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Another Approach to the Analysis of Isotropic Rectangular Thin Plates Subjected to a Concentrated Bending Moment Using the Fourier Series

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Article

Another Approach to the Analysis of Isotropic Rectangular Thin Plates Subjected to a Concentrated Bending Moment Using the Fourier Series

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Abstract: The aim of this paper is the analysis of isotropic rectangular thin plates simply supported or clamped along two opposite edges and subjected to a concentrated bending moment perpendicular to these edges, whereby the other edges have arbitrary support conditions. The standard approach to this problem is to replace the bending moment with a couple of forces infinitely close and to use the known expressions of efforts and deformations for the plate subjected to concentrated forces; the results are then related to the first derivatives of those efforts and deformations with respect to the position of application of the load. In this study, the concentrated moment was expanded into a Fourier series, leading to a distributed external bending moment, and the boundary conditions and continuity equations were set. Plates of infinite length were also analyzed and the results obtained were identical to those in the literature.

Keywords: Isotropic rectangular thin plate; concentrated bending moment; plates of infinite length; Lévy solution; Fourier sine series

1. Introduction

The Kirchhoff–Love plate theory (KLPT) was developed in 1888 by Love using assumptions proposed by Kirchhoff [1]. The KLPT is governed by the Germain–Lagrange plate equation; this equation was first derived by Lagrange in December 1811 in correcting the work of Germain [2] who provided the basis of the theory. Lévy [3] proposed in 1899 an approach for rectangular plates simply supported along two opposite edges; the applied load and the deflection were expressed in terms of Fourier components and simple trigonometric series, respectively. Many exact solutions for isotropic elastic thin plates were developed by Timoshenko [4] and Girkmann [5]; the simple trigonometric series of Lévy was mostly considered. Jian et al. [6] presented the equations for the lateral buckling analysis of fixed rectangular plates under the lateral concentrated load whereby the critical buckling strength of the plate calculated by finite element method was analyzed. Xu et al. [7] got exact solutions for rectangular anisotropic plates with four clamped edges through the state space method whereby the Fourier series in exponential form were adopted. Onyia et al. [8] presented the elastic buckling analysis of rectangular thin plates using the single finite Fourier sine integral transform method. Imrak et al. [9] presented an exact solution for a rectangular plate clamped along all edges in which each term of the series is trigonometric and hyperbolic, and identically satisfies the boundary conditions on all four edges. Fogang [10]

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used the flexibility method and a modified Lévy solution to analyze arbitrarily loaded isotropic rectangular thin plates with two opposite edges supported (simply supported or clamped), from which one or both are clamped, and the other edges with arbitrary support conditions. Mama et al. [11] presented the single finite Fourier sine integral transform method for the flexural analysis of rectangular Kirchhoff plate with opposite edges simply supported, and the other edges clamped for the case of triangular load distribution on the plate domain. Kamel [12] described the operational properties of the finite Fourier transform method, with the purpose of solving boundary value problems of partial differential equations.

In this paper, isotropic rectangular thin plates simply supported or clamped along two opposite edges and subjected to a concentrated bending moment perpendicular to these edges, whereby the other edges have arbitrary support conditions, were analyzed. The concentrated bending moment was expanded into a Fourier series, leading to a distributed external bending moment, and the boundary conditions and continuity equations were set.

2. Materials and methods

2.1 Governing equations of the plate

The Kirchhoff–Love plate theory (KLPT) [1] is used for thin plates whereby shear deformations are not considered. The spatial axis convention (X, Y, Z) is represented in figure 1 below.

