Concept Paper

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Finding an Unique and "Natural" Extension of the Expected Value That Is Finite for All Functions in Non-Shy Subset of the Set of All Measurable Functions

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Article

Finding an Unique and "Natural" Extension of the Expected Value That Is Finite for All Functions in Non-Shy Subset of the Set of All Measurable Functions

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Abstract: Suppose for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$. If set A is Borel; we want to find an unique and "natural" extension of the expected value, w.r.t the Hausdorff measure, that's a finite value for all f in a non-shy subset of B^*—the set of all Borel measurable functions in $\mathbb{R}^A$. The issue is current extensions of the expected value are finite for all functions in *only* a shy subset of B^*. Despite attempts at generalizing the expected value, we haven't found evidence suggesting mathematicians thought of this problem; however, it's assumed, in general, there's no meaningful way of averaging functions which cover an infinite expanse of space. Regardless, we'll attempt to solve the problem by defining a choice function—this shall choose a unique set of equivelant sequences of sets (i.e. $(F_k^{\star\star\star})_{k\in\mathbb{N}}$), where the set-theoretic limit of $F_k^{\star\star\star}$ is the graph of f; the measure H^h is the h-Hausdorff measure, where for each $k \in \mathbb{N}$, $0 < H^h(F_k^{\star\star\star}) < +\infty$; and $(f_k^{\star})_{k\in\mathbb{N}}$ is a sequence of functions, where $\{(x_1, \cdots, x_n, f_k^{\star}(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in \text{dom}(F_k^{\star\star\star})\} = F_k^{\star\star\star}$. Thus, if $(F_k^{\star\star\star})$ converges to A at a rate linear or super-linear to the rate non-equivalent sequences of sets converge, the extended expected value of f *or* $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ is: $\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N})\left(k \geq N \Rightarrow \left|\frac{1}{H^h(\text{dom}(F_k^{\star\star\star}))}\int_{\text{dom}(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}]\right| < \epsilon\right)$ which should be unique and "natural" extension of the expected value, for all f in a non-shy subset of B^*. Note we guessed the choice function using computer programming: we redefine linear and super-linear convergence in terms of partitions of equal h-Hausdorff measure, a sample point from each partition, pathways between points in the sample, length of segments in the pathway (except for the lengths that are anomalies), multiplying remaining lengths in the pathway by constant so they add up to one (i.e, a probability distribution), taking the entropy of the distribution, maximizing the entropy w.r.t all pathways, and creating limits—w.r.t the constant h-Hausdorff measure of the partitions and the index of the sequences of sets—which redefines linear and super-linear convergence. Despite this, we don't know if the choice function solves the problem. (Infact, we're unable to prove most of the concepts in the paper: we require assistance for proving certain statements.) Even then, we'll visualize the paper using examples in this paper and examples in sec. 3 & 4 of the paper "Mean of Unbounded Sets Using Conditional Expectation" [1]. The biggest use of this research is the extension of the expected value is unique and finite for a "non-negligible" amount of measurable functions: this is easier to use in application when finding the "average" of functions covering an infinite expanse of space.

Keywords: expected value; hausdorff measure; (exact) dimension function; measurable functions; function space; prevalent and shy sets; entropy; choice function

0. Introduction

According to an article in Quanta Magazine [2] Wood writes, "No known mathematical procedure can meaningfully average an infinite number of objects covering an infinite expanse of space in general. The path integral is more of a physics philosophy than an exact mathematical recipe." The cited paper [3] presents a constructive approach to Wood's statement using filters over families of

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finite set; however, the average in the approach is not unique: the method determines the average value of functions with a range that lies in any algebraic structure for which the finite averages make sense. In this paper, we will explore a more constructive approach where the average is unique, finite, and "natural" (defined in Section 2.3 & Section 2.4) for a non-shy subset [4] of the set of measurable functions. (Note the functions must be measurable for application purposes).

We begin by describing "the infinite objects" which cover "an infinite expanse of space" as unbounded functions, since these functions are approachable from a mathematical standpoint. Moreover, if we define $n \in \mathbb{N}$, where set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$; suppose a *prevalent* subset of a function-space means "almost all" functions are in that space, a *shy* subset of a function-space means "almost no" functions are in that space and B^* is the set of all Borel measurable functions in $\mathbb{R}^A$. We then get the set of unbounded f where the expected value is infinite or undefined, forms a non-shy (i.e., prevalent nor shy *or* prevelant) subset of B^*. Furthermore, the set of all f with a finite expected value forms *only* a shy subset of B^*, meaning only a "negligible" amount of measurable functions have finite expected values.

Therefore, when defining prevalent and shy sets using mathematics in Section 1.1; we'll define four attempts to answer the thesis [1] of the first paragraph of Section 1.2. Note neither attempts give complete answers: they extend the Hausdorff measure of A to be positive and finite for "most" subsets of $\mathbb{R}^n$ but don't guarantee that unbounded functions in a non-shy subset of measurable functions have finite expected values. Infact, the expected value from all attempts might be positive and finite for *only* a shy subset of B^*.

Hence, we define a sequence of sets called $\star$*-sequence of sets* (Definition 8) whose properties allow for finite expected values for a non-shy subset of B^*. Note these $\star$-sequences of sets converge to the graph of f i.e. $\{(x, f(x)) : x \in A\}$ rather than A; otherwise, the *generalized expected value* of f w.r.t to a $\star$-sequence (Definition 9) cannot, in general, be finite for unbounded functions. Moreover, since there are functions with multiple $\star$-sequences of sets, where generalized expected values of f w.r.t each $\star$-sequence are different and non-unique—we must have a choice function which chooses a unique set of equivalent $\star$-sequences with the same, unique expected value.

For defining the choice function, we ask a question in Section 2.4 where with previous sections; we define equivalent & non-equivalent $\star$-sequences of sets for Section 2.1, as well as "natural" expected values for Section 2.3. We attempt to answer the question in Section 2.4 by redefining linear/super-linear convergence (Definition 16) in terms of entropy, samples and "pathways" where the samples are derived by taking a point from each partition of a $\star$-sequence of sets, such the partitions have equal Hausdorff measure. Since all samples have finite points; we take a "pathway" of line segments between the nearest point to each start-point of all segments in the pathway (i.e., the pathway should intersect every point once), where in Definition 19 we *exclude* segments with lengths which are anomalies . **[?**]. The procedure is similar to the ones used in computers to graph functions [5]. We also take the length of each of the line segments in the "pathway", multiplying all lengths by a constant so they add up to one (i.e. a discrete probability distribution). We take the supremum of the Entropy of the distribution [6] w.r.t all "pathways" to redefine Definition 16 as Definition 20, where the redefined definition is used to create a choice function in Section 4.1.

1. Preliminary Definitons/Motivation

Other than integration with filters [3], there are few other constructive approaches to finding a unique and "natural" extension of the average that takes a finite value for additional functions. Before beginning, consider the following mathematical definitions:

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[1] We want to find an unique and "natural" extension of the expected value, w.r.t the Hausdorff measure, that takes finite values for all f in a non-shy subset of all Borel measurable functions in $\mathbb{R}^A$

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1.1. Preliminary Definitions

Let X be a completely metrizable topological vector space.

