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Stochastic Ordering Results on Implied Lifetime Distributions under a Specific Degradation Model

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Abstract: In this paper, a novel strategy is adopted in a degradation model to affect the implied lifetime distribution. The multiplicative degradation model is utilized as a postulate in the model. It will be established that the implied lifetime distribution forms a classical mixture model. In this mixture model, time-to-failure is lying with some probabilities between two first passage times of the degradation process to reach two specified levels. Stochastic comparisons in the model under a change in the probabilities are studied. Several examples are provided to highlight the applicability of the results in the cases when typical degradation models are candidate

Keywords: multiplicative degradation model; time-to-failure model; hazard rate; majorization.

MSC: 60E05; 62B10; 62N05; 94A17

1. Introduction

Reliability modelling and analysis of complex systems have been always an important topic in engineering sciences. Degradation-based modelling of failure time , as a fundamental process, has been a consistent method for analyzing the lifetime of complex systems in quite many practical situations (see, e.g., Nikulin et al. (2010), Pham (2011) Pellettier et al. (2017) and Chen et al. (2017) for a monograph on this topic). The items that are deteriorating with time having an observable process of the degradation can be entertained by a stochastic degradation model. To attain and produce high reliability systems as requested by the majority of consumers it is necessary to detect weaker systems. The association of failure time and the degradation process may not be deterministic and further investigation for the purpose of obtaining the distribution of levels of degradation and their impact on the failure time is warranted.

The stochastic process-deriven degradation model according to Albabtain et al. (2020) is considered to model the lifetime of a system. It is assumed that the stochastic process fluctuates around monotone sample paths. In the traditional definition it is assumed that the failure of an item corresponds with the time when degradation exceeds the predetermined threshold level $\mathfrak{D}_{\mathfrak{f}}$. Suppose that the degradation process is $\{W(t), t \geq 0\}, W(0) = 0$ with a monotonically increasing sample path as is frequently encountered in practice. The time-to-failure is denoted by T. Then T is the first passage time to the threshold $\mathfrak{D}_{\mathfrak{f}}$ i.e., $T = \inf\{t : W(t) > \mathfrak{D}_{\mathfrak{f}}\}$. The corresponding distribution function of the failures is denoted by F_T and the implied survival function is denoted by $\bar{F}_T = 1 - F_T$. We also denote by $F_{W(t)}$ and $f_{W(t)}$ the distribution and density functions of $W(t)$, respectively. We have

$$\bar{F}_T(t) = P(T > t) = P(W(t) \leq \mathfrak{D}_{\mathfrak{f}}) = F_{W(t)}(\mathfrak{D}_{\mathfrak{f}}). \tag{1}$$

If $\{W(t), t \geq 0\}, W(0) = 0$ possesses a monotonically decreasing sample path then the time-to-failure T is the first passage time to the threshold $\mathfrak{U}_{\mathfrak{f}}$ i.e., $T = \inf\{t : W(t) \leq \mathfrak{U}_{\mathfrak{f}}\}$. We obtain

$$\bar{F}_T(t) = P(T > t) = P(W(t) > \mathfrak{U}_{\mathfrak{f}}) = \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}}). \tag{2}$$

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Degradation models vary markedly across the fields of reliability modeling. In this section, the dynamic degradation-based model for analysis of failure time data which has been recently introduced by Albabtain et al. (2020) is reviewed. The methodology behind their model is applied in situations where a unit exhibits stochastic behaviour along the time during which the degradation is taking place and there is no certain value for the degradation amount upon which the unit is failed. The pliable aspect of the dynamic degradation-based failure time model is revealed when the failure of the unit under the degradation process is assumed to follow a stochastic rule in contrast to the traditional definition that the failure of the unit is considered to be deterministic once the degradation amount reaches a pre-determined threshold.

Suppose that the amount of degradation at time t is denoted by $W(t)$ with pdf $f_{W(t)}(\cdot)$ and cdf $F_{W(t)}(\cdot)$. It is considered as a postulate for increasing (decreasing) degradation paths that $W(t_1) \leq_{st} (\geq_{st}) W(t_2)$ for all $t_1 \leq t_2$. In the pervious literature, it was assumed that for a given threshold value D_f a system under degradation fails as soon as $W(t) > D_f$. This defines a stopping rule for T to be determined so that $T \equiv \inf\{t \geq 0 \mid W(t) > D_f\}$. This definition of the failure time was developed by Albabtain et al. (2020) so that an existing stochastic rule about the effect of degradation over time illustrates the process of the failure of the item. The failure time T under this modified setting has the sf

$$\bar{F}_T(t) = \int_0^\infty S(w;t) f_{W(t)}(w)\, dw = E(S(W(t);t)), \tag{3}$$

where $S(w;t)$ is limit of a conditional probability given, at the level of degradation w, by

$$S(w;t) = \lim_{\delta \to 0+} P(T > t \mid W(t) \in (w, w+\delta]).$$

To fulfill the degradation model, for an increasing (resp. decreasing) degradation path the bivariate function S must satisfy the following conditions:

(i) For all $w \geq 0$ and for all $t \geq 0$, $S(w;t) \in [0,1]$.
(ii) For any fixed $w \geq 0$, $S(w;t)$ is decreasing in $t \geq 0$.
(iii) For any fixed $t \geq 0$, $S(w;t)$ is decreasing (resp. increasing) in $w \geq 0$.

The conditions (i)-(iii) guarantee that $\bar{F}_T$ in (2.1) stands as a valid survival function. The model (2.1) is a dynamic failure-time model because the construction of the model is modified depending on how the survival rate of the item under degradation at a certain point of time may be influenced by the amount of the degradation. This influence is entertained by the formation of the function S.

The selection of S depends firstly on the knowledge of engineer or operator who controls the performance of the system. For example, when system degrades with time hardly (severe) then $S(w;t) = \exp\{-\gamma(w)t\}$ may be a proper choice. For a less severe degradation process, $S(w;t) = \frac{1}{(1+t)^{\gamma(w)}}$ may be more appropriate. However, if there is no information how the system degrades with time then everything depends on failure time data (observations on T) and a model selection strategy can be accomplished, i.e., make some choices candidate and select the best one of them using some possible model selection criterions such as AIC, BIC, ... measures.

It is assumed that data on $W(t)$ are not achievable for all $t \geq 0$ as usually the stochastic process $\{W(t), t \geq 0\}$ is partially observed referred to well-known sources of degradation models. To move forward along the line of reputable statistical survival models a common feature for $S(\cdot;t)$ can be adopted so that $S(w;t) = S_0^{\gamma(w)}(t)$ which is the characteristic of the proportional hazard rate model whenever $S(w;t)$ is a survival function in t for any $w \geq 0$, in which $\gamma > 0$. The function γ may depend on some parameters. The baseline probability (survival rate)

$$S_0(t) = \lim_{\delta \to 0+} P(T > t \mid W(t) \in (0, \delta])$$

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measures the probability of survival of the system at the time t at which the amount of degradation is zero. In the sequel, we may need to suppose that $S_0(t)$ is itself a survival function in $t \geq 0$. For $\lambda_0 > 0$, the exponential distribution may always be a good choice so that $S_0(t) = \exp(-\lambda_0 t)$ describes a no-ageing behaviour of the system under degradation. The Lomax distribution with survival function given by $S_0(t) = \frac{1}{1+\lambda_0 t}$ is also another choice for the baseline survival rate.