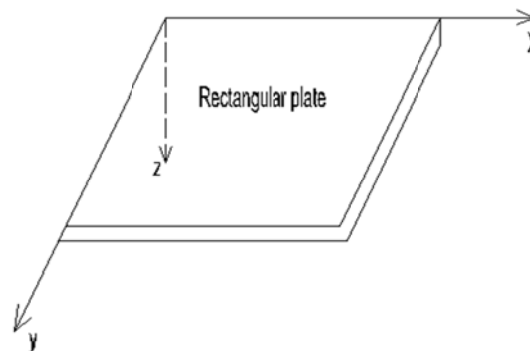


Figure 1. Spatial axis convention X, Y, Z

The equations of the present section are related to the KLPT. The governing equation of the isotropic Kirchhoff plate, derived by Lagrange, is given by

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x, y)}{D} \quad (1)$$

where $w(x, y, z)$ is the displacement in z -direction, $q(x, y)$ the applied transverse load per unit area, and D the flexural rigidity of the plate.

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The bending moments per unit length m_{xx} and m_{yy} , and the twisting moments per unit length m_{xy} are given by

$$\begin{aligned} m_{xx} &= -D \times \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), & m_{yy} &= -D \times \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \\ m_{xy} &= -D \times (1 - \nu) \times \frac{\partial^2 w}{\partial x \partial y}, \\ D &= \frac{Eh^3}{12(1 - \nu^2)} \end{aligned} \quad (2a-d)$$

The shear forces per unit length are given by

$$Q_x = -D \times \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right), \quad Q_y = -D \times \left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \right). \quad (3a-b)$$

The Kirchhoff shear forces per unit length used along the free edges combine shear forces and twisting moments, and are expressed as follows:

$$V_x = -D \times \left(\frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right), \quad V_y = -D \times \left(\frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right). \quad (4a-b)$$

In these equations, E is the elastic modulus of the plate material, h is the plate thickness, and ν is the Poisson's ratio.

2.2 Rectangular plate supported along two opposite edges and subjected to a concentrated bending moment

The plate dimensions in x - and y -direction are denoted by a and b , respectively. The rectangular plate is assumed simply supported or clamped along the opposite edges $x = 0$ and $x = a$. The concentrated bending moment denoted by M_{x0} is applied at the position (X_0, Y_0) and oriented along the $+x$ -axis (see Figure 2).

Bending moments parallel to the edges $x = 0$ and $x = a$ are satisfactorily treated in the literature and will not be analyzed in this paper.

First, the standard solution to this problem will be recalled. Second, the approach of this study will be presented whereby plates with two opposite edges simply supported and plates of infinite length will be considered. Third, plates with two opposite edges clamped will be treated.

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2.2.1 Standard solution to the problem

Let the concentrated moment M_{x0} be applied at the position (x_0, y_0) as shown in Figure 2

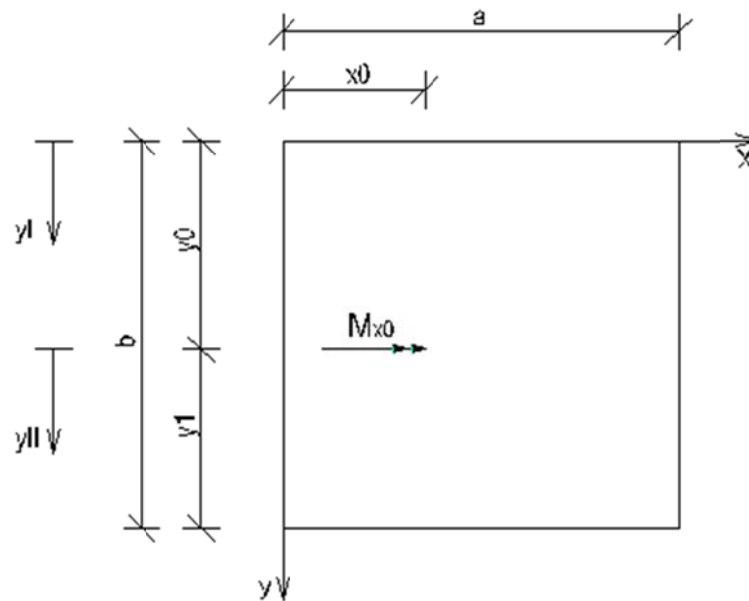


Figure 2 Rectangular plate subjected to a concentrated moment M_{x0} at (x_0, y_0)

The standard approach is to replace the bending moment with a couple of forces $-P$ and P applied at the positions (x_0, y_0) and $(x_0, y_0 + c)$, respectively, with c approaching zero and $M_{x0} = P \times c$.

