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# Phase Portraits of the Families VII and VIII of the Quadratic Systems

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**Abstract:** The quadratic polynomial differential systems in the plane are the easiest nonlinear differential systems. They have been studied intensively due to their nonlinearity and to their big number of applications. These systems can be classified in ten classes. Here we provide all the topologically different phase portraits in the Poincaré disc of two of these classes.

**Keywords:** quadratic vector fields; quadratic systems; phase portraits

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## 1. Introduction and statement of the main results

A *quadratic polynomial differential system* or simply a *quadratic system* is a differential system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $P$ and $Q$ are real polynomials in the variables $x$ and $y$ and the maximum degree of the polynomials $P$ and $Q$ is two.

At the beginning of the XX century started to be studied the quadratic systems. In [5] Coppel said that Büchel [3] in 1904 published the first work on quadratic systems. Two short surveys on quadratic systems were published by Coppel [5] In 1966, and by Chicone and Tian [4] in 1982.

During these last decades quadratic systems were studied intensively and many good results on them were obtained, see the books [2,12,13] and the references therein. Although quadratic systems have been studied in more than one thousand papers, we are far for a complete understanding of these systems.

In [8] it is proved that any quadratic system is affine-equivalent, scaling the time variable if necessary, to a quadratic system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y) = d + ax + by + \ell x^2 + mxy + ny^2,$$

where $\dot{x} = P(x,y)$ is one of the following ten:

| | | | |
|---|---|---|---|
| (I) | $\dot{x} = 1 + xy,$ | (VI) | $\dot{x} = 1 + x^2,$ |
| (II) | $\dot{x} = xy,$ | (VII) | $\dot{x} = x^2,$ |
| (III) | $\dot{x} = y + x^2,$ | (VIII) | $\dot{x} = x,$ |
| (IV) | $\dot{x} = y,$ | (IX) | $\dot{x} = 1,$ |
| (V) | $\dot{x} = -1 + x^2,$ | (X) | $\dot{x} = 0.$ |

Roughly speaking the Poincaré disc is the disc centered at the origin of $\mathbb{R}^2$ and radius one, where the interior of this disc is identified with the whole plane $\mathbb{R}^2$ and its boudary the circle $\mathbb{S}^1$ is identified with the infinity of the plane $\mathbb{R}^2$, because in the plane we can go to infinity in as many directions as points has the circle $\mathbb{S}^1$. For more details on the Poincaré compactification see subsection 2.2.

The aim of this paper is to classify the different phase portraits in the Poincaré disc of the classes of quadratic systems VII and VIII, i.e. of the systems

$$\dot{x} = x^2, \qquad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2, \tag{2}$$

and

$$\dot{x} = x, \qquad \dot{y} = d + ax + by + \ell x^2 + mxy + ny^2, \tag{3}$$

respectively.

Our main result is the following one.

**Theorem 1.** *The following two statements hold.*

(a) *The family of quadratic systems* VII *has* 27 *topologicvally different phase portraits in the Poincaré disc.*
(b) *The family of quadratic systems* VIII *has* 25 *topologicvally different phase portraits in the Poincaré disc.*

Statements (a) and (b) of Theorem 1 are proved in sections 3 and 4, respectively.

The paper is organized as follows. In section 2 we recall the basic results that we need about singular points and the Poincaré compactification. In section 3 and 4 first we study the local phase portraits of the finite singular points, after we study the local phase portraits of the infinite singular points, and finally we analyze the phase portraits of the quadratic systems (2) and (3) in the Poincaré disc, respectively.

## 2. Preliminary definitions

The study of the phase portraits of the quadratic systems always begin with the study of the local phase portraits of their singularities, finite and infinite, followed by the study of their separatrix connections and of their limit cycles.

In this section we introduce the basic definitions and notations that we will need for the analysis of the local phase portraits of the finite and infinite singular points.

### 2.1. Singular points

A point $q \in \mathbb{R}^2$ is said to be *singular point* of a polynomial differential system (1) if $P(q) = Q(q) = 0$. If the real part of the eigenvalues of the linear part of system (1) at the singular point $q$ are not zero, then $q$ is a *hyperbolic* singular point and its possible phase portraits are well known, see for instance Theorem 2.15 of [7]. If only one of the eigenvalues of the linear part of system (1) at the singular point $q$ is zero, then $q$ is called a *semi-hyperbolic* singular point, whose possible local phase portraits are also well known, see among others Theorem 2.19 of [7]. When both eigenvalues of the linear part of system (1) at the singular point $q$ are zero but the linear part is not identically null, then $q$ is a *nilpotent* singular point and again its local phase portraits are known, see for instant Theorem 3.5 of [7]. Finally when the linear part of system (1) at the singular point $q$ is identically zero, then we say that $q$ is *degenerate* or $q$ is *linearly zero*. The local phase portraits of a such singular point can be studied using the change of variables called blow-ups, see for instance [1].

### 2.2. Poincaré compactification

Let $X = (P, Q)$ be the vector field defined by the polynomial differential system (1). Roughly speaking the Poincaré compactification consists in creating a vector field $p(X)$ in a 2–dimensional sphere $\mathbb{S}^2$ such that its phase portrait in the open northern and southern hemispheres is a copy of the phase portrait of the vector field $X$, and the equator of the sphere plays the role of the infinity of the phase portrait of $X$, see for details [9], or Chapter 5 of [7]. In this way we can study the orbits of the vector field $X$ which goes to or comes from the infinity.

Let $\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the *Poincaré sphere*. We denote by $T_x\mathbb{S}^2$ the tangent plane to $\mathbb{S}^2$ at a point $x \in \mathbb{S}^2$. We consider the vector field $X$ defined on the plane $T_{(0,0,1)}\mathbb{S}^2$.

Then the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$ defines two copies of $X$ in $\mathbb{S}^2$, one in the northern hemisphere and the other in the southern hemisphere. Obviously the equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$, represents the *infinity* of $\mathbb{R}^2$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $x_3 = 0$ under $(x_1, x_2, x_3) \longmapsto (x_1, x_2)$ is called the *Poincaré disc,* and it is denoted by $\mathbb{D}^2$. As $\mathbb{S}^2$ is a differentiable manifold, we define six local charts $U_i = \{x \in \mathbb{S}^2 : x_i > 0\}$, and $V_i = \{x \in \mathbb{S}^2 : x_i < 0\}$ for $i = 1, 2, 3$; with the corresponding diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ which are the inverses of the central projections from the tangent planes at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively.

We denote by $(u, v)$ the value of $F_i(x)$ or $G_i(x)$ for any $i = 1, 2, 3$, so a few simple calculations lead for $p(X)$ the following formulae in the corresponding local charts (see Chapter 5 of [7]):

$$v^d \left( Q \left( \frac{1}{v}, \frac{u}{v} \right) - uP \left( \frac{1}{v}, \frac{u}{v} \right), -vP \left( \frac{1}{v}, \frac{u}{v} \right) \right) \quad \text{in } U_1,$$

$$v^d \left( P \left( \frac{u}{v}, \frac{1}{v} \right) - uQ \left( \frac{u}{v}, \frac{1}{v} \right), -vQ \left( \frac{u}{v}, \frac{1}{v} \right) \right) \quad \text{in } U_2,$$

$$(P(u, v), Q(u, v)) \quad \text{in } U_3,$$

where $d$ is the degree of the polynomial differential system (1). The formulae for $V_i$ are similar to the one for $U_i$ with a multiplicative factor $(-1)^{d-1}$. In these coordinates for $i = 1, 2$, the points $(u, v)$ of the infinity $\mathbb{S}^1$ satisfy $v = 0$.