2. Stepwise survival rate at interval degradation levels

In proceeding literature the correspondence between the randomness in degradation and randomness in the implied lifetime distribution has been assumed to be strong and direct so that failure occurs when the test item's degradation level reaches at a pre-determined threshold value (D_f). In such a case the resulting lifetime distribution follows from (3) if $S(w;t) = 1$ for $w < D_f$ and $S(w;t) = 0$ for $w > D_f$. However, the equation (2.1) stands valid as an sf of time to failure of an item under degradation when $0 < S(w;t) < 1$ in some time t and degradation w. The model (3) add the possibility to undertake situations where the deterioration of an item is not only due to the degradation. In real problems, as the time is elapsed and even the amount of degradation is not altered, the item under degradation is being ageing likewise. Therefore, the lifetime of a device subject to degradation may decrease as the degree of degradation increases. Therefore, the device is frailer at relatively high degradation levels so that a given threshold for the degree of degradation can easily be considered a deterministic rule for device failure. However, intervals of degradation levels can be set to develop a more dynamic time-to-failure degradation model.

Let us consider a degradation process with increasing degradation path and assume that $s_1 \geq s_2 \geq \ldots \geq s_k$ where $s_i \in [0,1]$ for $i = 1,2,\ldots,k$ are survival rates of a unit subject to degradation when $W(t) = w$, respectively, as the value w takes, lies in

$$(\mathfrak{D}_{\mathfrak{f}(0)}, \mathfrak{D}_{\mathfrak{f}(1)}], (\mathfrak{D}_{\mathfrak{f}(1)}, \mathfrak{D}_{\mathfrak{f}(2)}], \cdots, (\mathfrak{D}_{\mathfrak{f}(i-1)}, \mathfrak{D}_{\mathfrak{f}(i)}], \cdots, (\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-1)}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}]$$

where $\mathfrak{D}_{\mathfrak{f}(i-1)} \leq \mathfrak{D}_{\mathfrak{f}(i)}$ for every $i = 1,2,\ldots,k$ such that $\mathfrak{D}_{\mathfrak{f}(0)} = -\infty$ and $\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}+1)} = +\infty$. Note that $\mathfrak{k} = k$ throughout the paper. The degradation points that are adjacent to each other may induce a same amount of probability of failure, in the way that the survival rate at degradation level $W(t) = w$ takes the form

$$S(w;t) = \sum_{i=1}^{k} s_i I[w \in J_i], \tag{4}$$

where $I[A]$ is the indicator function of the set A and $J_i = (\mathfrak{D}_{\mathfrak{f}(i-1)}, \mathfrak{D}_{\mathfrak{f}(i)}]$. It is assumed that $s_i, i = 1,2,\ldots,k$ do not depend on w.

For instance in a multiplicative degradation model with increasing mean degradation path, we accept it as postulate that the probability for failure is not altered for degradation amounts in predetermined intervals, and as the degradation exceeds the last point (the greatest value) on each interval, the probability for failure is increased. For example, for high reliability products which 100 percent of them survive before the degradation level reaches $\mathfrak{D}_{\mathfrak{f}(1)}$, and as degradation reaches $\mathfrak{D}_{\mathfrak{f}(1)}$, 10 percent of products fail, and the remaining 90 percent survives before the degradation level reaches $\mathfrak{D}_{\mathfrak{f}(2)}$, and all of them fail as soon as the degradation level reaches $\mathfrak{D}_{\mathfrak{f}(2)}$, time-to-failure is modeled upon choosing $s(w;t) = I[w \in J_1] + 0.9I[w \in J_2]$.

By using (3) and taking $s_0 = 1$ and $s_{k+1} = 0$ we get

$$\begin{aligned} \bar{F}_T(t) &= \sum_{i=1}^{k} s_i P[W(t) \in J_i] \\ &= \sum_{i=1}^{k} s_i (F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(i)}) - F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(i-1)})). \end{aligned} \tag{5}$$

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Note that if $s_i = 1$ for every $i = 1, \dots, k$ and $\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})} = \mathfrak{D}_{\mathfrak{f}}$ where $\mathfrak{D}_{\mathfrak{f}}$ is the threshold for degradation in the standard model, then $\bar{F}_T(t) = \sum_{i=1}^{k} P[W(t) \in J_i] = F_{W(t)}(\mathfrak{D}_{\mathfrak{f}})$, i.e., (5) reduces to (1). The degradation process of a life unit does not always refer to products with high reliability, where gradual failure is foreseen. It also refers to situations where sudden failures are possible, with the probability of such failures increasing as the degree of degradation increases. The model (5) may contribute effectively in such situations. Let us suppose that $T_i := \inf\{t \geq 0 : W(t) > \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}\}$, $i = 0, 1, \dots, k+1$ is the first passage time of the stochastic process $\{W(t), t \geq 0\}$ to the value of $\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}$. By convention, $T_0 = 0$ and $T_{k+1} = +\infty$. If we denote by T the time-to-failure of the device degrading over time, then

$$p_i = s_i - s_{i+1} = P(T_i \leq T < T_{i+1}). \tag{6}$$

It is necessary that (5) and (8) have to be valid survival functions for time-to-failure T. For example, $F_{W(+\infty)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) = 0$, for all $i = 0, 1, \dots k$ and further, when $F_{W(0)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) = 1$, for every $i = 0, 1, \dots k$ then (5) defines a valid SF.

We can also consider a degradation process with decreasing degradation path and assume that $s_{k+1} \geq s_k \geq \dots \geq s_1$ where $s_i \in [0, 1]$ for $i = 1, 2, \dots, k+1$ are survival rates of a unit subject to degradation when $W(t) = w$, respectively, as the value w lies in

$$(\mathfrak{U}_{\mathfrak{f}(\mathfrak{k})}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1)}], (\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{k})}], \cdots, (\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}], \cdots, (\mathfrak{U}_{\mathfrak{f}(0)}, \mathfrak{U}_{\mathfrak{f}(1)}]$$

where $\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)} \leq \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}$ for every $i = 1, 2, \dots, k+1$ such that $\mathfrak{U}_{\mathfrak{f}(0)} = -\infty$ and $\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1)} = +\infty$. The survival rate at degradation level $W(t) = w$ is

$$S(w; t) = \sum_{i=1}^{k+1} s_i I[w \in J_i^\star], \tag{7}$$

where $J_i^\star = (\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}]$. By appealing to (3) when $s_{k+1} = 1$ and $s_0 = 0$ we can get

$$\begin{aligned} \bar{F}_T(t) &= \sum_{i=1}^{k+1} s_i P[W(t) \in J_i^\star] \\ &= \sum_{i=1}^{k+1} s_i (\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) - \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})})). \end{aligned} \tag{8}$$

In this case if $s_1 = 0$ and $s_i = 1$ for every $i = 2, \dots, k+1$ and $\mathfrak{U}_{\mathfrak{f}(1)} = \mathfrak{U}_{\mathfrak{f}}$ where $\mathfrak{U}_{\mathfrak{f}}$ is the threshold for degradation in the standard model, then $\bar{F}_T(t) = \sum_{i=2}^{k+1} P[W(t) \in J_i^\star] = \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}})$, i.e., (5) reduces to (2). Let us assume that $T_i^\star := \inf\{t \geq 0 : W(t) \leq \mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1-\mathfrak{i})}\}$, $i = 0, 1, \dots, k+1$ is the first passage time of the stochastic process $\{W(t), t \geq 0\}$ to the value of $\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}$. By convention, $T_0^\star = 0$ and $T_{k+1}^\star = +\infty$. The time-to-failure of the device is the random variable T, and

$$\pi_i = s_{i+1} - s_i = P(T_{k-i}^\star \leq T < T_{k-i+1}^\star). \tag{9}$$

The following lemma is essential in deriving future results. It shows that the SF of T, in the degradation model with increasing degradation path, is a convex transformation of $F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}), i = 0, 1, \dots, k$ as $p_i \geq 0$ and $\sum_{i=0}^{k} p_i = 1$. Further, the SF of T, in the degradation model with decreasing degradation path, is a convex transformation of $\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}), i = 0, 1, \dots, k$ as $\pi_i \geq 0$ and $\sum_{i=0}^{k} \pi_i = 1$.