Let $S(x, y, x_0, y_0)$ and $S^*(x, y, x_0, y_0)$ be values of interest (efforts, deformations ...) at positions (x, y) for the plate subjected to a load P and a unit load, respectively, at a position (x_0, y_0) . The values of interest are determined by combining the effect of the forces $-P$ and P as follows

$$\begin{aligned}
 & S(x, y, x_0, y_0 + c) - S(x, y, x_0, y_0) \\
 &= P \times S^*(x, y, x_0, y_0 + c) - P \times S^*(x, y, x_0, y_0) \\
 &= P \frac{\partial S^*(x, y, x_0, y_0)}{\partial y_0} c \\
 &= M_{x0} \frac{\partial S^*(x, y, x_0, y_0)}{\partial y_0}
 \end{aligned} \tag{5}$$

Hence the determination of a quantity of interest for the case of the plate subjected to a concentrated moment M_{x0} at a position (x_0, y_0) requires the analytical formulation of the quantity of interest for the case of the plate subjected to a unit load at the position (x_0, y_0) and its first derivative with respect to y_0 ; this result can be found in Girkmann [5].

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2.2.2 Solution of this study: Rectangular plate with two opposite edges simply supported

In this paper the concentrated moment M_{x0} is expanded into a Fourier series as follows

$$m_{x0}(x) = \frac{2M_{x0}}{a} \sum_m \sin \frac{m\pi x_0}{a} \sin \frac{m\pi x}{a} \quad (6)$$

Therefore, the concentrated moment M_{x0} is replaced with the distributed external moment $m_{x0}(x)$ along the line $y = y_0$. Referring to Figure 2, the efforts and deformations are represented with the subscripts I and II for the plate zones $0 \leq y_I \leq y_0$ and $0 \leq y_{II} \leq y_1$, respectively. The continuity equations along the line $y = y_0$ express the continuity of the deflection w and slope $\partial w / \partial y$ and the equilibrium of bending moment m_{yy} and shear force Q_y ; observing that the position $y = y_0$ corresponds to $y_I = y_0$ and $y_{II} = 0$, these equations are given by

$$\begin{aligned} w_I(x, y_I) \Big|_{y_I=y_0} &= w_{II}(x, y_{II}) \Big|_{y_{II}=0} \\ \frac{\partial w_I(x, y_I)}{\partial y_I} \Big|_{y_I=y_0} &= \frac{\partial w_{II}(x, y_{II})}{\partial y_{II}} \Big|_{y_{II}=0} \end{aligned} \quad (7a-d)$$

$$\begin{aligned} m_{yy,I}(x, y_I) \Big|_{y_I=y_0} - m_{yy,II}(x, y_{II}) \Big|_{y_{II}=0} &= -m_{x0}(x) \\ Q_{y,I}(x, y_I) \Big|_{y_I=y_0} - Q_{y,II}(x, y_{II}) \Big|_{y_{II}=0} &= 0 \end{aligned}$$

Assuming the rectangular plate simply supported along the edges $x = 0$ and $x = a$, the solution by Lévy [4] that satisfies the boundary conditions at these edges is considered for the deflection curve $w(x, y)$ as follows:

$$w(x, y) = \frac{1}{D} \sum_m F_m(y) \sin \alpha_m x, \quad \alpha_m = \frac{m\pi}{a} \quad (7e)$$

The solution to Equation (7e) as derived in the literature (see Timoshenko [4] and Girkmann [5]) is given by

$$w(x, y) = \frac{1}{D} \sum_m \left(\begin{aligned} &A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + \\ &C_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y \end{aligned} \right) \sin \alpha_m x \quad (7f)$$

