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Article

Functional Deutsch Uncertainty Principle

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Abstract: Let $\{f_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ be Parseval p-frames for a finite dimensional Banach space $\mathcal{X}$. Then we show that (1) $\log(nm) \geq S_f(x)+S_g(x) \geq -p\log\left(\sup_{y\in\mathcal{X}_f\cap\mathcal{X}_g,\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right)$, $\forall x \in \mathcal{X}_f\cap\mathcal{X}_g$, where $\mathcal{X}_f := \{z \in \mathcal{X} : f_j(z)\neq 0, 1\leq j\leq n\}$, $\mathcal{X}_g := \{w\in\mathcal{X} : g_k(w)\neq 0, 1\leq k\leq m\}$, $S_f(x) := -\sum_{j=1}^n\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p$, $S_g(x) := -\sum_{k=1}^m\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p$, $\forall x\in\mathcal{X}_g$. We call Inequality (1) as **Functional Deutsch Uncertainty Principle**. For Hilbert spaces, we show that Inequality (1) reduces to the uncertainty principle obtained by Deutsch *[Phys. Rev. Lett., 1983]*. We also derive a dual of Inequality (1).

Keywords: uncertainty principle; orthonormal basis; parseval frame; hilbert space; banach space

MSC: 42C15

1. Introduction

Let $d\in\mathbb{N}$ and $\widehat{\cdot}: \mathcal{L}^2(\mathbb{R}^d)\to\mathcal{L}^2(\mathbb{R}^d)$ be the unitary Fourier transform obtained by extending uniquely the bounded linear operator

$$\widehat{\cdot}: \mathcal{L}^1(\mathbb{R}^d)\cap\mathcal{L}^2(\mathbb{R}^d)\ni f\mapsto\widehat{f}\in C_0(\mathbb{R}^d);\quad \widehat{f}:\mathbb{R}^d\ni\xi\mapsto\widehat{f}(\xi):=\int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx\ \in\mathbb{C}.$$

The **Shannon entropy** at a function $f\in\mathcal{L}^2(\mathbb{R}^d)\setminus\{0\}$ is defined as

$$S(f) := -\int_{\mathbb{R}^d}\left|\frac{f(x)}{\|f\|}\right|^2\log\left|\frac{f(x)}{\|f\|}\right|^2 dx$$

(with the convention $0\log 0 = 0$) [1]. In 1957, Hirschman proved the following result [2].

Theorem 1.1. *[2] (**Hirschman Inequality**) For all $f\in\mathcal{L}^2(\mathbb{R}^d)\setminus\{0\}$,*

$$S(f)+S(\widehat{f})\geq 0. \tag{2}$$

In the same paper [2] Hirschman conjectured that Inequality (2) can be improved to

$$S(f)+S(\widehat{f})\geq d(1-\log 2),\quad f\in\mathcal{L}^2(\mathbb{R}^d)\setminus\{0\}. \tag{3}$$

Inequality (3) was proved independently in 1975 by Beckner [3] and Bialynicki-Birula and Mycielski [4].

Theorem 1.2. *[3,4] (**Hirschman-Beckner-Bialynicki-Birula-Mycielski Uncertainty Principle**) For all $f\in\mathcal{L}^2(\mathbb{R}^d)\setminus\{0\}$,*

$$S(f)+S(\widehat{f})\geq d(1-\log 2).$$

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Now one naturally asks whether there is a finite dimensional version of Shannon entropy and uncertainty principle. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Given an orthonormal basis $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$, the **(finite) Shannon entropy** at a point $h \in \mathcal{H}_\tau$ is defined as

$$S_\tau(h) := -\sum_{j=1}^{n}\left|\left\langle \frac{h}{\|h\|}, \tau_j\right\rangle\right|^2 \log\left|\left\langle \frac{h}{\|h\|}, \tau_j\right\rangle\right|^2 \geq 0,$$

where $\mathcal{H}_\tau := \{h \in \mathcal{H} : \langle h, \tau_j\rangle \neq 0, 1 \leq j \leq n\}$ [5]. In 1983, Deutsch derived following uncertainty principle for Shannon entropy which is fundamental to several developments in Mathematics and Physics [5].