Lemma 1. *Let $W(t)$, the degradation process,*

(i) stochastically increases with t. Then, $\bar{F}_T(t) = \sum_{i=0}^{k} p_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})})$.

(ii) stochastically decreases with t. Then, $\bar{F}_T(t) = \sum_{i=0}^{k} \pi_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})})$.

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Proof. We first prove (i). From (5), we can write

$$
\begin{aligned}
\bar{F}_T(t) &= \sum_{i=1}^{k} \Big(s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}) \Big) \\
&= \sum_{i=1}^{k} s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - s_{i-1} F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}) + s_{i-1} F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}) - s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}) \\
&= \sum_{i=1}^{k} s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - \sum_{i=1}^{k} s_{i-1} F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}) + \sum_{i=1}^{k} (s_{i-1} - s_i) F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}) \\
&= s_k F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}) - s_0 F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})}) + \sum_{i=0}^{k-1} (s_i - s_{i+1}) F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) \\
&= (s_k - s_{k+1}) F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}) + \sum_{i=0}^{k-1} (s_i - s_{i+1}) F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) \\
&= \sum_{i=0}^{k} p_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}),
\end{aligned}
$$

where $s_0 = 1, s_{k+1} = 0$, $F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})}) = F_{W(t)}(-\infty) = 0$ and $p_i = s_i - s_{i+1}$. The proof of (i) is complete. Now, let us prove (ii). In spirit of (8), one obtains

$$
\begin{aligned}
\bar{F}_T(t) &= \sum_{i=1}^{k+1} \Big(s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) - s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}) \Big) \\
&= \sum_{i=1}^{k+1} s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) - s_{i-1} \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) + s_{i-1} \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) - s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}) \\
&= \sum_{i=1}^{k+1} s_{i-1} \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) - \sum_{i=1}^{k+1} s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}) + \sum_{i=1}^{k+1} (s_i - s_{i-1}) \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) \\
&= s_0 \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{o})}) - s_{k+1} \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1)}) + \sum_{i=1}^{k+1} (s_i - s_{i-1}) \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) \\
&= \sum_{i=0}^{k} \pi_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}),
\end{aligned}
$$

where $s_0 = 0$, $\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{o})}) = \bar{F}_{W(t)}(-\infty) = 1$, $\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1)}) = \bar{F}_{W(t)}(+\infty) = 0$ and $\pi_i = s_{i+1} - s_i$. □

In the context of standard families of degradation models studied by Bae et al. (2007) we develop the failure-time model (3) under the multiplicative degradation model.

The general multiplicative degradation model is stated as

$$
W(t) = \eta(t) X, \tag{10}
$$

where η is the mean degradation path and X is random variation around $\eta(t)$ having PDF f_X, CDF F_X and SF $\bar{F}_X$. If the mean degradation path is considered as a monotonically increasing function, then we

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develop $\bar{F}_T$ under the multiplicative degradation model (10) Note that $F_{W(t)}(w) = F_X\left(\frac{w}{\eta(t)}\right)$, thus, it is deduced from Lemma 1(i) that

$$\begin{aligned}
\bar{F}_T(t) &= E(S(X\eta(t);t)) \\
&= E\left[\sum_{i=1}^{k} s_i I(X\eta(t) \in (\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}])\right] \\
&= \sum_{i=0}^{k} p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).
\end{aligned} \tag{11}$$

The PDF of T, time-to-failure under degradation model 10 when $\eta(t)$ is increasing in $t \geq 0$ ($\eta'(t) \geq 0$, for all $t \geq 0$), having SF (11) is obtained as follows:

$$f_T(t) = \frac{\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right). \tag{12}$$

The failure rate associated with the SF given in (11) is then derived as

$$r_T(t) = \frac{\eta'(t) \sum_{i=0}^{k} p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}{\eta^2(t) \sum_{i=0}^{k} p_i F_X(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}. \tag{13}$$

If the mean degradation path $\eta(t)$ is a monotonically decreasing function, then the time-to-failure is denoted by T_1 with SF $\bar{F}_{T_1}$. This SF can be obtained in the setting of the multiplicative degradation model (10). We see that $\bar{F}_{W(t)}(w) = \bar{F}_X\left(\frac{w}{\eta(t)}\right)$. Therefore, using Lemma 1(ii) we get

$$\begin{aligned}
\bar{F}_{T_1}(t) &= E\left[\sum_{i=1}^{k+1} s_i I(X\eta(t) \in (\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}])\right] \\
&= \sum_{i=0}^{k} \pi_i \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).
\end{aligned} \tag{14}$$

The PDF of T_1, time-to-failure under degradation model 10 when $\eta(t)$ is decreasing in $t \geq 0$ ($\eta'(t) \leq 0$, for all $t \geq 0$), having SF (14) is revealed to be:

$$f_{T_1}(t) = \frac{-\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right). \tag{15}$$

The failure rate of T with the SF given in (14) is

$$r_{T_1}(t) = \frac{-\eta'(t) \sum_{i=0}^{k} \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}{\eta^2(t) \sum_{i=0}^{k} \pi_i \bar{F}_X(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}. \tag{16}$$

3. Stochastic ordering results

In this section we study some stochastic ordering properties of time-to-failure distributions of two devices under multiplicative degradation model. It is acknowledged in industrial sciences that products may have different qualities, some of which are more reliable whereas the others fails earlier. The extent each subject under degradation resists not to fail can be evaluated by p_i's and π_i's in the models (5) and (8), respectively (see, e.g., Lemma 1).

Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_n^\star)$ denote two probability vectors assigning to a couple of devices working under a multiplicative degradation model with increasing mean degradation path.

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We suppose that P and $P^\star$ are associated with with random lifetimes T and $T^\star$, respectively, such that $p_i = P(T_i \leq T < T_{i+1})$ and also $p_i^\star = P(T_i \leq T^\star < T_{i+1})$ where $T_i := \inf\{t \geq 0 \mid W(t) > \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}\}$ for $i = 0, 1, \ldots, k+1$. In a similar manner, let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ denote other probability vectors related to a pair of devices working under a multiplicative degradation model with decreasing mean degradation path. It is assumed that Π and $\Pi^\star$ are associated with with random lifetimes T_1 and $T_1^\star$, respectively, such that $\pi_i = P(T_i^\star \leq T_1 < T_{i+1}^\star)$ and also $\pi_i^\star = P(T_i^\star \leq T_1^\star < T_{i+1}^\star)$ where $T_i^\star := \inf\{t \geq 0 | W(t) \leq \mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i}+1)}\}$ for $i = 0, 1, \ldots, k+1$. Suppose that $W(t) = \eta(t)X$ is the underlying degradation model. We impose a partial order condition among P and $P^\star$ or/and conditions on distribution of X (random variation around $\eta(t)$) such that some stochastic orders between T and $T^\star$ are procured. Further, we find some conditions on Π and $\Pi^\star$ and other conditions on distribution of X such that several stochastic orders between T_1 and $T_1^\star$ are fulfilled.

There are some concepts in applied probability that we need to introduce them before developing our stochastic comparison results. The following definition is due to Karlin (1968).

Definition 2. *The function w, as a transformation on (x, y), is said to be totally positive of order 2, TP_2, [reverse regular of order 2, RR_2] in $(x, y) \in \mathfrak{A} \times \mathfrak{B}$, if $w(x, y) \geq 0$ and*

$$\begin{vmatrix} w(x_1, y_1) & w(x_1, y_2) \\ w(x_2, y_1) & w(x_2, y_2) \end{vmatrix} \geq [\leq] 0,$$

for all $x_1 \leq x_2 \in \mathfrak{A}$ and for all $y_1 \leq y_2 \in \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are two subsets of $\mathbb{R}$.