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Hence the efforts and deformations needed for the continuity equations for the plate zone I ($0 \leq y_I \leq y_0$) can be expressed using Equations (7f), (2b), and (3b) as follows

$$w_I(x, y_I) = \frac{1}{D} \sum_m \left(A_{mI} \cosh \alpha_m y_I + B_{mI} \alpha_m y_I \sinh \alpha_m y_I + C_{mI} \sinh \alpha_m y_I + D_{mI} \alpha_m y_I \cosh \alpha_m y_I \right) \sin \frac{m\pi x}{a} \quad (8a-d)$$

$$\frac{\partial w_I(x, y_I)}{\partial y_I} = \frac{1}{D} \sum_m \alpha_m \left[A_{mI} \sinh \alpha_m y_I + B_{mI} (\sinh \alpha_m y_I + \alpha_m y_I \cosh \alpha_m y_I) + C_{mI} \cosh \alpha_m y_I + D_{mI} (\cosh \alpha_m y_I + \alpha_m y_I \sinh \alpha_m y_I) \right] \sin \alpha_m x$$

$$m_{yy,I} = - \sum_m \alpha_m^2 (1-\nu) \left[A_{mI} \cosh \alpha_m y_I + B_{mI} \left(\frac{2}{1-\nu} \cosh \alpha_m y_I + \alpha_m y_I \sinh \alpha_m y_I \right) + C_{mI} \sinh \alpha_m y_I + D_{mI} \left(\frac{2}{1-\nu} \sinh \alpha_m y_I + \alpha_m y_I \cosh \alpha_m y_I \right) \right] \sin \alpha_m x$$

$$Q_{y,I} = - \sum_m \alpha_m^3 [2B_{mI} \sinh \alpha_m y_I + 2D_{mI} \cosh \alpha_m y_I] \sin \alpha_m x$$

The efforts and deformations for the plate zone II ($0 \leq y_{II} \leq y_1$) are expressed by replacing the subscript I with II in Equations (8a-d). Recalling that the position $y = y_0$ corresponds to $y_I = y_0$ and $y_{II} = 0$, the continuity equations can be formulated using Equations (7a-d) and (8a-d) as follows

$$A_{mI} \cosh \alpha_m y_0 + B_{mI} \alpha_m y_0 \sinh \alpha_m y_0 + C_{mI} \sinh \alpha_m y_0 + D_{mI} \alpha_m y_0 \cosh \alpha_m y_0 - A_{mII} = 0$$

$$A_{mI} \sinh \alpha_m y_0 + B_{mI} (\sinh \alpha_m y_0 + \alpha_m y_0 \cosh \alpha_m y_0) + C_{mI} \cosh \alpha_m y_0 + D_{mI} (\cosh \alpha_m y_0 + \alpha_m y_0 \sinh \alpha_m y_0) - C_{mII} - D_{mII} = 0$$

(9a-d)

$$A_{mI} \cosh \alpha_m y_0 + B_{mI} \left(\frac{2}{1-\nu} \cosh \alpha_m y_0 + \alpha_m y_0 \sinh \alpha_m y_0 \right) + C_{mI} \sinh \alpha_m y_0 + D_{mI} \left(\frac{2}{1-\nu} \sinh \alpha_m y_0 + \alpha_m y_0 \cosh \alpha_m y_0 \right) - A_{mII} - \frac{2}{1-\nu} B_{mII} = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

$$2B_{mI} \sinh \alpha_m y_0 + 2D_{mI} \cosh \alpha_m y_0 - 2D_{mII} = 0$$

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The Kirchhoff shear forces are needed for boundary conditions at free edges; they are expressed using Equations (4b) and (8a)

$$V_{y,I} = -\sum_m \alpha_m^3 (\nu - 1) \left[\begin{aligned} &A_{mI} \sinh \alpha_m y_I + B_{mI} \left(\frac{\nu + 1}{\nu - 1} \sinh \alpha_m y_I + \alpha_m y_I \cosh \alpha_m y_I \right) + \\ &C_{mI} \cosh \alpha_m y_I + D_{mI} \left(\frac{\nu + 1}{\nu - 1} \cosh \alpha_m y_I + \alpha_m y_I \sinh \alpha_m y_I \right) \end{aligned} \right] \sin \alpha_m x \quad (10)$$

In summary, the coefficients A_{mI} , B_{mI} , C_{mI} , and D_{mI} and A_{mII} , B_{mII} , C_{mII} , and D_{mII} are determined by satisfying the boundary conditions at $y = 0$ and $y = b$ and the continuity conditions at $y = y_0$. They are calculated for various boundary conditions at $y = 0$ and $y = b$ in Appendix A.