Definition 1 (Prevalent Subset of X). *A Borel set $E \subset X$ is said to be **prevalent** if there exists a Borel measure μ on X such that:*

1. $0 < \mu(C) < \infty$ *for some compact subset C of X, and*
2. *the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.*

More generally, a subset F of X is prevalent if F contains a prevalent Borel Set. Also note:

Definition 2 (Shy Subset of X). *The complement of a prevalent set is called a shy set.*

such that we define:

Definition 3 (Non-Shy Subset of X). *A subset of X that is prevalent or neither prevalent nor shy.*

Furthermore, suppose we define:

Definition 4 (Hausdorff Measure). *Let (V, d) be a metric space, $\alpha \in [0, \infty)$. For every $C \in V$, define the diameter of C as:*

$$diam(C) := \sup \{d(x, y) : x, y \in C\}, \quad diam(\emptyset) := 0$$

We define:

$$H^{\alpha}_{\delta}(E) = \inf \left\{ \sum_{i=1}^{\infty} (diam(C_i))^{\alpha} : diam(C_i) \leq \delta, E \subseteq \bigcup_{i=1}^{\infty} C_i \right\}. \tag{1.1.1}$$

The Hausdorff Outer Measure is defined by

$$H^{\alpha}(E) = \sup_{\delta > 0} H^{\alpha}_{\delta}(E) = \lim_{\delta \to 0} H^{\alpha}_{\delta}(E)$$

If $i \in \mathbb{N}$ and $\delta \in \mathbb{R}$ such that $\delta > 0$, where the Euler's Gamma function is Γ and constant $\mathcal{N}_{\alpha}$ is:

$$\mathcal{N}_{\alpha} = \frac{\pi^{\alpha/2}}{2\Gamma\left(\frac{\alpha}{2} + 1\right)} \tag{1.1.2}$$

when $\alpha \in \mathbb{N}$ and E is a Borel set we have that

$$L^{\alpha}(E) = \frac{1}{2}\mathcal{N}_{\alpha} H^{\alpha}(E) \tag{1.1.3}$$

such that $H^{\alpha}(E)$ is related to the α-dimensional Lebesgue Measure.

Definition 5 (Hausdorff Dimension). *The Hausdorff Dimension of E is defined by $dim_H(E)$ where:*

$$H^{\alpha}(E) = \begin{cases} \infty & \text{if } 0 \leq \alpha < dim_H(E) \\ 0 & \text{if } dim_H(E) < \alpha < \infty \end{cases} \tag{1.1.4}$$

Therefore, we can use definitions 1, 2, 4 to prove or disprove:

Theorem 1. *The set of unbounded functions forms a prevalent subset of the set of all measurable functions.*

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Note 1 (**Notes on Theorem 1**). *By measurable function, we mean the pre-image of any subset of $\mathbb{R}$ (under a measurable function) is in the sigma-algebra of the Hausdorff measure. (Note function f on set A is unbounded when there is no $I \geq 0$ such that for all $x \in A$):*

$$|f(x)| \leq I$$

however, we're unsure if theorem 1 is correct. Despite this, we could prove or disprove theorem 1 using the paper on prevalence in [4].

We, therefore, define the expected value w.r.t the Hausdorff measure to be the following:

Definition 6 (**Expected Value of f**). *If $n \in \mathbb{N}$, where set $A \subseteq \mathbb{R}^n$, the expected value of function $f : A \to \mathbb{R}$ (using Definition 4 and 5) is*

$$\mathbb{E}[f] = \frac{1}{H^{\dim_H(A)}(A)} \int_A f \, dH^{\dim_H(A)}$$

where we can see there are cases where $\mathbb{E}[f]$ is undefined or infinite (e.g. $H^{\dim_H(A)}(A)$ is zero, $+\infty$ or f is unbounded). In this case, if topological vector space X is $\mathbb{R}^A$ (see Section 1.1) where we define B^* such that:

Definition 7 (**The set of all measurable functions**). *B^* is the set of all Borel measurable functions in $\mathbb{R}^A$.*

Then, we must prove:

Theorem 2. *The expected value $\mathbb{E}[f]$ is finite for all f in only a shy subset of B^*.*

Note 2 (**Note on Theorem 2**). *We're not sure how to prove theorem 2; however, we refer to an answer from @Mathe at the last page of this citation [7],*

"We can follow the argument presented in example 3.6 of [4]:

Because a function can always be represented as $f = f^+ - f^-$ we only consider whether positive functions have a mean value. We consider the case of a set A with finite positive measure. In this context having a mean means having a finite integral, and not being integrable means having an infinite integral.

Take $X := L^0(A)$ (measurable functions over A) let P denote the one-dimensional subspace of $L^0(A)$ consisting of constant functions (assuming the Hausdorff measure on A) and let $F := L^0(A) \setminus L^1(A)$ (measurable functions over A with no finite integral)

If λ_P denotes the Lebesgue measure over P, for any fixed $f \in F$

$$\lambda_P\left(\left\{\beta \in \mathbb{R} : \int_A (f+\beta)\mu < \infty\right\}\right) = 0$$

Meaning P is a 1-dimensional probe of F, so F is a 1-prevalent set. (In other terms, the set of measurable functions over A with no finite integral or mean, forms a prevalent subset of the set of all measurable functions in $\mathbb{R}^A$. Therefore, using Definition 2, the set of measurable functions with a finite integral or mean forms a shy subset of all Borel measurable functions in $\mathbb{R}^A$.)

1.2. Extended Expected Values

Four solutions to getting a finite expected value for "larger" subset of $\mathbb{R}^A$ is:

1. Defining a **dimension function**; i.e., $h : [0, +\infty) \to [0, +\infty]$, that's monotonically increasing, strictly positive and right continuous, such that when R denotes the radius of a ball in a covering for the definition of the Hausdorff Measure, we replace $R^{\dim_H(A)}$ with $h(R)$ so $H^h(A)$: the

h-**Hausdorff measure**, is positive and finite. This leads to the extended expected value $\mathbb{E}^{\star}[f]$, where:

$$\mathbb{E}^{\star}[f] = \frac{1}{H^h(A)} \int_A f \, dH^h$$

Note, however, not all A has dimension function h which leads to:

2. If A is fractal but has no gauge function, we could use this paper [8] which is an extension of the Lebesgue density theorem and this paper [9] which is an extension of the Hausdorff measure using Hyperbolic Cantor sets. Note, however, when A is non-fractal (e.g. countably infinite) or f is unbounded, there is a possibility that the expected value is infinite or undefined. Hence,
3. In the case f is unbounded and fractal, we could use [10] (p.19-47), which applies a Henstock-Kurzweil type integral (i.e., μ-HK integral) on a measure Metric Space. This coincides with unbounded functions with finite improper Riemman integrals, including bounded functions with finite Lebesgue integrals, bounded function with finite integrals w.r.t the Hausdorff measure, or function with finite Henstock-Kurzweil integrals.

1.3. Examples

If $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$, we want to apply the definitions of the next section for the following examples:

1. $A = \mathbb{R} \setminus \{0\}$ and $f(x) = 1/x$. This function is unbounded and has an undefined expected value since the average of $1/x$, using the improper Riemann integral on $\mathbb{R} \setminus \{0\}$:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)} \left(\int_{x_1}^{x_2} \frac{1}{x}\, dx + \int_{x_3}^{x_4} \frac{1}{x}\, dx \right) = \tag{1.3.1}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)} \left(\ln(|x|) + C\Big|_{x_1}^{x_2} + \ln(|x|) + C\Big|_{x_3}^{x_4} \right) = \tag{1.3.2}$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)} \left(\ln(|x_2|) - \ln(|x_1|) + \ln(|x_4|) - \ln(|x_3|) \right) \tag{1.3.3}$$

is $+\infty$ (when $x_2 = 1/x_1$, $x_3 = 1/x_4$, and $x_1 = \exp\left(x_4^2\right)$) or $-\infty$ (when $x_2 = 1/x_1$, $x_3 = 1/x_4$, and $x_4 = -\exp\left(x_1^2\right)$), making the average undefined.