## 3. Proof of statement $(a)$ of Theorem 1

### 3.1. Finite singular points

We are going to determine the local phase portrait at the finite singular points of the quadratic system (2).

Assume first $n \neq 0$. If $b^2 - 4dn > 0$ then the finite singular points of system (2) are

$$p_\pm = \left( 0, \frac{-b \pm \sqrt{b^2 - 4dn}}{2n} \right),$$

The eigenvalues of the Jacobian matrix of system (2) at $p_\pm$ are $0$ and $\pm\sqrt{b^2 - 4dn}$. So from Theorem 2.19 of [7] we have that $p_+$ and $p_-$ are semi-hyperbolic saddle-nodes.

If $b^2 - 4dn < 0$ there are no finite singular points.

If $b^2 - 4dn = 0$ then $d = b^2/(4n)$ and $p_+ = p_- = p = (0, -b/(2n))$. The Jacobian matrix of the differential system at $p$ is

$$\begin{pmatrix} 0 & 0 \\ a - \dfrac{bm}{2n} & 0 \end{pmatrix}.$$

If $a - bm/(2n) \neq 0$ then this singular point is nilpotent, and from Theorem 3.5 of [7], this singular point is a saddle-node.

If $a = bm/(2n)$ the linear part of the differential system at the equilibrium point $p$ is identically zero, and the differential system becomes a homogeneous quadratic differential system. Using the results of Date in [6] who classify the phase portraits of all the homogeneous quadratic systems we obtain that the phase portraits of system (3) when $(m-1)^2 - 4\ell n > 0$ are given in Figure 1 according the sign of $n$, If $(m-1)^2 - 4\ell n = 0$ then the phase portraits of system (3) are given in Figure 3 according with the sign of $n$. Finally if $(m-1)^2 - 4\ell n < 0$ the phase portraits of system (3) are given in Figure 2 according with the sign of $n$.
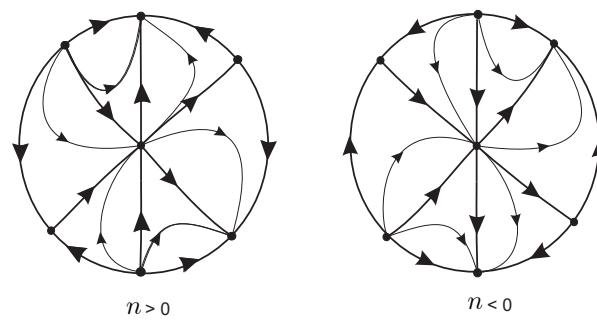
**Figure 1.** $(m-1)^2 - 4\ell n > 0$.
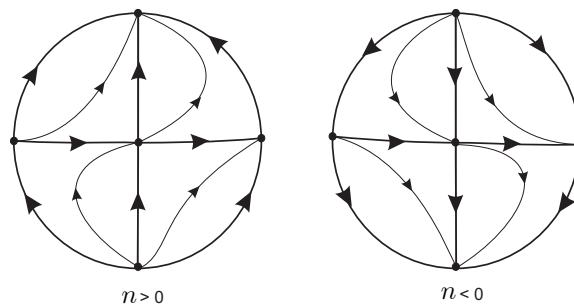


**Figure 2.** $(m-1)^2 - 4\ell n = 0$.
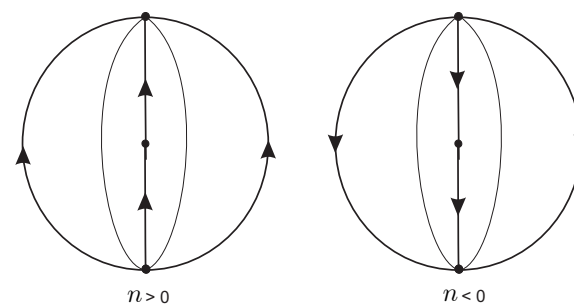


**Figure 3.** $(m-1)^2 - 4\ell n < 0$.

We assume now $n = 0$. In this case if $b \neq 0$ there exists a unique singular point namely $q = (0, -d/b)$, and the eigenvalues of the Jacobian matrix at $q$ are 0 and $b$. If $b \neq 0$ then $q$ is a semi-hyperbolic saddle-node by Theorem 2.19 of [7]. If $b = 0$ and $d \neq 0$ the differential system has no finite equilibria. If $b = d = 0$ then the system has a straight line filled with equilibria and we do not consider this kind of differential systems because this case can be reduced to a linear differential system doing a rescaling in the independent variable.

In summary we have proved the following proposition.

**Proposition 1.** *Assume that $n \neq 0$.*

(a) *If $b^2 - 4dn > 0$ the differential system (2) has two finite equilibria $p_\pm$ that are semi-hyperbolic saddle-nodes.*

(b) *If $b^2 - 4dn < 0$ the differential system (2) has no finite equilibria.*

(c) *$b^2 - 4dn = 0$.*

    (c.1) *If $a - bm/(2n) \neq 0$ the differential system (2) has one finite equilibrium point $p$ that is a nilpotent saddle-node.*

    (c.2) *$a - bm/(2n) = 0$.*

(c.2.1) *If $(m-1)^2 - 4\ell n > 0$ the phase portrait of the differential system (2) is topologically equivalent to the ones of Figure 1 according with the sign of n.*

(c.2.2) *If $(m-1)^2 - 4\ell n = 0$ the phase portrait of the differential system (2) is topologically equivalent to the ones of Figure 2 according with the sign of n.*

(c.2.3) *If $(m-1)^2 - 4\ell n < 0$ the phase portrait of the differential system (2) is topologically equivalent to the ones of Figure 3 according with the sign of n.*

*Assume that $n = 0$.*

(d) *If $b \neq 0$ the differential system (2) has one finite equilibria q which is a semi-hyperbolic saddle-node.*

(e) *If $b = 0$ then differential system (2) has no finite equilibria if $d \neq 0$, and one straight line of filled with equilibria if $d = 0$.*

We assume now $n = 0$. In this case if $b \neq 0$ there exists a unique singular point namely $q = (0, -d/b)$, and the eigenvalues of the Jacobian matrix at $q$ are 0 and $b$. If $b \neq 0$ then $q$ is a semi-hyperbolic saddle-node by Theorem 2.19 of [7]. If $b = 0$ and $d \neq 0$ the differential system has no finite equilibria. If $b = d = 0$ then the system has a straight line filled with equilibria and we do not consider this kind of differential systems because this case can be reduced to a linear differential system doing a rescaling in the independent variable.

### 3.2. The infinite singular points in the chart $U_1$

System (refe2x) in the local chart $U_1$ writes

$$\dot{u} = \ell - u + mu + nu^2 + av + buv + dv^2, \qquad \dot{v} = -v. \tag{4}$$

Assume $n \neq 0$ the infinite singular points are

$$P_\pm = \left(0, \frac{1 - m \pm \sqrt{(1-m)^2 - 4\ell n}}{2n}\right).$$

The eigenvalues of the Jacobian matrix at $P_\pm$ are $S_\pm = \left(-1, \pm\sqrt{(1-m)^2 - 4\ell n}\right)$. If they are real, then $(1-m)^2 - 4\ell n > 0$ and $P_+$ is a hyperbolic saddle and $P_-$ is a hyperbolic stable node. If $(1-m)^2 - 4\ell n = 0$ then $P_+ = P_- = P = (0, (1-m)/(2n))$. In this case the Jacobian matrix writes

$$\begin{pmatrix} 0 & a + \dfrac{b(1-m)}{2n} \\ 0 & -1 \end{pmatrix},$$

and the eigenvalues are -1 and 0, wich means that the unique singular point in the chart $U_1$ it is semi-hyperbolic and from Theorem 2.19 of [7] is a semi-hyperbolic saddle-node.