Then, the bending moments m_{yy} are calculated using Equation (8c), and the bending moments m_{xx} and twisting moments m_{xy} are calculated using Equations (2a, c) and (8a) as follows

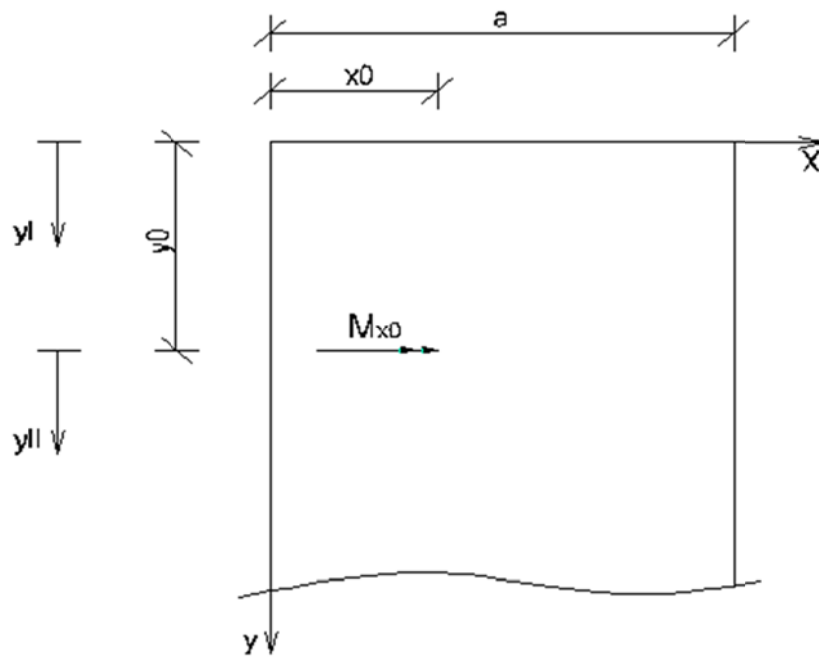
$$m_{xx,I} = -\sum_m \alpha_m^2 (\nu - 1) \left[\begin{aligned} &A_{mI} \cosh \alpha_m y_I + B_{mI} \left(\frac{2\nu}{\nu - 1} \cosh \alpha_m y_I + \alpha_m y_I \sinh \alpha_m y_I \right) + \\ &C_{mI} \sinh \alpha_m y_I + D_{mI} \left(\frac{2\nu}{\nu - 1} \sinh \alpha_m y_I + \alpha_m y_I \cosh \alpha_m y_I \right) \end{aligned} \right] \sin \alpha_m x \quad (11a-b)$$

$$m_{xy,I} = -(1 - \nu) \times \sum_m \alpha_m^2 \left[\begin{aligned} &A_{mI} \sinh \alpha_m y_I + B_{mI} (\sinh \alpha_m y_I + \alpha_m y_I \cosh \alpha_m y_I) + \\ &C_{mI} \cosh \alpha_m y_I + D_{mI} (\cosh \alpha_m y_I + \alpha_m y_I \sinh \alpha_m y_I) \end{aligned} \right] \cos \alpha_m x.$$

2.2.3 Solution of this study: Rectangular plate of infinite length with two opposite edges simply supported

Let us analyze a plate of infinite length with two opposite edges $x = 0$ and $x = a$ simply supported and subjected near its end to a concentrated moment M_{x0} , as shown in Figure 3

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Figure 3 Rectangular plate of infinite length subjected near its end to a concentrated moment M_{x0}

The efforts and deformation in plate zone I ($0 \leq y_I \leq y_0$) are formulated according to Section 2.2.2.