It is plain to verify that the TP_2 $[RR_2]$ property of w, as a transformation on (i, k), is equivalent to $\frac{w(i,k_2)}{w(i,k_1)}$ being non-decreasing [non-increasing] in i whenever $k_1 \leq k_2$ by considering the conventions $\frac{a}{0} = +\infty$ when $a > 0$ and $\frac{a}{0} = 0$ if $a = 0$.

The following lemma from Karlin (1968) known as general composition theorem (or basic composition formula) will be frequently used across the paper.

Lemma 3. *(i) (discrete case): Let g be TP_2 in $(j, i) \in \{1, 2\} \times \mathfrak{A}_k$ and also let w be TP_2 (respectively, RR_2) in $(i, t) \in \mathfrak{A}_k \times \mathfrak{B}$, where $\mathfrak{A}_k = \{0, 1, \ldots, k\}$. Then, the function $w^\star$, given by*

$$w^\star(j, t) := \sum_{i=0}^{k} g(j, i) w(i, t), \text{ is } TP_2 \text{ (respectively, } RR_2) \text{ in } (j, t) \in \{1, 2\} \times \mathfrak{B}.$$

(i) (continuous case): Let $g(j, y)$ is TP_2 in $(j, y) \in \{1, 2\} \times \mathfrak{Y}$ and let $w(y, x)$ is TP_2 (respectively, RR_2) in $(y, x) \in \mathfrak{Y} \times \mathfrak{B}$, where $\mathfrak{Y}$ and $\mathfrak{B}$ are two subsets of $[0, +\infty)$. Then,

$$w^\star(j, x) := \int_0^{+\infty} g(j, y) w(y, x)\, dy \text{ is } TP_2 \text{ (respectively, } RR_2) \text{ in } (j, x) \in \{1, 2\} \times \mathfrak{B}.$$

The following definition proposes some class of functions.

Definition 4. *Suppose that w, as a transformation of non-negative values, is a non-negative function. Then, w is said to have*
(i) one-sided scaled-ratio increasing (decreasing), OSSRI (OSSRD), property if $\frac{w(tx)}{w(x)}$ is increasing (decreasing) in $x \geq 0$ for every $t \geq 1$.
(ii) two-sided scaled-ratio increasing (decreasing), TSSRI (TSSRD), property if $\frac{w(t_2x) - w(t_1x)}{w(s_2x) - w(s_1x)}$ is increasing (decreasing) in $x \geq 0$ for every $t_i \geq s_i \geq 0, i = 1, 2$ with $t_1 \leq t_2$ and $s_1 \leq s_2$.

From Definition 4, it is apparent that if $t_1 = s_1 = 0$ and also $w(0) = 0$, then from assertion (ii) the ratio $\frac{w(t_2x)}{w(s_2x)}$ is increasing (decreasing) in x for every $t_2 > s_2 \geq 0$. Equivalently, this realizes that $\frac{w(tx)}{w(x)}$ is increasing (decreasing) in x for all $t \geq 1$. Therefore, every w with $w(0) = 0$ having TSSRI (TSSRD) property will also fulfill the OSSRI (OSSRD) property.

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Remark 5. *The properties in Definition 4(i) can be applied to produce reliability classes of lifetime distributions. It can be observed that X has increasing proportional likelihood ratio (IPLR) property if, and only if, f_X has the OSSRD property and also X has decreasing proportional likelihood ratio (DPLR) property if, and only if, f_X has the OSSRI property (see, Romero and Díaz (2001) for definitions of IPLR and DPLR). It can also be seen that X has increasing proportional hazard rate (IPHR) property if, and only if, $\bar{F}_X$ has the OSSRD property and, in parallel, X has decreasing proportional hazard rate (DPHR) property if, and only if, $\bar{F}_X$ has the OSSRI property (see, Belzunce et al. (2002) for IPHR and DPHR properties). It can also be verified that X has decreasing proportional reversed failure rate (DPRFR) property if, and only if, F_X has the OSSRD property and, further, X has increasing proportional failure rate (IPRFR) property if, and only if, F_X has the OSSRI property (see, Oliveira and Torrado (2014) for DPRFR and IPRFR classes).*

In applied probability, stochastic orders among random variables have been a useful approach for comparison of reliability of systems (see, e.g., Müller and Stoyan (2002), Osaki (2002), Shaked and Shanthikumar (2007) and Belzunce et al. (2015)). Stochastic orderings are considered a basic tool for making decisions under uncertainty (see, for instances, Mosler (1991) and Li and Li (2013)).

Let us assume that T and $T^\star$ are random variables with absolutely continuous CDFs F_T and $F_{T^\star}$, SFs $\bar{F}_T$ and $\bar{F}_{T^\star}$ and PDFs f_T and $f_{T^\star}$, respectively. We suppose that T and $T^\star$ have hazard rate functions h_T and $h_{T^\star}$, reversed hazard rate functions $\tilde{h}_T$ and $\tilde{h}_{T^\star}$, respectively. Then:

Definition 6. *We say that T is smaller than or equal with $T^\star$ in the*

(i) likelihood ratio order (denoted as $T \leq_{lr} T^\star$) if $\frac{f_{T^\star}(t)}{f_T(t)}$ is increasing in $t \geq 0$.
(ii) hazard rate order (denoted as $T \leq_{hr} T^\star$) if $\frac{\bar{F}_{T^\star}(t)}{\bar{F}_T(t)}$ is increasing in $t \geq 0$ or equivalently, $h_T(t) \geq h_{T^\star}(t)$, for all $t \geq 0$.
(iii) reversed hazard rate order (denoted as $T \leq_{rhr} T^\star$) if $\frac{F_{T^\star}(t)}{F_T(t)}$ is increasing in $t \geq 0$ or equivalently, $\tilde{h}_T(t) \leq \tilde{h}_{T^\star}(t)$, for all $t > 0$.
(iv) usual stochastic order (denoted as $T \leq_{st} T^\star$) if $\bar{F}_T(t) \leq \bar{F}_{T^\star}(t)$, for all $t \geq 0$.

As given in Shaked and Shanthikumar (2007) we have:

$$T \leq_{lr} T^\star \Rightarrow T \leq_{hr} T^\star \Rightarrow T \leq_{st} T^\star.$$

It is, furthermore, well-known that

$$T \leq_{lr} T^\star \Rightarrow T \leq_{rhr} T^\star \Rightarrow T \leq_{st} T^\star.$$

Two compare T and $T^\star$, according to the usual stochastic order, one sufficient conditions is found to be the well-known majorization order as given in the next definition. Majorization is a partial order relation of two probability vectors with same dimension inducing that the elements in one vector are less spread out or more nearly equal than the elements in another vector. The majorization order makes an elegant framework to compare two probability vectors (see, e.g., Marshall et al. (1965)).

We take $\mathbb{X} = (x_0, \ldots, x_k)$ and $\mathbb{Y} = (y_0, \ldots, y_k)$ as two vectors of real numbers such that $x_{(0)} \leq \ldots \leq x_{(k)}$ and $y_{(0)} \leq \ldots \leq y_{(k)}$ denote increasing arrangement of values of $\mathbb{X}$ and values of $\mathbb{Y}$, respectively, where $x_{(i)}$ is the ith smallest value among $x_0, \ldots, x_k$ and $y_{(i)}$ is the ith smallest value among $y_0, \ldots, y_k$, for $i = 1, \ldots, k$.