Theorem 1.3. *[5] (**Deutsch Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$2\log n \geq S_\tau(h) + S_\omega(h) \geq -2\log\left(\frac{1+\max\limits_{1\leq j,k\leq n}|\langle \tau_j, \omega_k\rangle|}{2}\right) \geq 0, \quad \forall h \in \mathcal{H}_\tau. \tag{4}$$

Recently, author derived Banach space versions of Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani uncertainty principle [6], Donoho-Stark approximate support uncertainty principle [7] and Ghobber-Jaming uncertainty principle [8]. We then naturally ask what is the Banach space version of Inequality (4)? In this paper, we are going to answer this question.

2. Functional Deutsch Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a finite dimensional Banach space over $\mathbb{K}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. We need the notion of Parseval p-frames for Banach spaces.

Definition 2.1. *[9,10] Let $\mathcal{X}$ be a finite dimensional Banach space over $\mathbb{K}$. A collection $\{f_j\}_{j=1}^n$ in $\mathcal{X}^*$ is said to be a **Parseval p-frame** $(1 \leq p < \infty)$ for $\mathcal{X}$ if*

$$\|x\|^p = \sum_{j=1}^{n}|f_j(x)|^p, \quad \forall x \in \mathcal{X}. \tag{5}$$

Note that (5) says that $\|f_j\| \leq 1$ for all $1 \leq j \leq n$. Given a Parseval p-frame $\{f_j\}_{j=1}^n$ for $\mathcal{X}$, we define the **(finite) p-Shannon entropy** at a point $x \in \mathcal{X}_f$ as

$$S_f(x) := -\sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p \log\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p \geq 0,$$

where $\mathcal{X}_f := \{x \in \mathcal{X} : f_j(x) \neq 0, 1 \leq j \leq n\}$. Following is the fundamental result of this paper.

Theorem 2.2. *(**Functional Deutsch Uncertainty Principle**) Let $\{f_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ be Parseval p-frames for a finite dimensional Banach space $\mathcal{X}$. Then*

$$\frac{1}{(nm)^{\frac{1}{p}}} \leq \sup_{y\in\mathcal{X}, \|y\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(y)g_k(y)|\right)$$

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and

$$\log(nm) \geq S_f(x) + S_g(x) \geq -p\log\left(\sup_{y\in\mathcal{X}_f\cap\mathcal{X}_g,\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right) > 0, \quad \forall x \in \mathcal{X}_f\cap\mathcal{X}_g. \tag{6}$$

Proof. Let $z \in \mathcal{X}$ be such that $\|z\| = 1$. Then

$$\begin{aligned}
1 &= \left(\sum_{j=1}^{n}|f_j(z)|^p\right)\left(\sum_{k=1}^{m}|g_k(z)|^p\right) = \sum_{j=1}^{n}\sum_{k=1}^{m}|f_j(z)g_k(z)|^p\\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{m}\left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right)^p\\
&= \left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right)^p mn
\end{aligned}$$

which gives

$$\frac{1}{mn} \leq \left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right)^p.$$

Since $1 = \sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p$ for all $x \in \mathcal{X}\setminus\{0\}$, $1 = \sum_{k=1}^{m}\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p$ for all $x \in \mathcal{X}\setminus\{0\}$ and log function is concave, using Jensen's inequality (see [11]) we get

$$\begin{aligned}
S_f(x) + S_g(x) &= \sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\log\left(\frac{1}{\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p}\right) + \sum_{k=1}^{m}\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left(\frac{1}{\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p}\right)\\
&\leq \log\left(\sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\frac{1}{\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p}\right) + \log\left(\sum_{k=1}^{m}\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\frac{1}{\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p}\right)\\
&= \log n + \log m = \log(nm), \quad \forall x \in \mathcal{X}_f\cap\mathcal{X}_g.
\end{aligned}$$