2. $A = \mathbb{Q}$, gcd is the greatest common divisor, and $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ where:

$$f(x) = \begin{cases} f_1(x) & x \in A_1 := \{r/q : r \in \text{odd } \mathbb{Z}, q \in \text{even } \mathbb{Z}, q \neq 0, \gcd(r,q) = 1\} \\ f_2(x) & x \in A_2 := \{r_1/(q_1) : r_1 \in \mathbb{Z}, q_1 \in \text{odd } \mathbb{Z}, \gcd(r_1,q_1) = 1\} \end{cases} \tag{1.3.4}$$

For instance, point $(1/4, 1)$ is a point in the graph of f (since $1/4 \in \mathbb{Q}$ and $1/4 \in A_1$, making $f(1/4) = f_1(1/4)$). Also, point $(1/3, 0)$ is a point in the graph of f (since $1/3 \in \mathbb{Q}$ and $1/3 \in A_2$, making $f(1/3) = f_2(1/3)$); however, point $(\sqrt{2}, 1)$ is not in the graph of f (since $\sqrt{2} \notin \mathbb{Q}$).

Note the function in Equation (1.3.4) is bounded; however, the expected value & extensions are undefined. (Using Definition 6, we know $\dim_{\mathrm{H}}(A) = 0$ but $H^{\dim_H(A)}(A) = +\infty$, which makes $\mathbb{E}[f]$:

$$\mathbb{E}[f] = \frac{1}{H^{\dim_{\mathrm{H}}(A)}(A)} \int_A f \, dH^{\dim_{\mathrm{H}}(A)}$$

undefined by division of $+\infty$.) Further, we assume using Section 1.2, crit. 1, there is no (exact) dimension function of A nor could A be "fractal" enough for extensions of the Lebesgue Density Theorem [8], extensions of the Hausdorff measure using Hyperbolic Cantor Sets [9], or extension of the Henstock-Kurzweil integral on the Metric Space [10] (p.19-47).

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2. Attempt to Answer Thesis

Suppose for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$. Moreover, H^h is the h-Hausdorff measure (Section 1.2, crit. 1) where h is the dimension function, and B^* is the set of all Borel measurable functions in $\mathbb{R}^A$.

Definition 8 (**⋆-Sequence of Sets**). *If we define a sequence of sets $(F_r^\star)_{r\in\mathbb{N}}$, where h is the dimension function, then when:*

1. *The set theoretic limit of $(F_r^\star)_{r\in\mathbb{N}}$ is the graph of f (i.e., $(F_r^\star)_{r\in\mathbb{N}}$ **converges** to the graph of f) where*

$$\limsup_{r\to\infty} F_r^\star = \bigcap_{r\geq 1}\bigcup_{q\geq r} F_q^\star$$
$$\liminf_{r\to\infty} F_r^\star = \bigcup_{r\geq 1}\bigcap_{q\geq r} F_q^\star$$

   with the graph of f as:

$$\{(x_1,\cdots,x_n,f(x_1,\cdots,x_n)) : (x_1,\cdots,x_n) \in A\}$$

   the set-theoretic limit should be:

$$\limsup_{r\to\infty} F_r^\star = \liminf_{r\to\infty} F_r^\star = \{(x_1,\cdots,x_n,f(x_1,\cdots,x_n)) : (x_1,\cdots,x_n) \in A\}$$

2. *For all $r \in \mathbb{N}$, where H^h is the h-Hausdorff measure (Section 1.2, crit. 1),*

$$0 < H^h(F_r^\star) < +\infty$$

3. *we define sequence of functions $(f_r^\star)_{r\in\mathbb{N}}$ where $f_r^\star : dom(F_r^\star) \to range(F_r^\star)$ such that:*

$$\{(x_1,\cdots,x_n,f_r^\star(x_1,\cdots,x_n)) : (x_1,\cdots,x_n) \in dom(F_r^\star)\} = F_r^\star$$

*we have $(F_r^\star)$ is a **⋆-sequence of sets** or **starred-sequence of sets**.*

Example 1. *One ⋆-sequence of sets of $f(x) = 1/x$ on $\mathbb{R}\setminus\{0\}$ (Section 1.3, crit. 1) is:*

$$(F_r^\star)_{r\in\mathbb{N}} = (\{(x,1/x) : x \in [-r,-1/r] \cup [1/r,r]\})_{r\in\mathbb{N}}$$

Example 2. *Another example of a ⋆-sequence of sets of $f : \mathbb{Q} \to \mathbb{R}$ where:*

$$f(x) = \begin{cases} 1 & x \in A_1 := \{r/q : r \in odd\ \mathbb{Z}, q \in even\ \mathbb{N}, q \neq 0, \gcd(r,q) = 1\} \\ 0 & x \in A_2 := \{r_1/(q_1) : r_1 \in \mathbb{Z}, q_1 \in odd\ \mathbb{N}, \gcd(r_1,q_1) = 1\} \end{cases} \tag{2.0.1}$$

using (Section 1.3, crit. 2) is the following:

$$(F_r^\star)_{r\in\mathbb{N}} = ((x,f(x)) : x \in \{c/(r!) : -r\cdot r! \leq c \leq r\cdot r!\})_{r\in\mathbb{N}} \tag{2.0.2}$$

another example is:

$$(F_r^\star)_{r\in\mathbb{N}} = ((x,f(x)) : x \in \{c/d : d \leq r,\ -d\cdot r \leq c \leq d\cdot r\})_{r\in\mathbb{N}} \tag{2.0.3}$$

Note this leads to a new extension of the expected value where when there's at least one starred-sequence of sets where the extension is finite, the extension could be finite for all f in a non-shy subset of all Borel measurable functions in $\mathbb{R}^A$.

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Definition 9 (Generalized Expected Value). *If $(F_r^\star)_{r\in\mathbb{N}}$ is a $\star$-sequence of sets (Definition 8), the generalized expected value of f w.r.t $(F_r^\star)_{r\in\mathbb{N}}$ is $E^{**}[f, F_r^\star]$ (when it exists) where:*

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\frac{1}{H^h\left(dom\left(F_r^\star\right)\right)}\int_{dom(F_r^\star)} f_r^\star \, dH^h - \mathbb{E}^{**}[f, F_r^\star]\right| < \epsilon\right) \quad (2.0.4)$$

Example 3. *Using example 1, we find that when $(F_r^\star)_{r\in\mathbb{N}} = (\{(x, 1/x) : x \in [-r, -1/r] \cup [1/r, r]\})_{r\in\mathbb{N}}$:*

1. *$dom(F_r^\star) = ([-r, -1/r] \cup [1/r, r])_{r\in\mathbb{N}}$*
2. *$f_r(x) = 1/x$ for $x \in [-r, -1/r] \cup [1/r, r]$*

and the generalized expected value is:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4 - x_3) + (x_2 - x_1)}\left(\int_{x_1}^{x_2}\frac{1}{x}\,dx + \int_{x_3}^{x_4}\frac{1}{x}\,dx\right) = \quad (2.0.5)$$

$$\lim_{r\to\infty}\frac{1}{(r - 1/r) + (-1/r - (-r))}\left(\int_{-r}^{-1/r}\frac{1}{x}\,dx + \int_{1/r}^{r}\frac{1}{x}\,dx\right) = \quad (2.0.6)$$

$$\lim_{r\to\infty}\frac{1}{(r - 1/r) + (-1/r + r)}\left(\ln(|x|) + C\Big|_{-r}^{-1/r} + \ln(|x|) + C\Big|_{1/r}^{r}\right) = \quad (2.0.7)$$

$$\lim_{r\to\infty}\frac{1}{(r - 1/r) + (-1/r + r)}\left(\ln(|-r|) - \ln(|-1/r|) + \ln(|r|) - \ln(|1/r|)\right) = \quad (2.0.8)$$

$$\lim_{r\to\infty}\frac{1}{2r - 2/r}\cdot 4\ln(r) = \quad (2.0.9)$$

$$0 \quad (2.0.10)$$

We can see from example 1, the average was once undefined but now we've "chosen" a $\star$-sequence which gives a finite expected value.