Definition 7. *It is said that $\mathbb{X}$ is majorized by $\mathbb{Y}$, written as $\mathbb{X} \preceq \mathbb{Y}$ whenever $\sum_{i=0}^k x_i = \sum_{i=0}^k y_i$, and $\sum_{i=0}^j x_{(k-i)} \leq \sum_{i=0}^j y_{(k-i)}$, for every $j = 0, \ldots, k-1$.*

In the sequel of the paper, we will assume that T and $T^\star$ are two random variables denoting time-to-failure under the dynamic multiplicative degradation model $W(t) = X\eta(t)$ where η is an increasing function with SFs

$$\bar{F}_T(t) = \sum_{i=0}^k p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } \bar{F}_{T^\star}(t) = \sum_{i=0}^k p_i^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

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The corresponding PDFs are derived as

$$f_T(t) = \frac{\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } f_{T^\star}(t) = \sum_{i=0}^{k} p_i^\star \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

We will also suppose that T_1 and $T_1^\star$ are two random variables denoting time-to-failure under the multiplicative degradation model $W(t) = X\eta(t)$ where η is a decreasing function with SFs

$$\bar{F}_{T_1}(t) = \sum_{i=0}^{k} \pi_i \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } \bar{F}_{T_1^\star}(t) = \sum_{i=0}^{k} \pi_i^\star \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

The associated PDFs are obtained as

$$f_{T_1}(t) = \frac{-\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } f_{T_1^\star}(t) = \frac{-\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} \pi_i^\star \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

We utilize the following technical lemma.

Lemma 8. *(i) Let $w_0, w_1, \ldots, w_k$ be a set of real numbers satisfying $\sum_{i=0}^{k} w_i = 0$. If $h(i)$ is non-decreasing in $i = 0, 1, \ldots, k$, then*

$$W_j = \sum_{i=j}^{k} w_i \geq 0, \text{ for all } j = 1, 2, \ldots, k \text{ implies that } \sum_{i=0}^{k} h(i) w_i \geq 0.$$

(ii) Let $w_0, w_1, \ldots, w_k$ be real numbers. If $h(i) \geq 0$ is non-increasing for $i = 0, 1, \ldots, k$, then

$$W_j = \sum_{i=0}^{j} w_i \geq 0, \text{ for all } j = 0, 1, \ldots, k \text{ implies that } \sum_{i=0}^{k} h(i) w_i \geq 0.$$

The next result discusses sufficient conditions for stochastic comparison of T and $T^\star$ and also stochastic ordering of T_1 and $T_1^\star$ according to the usual stochastic order.

Theorem 9. *(i) Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_k^\star)$ be two probability vectors satisfying $p_0 \leq \ldots \leq p_k$ and $p_0^\star \leq \ldots \leq p_k^\star$ such that $P \preceq P^\star$. Then, $T \leq_{st} T^\star$.*
(ii) Let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ be two probability vectors with $\pi_0 \geq \ldots \geq \pi_k$ and $\pi_1^\star \geq \ldots \geq \pi_k^\star$ such that $\Pi^\star \preceq \Pi$. Then, $T_1 \leq_{st} T_1^\star$.

Proof. Firstly, we prove assertion (i). Note that for any $t \geq 0$,

$$F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})}}{\eta(t)}\right) \leq F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}}{\eta(t)}\right) \leq \cdots \leq F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}}{\eta(t)}\right). \tag{17}$$

By appealing to Eq. (11) and since $p_i \leq p_j$, for every $i < j$ and also from (17), $F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \leq F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{j})}}{\eta(t)}\right)$ for every $i < j$, as $i, j = 0, 1, \ldots, k$ we thus by rearranging[1] the elements in sigma in Eq. (11) conclude that

$$\bar{F}_T(t) = \sum_{i=0}^{k} p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k} p_{(k-i)} F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

Similarly,

$$\bar{F}_{T^\star}(t) = \sum_{i=0}^{k} p_i^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k} p_{(k-i)}^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

[1] It is straightforward that if $a_0 \leq a_1 \leq \cdots \leq a_k$ and also $b_0 \leq b_1 \leq \cdots \leq b_k$, then $\sum_{i=0}^{k} a_i b_i = \sum_{i=0}^{k} a_{k-i} b_{k-i} = \sum_{i=0}^{k} a_{(k-i)} b_{k-i}$ in which $a_{(0)} \leq a_{(1)} \leq \cdots \leq a_{(k)}$ denote the ordered values of $a_0, a_1, \ldots, a_k$.

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Let us take $h(i) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$ which, by (17), is a non-increasing function in $i = 0, 1, \ldots, k$. Since $P \preceq P^\star$, thus $\sum_{i=0}^{j}(p^\star_{(k-i)} - p_{(k-i)}) \geq 0$, for all $j = 0, 1, \ldots, k$. Therefore, from Lemma 8(ii),

$$\bar{F}_{T^\star}(t) - \bar{F}_T(t) = \sum_{i=0}^{k}(p^\star_{(k-i)} - p_{(k-i)})F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$$

is non-negative, which means that $T \leq_{st} T^\star$. We now prove assertion (ii). For each fixed $t \geq 0$, we have:

$$\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{o})}}{\eta(t)}\right) \geq \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{1})}}{\eta(t)}\right) \geq \cdots \geq \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k})}}{\eta(t)}\right). \tag{18}$$

By applying Eq. (14) and since $\pi_i \geq \pi_j$, for every $i < j$ and also from (18), $\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \geq \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{j})}}{\eta(t)}\right)$ for every $i < j$, when $i, j = 0, 1, \ldots, k$ we thus by rearranging[2] the elements of sigma in Eq. (14) can get

$$\bar{F}_{T_1}(t) = \sum_{i=0}^{k}\pi_i\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k}\pi_{(k-i)}\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

In parallel,

$$\bar{F}_{T_1^\star}(t) = \sum_{i=0}^{k}\pi^\star_i\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k}\pi^\star_{(k-i)}\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

We set $h(i) = \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$ which by (18), is a non-decreasing function in $i = 0, 1, \ldots, k$. Since $\Pi^\star \preceq \Pi$, thus $\sum_{i=j}^{k}(\pi^\star_{(k-i)} - \pi_{(k-i)}) \geq 0$, for all $j = 1, \ldots, k$ and $\sum_{i=0}^{k}(\pi^\star_{(k-i)} - \pi_{(k-i)}) = 0$. Hence, an application of Lemma 8(i) yields

$$\bar{F}_{T_1^\star}(t) - \bar{F}_{T_1}(t) = \sum_{i=0}^{k}(\pi^\star_{(k-i)} - \pi_{(k-i)})\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$$

is non-negative, which means that $T_1 \leq_{st} T_1^\star$. The proof is complete. □