Assume now $n = 0$. Then the unique infinite singular point in the local chart $U_1$ is $P = (-\ell/(m-1), 0)$, and the eigenvalues of the Jacobian matrix of system (4) at $P$ are $-1$ and $m - 1$. If $m \neq 1$, from Theorem 2.15 of [7] $P$ is a hyperbolic saddle if $m > 1$ and a hyperbolic node if $m < 1$. If $m = 1$ there are no singular points in $U_1$.

### 3.3. The infinite singular point at the origin of the chart $U_2$

Studying the infinite singular points in the local chart $U_1$ we also have studied the infinite singular points in the local chart $V_1$. So only remains to see if the origins of the local charts $U_2$ and $V_2$ are infinite singular points or not.

System (2) in the local chart $U_2$ writes

$$\dot{u} = -u(n + (m-1)u + bv + \ell u^2 + auv + dv^2) = P(u, v),$$
$$\dot{v} = -v(n + mu + bv + \ell u^2 + auv + dv^2) = Q(u, v), \tag{5}$$

so the origin of $U_2$ always is an infinite singular point. The eigenvalues of the Jacobian matrix of system (8) at the origin are $-n$ with multiplicity two. Therefore the origin is a hyperbolic stable node if $n > 0$, and unstable node if $n < 0$.

If $n = 0$ then the Jacobian matrix of the system at the origin of the local chart $U_2$ is the zero matrix and we need to make blow-ups in order to study its local phase portrait. Before doing a vertical blow-up we need to be sure that $u = 0$ is not a characteristic direction. If $u = 0$ is a characteristic direction then $u$ is a factor of the polynomial $\Pi = vP_2(u,v) - uQ_2(u,v)$, where $P_2(u,v)$ and $Q_2(u,v)$ are the terms of degree two of $P(u,v)$ and $Q(u,v)$. In our case $\Pi = u^2v$. So $u = 0$ is a characteristic direction and consequently before doing a vertical blow-up, we must do a twist in order that $u = 0$ do not be a characteristic direction. This is done with the change of variables $(u,v) \rightarrow (u_1,v_1)$ where $u_1 = u + v$, $v_1 = v$. Doing this change of variables the differential system (8) writes

$$\dot{u}_1 = (1 - m)u_1^2 - (b + 2 - m)u_1v_1 + v_1^2 - \ell u_1^3 - (a - 2\ell)u_1^2v_1 + (a - d - \ell)u_1v_1^2$$
$$\dot{v}_1 = -mu_1v_1 + (m - b)v_1^2 - \ell u_1^2v_1 + (2\ell - a)u_1v_1^2 + (a - d - \ell)v_1^3, \tag{6}$$

Since $u_1 = 0$ is not a characteristic direction we can do a vertical blow-up. This vertical blow-up is given by the change of variables $(u_1,v_1) \rightarrow (u_2,v_2)$ where $u_2 = u_1, v_2 = v_1/u_1$. Then system (6) becomes

$$\dot{u}_2 = u_2^2\big(m - 1 + \ell u_2 + (b - m + 2)v_2 + (a - 2\ell)u_2v_2 - v^2 - (a - d - \ell)u_2v_2^2\big),$$
$$\dot{v}_2 = -u_2v_2(-1 + v_2)^2.$$

Now doing a rescaling of the time with the factor $u_2$ we obtain the system

$$\dot{u}_2 = u_2\big(m - 1 + \ell u_2 + (b - m + 2)v_2 + (a - 2\ell)u_2v_2 - v^2 - (a - d - \ell)u_2v_2^2\big),$$
$$\dot{v}_2 = -v_2(-1 + v_2)^2.$$

The singular points of system (11) on $u_2 = 0$ are $(0,0)$ and $(0,1)$ which is double. The eigenvalues of the Jacobian matrix at $(0,0)$ are $-1$ and $1 - m$. So the point $(0,0)$ is a hyperbolic stable node if $m > 1$, a hyperbolic saddle if $m < 1$, and for $m = 1$ a sami-hyperbolic saddle-node according to Theorem 2.19 of [7]. The eigenvalues of the Jacobian matrix at $(0,1)$ are $0$ and $-b$. So the local phase portrait of the origin of the local chart $U_2$ is shown in Figure 4(a) when $n = 0$, $b > 0$ and $m > 1$.
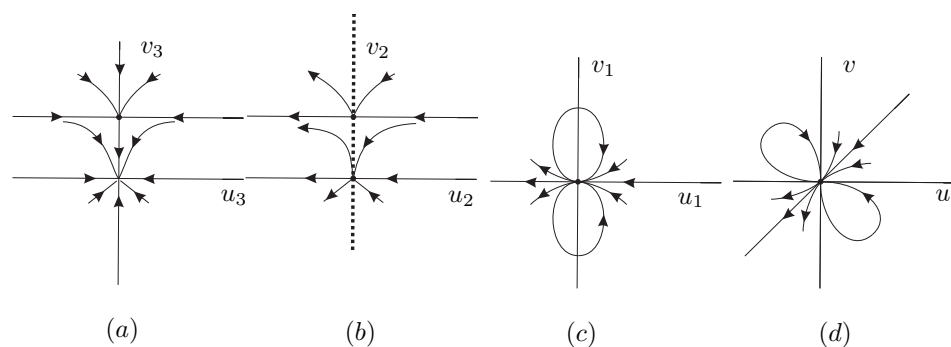


**Figure 4.** The sequences of blow-ups for obtaining the local phase portrait at the origin of the local chart $U_2$ when $n = 0$, $b > 0$ and $m > 1$.

Starting from Figure 4(a) we obtain the local phase portrait at the axis $u_2 = 0$ of system (10): see Figure 4(b) and going back through the vertical blow, taking into account the value of $\dot{u}_1|_{u_1=0} = v_1^2$ we obtain the local phase portrait at the origin of system (5) in Figure 4(c). Finally undoing the twist we get the local phase portrait at the origin of the local chart $U_2$ which is shown in 4(d) and 5(a).

Working in a similar way to the preceding case, doing the convenient blow ups and using Theorems 2.15 and 2.19 of [7] we obtain all the local phase portraits at the origin of the local chart $U_2$ in Figure 5. All those local phase portraits are then the following

$n = 0$, $b > 0$ and $m > 1$ in Figure 5(a);

$n = 0$, $b < 0$ and $m > 1$ in Figure 5(b);

$n = 0$, $b > 0$ and $m < 1$ in Figure 5(c);

$n = 0$, $b < 0$ and $m < 1$ in Figure 5(d);

$n = 0$, $b > 0$, $m = 1$ and $l \neq 0$ in Figure 5(e);

$n = 0$, $b > 0$, $m = 1$ and $l = 0$ then $v = 0$ is a stright line of singular points;

$n = 0$, $b < 0$, $m = 1$ and $l \neq 0$ in Figure 5(f);

$n = 0$, $b < 0$, $m = 1$ and $l = 0$ then $v = 0$ is a stright line of singular points;

$n = 0$, $b = 0$, $d > 0$ and $m > 1$ in Figure 5(g);

$n = 0$, $b = 0$, $d < 0$ and $m > 1$ in Figure 5(h);

$n = 0$, $b = 0$, $d > 0$ and $m < 1$ in Figure 5(i);