Let $x \in \mathcal{X}_f\cap\mathcal{X}_g$. Then

$$\begin{aligned}
S_f(x) + S_g(x) &= -\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\left[\log\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p + \log\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\right]\\
&= -\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_j\left(\frac{x}{\|x\|}\right)g_k\left(\frac{x}{\|x\|}\right)\right|^p\\
&= -p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_j\left(\frac{x}{\|x\|}\right)g_k\left(\frac{x}{\|x\|}\right)\right|\\
&\geq -p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left(\sup_{y\in\mathcal{X}_f\cap\mathcal{X}_g,\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right)\\
&= -p\log\left(\sup_{y\in\mathcal{X}_f\cap\mathcal{X}_g,\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right)\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\\
&= -p\log\left(\sup_{y\in\mathcal{X}_f\cap\mathcal{X}_g,\|y\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|f_j(y)g_k(y)|\right)\right).
\end{aligned}$$

□

Corollary 2.3. *Theorem 1.3 follows from Theorem 2.2.*

Proof. Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Define

$$f_j : \mathcal{H} \ni h \mapsto \langle h, \tau_j\rangle \in \mathbb{K}; \quad g_j : \mathcal{H} \ni h \mapsto \langle h, \omega_j\rangle \in \mathbb{K}, \quad \forall 1 \leq j \leq n.$$

Now by using Buzano inequality (see [12,13]) we get

$$\begin{aligned}\sup_{h\in\mathcal{H},\|h\|=1}\left(\max_{1\leq j,k\leq n}|f_j(h)g_k(h)|\right) &= \sup_{h\in\mathcal{H},\|h\|=1}\left(\max_{1\leq j,k\leq n}|\langle h,\tau_j\rangle||\langle h,\omega_k\rangle|\right)\\ &\leq \sup_{h\in\mathcal{H},\|h\|=1}\left(\max_{1\leq j,k\leq n}\left(\|h\|^2\frac{\|\tau_j\|\|\omega_k\|+|\langle\tau_j,\omega_k\rangle|}{2}\right)\right)\\ &= \frac{1+\max\limits_{1\leq j,k\leq n}|\langle\tau_j,\omega_k\rangle|}{2}.\end{aligned}$$

□

Theorem 2.2 brings the following question.

Question 2.4. *Given p, m, n and a Banach space $\mathcal{X}$, for which pairs of Parseval p-frames $\{f_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ for $\mathcal{X}$, we have equality in Inequality (6)?*

Next we derive a dual inequality of (6). For this we need dual of Definition 2.1.

Definition 2.5. *[14–16] Let $\mathcal{X}$ be a finite dimensional Banach space over $\mathbb{K}$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{X}$ is said to be a **Parseval p-frame** $(1 \leq p < \infty)$ for $\mathcal{X}^*$ if*

$$\|f\|^p = \sum_{j=1}^n |f(\tau_j)|^p, \quad \forall f \in \mathcal{X}^*. \tag{7}$$

Note that (7) says that

$$\|\tau_j\| = \sup_{f\in\mathcal{X}^*,\|f\|=1}|f(\tau_j)| \leq \sup_{f\in\mathcal{X}^*,\|f\|=1}\left(\sum_{j=1}^n|f(\tau_j)|^p\right)^{\frac{1}{p}} = \sup_{f\in\mathcal{X}^*,\|f\|=1}\|f\| = 1, \quad \forall 1 \leq j \leq n.$$

Given a Parseval p-frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{X}^*$, we define the **(finite) p-Shannon entropy** at a point $f \in \mathcal{X}_\tau^*$ as

$$S_\tau(f) := -\sum_{j=1}^n\left|\frac{f(\tau_j)}{\|f\|}\right|^p\log\left|\frac{f(\tau_j)}{\|f\|}\right|^p \geq 0,$$

where $\mathcal{X}_\tau^* := \{f \in \mathcal{X}^* : f(\tau_j) \neq 0, 1 \leq j \leq n\}$. We now have the following dual to Theorem 2.2.