2.1. Equivalent and Non-Equivalent $\star$-sequences of Sets

Suppose we define the following:

Definition 10 (Set V′). *Set V' is the set of all f, where the generalized expected value—w.r.t at least one starred sequence—exists.*

The following are definitions of equivelant and non-equivelant starred-sequences of sets:

Definition 11 (Non-Equivalent Starred-Sequences of Sets). *All starred-sequences of sets (in a set of $\star$-sequences of sets) are non-equivalent, if there exists an $f \in V'$ (Definition 10), where the generalized expected values of f (Definition 9) w.r.t each starred-sequence of sets has two or more different values (e.g., defined and undefined values are different).*

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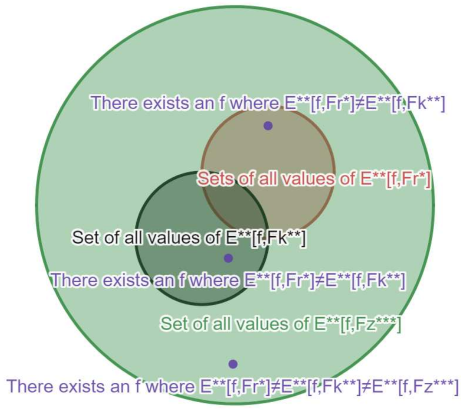

Figure 1. Below $F_r^\star$, $F_k^{\star\star}$, $F_z^{\star\star\star}$ are *non-equivalent* starred sequences of sets, where V' is all circles and E^{**} is the generalized expected value of f w.r.t either $\star$-sequence of sets (Definition 8)

Definition 12 (Equivalent Starred-Sequences of Sets). *All starred-sequences of sets (in the set of $\star$-sequences of sets) are equivalent, if we get for all $f \in V'$ (Definition 10); the generalized expected value of f (Definition 9) w.r.t each starred-sequence of sets has the same value.*

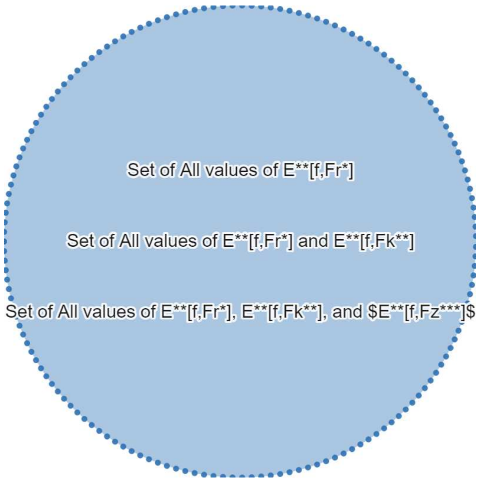

Figure 2. Below $F_r^\star$, $F_k^{\star\star}$, $F_z^{\star\star\star}$ are equivalent starred sequences of sets, where V' is the entire circle and E^{**} is the generalized expected value of f w.r.t either $\star$-sequence of sets (Definition 8)

However, proving that two or more starred-sequences of sets are non-equivalent or equivalent (using Definition 14 or 11) is tedious. Therefore, we ask the following:

2.1.1. Question 1

Is there are a simpler definition of equivalent and non-equivalent $\star$-sequences of sets.

2.1.2. Possible Answer

For the sake of brevity, suppose starred-sequences (Definition 8) $F_{r_1}^{(1)} = F_{r_1}^{\star}$, such that $F_{r_2}^{(2)} = F_{r_2}^{\star\star}$, $F_{r_3}^{(3)} = F_{r_3}^{\star\star\star}$, and $F_{r_s}^{(s)} = F_{r_s}^{\star\text{-s times}}$

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Definition 13 (Equivalent Starred-Sequences of Sets). *Starred-sequence of sets* $(F^{\star}_{r_1})_{r_1\in\mathbb{N}}$ *and* $(F^{\star\star}_{r_2})_{r_2\in\mathbb{N}}$ *are equivalent, if there exists a* $N'\in\mathbb{N}$*, where for all* $r_1\geq N'$*, there exists a* $r_2\in\mathbb{N}$*, where if* h_1 *is the dimension function (Section 1.2, crit. 1) of* $F^{\star}_{r_1}$*,*

$$H^{h_1}(F^{\star}_{r_1}\Delta F^{\star\star}_{r_2})=0$$

and also for all $r_2\geq N'$*, there exists a* $r_1\in\mathbb{N}$*, where if* h_2 *is the dimension function of* $F^{\star\star}_{r_2}$ *then:*

$$H^{h_2}(F^{\star}_{r_1}\Delta F^{\star\star}_{r_2})=0$$

Note we denote these equivalent starred-sequence of sets as

$$(F^{\star}_{r_1})_{r_1\in\mathbb{N}}\sim(F^{\star\star}_{r_2})_{r_2\in\mathbb{N}}$$

Definition 14 (Multiple Equivalent Starred-Sequences of Sets). *All starred-sequences of sets in:*

$$\left\{(F^{\star}_{r_1})_{r_1\in\mathbb{N}},(F^{\star\star}_{r_2})_{r_1\in\mathbb{N}},\cdots,(F^{(j)}_{r_j})_{r_1\in\mathbb{N}}\right\}$$

are equivalent, if for all $k,v\in\{1,\cdots,j\}$ *where* $k\neq v$*,* $(F^{(k)}_{r_k})_{r_k\in\mathbb{N}}$ *and* $(F^{(v)}_{r_v})_{r_v\in\mathbb{N}}$ *are equivelant (Definition 13). We also state the former as:*

$$(F^{(k)}_{r_k})_{r_k\in\mathbb{N}}\sim(F^{(v)}_{r_v})_{r_v\in\mathbb{N}}$$

Theorem 3. *If starred-sequences of sets in:*

$$\left\{(F^{\star}_{r_1})_{r_1\in\mathbb{N}},(F^{\star\star}_{r_2})_{r_1\in\mathbb{N}},\cdots,(F^{(j)}_{r_j})_{r_1\in\mathbb{N}}\right\}$$

are equivalent (Definition 14), then for all $k,v\in\{1,\cdots,j\}$ *where* $k\neq v$*, the generalized means of A w.r.t the* $\star$*-sequences (Definition 9) have the same mean value. In other words:*

$$\mathbb{E}^{**}[f,F^{(k)}_{r_k}]=\mathbb{E}^{**}[f,F^{(v)}_{r_v}]$$

Definition 15 (Non-Equivalent Starred-Sequences of Sets). *All starred-sequences of sets in* $\left\{(F^{(s)}_{r_s})_{r_s\in\mathbb{N}}:s\in\mathbb{N}\right\}$ *are non-equivalent, if Definition 14 is false.*

2.2. Motivation For Section 2.4

For all f in a non-shy subset of B^* (Definition 7), we may choose a $\star$-sequence of sets $(F^{\star}_{r})_{r\in\mathbb{N}}$ where the generalized expected value of f w.r.t least one starred-sequence is finite. However, consider the following problem:

Theorem 4. *The set of all* f*, where the generalized expected values of* f *w.r.t two or more non-equivalent* $\star$*-sequences of sets has different values, form a non-shy subset of all Borel measurable functions in* $\mathbb{R}^A$*.*

This means "almost all" measurable functions have *several* generalized expected values depending on the starred-sequence chosen. Therefore, we need to choose a unique $\star$-sequence of sets where the new extended expected value is an "natural" extension of the original expected value.

2.3. Essential Definitions for a "Natural" Expected Value

Suppose $(F^{\star}_{r})_{r\in\mathbb{N}}$ and $(F^{\star\star}_{j})_{j\in\mathbb{N}}$ are non-equivelant starred-sequences of sets (def. 8 & 11): we have the following is essential for a "natural" expected value.