$n = 0$, $b = 0$, $d < 0$ and $m < 1$ in Figure 5(j);

$n = 0$, $b = 0$, $d > 0$ and $m = 1$ in Figure 5(k);

$n = 0$, $b = 0$, $d < 0$ and $m = 1$ in Figure 5(l);

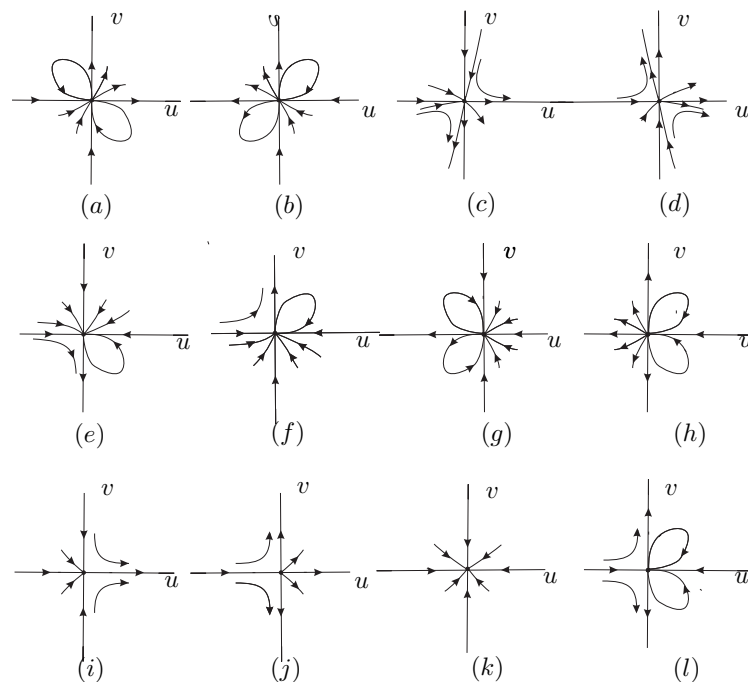$n = 0$, $b = 0$ and $d = 0$ then $u = 0$ is a stright line of singular points.



**Figure 5.** The distinct topological local phase portraits at the origin of the local chart $U_2$.

*3.4. The global phase portaits*

The preceding results for the finite and infinite singular points, allow to obtain the global phase portraits quite easily, taking into account that the straight line $x = 0$ is invariant.

First we consider the case satisfying the following conditions: $n > 0$, $b^2 - 4dn > 0$ and $(1 - m)^2 > 4\ell n$. We have seen that if $n > 0$ then there is a stable hyperbolic node at the origin of the chart $U_2$. Since $b^2 - 4dn > 0$ there exist two real finite singular points $p_+$ and $p_-$ that are semi-hyperbolic saddle-nodes. Finitely since $(1 - m)^2 > 4\ell n$ imply the existence of two infinite singular points in the chart $U_1$ ($P_+$ is a hyperbolic saddle and $P_-$ a hyperbolic node). The local phase portraits at all

these singular points are shown in Figure 6. The tools for studying the phase portraits in this case are employed for all possible configurations which appears in Figure 7.
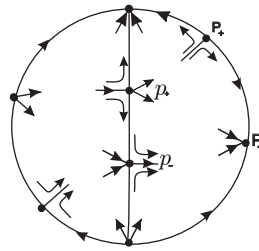


**Figure 6.** The local phase portraits at the finite and infinite singular points for the case $n > 0$, $b^2 - 4dn > 0$ and $(1 - m)^2 > 4\ell n$.
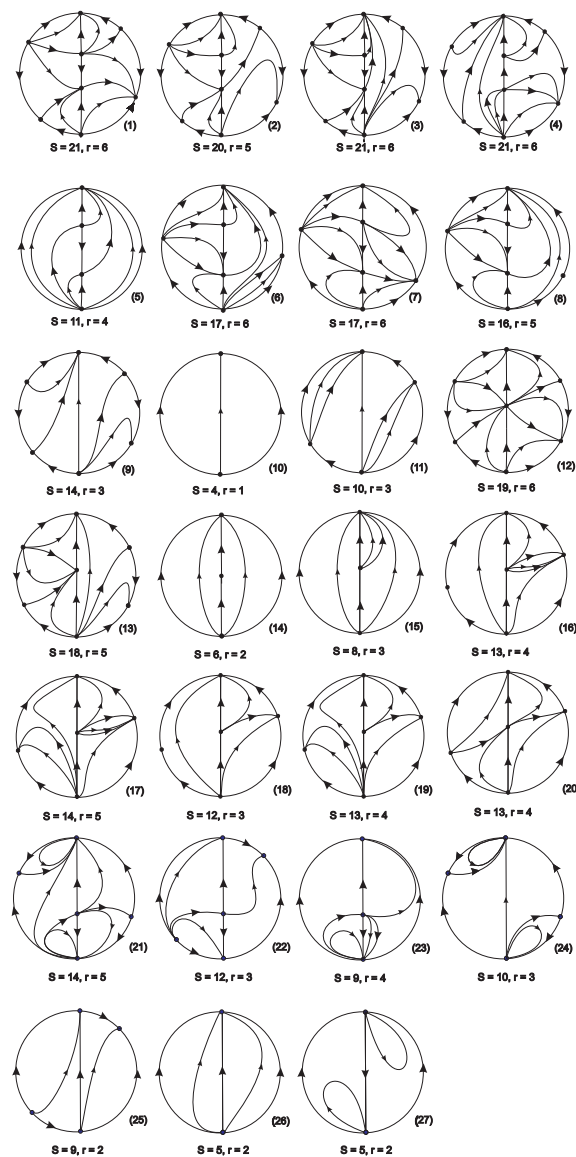


**Figure 7.** All the distinct topological phase portraits of the quadratic systems VII. Here $s$ (respectively $r$) denotes the number of separatrices of a phase portrait in the Poincaré disc (respectively canonical regions).

$n > 0$, $b^2 - 4dn > 0$ and $(1 - m)^2 > 4\ell n$ in Figures 7(1) to 7(4), but the phase portrait in Figure 7(2) appears by continuity between the phase portraits in Figure 7(1) to 7(3);

$n > 0$, $b^2 - 4dn > 0$ and $(1 - m)^2 < 4\ell n$ from Figure 7(5);

$n > 0$, $b^2 - 4dn > 0$ and $(1 - m)^2 = 4\ell n$ in Figures 7(6) to 7(8);

$n > 0$, $b^2 - 4dn < 0$, and $(1 - m)^2 > 4\ell n$ in Figure 7(9);

$n > 0$, $b^2 - 4dn < 0$, and $(1 - m)^2 < 4\ell n$ in Figure 7(10);

$n > 0$, $b^2 - 4dn < 0$, $(1 - m)^2 = 4\ell n$ in Figure 7(11);

$n > 0$, $b^2 - 4dn = 0$, and $(1 - m)^2 > 4\ell n$ from Figure 7(12) and 7(13);

$n > 0$, $b^2 - 4dn = 0$, and $(1 - m)^2 < 4\ell n$ in Figures 7(14) and 7(15);

$n > 0$, $b^2 - 4dn = 0$, $(1 - m)^2 = 4\ell n$ in Figures 7(16) to 7(20).

The phase portraits with $n < 0$ are symmetric with respect to the origin of coordinates of the preceding eight cases.

Now we study the phase portraits when $n = 0$.