Keeping in mind that the edges $x = 0$ and $x = a$ are simply supported, the deflection curve for the plate of infinite length (plate zone $y_{II} \geq 0$ in Figure 3) as derived in the literature (see Timoshenko [4] and Girkmann [5]) is given by

$$w_{II}(x, y_{II}) = \frac{1}{D} \sum_m (A_{mII} + B_{mII} \alpha_m y_{II}) e^{-\alpha_m y_{II}} \sin \alpha_m x \quad (12)$$

It yields as follows the slope $\partial w_{II} / \partial y_{II}$, the bending moment m_{yy} and the shear force Q_y needed for the continuity conditions along $y = y_0$

$$\frac{\partial w_{II}(x, y_{II})}{\partial y_{II}} = \frac{1}{D} \sum_m \alpha_m (B_{mII} - A_{mII} - B_{mII} \alpha_m y_{II}) e^{-\alpha_m y_{II}} \sin \alpha_m x \quad (13a-c)$$

$$m_{yy,II}(x, y_{II}) = - \sum_m \alpha_m^2 \left[(1-\nu) A_{mII} - 2 B_{mII} + (1-\nu) B_{mII} \alpha_m y_{II} \right] e^{-\alpha_m y_{II}} \sin \alpha_m x$$

$$Q_{y,II}(x, y_{II}) = - \sum_m 2 \alpha_m^3 B_{mII} e^{-\alpha_m y_{II}} \sin \alpha_m x$$

The Kirchhoff shear forces useful for boundary conditions at free edges are given by

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$$V_{y,II} = - \sum_m \alpha_m^3 \left[(1-\nu) A_{mII} + (1+\nu) B_{mII} + (1-\nu) B_{mII} \alpha_m y_{II} \right] e^{-\alpha_m y_{II}} \sin \alpha_m x \quad (14)$$

Reminding that the position $y = y_0$ corresponds to $y_I = y_0$ and $y_{II} = 0$, the continuity equations between the plate zone I ($0 \leq y_I \leq y_0$) and the plate zone II of infinite length ($y_{II} \geq 0$) can be expressed using Equations (7a-d), (8a-d), (12), and (13a-c)

$$A_{mI} \cosh \alpha_m y_0 + B_{mI} \alpha_m y_0 \sinh \alpha_m y_0 + C_{mI} \sinh \alpha_m y_0 + D_{mI} \alpha_m y_0 \cosh \alpha_m y_0 - A_{mII} = 0$$

$$A_{mI} \sinh \alpha_m y_0 + B_{mI} (\sinh \alpha_m y_0 + \alpha_m y_0 \cosh \alpha_m y_0) + C_{mI} \cosh \alpha_m y_0 + D_{mI} (\cosh \alpha_m y_0 + \alpha_m y_0 \sinh \alpha_m y_0) + A_{mII} - B_{mII} = 0 \quad (15a-d)$$

$$A_{mI} \cosh \alpha_m y_0 + B_{mI} \left(\frac{2}{1-\nu} \cosh \alpha_m y_0 + \alpha_m y_0 \sinh \alpha_m y_0 \right) + C_{mI} \sinh \alpha_m y_0 + D_{mI} \left(\frac{2}{1-\nu} \sinh \alpha_m y_0 + \alpha_m y_0 \cosh \alpha_m y_0 \right) - A_{mII} + \frac{2}{1-\nu} B_{mII} = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

$$2B_{mI} \sinh \alpha_m y_0 + 2D_{mI} \cosh \alpha_m y_0 - 2B_{mII} = 0$$

In summary, the coefficients A_{mI} , B_{mI} , C_{mI} , D_{mI} , A_{mII} , and B_{mII} are determined by satisfying the boundary conditions at $y = 0$ and the continuity conditions at $y = y_0$. They are calculated for various support conditions at $y = 0$ in Appendix B.

