2.2)$$

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provided $Re(\beta) > 0,\ Re(2\gamma - \beta) > 0,\ Re(2\gamma - \alpha - \beta) > 0.$
Also Ω is the same as given in (1.6).

$$\int_0^1 x^{\beta-1}(1-x)^{\frac{1}{2}(\alpha-\beta+3)-1}[1+ax+b(1-x)]^{\frac{1}{2}(\alpha+\beta+3)}$$
$$\times\ {}_3F_2\left[\begin{matrix}\alpha, & \gamma, & \delta+1 \\ & & \\ 2\gamma, & \delta & \end{matrix} ;\ \frac{(1+a)x}{1+ax+b(1-x)}\right]dx$$
$$= \frac{1}{(1+a)^{\beta}(1+b)^{\frac{1}{2}(\alpha-\beta+3)}}\frac{\Gamma(\beta)\Gamma\left(\frac{1}{2}(\alpha-\beta+3)\right)}{\Gamma\left(\frac{1}{2}(\alpha+\beta+3)\right)}\Omega, \tag{2.3}$$

provided $Re(\beta) > 0,\ Re(\alpha - \beta) > -3,\ Re(2\gamma - \alpha - \beta) > 0.$
Also Ω is the same as given in (1.6).

$$\int_0^1 x^{\delta}(1-x)^{2\gamma-\delta-2}[1+ax+b(1-x)]^{-2\gamma}$$
$$\times\ {}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma \\ & & \\ \frac{1}{2}(\alpha+\beta+3), & \delta & \end{matrix} ;\ \frac{(1+a)x}{1+ax+b(1-x)}\right]dx$$
$$= \frac{1}{(1+a)^{\delta}(1+b)^{2\gamma-\delta-1}}\frac{\Gamma(\delta+1)\Gamma(2\gamma-\delta-1)}{\Gamma(2\gamma)}\Omega, \tag{2.4}$$

provided $Re(\delta) > 0,\ Re(2\gamma - \delta) > 1,\ Re(2\gamma - \alpha - \beta) > 1.$

Proof. In order to establish the results (3.1) to (3.4) we proceed as follows:
By using I to represent the left side of (3.1), we have

$$I = \int_0^1 x^{\gamma-1}(1-x)^{\gamma-1}[1+ax+b(1-x)]^{-2\gamma}$$
$$\times\ {}_3F_2\left[\begin{matrix}\alpha, & \beta, & \delta+1 \\ & & \\ \frac{1}{2}(\alpha+\beta+3), & \delta & \end{matrix} ;\ \frac{(1+a)x}{1+ax+b(1-x)}\right]dx.$$

Expressing ${}_3F_2$ as series, change the order of integration and summation, we get after some algebra.

$$I = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\delta+1)_n(1+a)^n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n(\delta)_n n!}\int_0^{\infty} x^{\gamma+n-1}(1-x)^{\gamma-1}[1+ax+b(1-x)]^{-2\gamma-n}dx.$$

Now, evaluating the integral with the help of the known integral (1.11) due to MacRobert, we have

$$I = \frac{\Gamma(\gamma)}{(1+a)^{\gamma}(1+b)^{\gamma}}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n(\delta)_n n!}\frac{\Gamma(\gamma+n)}{\Gamma(2\gamma+n)}.$$

Using the relation (1.10), we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{(1+a)^{\gamma}(1+b)^{\gamma}}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\gamma)_n(\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n(2\gamma)_n(\delta)_n n!}.$$

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Summing up the series, we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{(1+a)^{\gamma}(1+b)^{\gamma}\Gamma(2\gamma)} {}_4F_3\left[\begin{matrix} \alpha, & \beta, & \gamma, & \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), & 2\gamma, & \delta & \end{matrix} ; 1\right]$$

Now, it's easy to see that the ${}_4F_3$ appearing can be evaluated with the help of the result (1.6) and we easily arrive at the right-hand side of our first integral (3.1). In exactly, the same manner, the other integrals (3.2) to (3.4) can be established. So we left this to the interested reader as an exercise. □

Remark 1. For a similar proof, we refer a recent paper by Jun and Kilicman [4].

3. Special Cases

We will discuss a few of the remarkable specific cases of our major results in this section. For this, it's easily seen that in the integrals (3.1) to (3.4), if n is zero or a positive integer and $\beta = -2n$ and α is replaced by $\alpha + 2n$ or $\beta = -2n - 1$, α is replaced by $\alpha + 2n + 1$. In both cases, one of the two terms appearing in the right-hand side of the integrals (3.1) to (3.4) vanish and we can easily obtain eight new and interesting results. But due to lack of space we are not given here.

4. Concluding Remark

The fact that generalized hypergeometric functions are always reduced to gamma functions and that the results are always significant from an application standpoint should be noteworthy at this point. In this note, we have given a new derivation of the extended Watson theorem due to Kim et al.. As an application, we evaluated four interesting integrals involving generalized hypergeometric functions. In the end, we mention outlines of the special cases. In order to wrap up this note, we would like to mention that evaluation of finite single and double integrals is now being studied as an application and will be covered in our upcoming article in this area.

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