$n = 0$, $b > 0$ and $m > 1$ in Figure 7(21);

$n = 0$, $b > 0$ and $m < 1$ in Figure 7(22);

$n = 0$, $b > 0$ and $m = 1$ in Figure 7(23);

The cases with $b < 0$ are the symmetric with respect to the origin of coordinates of the preceding three cases.

$n = 0$, $b = 0$, $d > 0$ and $m > 1$ in Figure 7(24);

The cases with $d < 0$ are the symmetric with respect to the origin of coordinates of all the preceding case.

$n = 0$, $b = 0$, $d > 0$ and $m < 1$ in Figure 7(25);

The cases with $d < 0$ are the symmetric with respect to the origin of coordinates of all the preceding case.

$n = 0$, $b = 0$, $d > 0$ and $m = 1$ in Figure 7(26);

$n = 0$, $b = 0$, $d < 0$ and $m = 1$ in Figure 7(27).

Of course from the Table 1 the phase portraits with different number of separatrices and canonical regions are topologically distinct. Now we shall see that the phase portraits with the same number of separatrices and canonical regions of the Table 1 also are topologically different.

**Table 1.** Here p.p. denotes phase portrait in the Poincaré disc, $s$ denotes the number of separaratrices of the phase portrait, and $r$ denotes the number of canonical regions of the phase portrait.

| s | 4 | 5 | 6 | 8 | 9 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|----|----|----|
| r | 1 | 2 | 2 | 3 | 4 | 2 | 3 | 4 | 3 |
| p.p. | 10 | 26,27 | 14 | 15 | 23 | 25 | 11,24 | 5 | 18,22 |
| s | 13 | 14 | 14 | 16 | 17 | 18 | 19 | 20 | 21 |
| r | 4 | 3 | 5 | 5 | 6 | 5 | 6 | 5 | 6 |
| p.p. | 16,19,20 | 9 | 17,21 | 8 | 6,7 | 13 | 12 | 2 | 1,3,4 |

The phase portraits 26 and 27 of Figure 7 are topologically different because the phase portrait 27 has two elliptic sectors and the phase portrait 26 has no elliptic sectors.

The phase portraits 11 and 24 of Figure 7 are topologically different because the phase portrait 24 has two elliptic sectors and the phase portrait 11 has no elliptic sectors.

The phase portraits 18 and 22 of Figures 7 are topologically different because the phase portrait 18 have orbits starting at the origin of the local chart $U_2$ and ending at the origin of the local chart $U_1$, and this kind of orbits do not exist in the phase portrait 22.

The phase portraits 16, 19 and 20 of Figure 7 are topologically different. First the phase portrait 16 have orbits starting at the origin of the local chart $U_2$ and ending at the origin of the local chart $U_1$, and this kind of orbits do not exist in the phase portrait 19 and 20. The phase portrait 19 has a separatrix

starting at the origin of the local chart $U_2$ and ending at an infinite equilibrium point in the local chart $V_1$, and such a kind of separatrix does not exist in the phase portrait 20.

The phase portraits 17 and 21 of Figure 7 are topologically different because the phase portrait 21 has two elliptic sectors and the phase portrait 17 has no elliptic sectors.

The phase portraits 1, 3 and 4 of Figure 7 are topologically different because the unstable separatrix of the lower equilibrium point on the straight line $x = 0$ contained in $x > 0$ has different ending infinite equilibrium point in the these three phase portraits.

## 4. Proof of statement ($b$) Theorem 1

### 4.1. Finite singular points

We are going to analyze the singular points of the quadratic system (3).

Assume first $n \neq 0$. The finite singular points of system (3) are

$$p_\pm = \left(0, \frac{-b \pm \sqrt{b^2 - 4dn}}{2n}\right).$$

If $b^2 - 4dn > 0$, the eigenvalues of the Jacobian matrix of system (3) at $p_\pm$ are 1 and $\pm\sqrt{b^2 - 4dn}$. So from Theorem 2.15 of [7] we have $p_+$ is a hyperbolic unstable node and $p_-$ is a hyperbolic saddle. If $b^2 - 4dn = 0$ then $p_+ = p_- = p = (0, -b/(2n))$. The eigenvalues of the Jacobian matrix of system (3) at $p$ are $1, 0$ therefore by Theorem 2.19 of [7] we obtain that $p$ is a semi-hyperbolic saddle-node. Of course, if $b^2 - 4dn < 0$ there are no finite singular points.

We assume now $n = 0$. In this case if $b \neq 0$ there exists a unique singular point namely $p = (0, -d/b)$, and the eigenvalues of the Jacobian matrix at $p$ are 1 and $b$. If $b > 0$ then $p$ is a hyperbolic unstable node. If $b < 0$ then $p$ is a hyperbolic saddle. If $b = 0$ there are no finite singular points.

### 4.2. The infinite singular points in the chart $U_1$

System (3) in the local chart $U_1$ writes

$$\dot{u} = \ell + mu + av + nu^2 + (b - 1)uv + dv^2, \qquad \dot{v} = -v^2, \qquad (7)$$

Assume $n \neq 0$ the infinite singular points are

$$P_\pm = \left(0, \frac{-m \pm \sqrt{m^2 - 4\ell n}}{2n}\right),$$

if $m^2 - 4\ell n > 0$. If $m^2 - 4\ell n = 0$ then $P_+ = P_- = P = (0, -m/(2n))$. The eigenvalues of the Jacobian matrix at $P_\pm$ are 0 and $\pm\sqrt{m^2 - 4\ell n})$. By Theorem 2.19 of [7] we get $P_\pm$ are semi-hyperbolic saddle-nodes. The Jacobian matrix at $P$ is

$$\begin{pmatrix} 0 & \dfrac{2an - bm + m}{2n} \\ 0 & 0 \end{pmatrix}.$$

If $2an + (1 - b)m \neq 0$ then $P$ is a nilpotent singular point, and by Theorem 3.5 of [7] is a saddle-node. If $2an + (1 - b)m = 0$, then $P$ is degenerate. If we translate the singular point $P$ to the origin it becomes a homogeneous quadratic system and their phase portraits have been classified by Date in [6]. It follows that if $b^2 - 4dn \geq 0$ we obtain that the local phase portrait at $P$ on the Poincaré sphere is formed by two hyperbolic sectors separated by two parabolic ones, the infinity separates the two hyperbolic sectors which have one separatrix at infinity. If $b^2 - 4dn < 0$, then the local phase portrait at $P$ is a node, unstable if $n < 0$, and stable if $n > 0$.

Assume now $n = 0$. Then the unique infinite singular point in the local chart $U_1$ is $P = (-l/m, 0)$, and the eigenvalues of the Jacobian matrix of system (7) at $P$ are 0 and $m$. If $m \neq 0$, from Theorem 2.19 of [7] $P$ is a semi-hyperbolic saddle-node. If $m = 0$ there are no infinite singular points in the local chart $U_1$.

### 4.3. The infinite singular point at the origin of the chart $U_2$

Studying the infinite singular points in the local chart $U_1$ we also have studied the infinite singular points in the local chart $V_1$. So only remains to see if the origins of the local charts $U_2$ and $V_2$ are infinite singular points or not.

System (3) in the local chart $U_2$ writes

$$
\begin{aligned}
\dot{u} &= -u(n + mu + (b-1)v + \ell u^2 + auv + dv^2) = P(u,v), \\
\dot{v} &= -v(n + mu + bv + \ell u^2 + auv + dv^2) = Q(u,v),
\end{aligned}
\tag{8}
$$

so the origin of $U_2$ always is an infinite singular point. The eigenvalues of the Jacobian matrix of the system at the origin are $-n$ with multiplicity two. Therefore the origin is a hyperbolic node, stable if $n > 0$, and unstable if $n < 0$.