Theorem 2.6. *(**Functional Deutsch Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$ and $\{\omega_k\}_{k=1}^m$ be two Parseval p-frames for the dual $\mathcal{X}^*$ of a finite dimensional Banach space $\mathcal{X}$. Then*

$$\frac{1}{(nm)^{\frac{1}{p}}} \leq \sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\max_{1\leq j\leq n,1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)$$

and

$$\log(nm) \geq S_\tau(f) + S_\omega(f) \geq -p\log\left(\sup_{g\in\mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right) > 0, \quad \forall f \in \mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*. \tag{8}$$

Proof. Let $h \in \mathcal{X}^*$ be such that $\|h\| = 1$. Then

$$\begin{aligned}
1 &= \left(\sum_{j=1}^{n}|h(\tau_j)|^p\right)\left(\sum_{k=1}^{m}|h(\omega_k)|^p\right) = \sum_{j=1}^{n}\sum_{k=1}^{m}|h(\tau_j)h(\omega_k)|^p \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{m}\left(\sup_{g\in\mathcal{X}^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right)^p \\
&= \left(\sup_{g\in\mathcal{X}^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right)^p mn
\end{aligned}$$

which gives

$$\frac{1}{mn} \leq \left(\sup_{g\in\mathcal{X}^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right)^p.$$

Since $1 = \sum_{j=1}^{n}\left|\frac{f(\tau_j)}{\|f\|}\right|^p$ for all $f \in \mathcal{X}^*\setminus\{0\}$, $1 = \sum_{k=1}^{m}\left|\frac{f(\omega_k)}{\|f\|}\right|^p$ for all $f \in \mathcal{X}^*\setminus\{0\}$ and log function is concave, using Jensen's inequality we get

$$\begin{aligned}
S_\tau(f) + S_\omega(f) &= \sum_{j=1}^{n}\left|\frac{f(\tau_j)}{\|f\|}\right|^p \log\left(\frac{1}{\left|\frac{f(\tau_j)}{\|f\|}\right|^p}\right) + \sum_{k=1}^{m}\left|\frac{f(\omega_k)}{\|f\|}\right|^p \log\left(\frac{1}{\left|\frac{f(\omega_k)}{\|f\|}\right|^p}\right) \\
&\leq \log\left(\sum_{j=1}^{n}\left|\frac{f(\tau_j)}{\|f\|}\right|^p \frac{1}{\left|\frac{f(\tau_j)}{\|f\|}\right|^p}\right) + \log\left(\sum_{k=1}^{m}\left|\frac{f(\omega_k)}{\|f\|}\right|^p \frac{1}{\left|\frac{f(\omega_k)}{\|f\|}\right|^p}\right) \\
&= \log n + \log m = \log(nm), \quad \forall f \in \mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*.
\end{aligned}$$

Let $f \in \mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*$. Then

$$\begin{aligned}
S_\tau(f) + S_\omega(f) &= -\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p\left[\log\left|\frac{f(\tau_j)}{\|f\|}\right|^p + \log\left|\frac{f(\omega_k)}{\|f\|}\right|^p\right] \\
&= -\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p \log\left|\frac{f(\tau_j)}{\|f\|}\frac{f(\omega_k)}{\|f\|}\right|^p \\
&= -p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p \log\left|\frac{f(\tau_j)}{\|f\|}\frac{f(\omega_k)}{\|f\|}\right| \\
&\geq -p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p \log\left(\sup_{g\in\mathcal{X}^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right) \\
&= -p\log\left(\sup_{g\in\mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right)\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p \\
&= -p\log\left(\sup_{g\in\mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g(\tau_j)g(\omega_k)|\right)\right).
\end{aligned}$$

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□

Theorem 2.6 again gives the following question.