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Definition 16 (**Linear & Super-linear Convergence of a ⋆-Sequence of Sets To That Of Another ⋆-Sequence of Sets).** *If we define function $S : \mathbb{R} \to \mathbb{R}$, where $r \in \mathbb{N}$ and for any linear $j_1 : \mathbb{N} \to \mathbb{N}$, where $j = j_1(r)$, $\mathcal{O}$ is the Big-O notation, and:*

$$H^h(F_r^{\star}) = \mathcal{O}(S(H^h(F_j^{\star\star})))$$

where if the following is true:

$$0 < \lim_{x \to \infty} S(x)/x$$

then $(F_r^{\star})_{r \in \mathbb{N}}$ converges to the graph of f: i.e.,

$$\{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$$

*at a **linear** or **super-linear** rate compared to that of $(F_j^{\star\star})_{j \in \mathbb{N}}$.*

Now we may combine the previous definitions into a main question with an answer that solves the thesis [2].

2.4. Main Question

Does there exist a choice function that chooses a unique set (of equivalent ⋆-sequences of sets—Definition 13) such that:

1. The chosen starred-sequences of sets converge to $\{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$ at a rate *linear* or *super-linear* (Definition 16) to the rate non-equivalent ⋆-sequences of sets (Definition 11) converge to $\{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$
2. The *generalized expected value* (Definition 9) of f w.r.t the chosen (and equivalent) starred-sequences of sets (Definition 13) is finite.
3. The choice function chooses a unique set of equivalent ⋆-sequences of sets which satisfy (1) and (2), for all $f \in Q$ such that Q is a non-shy subset (Definition 5) of B^* (i.e., the set of all Borel measurable functions in $\mathbb{R}^A$).
4. Out of all the choice functions which satisfy (1), (2) and (3), we choose the one with the *simplest form*, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers (excluding those with quantifiers)?

Note 3 (**Notes On Question).** *Note, the unique set of equivalent and chosen starred-sequences of sets is defined using notation $\sim (F_k^{\star\star\star})_{k \in \mathbb{N}}$, where $(F_k^{\star\star\star})_{k \in \mathbb{N}}$ is a starred-sequence in $\sim (F_k^{\star\star\star})_{k \in \mathbb{N}}$. Therefore, after we define the choice function, the answer should be $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$—using Definition 9 (when it exists):*

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N})\left(k \geq N \Rightarrow \left|\frac{1}{H^h\left(dom\left(F_k^{\star\star\star}\right)\right)}\int_{dom(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}]\right| < \epsilon\right) \tag{2.4.1}$$

Also, consider the following: if the solution to the main question is extraneous, what other criteria can be included to get a unique choice function? (Note if the solution is always extraneous, we want to replace "equivelant starred-sequences of sets" with the following: "the set of all ⋆-sequences of sets, where the generalized expected values of f w.r.t each starred-sequence is the same".)

---

[2] We want to find unique and "natural" extension of the expected value, w.r.t the Hausdorff measure, that takes finite values for all f in a non-shy subset of all Borel measurable functions in $\mathbb{R}^A$

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3. Solution To The Main Question Of Section 2.4

Suppose h is the dimension function, H^h is the h-Hausdorff measure (Section 1.2, crit. 1), and $(F_r^\star)_{r\in\mathbb{N}}$ is the starred-sequence of sets (Definition 8). We will use an alternative approach to definition 16 or Definition 20 so we can define a choice function which solves the main question. Read from the second sentence of the last paragraph of the intro of Section 0 for a summary. Also, refer to sec. 3 and 4 of [1] for examples: (the cited paper uses sets instead of functions).

3.1. Preliminary Definitions

Definition 17 (Uniform ε coverings of each term of a $\star$-sequence of sets). *We define uniform ε coverings of each term of $(F_r^\star)_{r\in\mathbb{N}}$ as a group of pair-wise disjoint sets which cover $F_r^\star$ (for some $r\in\mathbb{N}$), such when taking dimension function h of $F_r^\star$, we want H^h of each pair-wise disjoint set to have the same value $\varepsilon\in \text{range}(H^h)$, where $\varepsilon>0$ and the total sum of H^h of the coverings is minimized. In shorter notation, if*

- *The element $t\in\mathbb{N}$*
- *The set $T\supset\mathbb{N}$ is arbitrary and uncountable.*

and set Ω is defined as:

$$\Omega=\begin{cases}\{1,\cdots,t\} & \text{if there are } t \text{ ways of writing uniform } \varepsilon \text{ coverings of } F_r^\star\\ \mathbb{N} & \text{if there are countably infinite ways of writing uniform } \varepsilon \text{ coverings of } F_r^\star\\ T & \text{if there are uncountable ways of writing uniform } \varepsilon \text{ coverings of } F_r^\star\end{cases} \tag{3.1.1}$$

then for every $\omega\in\Omega$, the set of uniform ε coverings is defined using $\mathscr{U}(\varepsilon,F_r^\star,\omega)$ where ω "enumerates" all possible uniform ε coverings of $F_r^\star$ for every $r\in\mathbb{N}$.

Definition 18 (Sample of the uniform ε coverings of each term of a $\star$-sequence of sets). *The sample of uniform ε coverings of each term of $(F_r^\star)_{r\in\mathbb{N}}$ is the set of points where for every $\varepsilon\in\text{range}(H^h)$ and $r\in\mathbb{N}$, we take a point from each pair-wise disjoint set in the uniform ε coverings of $F_r^\star$ (Definition 17). In shorter notation, if*

- *The element $k\in\mathbb{N}$*
- *The set $\mathcal{K}\supset\mathbb{N}$ is arbitrary and uncountable.*

and set Ψ_ω is defined as:

$$\Psi_\omega=\begin{cases}\{1,\cdots,k\} & \text{if there are } k \text{ ways of writing the sample of uniform } \varepsilon \text{ coverings of } F_r^\star\\ \mathbb{N} & \text{if there are countably infinite ways of writing the sample of uniform } \varepsilon \text{ coverings of } F_r^\star\\ \mathcal{K} & \text{if there are uncountable ways of writing the sample of uniform } \varepsilon \text{ coverings of } F_r^\star\end{cases} \tag{3.1.2}$$

then for every $\psi\in\Psi_\omega$, the set of all samples of the set of uniform ε coverings is defined using $\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi)$, such that ψ "enumerates" all possible samples of $\mathscr{U}(\varepsilon,F_r^\star,\omega)$.

Definition 19 (Entropy on the sample of uniform coverings of each term of $\star$-sequence of sets). *Since there are finitely many points in the sample of the uniform ε coverings of each term of $(F_r^\star)_{r\in\mathbb{N}}$ (Definition 18), we:*

1. *Take a "pathway" of line segments between all points in each sample (Definition 18), such that if we define the following:*
   (a) *$\lceil\cdot\rceil$ is the ceiling function*
   (b) *$d(Q,R)$ is the Euclidean-distance between points $Q\in\mathbb{R}^n$ and $R\in\mathbb{R}^n$*
   (c) *The sequence:*

$$\{x_{i-1}\}_{i=1}^{\lceil H^h(F_r^\star)/\varepsilon\rceil-1}$$

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contains all points in the "original" sample $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)$ where we define a "pathway" for which we:

i. *Choose a point $x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)$*
ii. *Take a point from $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)$ (excluding x_0) with smallest euclidean distance from point $x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)$. We denote this point x_1 where we take $d(x_0, x_1)$. (If more than one point has the smallest Euclidean distance from x_0, we take either point).*
iii. *Take a point in $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)$ (excluding x_0 and x_1) with smallest euclidean distance from x_1. We denote this point x_2, where we take $d(x_1, x_2)$. (If more than one point has the smallest Euclidean distance from x_1, we take either point).*
iv. *Take a point in $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)$ (excluding x_0, x_1, and x_2) with smallest euclidean distance from x_2. We denote this point x_3 then take $d(x_2, x_3)$. (If more than one point has the smallest Euclidean distance from x_2, we take either point).*
v. *Repeat the process excluding points x_0, x_1, x_2, x_3, etc. until all points in the sample are "denoted". (This should occur $\lceil H^h(F_r^{\star})/\varepsilon \rceil - 1$ times.)*