If $n = 0$ then the Jacobian matrix of the system at the origin is the zero matrix and we need to make blow-ups in order to study the local phase portrait at the origin of $U_2$. Before doing a vertical blow-up we need to be sure that $u = 0$ is not a characteristic direction. If $u = 0$ is a characteristic direction then $u$ is a factor of the polynomial $\Pi = vP_2(u,v) - uQ_2(u,v)$, where $P_2(u,v)$ and $Q_2(u,v)$ are the terms of lowest degree of $P(u,v)$ and $Q(u,v)$. In our case $\Pi = uv^2$. So $u = 0$ is characteristic direction and consequently before doing a vertical blow-up, we must do a twist in order that $u = 0$ do not be a characteristic direction. This is done with the change of variables $(u,v) \rightarrow (u_1, v_1)$ where $u_1 = u + v$, $v_1 = v$. Doing this change of variables the differential system (8) writes

$$
\begin{aligned}
\dot{u}_1 &= -mu_1^2 - v_1^2 + (1 - b + m)u_1v_1 + (a - d - \ell)u_1v_1^2 - \ell u_1^3 + (2\ell - a)u_1^2 v_1 \\
\dot{v}_1 &= -v_1\big(mu_1 + (b - m)v_1 + \ell u_1^2 + (a - 2\ell)u_1v_1 + (d - a + \ell)v_1^2\big),
\end{aligned}
\tag{9}
$$

The characteristic directions of this system are given by the polynomial $\Pi = (u_1 - v_1)v_1^2$, so $u_1 = 0$ is not a characteristic direction and we can do a vertical blow-up. This vertical blow-up is given by the change of variables $(u_1, v_1) \rightarrow (u_2, v_2)$ where $u_2 = u_1$, $v_2 = v_1/u_1$. Then system (9) becomes

$$
\begin{aligned}
\dot{u}_2 &= u_2^2\big(-m - \ell u_2 + (1 - b + m)v_2 + (2\ell - a)u_2v_2 - v_2^2 + (a - d - \ell)u_2v_2^2\big), \\
\dot{v}_2 &= u_2(-1 + v_2)v_2^2.
\end{aligned}
\tag{10}
$$

Now doing a rescaling of the time with the factor $u_2$ we obtain the system

$$
\begin{aligned}
\dot{u}_2 &= u_2\big(-m - \ell u_2 + (1 - b + m)v_2 + (2\ell - a)u_2v_2 - v_2^2 + (a - d - \ell)u_2v_2^2\big), \\
\dot{v}_2 &= (-1 + v_2)v_2^2.
\end{aligned}
\tag{11}
$$

The singular points of system (11) on $u_2 = 0$ are $(0,0)$ which is double and $(0,1)$. The eigenvalues of the Jacobian matrix at $(0,0)$ are 0 and $-m$. So the $(0,0)$ is a semi-hyperbolic singular point, applying to it Theorem 2.19 of [7] it is a saddle-node. The eigenvalues of the Jacobian matrix at $(0,1)$ are 1 and $-b$. So this singular point is hyperbolic, a saddle if $b > 0$, an unstable node if $b < 0$, see Figure 8(a) when $n = 0$, $m < 0$ and $b < 0$.

From Figure 8(a) we obtain that the local phase portrait at the axis $u_2 = 0$ of system (10) is given in Figure 8(b). Now going back through the vertical blow up and taking into account the value of $\dot{u}_1|_{u_1=0} = -v_1^2$ we obtain the local phase portrait at the origin of system (8) in Figure 8(c). Finally

unding the twist we get the local phase portrait at the origin of the local chart $U_2$ which is shown in Figures 8(d) and 9(a).
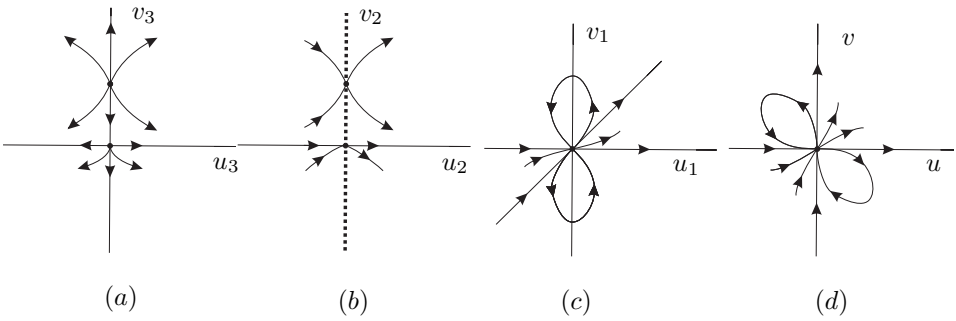
**Figure 8.** The sequences of blow-ups for obtaining the local phase portrait at the origin of the local chart $U_2$ when $n = 0$, $b < 0$ and $m < 0$.
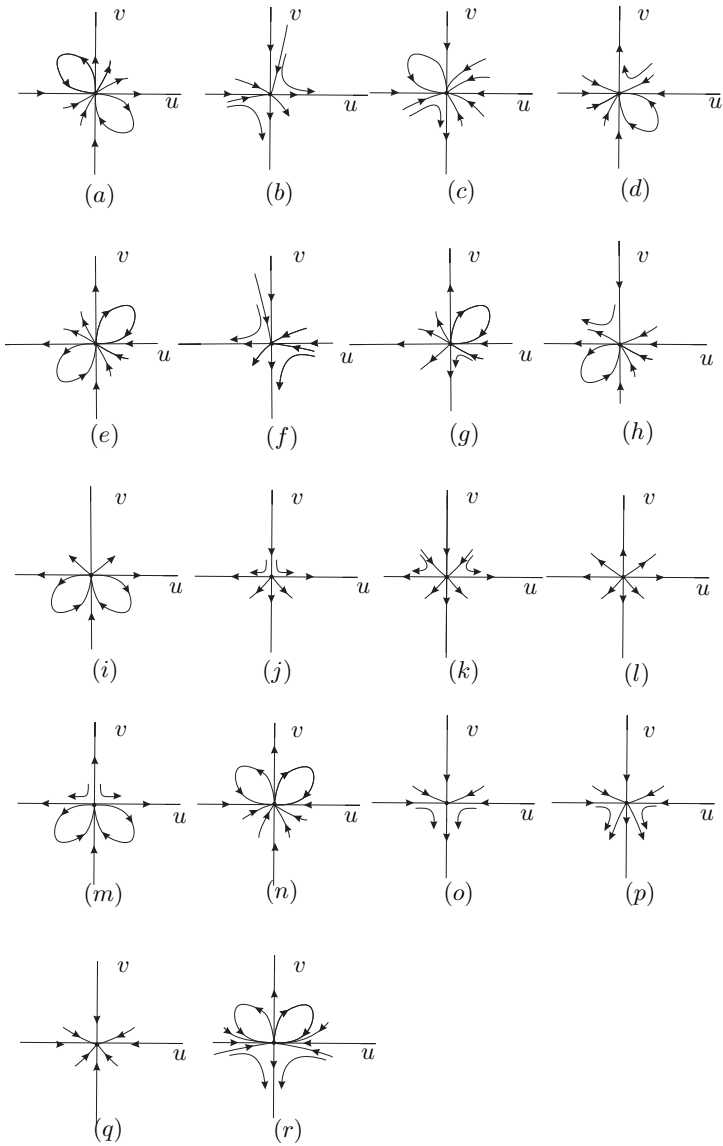
**Figure 9.** The distinct topological local phase portraits at the origin of the local chart $U_2$.