Question 2.7. *Given p, m, n and a Banach space $\mathcal{X}$, for which pairs of Parseval p-frames $\{\tau_j\}_{j=1}^n$ and $\{\omega_k\}_{k=1}^m$ for $\mathcal{X}^*$, we have equality in Inequality (8)?*

Author is aware of the improvement of Theorem 1.3 by Maassen and Uffink [17] (cf. [18]) (motivated from a conjecture of Kraus [19]) but unable to derive Maassen-Uffink uncertainty principle from Theorem 2.2.

References

1. Shannon, C.E. A mathematical theory of communication. *Bell System Tech. J.* **1948**, *27*, 379–423, 623–656. doi:10.1002/j.1538-7305.1948.tb01338.x.
2. Hirschman, Jr., I.I. A note on entropy. *Amer. J. Math.* **1957**, *79*, 152–156. doi:10.2307/2372390.
3. Beckner, W. Inequalities in Fourier analysis. *Ann. of Math. (2)* **1975**, *102*, 159–182. doi:10.2307/1970980.
4. Bialynicki-Birula, I.; Mycielski, J. Uncertainty relations for information entropy in wave mechanics. *Comm. Math. Phys.* **1975**, *44*, 129–132.
5. Deutsch, D. Uncertainty in quantum measurements. *Phys. Rev. Lett.* **1983**, *50*, 631–633. doi:10.1103/PhysRevLett.50.631.
6. Krishna, K.M. Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle. *arXiv: 2304.03324v1 [math.FA] 5 April* **2023**.
7. Krishna, K.M. Functional Donoho-Stark Approximate-Support Uncertainty Principle. *arXiv:2307.01215v1 [math.FA] 1 July* **2023**.
8. Krishna, K.M. Functional Ghobber-Jaming Uncertainty Principle. *arXiv:2306.01014v1 [math.FA] 1 Jun* **2023**.
9. Aldroubi, A.; Sun, Q.; Tang, W.S. p-frames and shift invariant subspaces of L^p. *J. Fourier Anal. Appl.* **2001**, *7*, 1–21. doi:10.1007/s00041-001-0001-2.
10. Christensen, O.; Stoeva, D.T. p-frames in separable Banach spaces. *Adv. Comput. Math.* **2003**, *18*, 117–126. doi:10.1023/A:1021364413257.
11. Steele, J.M. *The Cauchy-Schwarz master class : An introduction to the art of mathematical inequalities*; AMS/MAA Problem Books Series, Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 2004; pp. x+306. doi:10.1017/CBO9780511817106.
12. Buzano, M.L. Generalizzazione della diseguaglianza di Cauchy-Schwarz. *Rend. Sem. Mat. Univ. e Politec. Torino* **1971/73**, *31*, 405–409 (1974).
13. Fujii, M.; Kubo, F. Buzano's inequality and bounds for roots of algebraic equations. *Proc. Amer. Math. Soc.* **1993**, *117*, 359–361. doi:10.2307/2159168.
14. Terekhin, P.A. Frames in a Banach space. *Funktsional. Anal. i Prilozhen.* **2010**, *44*, 50–62. doi:10.1007/s10688-010-0024-z.
15. Casazza, P.; Christensen, O.; Stoeva, D.T. Frame expansions in separable Banach spaces. *J. Math. Anal. Appl.* **2005**, *307*, 710–723. doi:10.1016/j.jmaa.2005.02.015.
16. Terekhin, P.A. Representation systems and projections of bases. *Mat. Zametki* **2004**, *75*, 944–947. doi:10.1023/B:MATN.0000030997.11701.c0.
17. Maassen, H.; Uffink, J.B.M. Generalized entropic uncertainty relations. *Phys. Rev. Lett.* **1988**, *60*, 1103–1106. doi:10.1103/PhysRevLett.60.1103.

18. Dembo, A.; Cover, T.M.; Thomas, J.A. Information-theoretic inequalities. *IEEE Trans. Inform. Theory* **1991**, *37*, 1501–1518. doi:10.1109/18.104312.
19. Kraus, K. Complementary observables and uncertainty relations. *Phys. Rev. D (3)* **1987**, *35*, 3070–3075. doi:10.1103/PhysRevD.35.3070.