Remark 10. *The result of Theorem 9 indicates that the usual stochastic order between T and $T^\star$ and also that of T_1 and $T_1^\star$ do not depend on the distribution of random variation X. The conditions imposed to get $T \leq_{st} T^\star$ in Theorem 9(i) consist of an order relation among p_i's (i.e., $p_0 \leq \ldots \leq p_k$) and the same order relation among $p_i^\star$'s (i.e., $p_0^\star \leq \ldots \leq p_k^\star$) and a condition of majorization order of P and $P^\star$. The probability vector ($P^\star$) which majorizes the other probability vector (P) will lead to a more reliable product under multiplicative degradation model with increasing $\eta(t)$. The order relations $p_0 \leq \ldots \leq p_k$ and $p_0^\star \leq \ldots \leq p_k^\star$ are valid assumptions in practical works. This is because in a multiplicative degradation model with increasing $\eta(t)$, as the time t is elapsed, the degradation amount $W(t)$ is increased and, therefore, the probability for failure is correspondingly grown. Note that the first elements of P and $P^\star$ are associated with smaller amounts of $W(t)$. The conditions found to obtain $T \leq_{st} T^\star$ in Theorem 9(ii) are, firstly, an order relation of π_i's (i.e., $\pi_0 \geq \ldots \geq \pi_k$) and an analogues order relation of $\pi_i^\star$'s (i.e., $\pi_0^\star \geq \ldots \geq \pi_k^\star$) and, secondly, the majorization order of $\Pi^\star$ and Π. The probability vector (Π) which majorizes the other probability vector ($\Pi^\star$) will lead to a less reliable product under multiplicative degradation model with decreasing $\eta(t)$. The order constraints $\pi_0 \geq \ldots \geq \pi_k$ and $\pi_0^\star \geq \ldots \geq \pi_k^\star$ are also valid assumptions in practical situations. This is because in a multiplicative degradation model with decreasing $\eta(t)$, as the time t is elapsed, the factor $W(t)$ for degradation is decreased and, therefore, the probability for failure of the product is correspondingly going up. Notice that the first elements of Π and $\Pi^\star$ are associated with smaller amounts $W(t)$ take.*

[2] It is plain to see if $a_0 \geq a_1 \geq \cdots \geq a_k$ and also $b_0 \geq b_1 \leq \cdots \geq b_k$, then $\sum_{i=0}^{k} a_i b_i = \sum_{i=0}^{k} a_{k-i} b_{k-i} = \sum_{i=0}^{k} a_{(k-i)} b_{k-i}$.

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The following theorems present some conditions to make the order $\leq_{lr}$ between time-to-failure random variables in the dynamic multiplicative degradation model with increasing mean degradation path $\eta(t)$ (Theorem 11(i)) and the dynamic multiplicative degradation model with decreasing mean degradation path $\eta(t)$ (Theorem 11(ii)).

Theorem 11. *(i) Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_k^\star)$ be two probability vectors so that $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$. If f_X is OSSRD (OSSRI), then $T \leq_{lr} (\geq_{lr}) T^\star$.*
(ii) Let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ be two probability vectors so that $\frac{\pi_i^\star}{\pi_i}$ is non-decreasing in $i = 0, 1, \ldots, k$. If f_X is OSSRI (OSSRD), then $T_1 \leq_{lr} (\geq_{lr}) T_1^\star$.

Proof. To prove (i) it suffices to demonstrate that

$$\frac{f_{T^\star}(t)}{f_T(t)} = \frac{\sum_{i=0}^k p_i^\star \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^k p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t > 0$. Set $g(j,i) = p_i$, for $j = 1$ and $g(j,i) = p_i^\star$, for $j = 2$ and also $w(i,t) = \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. Therefore, $T \leq_{lr} (\geq_{lr}) T^\star$ if, and only if, $w^\star(j,t) := \sum_{i=0}^k g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. Note that, by assumption, $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$, hence, $g(j,i)$ is TP_2 in $(j,i) \in \{1,2\} \times \{0,1,\ldots,k\}$ and also since f_X is OSSRD (OSSRI) and $\eta(t)$ is non-decreasing in $t \geq 0$, thus, for every $i_1 < i_2 \in \{0,1,\ldots,k\}$

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t \geq 0$. This means $w(i,t)$ is TP_2 (RR_2) in $(i,t) \in \{0,1,\ldots,k\} \times \{1,2\}$. By Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this completes the proof of (i). To prove (ii) one needs to show that

$$\frac{f_{T_1^\star}(t)}{f_{T_1}(t)} = \frac{\sum_{i=0}^k \pi_i^\star \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^k \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t > 0$. We take $g^\star(j,i) = \pi_i$, for $j = 1$ and $g^\star(j,i) = \pi_i^\star$, for $j = 2$ and also set $w_1(i,t) = \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. Thus, $T_1 \leq_{lr} (\geq_{lr}) T_1^\star$ if, and only if, $w_2(j,t) := \sum_{i=0}^k g^\star(j,i) w_1(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. From assumption, $\frac{\pi_i^\star}{\pi_i}$ is non-decreasing in $i = 0, 1, \ldots, k$, hence, $g^\star(j,i)$ is TP_2 in $(j,i) \in \{1,2\} \times \{0,1,\ldots,k\}$ and also since f_X is OSSRI (OSSRD) and $\eta(t)$ is non-increasing in $t \geq 0$, thus, for every $i_1 < i_2 \in \{0,1,\ldots,k\}$

$$\frac{w_1(i_2,t)}{w_1(i_1,t)} = \frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_2)} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_1)} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t \geq 0$. This means $w_1(i,t)$ is TP_2 (RR_2) in $(i,t) \in \{0,1,\ldots,k\} \times \{1,2\}$. By Lemma 3(i), $w_2(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, which validates the proof of (ii). □

The following theorem presents conditions to make the order $\leq_{hr}$ between time-to-failure random variables in the dynamic multiplicative degradation model with increasing mean degradation path $\eta(t)$.

Theorem 12. *Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_k^\star)$ be two probability vectors such that*

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(i) $\frac{p_i^\star}{p_i}$ *is non-decreasing in* $i = 0, 1, \ldots, k$. *If* F_X *is OSSRD (OSSRI), then we have* $T \le_{hr} (\ge_{hr}) T^\star$.
(ii) $\frac{s_i^\star}{s_i}$ *is non-decreasing in* $i = 1, 2, \ldots, k$. *If* F_X *is TSSRD (TSSRI), then we have* $T \le_{hr} (\ge_{hr}) T^\star$.

Proof. For assertion (i) to be proved it is enough to show that

$$\frac{\bar{F}_{T^\star}(t)}{\bar{F}_T(t)} = \frac{\sum_{i=0}^{k} p_i^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^{k} p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t \ge 0$. Let us take $g(j,i) = p_i$, for $j = 1$ and $g(j,i) = p_i^\star$, for $j = 2$ and also $w(i,t) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. Thus, $T \le_{hr} (\ge_{hr}) T^\star$ if, and only if, $w^\star(j,t) := \sum_{i=0}^{k} g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. By assumption, $\frac{p_i^\star}{p_i}$ is non-decreasing in i hence, $g(j,i)$ is TP_2 in (j,i) and further, since F_X is OSSRD (OSSRI) and $\eta(t)$ is non-decreasing in $t \ge 0$, thus, for every $i_1 < i_2$, in domain of i,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t \ge 0$. This is equivalent to saying that $w(i,t)$ is TP_2 (RR_2) in (i,t). By Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this ends the proof of (i). For the proof of assertion (ii) one needs to prove that

$$\frac{\bar{F}_{T^\star}(t)}{\bar{F}_T(t)} = \frac{\sum_{i=1}^{k} s_i^\star \left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)\right)}{\sum_{i=1}^{k} s_i \left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)\right)}$$

is non-decreasing (non-increasing) in $t \ge 0$. We can set $g(j,i) = s_i$, for $j = 1$ and $g(j,i) = s_i^\star$, for $j = 2$ and also take $w(i,t) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)$ which is non-negative since $\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} \ge \mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}$. By these notations $T \le_{hr} (\ge_{hr}) T^\star$ if, and only if, $w^\star(j,t) := \sum_{i=1}^{k} g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. From assumption, $\frac{s_i^\star}{s_i}$ is non-decreasing in i hence, $g(j,i)$ is TP_2 in (j,i) and moreover, since F_X is TSSRD (TSSRI) and $\eta(t)$ is non-decreasing in $t \ge 0$, thus, for every $i_1 < i_2$,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2-1)}}{\eta(t)}\right)}{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1-1)}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t \ge 0$. This is equivalent to $w(i,t)$ being TP_2 (RR_2) in (i,t). On applying Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this gives the required result in assertion (ii). □