(d) ***V** is a subset of $\left\{ i \in \mathbb{N} : 1 \le i \le \lceil H^h(F_r^{\star})/\varepsilon \rceil - 1 \right\}$ with the largest cardinality, where that we take the subset of i-values where x_i has the r_i-th smallest Euclidean distance from x_{i-1} (compared to every point in $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi) \setminus \{x_{i-1}\}$) such that r_i is not an outlier [11] of*

$$\left\{ r_t : t \in \mathbb{N}, 1 \le t \le \lceil H^h(F_r^{\star})/\varepsilon \rceil - 1 \right\}$$

In other words:

i. *For all $w \in \mathbf{V}$, we want $\mathbf{V}$ to be the largest subset of $\left\{ i \in \mathbb{N} : 1 \le i \le \lceil H^h(F_r^{*})/\varepsilon \rceil - 1 \right\}$ for which w-values are all i-values satisfying criteria 1d.*
ii. *Combining everything in (1), we ultimately want all lengths between every point in the "pathway" (Definition 18) satisfying crit. 1d. We call this:*

$$\mathcal{D}\left(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)\right) = \{d(x_w, x_{w-1}) : w \in \mathbf{V}\}$$

iii. *Using Definition 19, crit. 1(d)ii, normalize $\mathcal{D}$ into a discrete probability distribution. This is defined as:*

$$\mathbb{P}\left(\mathcal{D}\left(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)\right)\right) = \left\{ y \Big/ \left(\sum_{z \in \mathcal{D}(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi))} z \right) : y \in \mathcal{D}\left(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\epsilon, F_r^{\star}, \omega), \psi)\right) \right\} \tag{3.1.3}$$

iv. *Take the entropy of (2), (for further reading, see [6] (p.61-95)). This is defined as:*

$$E\left(\mathcal{D}\left(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)\right)\right) = -\sum_{x \in \mathbb{P}(\mathcal{D}(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)))} x \log_2 x \tag{3.1.4}$$

v. *Take $x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi))$ where $E\left(\mathcal{D}\left(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)\right)\right)$ is maximized. Call this, $E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)))$ where:*

$$E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi))) = \sup_{x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)} E\left(\mathcal{D}\left(x_0, \{x_{w-1}\}_{w \in \mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)\right)\right) \tag{3.1.5}$$

with Equation (3.1.5) the entropy of the sample of uniform ε coverings of $F_r^{\star}$.

Definition 20 (Starred-Sequence of sets converging Sublinearly, Linearly, or Superlinearly to A compared to that of another ⋆-Sequence). *Suppose we define starred-sequences of sets $(F_r^{\star})_{r \in \mathbb{N}}$ and $(F_j^{\star\star})_{j \in \mathbb{N}}$, where for a **constant** $\varepsilon \in range(H^h)$ greater than zero and variable $r \in \mathbb{N}$, we say:*

(a) *Using Definition 18 and 19, suppose we have:*

$$\underline{|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)|} = \tag{3.1.6}$$
$$\sup\left\{\left|\mathcal{S}(\mathscr{U}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right| : j\in\mathbb{N},\ \omega'\in\Omega,\ \psi'\in\Psi_\omega,\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon,F_j^{\star\star},\omega'),\psi'))) \le E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)))\right\}$$

then (using $\underline{|\mathcal{S}(\mathscr{U}(\epsilon,F_r^{\star},\omega),\psi)|}$) we get

$$\overline{\alpha}(\varepsilon,r,\omega,\psi) = |\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi))| \,/\, \sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)|} \tag{3.1.7}$$

(b) *From Definition 18 and 19, suppose we have:*

$$\overline{|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)|} = \tag{3.1.8}$$
$$\inf\left\{\left|\mathcal{S}(\mathscr{U}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right| : j\in\mathbb{N},\ \omega'\in\Omega,\ \psi'\in\Psi_\omega,\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon,F_j^{\star\star},\omega'),\psi'))) \ge E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)))\right\}$$

then (using $\overline{|\mathcal{S}(\mathscr{U}(\epsilon,F_r^{\star},\omega),\psi)|}$) we have:

$$\underline{\alpha}(\varepsilon,r,\omega,\psi) = |\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)| \,/\, \sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^{\star},\omega),\psi)|} \tag{3.1.9}$$

1. *If using $\overline{\alpha}(\epsilon,r,\omega,\psi)$ and $\underline{\alpha}(\epsilon,r,\omega,\psi)$ we have that:*

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\overline{\alpha}(\varepsilon,r,\omega,\psi) = \inf_{\omega\in\Omega}\inf_{\psi\in\Psi_\omega}\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\underline{\alpha}(\varepsilon,r,\omega,\psi) = 0$$

we say $(F_r^{\star})_{r\in\mathbb{N}}$ converges to A at a rate ***superlinear*** *to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

2. *If using equations $\overline{\alpha}(\varepsilon,j,\omega,\psi)$ and $\underline{\alpha}(\varepsilon,j,\omega,\psi)$ (where we swap $(F_r^{\star})_{r\in\mathbb{N}}$ in $\overline{\alpha}(\epsilon,r,\omega,\psi)$ and $\underline{\alpha}(\epsilon,r,\omega,\psi)$ with $(F_j^{\star\star})_{j\in\mathbb{N}}$) we have that:*

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\limsup_{\varepsilon\to 0}\limsup_{j\to\infty}\overline{\alpha}(\varepsilon,j,\omega,\psi) = \inf_{\omega\in\Omega}\inf_{\psi\in\Psi_\omega}\liminf_{\varepsilon\to 0}\liminf_{j\to\infty}\underline{\alpha}(\varepsilon,j,\omega,\psi) = 0$$

we then say $(F_r^{\star})_{r\in\mathbb{N}}$ converges to A at a rate ***sublinear*** *to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

3. *If using equations $\overline{\alpha}(\varepsilon,r,\omega,\psi)$, $\underline{\alpha}(\varepsilon,r,\omega,\psi)$, $\overline{\alpha}(\varepsilon,j,\omega,\psi)$, and $\underline{\alpha}(\varepsilon,j,\omega,\psi)$ (such for the two latter, we swap $(F_r^{\star})_{r\in\mathbb{N}}$ in $\overline{\alpha}(\varepsilon,r,\omega,\psi)$ and $\underline{\alpha}(\varepsilon,r,\omega,\psi)$ with $(F_j^{\star\star})_{j\in\mathbb{N}}$) we have* ***both****:*

   (a) $\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\overline{\alpha}(\varepsilon,r,\omega,\psi)$ *or* $\inf_{\omega\in\Omega}\inf_{\psi\in\Psi_\omega}\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\underline{\alpha}(\varepsilon,r,\omega,\psi)$ *does not equal zero*

   (b) $\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\limsup_{\varepsilon\to 0}\limsup_{j\to\infty}\overline{\alpha}(\varepsilon,j,\omega,\psi)$ *or* $\inf_{\omega\in\Omega}\inf_{\psi\in\Psi_\omega}\liminf_{\varepsilon\to 0}\liminf_{j\to\infty}\underline{\alpha}(\varepsilon,j,\omega,\psi)$ *does not equal zero*

and say $(F_r^{\star})_{r\in\mathbb{N}}$ converges to A at a rate ***linear*** *to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

4. Attempt to Answer Main Question Of Section 2.4

4.1. Choice Function

Suppose we define the following:

1. $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ is a starred-sequence of sets (Definition 8) which satisfies (1), (2), and (3) of the main question in Section 2.4
2. $\mathbb{S}'(G)$, where G is the graph of f; i.e.,

$$G = \{(x_1,\cdots,x_n,f(x_1,\cdots,x_n)) : (x_1,\cdots,x_n)\in A\}$$

is the set of the starred-sequences of sets that have finite *generalized mean* (Definition 9).