Working in a similar way to the case $n = 0$, $b < 0$ and $m < 0$, i.e. doing the convenient blow ups and using Theorems 2.15 and 2.19 of [7] we obtain all the local phase portraits at the origin of the local chart $U_2$ in Figure 9 for the following cases:

$n = 0$, $m < 0$ and $b > 0$ in Figure 9(b);
$n = 0$, $m < 0$, $b = 0$, and $d < 0$ in Figure 9(c);
$n = 0$, $m < 0$ $b = 0$ and $d > 0$, in Figure 9(d);
$n = 0$, $m > 0$ and $b < 0$ in Figure 9(e);
$n = 0$, $m > 0$ and $b > 0$ in Figure 9(f);
$n = 0$, $m > 0$, $b = 0$ and $d < 0$ in Figure 9(g);
$n = 0$, $m > 0$, $b = 0$ and $d > 0$ in Figure 9(h);
$n = 0$, $m = 0$, $\ell < 0$ and $b < 0$ in Figure 9(i);
$n = 0$, $m = 0$, $\ell < 0$ and $0 < b \leq \ell + 2$ in Figure 9(j);
$n = 0$, $m = 0$, $\ell < 0$ and $b > \ell + 2$ in Figure 9(k);
$n = 0$, $m = 0$, $\ell < 0$, $b = 0$ and $d < 0$ in Figure 9(l);
$n = 0$, $m = 0$, $\ell < 0$, $b = 0$ and $d > 0$ in Figure 9(m);
$n = 0$, $m = 0$, $\ell > 0$, $b < 0$ and in Figure 9(n);
$n = 0$, $m = 0$, $\ell > 0$ and $0 < b \leq \ell + 2$ in Figure 9(o);
$n = 0$, $m = 0$, $\ell > 0$ and $b > \ell + 2$ in Figure 9(p);
$n = 0$, $m = 0$, $\ell > 0$, $b = 0$ and $d < 0$ in Figure 9(q);
$n = 0$, $m = 0$, $\ell > 0$ $b = 0$ and $d > 0$ in Figure 9(r).

### 4.4. The global phase portaits

The preceding results for the finite and infinite singular points, allow to obtain the global phase portraits quite easily, taking into account that the straight line $x = 0$ is invariant.

First we consider the case satisfying the following conditions: $n > 0$, $b^2 - 4dn > 0$ and $m^2 > 4\ell n$. We have seen that $n > 0$ denotes a stable hyperbolic node at the origin of the chart $U_2$, $b^2 - 4dn > 0$ indicates the existence of two real finite singular points ($p_+$ a hyperbolic unstable node and $p_-$ a hyperbolic saddle), and $m^2 > 4\ell n$ imply two infinite singular points in the chart $U_1$ ($P_+$ and $P_-$ which are nilpotent saddle-nodes). The local phase portraits at all these singular points are shown in Figure 10.
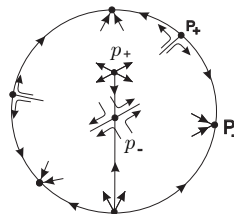


**Figure 10.** The local phase portraits at the finite and infinite singular points for the case $n > 0$, $b^2 - 4dn > 0$ and $m^2 > 4\ell n$.

With the help of Mathematica we have proved that in order that the conditions $n > 0$, $b^2 - 4dn > 0$ and $m^2 > 4\ell n$ hold, the parameters of the differential system (3) must satisfy one of the conditions:

  (i) $b < 0$, $d \leq 0$, $\ell < 0$ and $n > 0$;
  (ii) $b < 0$, $d \leq 0$, $\ell \geq 0$, $n > 0$ and $m < -2\sqrt{\ell n}$;
 (iii) $b < 0$, $d \leq 0$, $\ell \geq 0$, $n > 0$ and $m > -2\sqrt{\ell n}$;
 (iv) $b < 0$, $d > 0$, $\ell < 0$ and $0 < n < b^2/(4d)$;
  (v) $b < 0$, $d > 0$, $\ell \geq 0$, $0 < n < b^2/(4d)$ and $m < -2\sqrt{\ell n}$;
 (vi) $b < 0$, $d > 0$, $\ell \geq 0$, $0 < n < b^2/(4d)$ and $m > -2\sqrt{\ell n}$;
 (vii) $b = 0$, $d < 0$, $\ell < 0$ and $n > 0$;
(viii) $b = 0$, $d < 0$, $\ell \geq 0$, $n > 0$ and $m < -2\sqrt{\ell n}$;
 (ix) $b = 0$, $d < 0$, $\ell \geq 0$, $n > 0$) and $m > -2\sqrt{\ell n}$;

(x)   $b > 0$, $d \leq 0$, $\ell < 0$ and $n > 0$;
(xi)   $b > 0$, $d \leq 0$, $\ell \geq 0$, $n > 0$ and $m < -2\sqrt{\ell n}$;
(xii)   $b > 0$, $d \leq 0$, $\ell \geq 0$, $n > 0$ and $m > -2\sqrt{\ell n}$;
(xiii)   $b > 0$, $d > 0$, $\ell < 0$ and $0 < n < b^2/(4d)$;
(xiv)   $b > 0$, $d > 0$, $\ell \geq 0$, $0 < n < b^2/(4d)$ and $m < -2\sqrt{\ell n}$;
(xv)   $b > 0$, $d > 0$, $\ell \geq 0$, $0 < n < b^2/(4d)$ and $m > -2\sqrt{\ell n}$.

We have proved that in the cases (i), (ii), (iv), (vii) and from (ix) to (xv) we get the phase portrait (1) of Figure 11; in the cases (iii), (vi) and (viii) we obtain the phase portrait (2) of Figure 11; and finally in the case (v) we get the phase portrait symetric to the phase portrait (2) with respect to the straigt line $x = 0$. For instance: the phase portrait (1) of Figure 11 is obtained when the parameters of system (3) are $d = a = 0$, $b = -1$, $\ell = -1$, $m = -3$ and $n = 1$; the phase portrait (2) of Figure 11 is obtained when the parameters are $d = a = 0$, $b = -1$, $\ell = 1$, $m = 3$ and $n = 1$.The phase portrait (3) of Figure 11 exists by continuity going from the phase portrait (1) to the phase portrait (2).
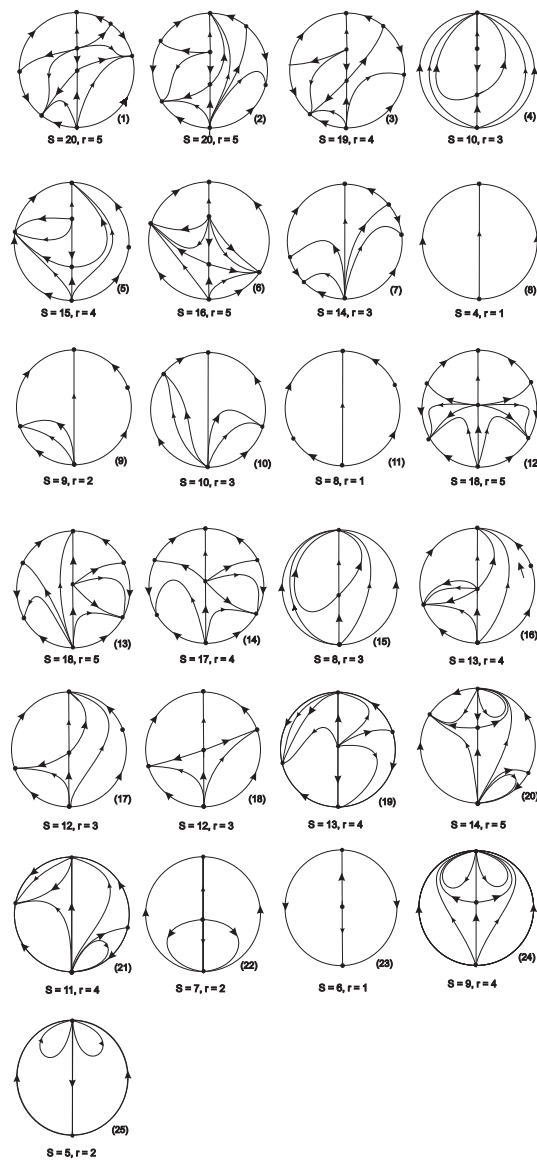