In the context of Theorem 12, if $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$, then $\frac{s_i^\star}{s_i}$ is also non-decreasing in $i = 1, 2, \ldots, k$. We can use Lemma 3(i) to prove it. Let us take $g(j,i) = p_i^\star$, for $j = 2$ and $g(j,i) = p_i$ for $j = 1$ when $i = 0, 1, \ldots, k$. Set $w(i,t) = I[i \ge t]$ where $t = 1, 2, \ldots, k$ and $i = 0, 1, \ldots, k$. Since $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$, thus $g(j,i)$ is TP_2 in (j,i) and also it is straightforward to show that $w(i,t) = I[i \ge t]$ is TP_2 in (i,t). Hence, $w^\star(j,i) = \sum_{i=0}^{k} g(j,i) w(i,t)$ is TP_2 in (j,t), i.e., $\frac{s_i^\star}{s_i}$ is non-decreasing in $i = 1, 2, \ldots, k$. Therefore, the condition on probabilities in Theorem 12(ii) is weaker than the condition imposed on probabilities in Theorem 12(i). It is also plain to show that if F_X is TSSRD (TSSRI) then F_X is OSSRD (OSSRI). Therefore, the condition on random effect distribution in Theorem 12(ii) is stronger than the condition on random effect distribution in Theorem 12(i).

The theorem below presents conditions to make the order $\le_{hr}$ between time-to-failure random variables in the dynamic multiplicative degradation model with decreasing mean degradation path $\eta(t)$. The proof being similar to the proof of Theorem 12 has been omitted.

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Theorem 13. *Let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ be two probability vectors such that*

(i) $\frac{\pi_i^\star}{\pi_i}$ is non-decreasing in $i = 0, 1, \ldots, k$. If $\bar{F}_X$ is OSSRI (OSSRD), then we have $T_1 \leq_{hr} (\geq_{hr}) T_1^\star$.
(ii) $\frac{s_i^\star}{s_i}$ is non-decreasing in $i = 1, 2, \ldots, k+1$. If F_X is TSSRI (TSSRD), then we have $T_1 \leq_{hr} (\geq_{hr}) T_1^\star$.

The next result presents conditions under which the order $\leq_{rhr}$ is fulfilled by time-to-failure random variables in the dynamic multiplicative degradation model with increasing mean degradation path $\eta(t)$.

Theorem 14. *Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_k^\star)$ be two probability vectors such that*

(i) $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$. If $\bar{F}_X$ is OSSRD (OSSRI), then $T \leq_{rhr} (\geq_{rhr}) T^\star$.
(ii) $\frac{1-s_i^\star}{1-s_i}$ is non-decreasing in $i = 1, 2, \ldots, k+1$. If F_X is TSSRD (TSSRI), then $T \leq_{rhr} (\geq_{rhr}) T^\star$.

Proof. The assertion (i) is established if one shows that

$$\frac{F_{T^\star}(t)}{F_T(t)} = \frac{\sum_{i=0}^{k} p_i^\star \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^{k} p_i \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t > 0$. Let $g(j,i) = p_i$, for $j = 1$ and $g(j,i) = p_i^\star$, for $j = 2$ and also $w(i,t) = \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. As a result, $T \leq_{rhr} (\geq_{rhr}) T^\star$ if, and only if, $w^\star(j,t) := \sum_{i=0}^{k} g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0, +\infty)$. By assumption, $\frac{p_i^\star}{p_i}$ is non-decreasing in i hence, $g(j,i)$ is TP_2 in (j,i) and further, since $\bar{F}_X$ is OSSRD (OSSRI) and $\eta(t)$ is non-decreasing in $t \geq 0$, thus, for every $i_1 < i_2$,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{\bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{\bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is non-decreasing (non-increasing) in $t \geq 0$, which means $w(i,t)$ is TP_2 (RR_2) in (i,t). Using Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0, +\infty)$, and this provides the proof of (i). For assertion (ii) we need to demonstrate that

$$\frac{F_{T^\star}(t)}{F_T(t)} = \frac{\sum_{i=1}^{k+1} (1 - s_i^\star) \left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)\right)}{\sum_{i=1}^{k+1} (1 - s_i) \left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)\right)}$$

is non-decreasing (non-increasing) in $t > 0$. Let us define $g(j,i) = 1 - s_i$, for $j = 1$ and $g(j,i) = 1 - s_i^\star$, for $j = 2$ and also define $w(i,t) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)$. Now, $T \leq_{rhr} (\geq_{rhr}) T^\star$ if, and only if, $w^\star(j,t) := \sum_{i=1}^{k} g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0, +\infty)$. By assumption, $\frac{1-s_i^\star}{1-s_i}$ is non-decreasing in i hence, $g(j,i)$ is TP_2 in (j,i) and in addition, since $\bar{F}_X$ is TSSRD (TSSRI) and $\eta(t)$ is non-decreasing in $t \geq 0$, thus, $w(i,t)$ is TP_2 (RR_2) in (i,t). By Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0, +\infty)$, and this proves assertion (ii). □

In the setting of Theorem 14, if $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$, then $\frac{1-s_i^\star}{1-s_i}$ is non-decreasing in $i = 1, 2, \ldots, k+1$. Lemma 3(i) can be used to prove it. Let us set $g(j,i) = p_i^\star$, for $j = 2$ and $g(j,i) = p_i$ for $j = 1$ when $i = 0, 1, \ldots, k$. Set $w(i,t) = I[i \leq t-1]$ where $t = 1, 2, \ldots, k+1$ and $i = 0, 1, \ldots, k$. Since $\frac{p_i^\star}{p_i}$ is non-decreasing in $i = 0, 1, \ldots, k$, thus $g(j,i)$ is TP_2 in (j,i) and also $w(i,t) = I[i \leq t]$ is TP_2 in (i,t). Thus, $w^\star(j,i) := \sum_{i=0}^{k} g(j,i) w(i,t)$ is TP_2 in (j,t), i.e., $\frac{1-s_i^\star}{1-s_i}$ is non-decreasing in $i = 1, 2, \ldots, k+1$. Hence, the condition on probabilities in Theorem 14(ii) is weaker than the condition on probabilities in Theorem 14(i). Furthermore, if F_X is TSSRD (TSSRI) then $\bar{F}_X$ is OSSRD (OSSRI). This means that the

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condition on random effect distribution in Theorem 14(ii) is stronger than the condition on random effect distribution in Theorem 14(i).

The theorem given next presents conditions to make the order $\leq_{rhr}$ between time-to-failure random variables in the dynamic multiplicative degradation model with decreasing mean degradation path $\eta(t)$. The proof being similar to the proof of Theorem 14 has been omitted.

Theorem 15. *Let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ be two probability vectors such that*

(i) $\frac{\pi_i^\star}{\pi_i}$ is non-decreasing in $i = 0, 1, \ldots, k$. If F_X is OSSRI (OSSRD), then we have $T_1 \leq_{rhr} (\geq_{rhr}) T_1^\star$.
(ii) $\frac{1-s_i^\star}{1-s_i}$ is non-decreasing in $i = 1, 2, \ldots, k+1$. If F_X is TSSRI (TSSRD), then we have $T_1 \leq_{rhr} (\geq_{rhr}) T_1^\star$.

4. Examples

In this section, we examine and scrutinize the conditions on random effect distribution to fulfill the ordering properties in Section 3 with some typical random effect distribution functions according to the ones listed in Bae et al. (2007). These functions, as remarked in Bae et al. (2007), are proper functions arisen in most practical situations. We prove that the standard applicative distributions for random variation X lie in the framework of theorems in Section 3.

Before stating the examples we state the following lemma.