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3. $(F_j^{\star\star})_{j\in\mathbb{N}}$ is an element $\mathbb{S}'(G)$ but **not** an element in the set of equivalent starred-sequences of sets (Definition 14) of $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ where using note 3, we can represent this criteria as:

$$(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(G) \setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}} \tag{4.1.1}$$

Further note, from Definition 20, if we take:

$$\overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \tag{4.1.2}$$
$$\inf\left\{|\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_{\omega},\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \geq E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)))\right\}$$

and from Definition 20, we take:

$$\underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \tag{4.1.3}$$
$$\sup\left\{|\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_{\omega},\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \leq E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)))\right\}$$

Then, when we write Definition 18, Equation (4.1.2) and Equation (4.1.3) as:

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega} |\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)| = |\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| = |\mathcal{S}'| \tag{4.1.4}$$

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega} \overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \overline{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|} = \overline{|\mathcal{S}'|} \tag{4.1.5}$$

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega} \underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \underline{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|} = \underline{|\mathcal{S}'|} \tag{4.1.6}$$

the choice function (which we'll later define on pg. 15, thm. 5) should immediately choose $F_k^{\star\star\star}$ when:

1. For all $m \in \{1, \cdots, n\}$ when defining the set of all values of the m-th coordinate of $(c_1, c_2, \cdots, c_n) \in F_k^{\star\star\star}$ (i.e., $F_{k,m}^{\star\star\star}$—where, unlike cit. [1] (§4), we focus on the domain of $F_k^{\star\star\star}$ to get "$n-1$" instead of "n"), then when $z > 0$, we either want:
   (a) $\sup(F_{k+1,m}^{\star\star\star}) - \sup(F_{k,m}^{\star\star\star}) = z$ and $\inf(F_{k+1,m}^{\star\star\star}) - \inf(F_{k,m}^{\star\star\star}) = -z$.
   (b) $\sup(F_{k+1,m}^{\star\star\star}) - \sup(F_{k,m}^{\star\star\star}) = 0$ and $\inf(F_{k+1,m}^{\star\star\star}) - \inf(F_{k,m}^{\star\star\star}) = -z$.
   (c) $\sup(F_{k+1,m}^{\star\star\star}) - \sup(F_{k,m}^{\star\star\star}) = z$ and $\inf(F_{k+1,m}^{\star\star\star}) - \inf(F_{k,m}^{\star\star\star}) = 0$.
   (d) $\sup(F_{k+1,m}^{\star\star\star}) - \sup(F_{k,m}^{\star\star\star}) = 0$ and $\inf(F_{k+1,m}^{\star\star\star}) - \inf(F_{k,m}^{\star\star\star}) = 0$.
2. If the center of the universe is a chosen point $Z \in \mathbb{R}^n$, where:

$$Z = (z_1, z_2, \cdots, z_n) \tag{4.1.7}$$

then for all $m \in \{1, \cdots, n\}$, there exists $q \in \mathbb{N}$, s.t. for all $k \geq q$, when set $F_{k,m}^{\star\star\star}$ is a collection of all the values of the m-th co-ordinate of $(c_1, c_2, \cdots, c_n) \in F_k^{\star\star\star}$, such that $x_1 \in F_{k,m}^{\star\star\star}$ (again, unlike cit. [1] (§4), we focus on the domain of $F_k^{\star\star\star}$ to get "$n-1$" instead of "n"), we must get:

$$\frac{1}{H^h(F_{k,m}^{\star\star\star})}\int_{F_{k,m}^{\star\star\star}} x_1\, dH^h = z_m \tag{4.1.8}$$

where, using absolute value function $||\cdot||$ and $m \in \{1, 2, \cdots, n\}$, when set $F_{k,m}^{\star\star\star}$ is a collection of all the values of the m-th co-ordinate of $(c_1, c_2, \cdots, c_n) \in F_k^{\star\star\star}$, for $z > 0$, when we define:

$$S(z,k,m) = \Big|\Big|z - \left(\sup\left(F_{k+1,m}^{\star\star\star}\right) - \sup\left(F_{k,m}^{\star\star\star}\right)\right)\left(\inf\left(F_{k,m}^{\star\star\star}\right) - \inf\left(F_{k+1,m}^{\star\star\star}\right)\right) \tag{4.1.9}$$
$$\Big|\Big|\left(\inf\left(F_{k,m}^{\star\star\star}\right) - \inf\left(F_{k+1,m}^{\star\star\star}\right)\right)\left(\sup\left(F_{k+1,m}^{\star\star\star}\right) - \sup\left(F_{k,m}^{\star\star\star}\right) - 1\right)\Big|\Big|\Big|\Big|$$

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and

$$T(z_m,k,m) = \left[\left(\sup\left(F^{\star\star\star}_{k+1,m}\right) - z_m\right)\left(\inf\left(F^{\star\star\star}_{k,m}\right) - z_m\right) - \left(\sup\left(F^{\star\star\star}_{k,m}\right) - z_m\right)\left(\inf\left(F^{\star\star\star}_{k+1,m}\right) - z_m\right)\right]$$
$$\left[\left(\inf\left(F^{\star\star\star}_{k,m}\right) - z_m\right) - \left(\inf\left(F^{\star\star\star}_{k+1,m}\right) - z_m\right) + \left(\sup\left(F^{\star\star\star}_{k+1,m}\right) - z_m\right) - \left(\sup\left(F^{\star\star\star}_{k,m}\right) - z_m\right) - 1\right]$$
$$\left[\left(\inf\left(F^{\star\star\star}_{k,m}\right) - z_m\right) - \left(\inf\left(F^{\star\star\star}_{k+1,m}\right) - z_m\right)\right]\left[\left(\sup\left(F^{\star\star\star}_{k+1,m}\right) - z_m\right) - \left(\sup\left(F^{\star\star\star}_{k,m}\right) - z_m\right)\right] \tag{4.1.10}$$

criteria (1) is achieved, using Equation (4.1.9), when:

$$S'(z,k) = \frac{1}{n-1}\sum_{m=1}^{n-1} S(z,k,m) \tag{4.1.11}$$

such that, for all $k \in \mathbb{N}$:

$$S'(z,k) = 1 \tag{4.1.12}$$

and criteria (2) is achieved, using Equation (4.1.7) and 4.1.10, when:

$$T'(Z,k) = \frac{1}{n-1}\sum_{m=1}^{n-1} T(z_m,k,m) \tag{4.1.13}$$

such that, for all $k \in \mathbb{N}$:

$$T'(Z,k) = 0 \tag{4.1.14}$$

where we consider the following:

4.2. Question:

How do we create a choice function which solves the question in sec. 2.4 using $\mathcal{S}'$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, $S'(z,k)$, and $T'(Z,k)$ or equations 4.1.4, 4.1.5, 4.1.6, 4.1.11 and 4.1.13 resp.?

4.3. "Attempt" to answer the Question

(Note the attempt might be wrong but could offer hints to how the solution would appear).

Suppose $z = 1$ and the chosen coordinate for the center of the universe (i.e., Equation (4.1.7)) is the origin, where $z_m = 0$ for all $m \in \{1, \cdots, n\}$:

$$Z = (z_1, z_2, \cdots, z_n) \Rightarrow$$
$$Z = O = (\underbrace{0, 0, \cdots, 0}_{n \text{ times}}) \tag{4.3.1}$$

Using equations $\mathcal{S}'$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, $S'(z,k)$, and $T'(Z,k)$ (i.e., Equation (4.1.4), 4.1.5, 4.1.6, 4.1.11 and 4.1.13) with the absolute value function $||\cdot||$ and the nearest integer function $[\cdot]$, we define:

$$K(\varepsilon, F^{\star\star\star}_k) =$$
$$S'(1,k)\left(\left\|\frac{|\mathcal{S}'|\left(1+\left[\frac{|\mathcal{S}'|\left(\underline{|\mathcal{S}'|}+2|\mathcal{S}'|\right)}{\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|\right)\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|+\overline{|\mathcal{S}'|}\right)}\right]\right)\left(1+\left[\underline{|\mathcal{S}'|}/|\mathcal{S}'|\right]\right)}{\left(1+\left[|\mathcal{S}'|/\overline{|\mathcal{S}'|}\right]\right)\left(1+\left[\underline{|\mathcal{S}'|}/\overline{|\mathcal{S}'|}\right]\right)} - |\mathcal{S}'|\right\| + |\mathcal{S}'|\right) - T'(O,k) \tag{4.3.2}$$

where using $K(\varepsilon, F^{\star\star\star}_k)$, the choice function should be the following:

Theorem 5. *If we define:*

$$\mathcal{M}(\varepsilon, F^{\star\star\star}_k) = |\mathcal{S}'(\varepsilon, F^{\star\star\star}_k)|\left(K(\varepsilon, F^{\star\star\star}_k) - |\mathcal{S}'(\varepsilon, F^{\star\star\star}_k)|\right)$$

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$$\mathcal{M}(\varepsilon, F_j^{\star\star}) = |\mathcal{S}'(\varepsilon, F_j^{\star\star})|\left(K(\varepsilon, F_j^{\star\star}) - |\mathcal{S}'(\varepsilon, F_j^{\star\star})|\right)$$

where for $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$*, we define* $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$ *to be the same as* $\mathcal{M}(\varepsilon, F_j^{\star\star})$ *when swapping "*$j \in \mathbb{N}$*" with "*$k \in \mathbb{N}$*" (for Equation (4.1.5) & 4.1.6) and sets* $F_k^{\star\star\star}$ *with* $F_j^{\star\star}$ *(for Equation (4.1.4)–4.3.2), then for constant* $v > 0$ *and variable* $v^* > 0$*, if:*

$$\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) = \inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \geq \mathcal{M}(\varepsilon, F_k^{\star\star\star}) \geq v^*\right\} \cup \{v^*\}\right) + v \qquad (4.3.3)$$

and:

$$\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) = \sup\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, F_j^{\star\star}) \leq \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\} \cup \{-v^*\}\right) + v \qquad (4.3.4)$$

then for all $(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(G) \setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$ *(Section 4.1, crit. 3), if:*

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \lim_{v^*\to\infty} \lim_{k\to\infty} \frac{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v}{\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})} = \\ &\limsup_{\varepsilon\to 0} \lim_{v^*\to\infty} \lim_{k\to\infty} \frac{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v}{\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})} = 0 \end{aligned} \qquad (4.3.5)$$

we choose $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ *satisfying Equation (4.3.5). (Note, we want* $\sup\emptyset = -\infty$, $\inf\emptyset = +\infty$*, and* $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ *to answer the main question of Section 2.4) where the answer to the focus*[3] *is* $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ *in Equation (4.3.6)—using Definition 9 (when it exists):*

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N})\left(k \geq N \Rightarrow \left|\frac{1}{H^h\left(\text{dom}\left(F_k^{\star\star\star}\right)\right)}\int_{\text{dom}(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}]\right| < \epsilon\right) \qquad (4.3.6)$$

Note 4 (Explanation of Theorem 5). *The theorem 5 is similar to the methods used in Definition 20 crit. 0a and 0b or* $\overline{\alpha}(\varepsilon, r, \omega, \psi)$ *and* $\underline{\alpha}(\varepsilon, r, \omega, \psi)$ *and Definition 20 crit. 1, where:*

$$\sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \limsup_{\varepsilon\to 0} \limsup_{r\to\infty} \overline{\alpha}(\varepsilon, r, \omega, \psi) = \inf_{\omega\in\Omega} \inf_{\psi\in\Psi_\omega} \liminf_{\varepsilon\to 0} \liminf_{r\to\infty} \underline{\alpha}(\varepsilon, r, \omega, \psi) = 0$$

such that we replace:

$$\begin{aligned} &E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi))) \mapsto \mathcal{M}(\varepsilon, F_k^{\star\star\star}) \\ &E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega), \psi))) \mapsto \mathcal{M}(\varepsilon, F_j^{\star\star}) \\ &|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi))| \mapsto |\mathcal{S}'(\varepsilon, F_j^{\star\star})| \\ &\sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)|} \mapsto \underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) \\ &\sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^{\star}, \omega), \psi)|} \mapsto \overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) \end{aligned}$$

note the changes to Definition 20, crit. 1 were made, so $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$ *is "large enough" compared to* $\mathcal{M}(\varepsilon, F_j^{\star\star})$*, with* $(F_j^{\star\star})_{j\in\mathbb{N}}$ *non-equivalent to* $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ *(e.g. when* $A = \mathbb{Q}$*,* $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ *should be* $(\{c/k! : c \in \mathbb{N}, 1 \leq c \leq k!\})_{k\in\mathbb{N}}$ *and never give* $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$ *smaller than "small"* $\mathcal{M}(\varepsilon, F_j^{\star\star})$*, e.g.:*

$$(F_j^{\star\star})_{j\in\mathbb{N}} = (\{u/w : u \in \mathbb{Z}, w \in \mathbb{N}, w \leq j, -w \cdot j \leq u \leq w \cdot j\})_{j\in\mathbb{N}}$$

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[3] We want to find an unique and "natural" extension of the expected value, w.r.t the Hausdorff measure, that takes finite values for all f in a non-shy subset of all Borel measurable functions in $\mathbb{R}^A$

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or larger than "large" $\mathcal{M}(\varepsilon, F_j^{\star\star})$; e.g., $(F_j^{\star\star})_{j\in\mathbb{N}} = (\{u_1/(6(j!)) : u_1 \in \mathbb{Z}, -6j \cdot j! \le u_1 \le 6j \cdot j!\})_{j\in\mathbb{N}}$)

Moreover, in $\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})$ and $\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})$ of thm. 5, we add constant $v > 0$ and variable $v^* > 0$ so if either

1. *$\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) - v = 0$ (i.e., using a related limit to Equation (4.3.5), division by zero is undefined).*
2. *$\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) - v = 0$ (i.e., using a related limit to Equation (4.3.5), division by zero is undefined).*
3. *$\inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \ge \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\}\right) = +\infty$ (i.e., similar to $\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})$ of Equation (4.3.3), with no variable v^* such that $\mathcal{M}(\varepsilon, F_k^{\star\star\star}) = 0$ and $\exists(J > 0)\forall(j_1 > 0)\exists(j \ge j_1)(\mathcal{M}(\varepsilon, F_j^{\star\star}) \le J)$, where we apply a related limit to Equation (4.3.5) that's undefined due to division by infinity.)*
4. *$\inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \ge \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\}\right) = \emptyset$ (i.e., similar to $\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})$ of Equation (4.3.3), with no variable v^* and $\mathcal{M}(\varepsilon, F_j^{\star\star}) = 0$, where we apply a related limit to Equation (4.3.5) that's undefined since $\inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \ge \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\}\right)$ is an undefined empty set.)*
5. *$\sup\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \le \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\}\right) = +\infty$ (i.e., similar to $\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})$ of Equation (4.3.4), with no variable v^* and $\mathcal{M}(\varepsilon, F_k^{\star\star}) = 0$, where we apply a related limit to Equation (4.3.5) that's undefined due to division by infinity.)*
6. *$\sup\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \le \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\}\right) = \emptyset$ (i.e., similar to $\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})$ of Equation (4.3.3), with no variable v^* and $\mathcal{M}(\varepsilon, F_k^{\star\star}) = 0$, where we apply a related limit to Equation (4.3.5) that's undefined since $\inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \ge \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\}\right)$ is an undefined empty set.)*
7. *$|\{z : j, z \in \mathbb{N}, \mathcal{M}(\varepsilon, F_{j+z}^{\star\star}) \le \mathcal{M}(\varepsilon, F_j^{\star\star})\}| = +\infty$ (i.e., infinite number succeeding F_j are smaller than original F_j, where such F_j should be eliminated).*

the limit in Equation (4.3.5) still exists.

4.4. Question:

How do we use mathematica code to illustrate Sections 3 and 4?

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