**Figure 11.** All the distinct topological phase portraits of the quadratic systems VIII. Here $s$ (respectively $r$) denotes the number of separatrices of a phase portrait in the Poincaré disc (respectively canonical regions).

We recall that the *separatrices* of a polynomial differential system in the Poincaré disc are all the orbits at infinity, the finite singularities and the two orbits at the boundary of an hyperbolic sector. Also the limit cycles are separatrices but the quadratic system VIII has no limit cycles. If in a phase portrait of the Poincaré disc we remove all the separatrices the open components which remain are called the *the canonical regions* of the phase portrait. For more details on the separatrices and canonical regions see [10,11].

The tools for studying the phase portraits of system (3) for the case $n > 0$, $b^2 - 4dn > 0$ and $m^2 > 4\ell n$ are used in the following cases, leading to the next results:

$n > 0$, $b^2 - 4dn > 0$, and $m^2 < 4\ell n$ in Figure 11(4);

$n > 0$, $b^2 - 4dn > 0$, $m^2 = 4\ell n$ and $2an + (1-b)m > 0$ in Figure 11(5);

$n > 0$, $b^2 - 4dn > 0$, $m^2 = 4\ell n$ and $2an + (1-b)m < 0$ in this case the phase portrait is symmetric with respect to the straight line $x = 0$ of the phase portrait of the previous case;

$n > 0$, $b^2 - 4dn > 0$, $m^2 = 4\ell n$ and $2an + (1-b)m = 0$ in Figure 11(6);

$n > 0$, $b^2 - 4dn < 0$, and $m^2 > 4\ell n$ in Figure 11(7);

$n > 0$, $b^2 - 4dn < 0$, and $m^2 < 4\ell n$ in Figure 11(8);

$n > 0$, $b^2 - 4dn < 0$, $m^2 = 4\ell n$ and $2an + (1-b)m > 0$ in Figure 11(9);

$n > 0$, $b^2 - 4dn < 0$, $m^2 = 4\ell n$ and $2an + (1-b)m < 0$ this case is a symetric phase portrait with respect to the straight line $x = 0$ of the previous phase phase portrait;

$n > 0$, $b^2 - 4dn < 0$, $m^2 = 4\ell n$ and $2an + (1-b)m = 0$ in Figures 11(10) and11(11);

$n > 0$, $b^2 - 4dn = 0$, and $m^2 > 4\ell n$ from Figure 11(12) to 11(14);

$n > 0$, $b^2 - 4dn = 0$, and $m^2 < 4\ell n$ in Figure 11(15);

$n > 0$ $b^2 - 4dn = 0$, $m^2 = 4\ell n$ and $2an + (1-b)m > 0$ in Figures 11(16) and 11(17);

$n > 0$, $b^2 - 4dn = 0$, $m^2 = 4\ell n$ and $2an + (1-b)m = 0$ in Figure 11(18);

The cases with $n < 0$ are the symetric with respect to the straight line $y = 0$ to all the preceding cases;

$n = 0$, $m > 0$, $b > 0$ in Figure 11(19);

$n = 0$, $m > 0$, $b < 0$ in Figure 11(20);

$n = 0$, $m > 0$, $b = 0$ and $d > 0$ in Figure 11(21);

$n = 0$, $m > 0$, $b = 0$ and $d < 0$ this case has a symmetric phase portrait with respect to $y = 0$ to the previous case;

The phase portraits of the cases $n = 0$ and $m < 0$ are symmetric with respect to the straight line $x = 0$ of the phase portraits of the cases $n = 0$ and $m > 0$;

$n = 0$, $m = 0$, $b > 2 + l$ and $\ell > 0$ in Figure 11(22);

$n = 0$, $m = 0$, $b > 2 + l$ and $\ell < 0$ this case has a symetric phase portrait with respect to the $y = 0$ axis;

$n = 0$, $m = 0$, $0 < b \le 2 + \ell$ and $\ell \ne 0$ in Figure 11(23);

$n = 0$, $m = 0$, $b < 0$ and $\ell > 0$ in Figure 11(24);

$n = 0$, $m = 0$, $b < 0$ and $\ell < 0$ the phase portrait of this case is symetric with respect to the straight line $y = 0$ of the previous phase portrait;

$n = 0$, $m = 0$, $b = 0$, $\ell > 0$ and $d < 0$ in the Figure 11(25);

$n = 0$, $m = 0$, $b = 0$, $\ell < 0$ and $d > 0$ the phase portrait of this case is symetric with respect to the straight line $y = 0$ of the previous phase portrait;

$n = 0$, $m = 0$, $b = 0$, $\ell > 0$ and $d > 0$ this case has the same phase portrait of Figure 11(8);

$n = 0$, $m = 0$, $b = 0$, $\ell < 0$ and $d < 0$ this case has the symetric phase portrait with respect to the straight line $y = 0$ to the phase portrait of Figure 11(8).

Of course from the Table 2 the phase portraits with different number of separatrices and canonical regions are topologically distinct. Now we shall see that the phase portraits with the same number of separatrices and canonical regions of the Table 2 also are topologically different.

The phase portraits 4 and 10 of Figure 12 are topologically different because the phase portrait 4 has two finite singular points and the phase portrait 10 has no finite singular points.

**Table 2.** Here p.p. denotes phase portrait in the Poincaré disc, $s$ denotes the number of separaratrices of the phase portrait, and $r$ denotes the number of canonical regions of the phase portrait.

| s | 4 | 5 | 6 | 7 | 8 | 8 | 9 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| r | 1 | 2 | 1 | 2 | 1 | 3 | 2 | 4 | 3 | 4 |
| p.p. | 8 | 25 | 23 | 22 | 11 | 15 | 9 | 24 | 4,10 | 21 |
| s | 12 | 13 | 14 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| r | 3 | 4 | 3 | 5 | 4 | 5 | 4 | 5 | 4 | 5 |
| p.p. | 17,18 | 16,19 | 7 | 20 | 5 | 6 | 14 | 12,13 | 3 | 1,2 |

The phase portraits 17 and 18 (respectively 16 and 19) of Figure 12 are topologically different because the phase portrait 17 (respectively 16) has two orbits going to the origin of the chart $U_2$, and such orbits do not exist in the phase portrait 18 (respectively 19).

The phase portrait 14 of Figure 12 has three pairs of infinite singular points while the phase portrats 16 and 19 only have only two pairs of infinite singumlar, so the phase portrait 14 is different from the phase portraits 16 and 19.

We note that the phase portrait 13 of Figure 12 has three pairs of infinite singular points, while the phase portrait 20 has only two pairs, so these two phases portraits are topologically disctinct.

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