Lemma 16. *Let f_X, F_X and $\bar{F}_X$ be the PDF, CDF and SF of random variation X around $\eta(t)$. Then,*

(i) If F_X is TSSRD (TSSRI), then F_X is OSSRD (OSSRI).
(ii) F_X is TSSRD (TSSRI), if and only if, $\bar{F}_X$ is TSSRD (TSSRI).
(iii) If f_X is OSSRD (OSSRI), then F_X and $\bar{F}_X$ are TSSRD (TSSRI).

Proof. The proof of (i) is obvious (see the lines after Definition 4). To prove (ii), it is enough to observe that for all $t_i \geq s_i \geq 0, i = 1, 2$ and $t_2 \geq t_1$ and $s_2 \geq s_1$, it holds that:

$$\frac{F_X(t_2x) - F_X(t_1x)}{F_X(s_2x) - F_X(s_1x)} = \frac{\bar{F}_X(t_2x) - F_X(t_1x)}{\bar{F}_X(s_2x) - \bar{F}_X(s_1x)}.$$

To prove assertion (iii), it suffices to establish that if f_X is OSSRD (OSSRI), then F_X is TSSRD (TSSRI) because this is equivalent to $\bar{F}_X$ being TSSRD (TSSRI) from assertion (ii). We have

$$\frac{F_X(t_2x) - F_X(t_1x)}{F_X(s_2x) - F_X(s_1x)} = \frac{\int_{t_1x}^{t_2x} f_X(u)du}{\int_{s_1x}^{s_2x} f_X(u)du} = \frac{\int_{t_1}^{t_2} f_X(xy)dy}{\int_{s_1}^{s_2} f_X(xy)dy}.$$

The ratio $\frac{F_X(t_2x)-F_X(t_1x)}{F_X(s_2x)-F_X(s_1x)}$ is non-increasing (non-decreasing) in $x \geq 0$ for all $t_i \geq s_i \geq 0, i = 1, 2$ and $t_2 \geq t_1$ and $s_2 \geq s_1$, if and only if, $w^\star(j, x) := \int_0^{+\infty} g(j, y) w(y, x)\, dy$ is RR_2 (TP_2) in $(j, x) \in \{1, 2\} \times [0, +\infty)$, where $g(j, y) = I[s_1 < y \leq s_2]$ for $j = 1$ and $g(j, y) = I[t_1 < y \leq t_2]$ for $j = 2$ and $w(y, x) = f_X(xy)$. It is not hard to prove that $g(j, y)$ is TP_2 in (j, y) and also since f_X is OSSRD (OSSRI) thus $w(y, x)$ is RR_2 (TP_2) in (y, x). Hence, by Lemma 3(ii) the required result follows. □

The following examples show that the results of Theorems 11, 12, 13, 14 and Theorem 15 can be applied for several standard typical distributions for random variation X.

Example 17. *(X is Weibull-distributed). Suppose that X has SF $\bar{F}_X(x) = \exp(-(\lambda x)^\alpha)$ where $\lambda > 0$ and $\alpha > 0$. The PDF of X is $f_X(x) = \alpha \lambda^\alpha x^{\alpha-1} \exp(-(\lambda x)^\alpha)$. Thus,*

$$\frac{f_X(tx)}{f_X(x)} = t^{\alpha-1} \exp((\lambda x)^\alpha (1 - t^\alpha))$$

which is decreasing in $x \geq 0$, for all $t > 1$, thus, f_X is OSSRD and as a result of Lemma 16(iii), F_X is TSSRD and $\bar{F}_X$ is TSSRD.

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Example 18. *(X is gamma-distributed). Assume that X has PDF* $f_X(x) = \frac{\lambda^\gamma x^{\gamma-1}\exp(-\lambda x)}{\Gamma(\gamma)}$ *where* $\gamma > 0$ *and* $\lambda > 0$. *We obtain*

$$\frac{f_X(tx)}{f_X(x)} = t^{\gamma-1}\exp((\lambda x)(1-t))$$

which is decreasing in $x \geq 0$, *for every* $t > 1$, *i.e.* f_X *is OSSRD and by Lemma 16(iii),* F_X *is TSSRD and* $\bar{F}_X$ *is also TSSRD.*

Example 19. *(X is log-logistically distributed). Let us take X as a random variable with PDF* $f_X(x) = \frac{\beta e^\alpha x^{\beta-1}}{(1+e^\alpha x^\beta)^2}$, *for* $\beta > 0$. *We can derive*

$$\frac{f_X(tx)}{f_X(x)} = t^{\beta-1}\left(\frac{1+e^\alpha x^\beta}{1+e^\alpha(tx)^\beta}\right)^2$$

which is decreasing in $x \geq 0$, *for every* $t > 1$, *and this means* f_X *is OSSRD which by Lemma 16(iii) implies that* F_X *is TSSRD and* $\bar{F}_X$ *is also TSSRD.*

The following example makes an application of Theorem 9.

Example 20. *Suppose* $W(t)$ *is a degradation process with increasing mean degradation path. Let us assume that T denotes the time-to-failure of a device and that* $T^\star$ *denotes the time-to-failure after a burn-in strategy is adopted. In this strategy the items which fail before their degradation reaches* $\mathfrak{D}_{\mathfrak{f}(1)}$, *are omitted. If* $T_1 := \inf\{t \geq 0 \mid W(t) > \mathfrak{D}_{\mathfrak{f}(1)}\}$ *then*

$$p_0^\star = P(0 \leq T^\star < T_1) = 0, p_1^\star = P(T_1 \leq T^\star < +\infty) = 1$$

and also we assume that

$$p_0 = P(0 \leq T < T_1) > 0, p_1 = P(T_1 \leq T < +\infty) < 1.$$

Since, $P \preceq P^\star$, *with* $P = (p_0, p_1)$ *and* $P^\star = (0,1)$, *thus, according to Theorem 9(i),* $T \leq_{st} T^\star$. *Note that* $\frac{p_0^\star}{p_0} < \frac{p_1^\star}{p_1}$, *therefore, if X is OSSRD then by Theorem 11(i),* $T \leq_{lr} T^\star$.

5. Concluding remarks

In this paper we have achieved two goals. The first one was developing a novel time-to-failure model to fit to the lifetime of devices under a typical degradation process namely the multiplicative degradation model $W(t) = X\eta(t)$. The basic idea was that the probabilities of failure of the device is constant in consecutive intervals when degradation amounts are increased (decreased). It was shown that the time-to-failure according to the model follows a well-known classical mixture model (Lemma 1). The second goal was to get some stochastic ordering properties under variation of probabilities in two different settings and to obtain conditions under which the device which has a stochastic greater lifetime is identified. The degradation intervals were assumed to be fixed in the two cases, the mean degradation function $\eta(t)$ was also fixed and the random variation X around $\eta(t)$ was assumed to follow a common distribution function in the two settings. The usual stochastic order holds true if a majorization property between probability vectors is satisfied, by which one concludes that when the probabilities is more spread out in one case in comparison with the other cases, the reliability of the device under degradation is, correspondingly, decreased. For the stronger stochastic orderings such as the likelihood ratio order, the hazard rate order and the reversed hazard rate order, it was clarified that further conditions on the distribution function of X are needed in addition to the ones that are necessary to be imposed to classify the probability vectors arisen from the two settings. We demonstrated by some examples that the conditions on distribution function of X are fulfilled for some typical applicative standard distributions.

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In the future of this work, we may consider other settings or frameworks to detect devices under degradation which have more reliability. For instance, the lower and the upper bounds of the degradation intervals may be selected to be (random or non-random) variables, the distribution function of X as well as the mean degradation amount around it may vary. Aging properties of the new time-to-failure model can also be investigated which is useful in model selection geostrategies.

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