Then, in the plate zone of infinite length the bending moments m_{yy} are calculated using Equation (13b), and the bending moments m_{xx} and twisting moments m_{xy} are calculated using Equations (2a, c) and (12) as follows

$$m_{xx,II}(x, y_{II}) = \sum_m \alpha_m^2 \left[(1-\nu) A_{mII} + 2\nu B_{mII} + (1-\nu) B_{mII} \alpha_m y_{II} \right] e^{-\alpha_m y_{II}} \sin \alpha_m x \quad (15e-f)$$

$$m_{xy,II} = -(1-\nu) \times \sum_m \alpha_m^2 \left[-A_{mII} + B_{mII} - B_{mII} \alpha_m y_{II} \right] e^{-\alpha_m y_{II}} \cos \alpha_m x$$

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2.2.4 Solution of this study: Rectangular plate with one or two opposite edges clamped

It is assumed that from the two opposite edges supported, one or both are clamped.

The analysis can be conducted with the flexibility method. The primary system is the rectangular plate simply supported along the edges $x = 0$ and $x = a$, and is treated according to the previous sections. The flexibilities δ_{j0} (slopes at positions j of the opposite edges where the compatibility equations will be set) for the primary problem are calculated for an ordinary plate and a plate of infinite length, respectively, as follows

$$\delta_{j0} = \frac{\partial w(x, y)}{\partial x} \bigg|_{\substack{x=x_j \\ y=y_j}} = \frac{1}{D} \sum_m \alpha_m \left[\begin{aligned} &A_{ml} \cosh \alpha_m y_j + B_{ml} \alpha_m y_j \sinh \alpha_m y_j + \\ &C_{ml} \sinh \alpha_m y_j + D_{ml} \alpha_m y_j \cosh \alpha_m y_j \end{aligned} \right] \cos \alpha_m x_j \quad (16a-b)$$

$$\delta_{j0} = \frac{\partial w(x, y)}{\partial x} \bigg|_{\substack{x=x_j \\ y=y_j}} = \frac{1}{D} \sum_m \alpha_m (A_{ml} + B_{ml} \alpha_m y_j) e^{-\alpha_m y_j} \cos \alpha_m x_j$$

The redundant system is the plate simply supported along the opposite edges and subjected to bending moments along those edges. The analysis is done according to Fogang [10].

3 Results and discussion

3.1 Plate of infinite length subjected to a concentrated moment in the middle

The plate of infinite length is subjected to a concentrated moment in the middle, as represented in Figure 4.

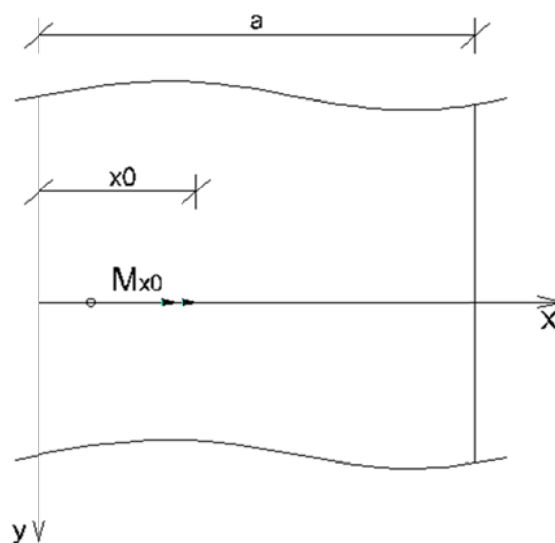


Figure 4 Rectangular plate of infinite length subjected to a concentrated moment M_{x0} in the middle

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Given the anti-symmetrical nature of the load, the deflections are zero along the line $y = 0$. And it is observed that the load is equally divided between the two halves of the plate $\rightarrow m_{yy}(x, y = 0) = m_{x0}(x)/2$. Substituting these conditions into Equations (12), (13b), and (6) yields for $y \geq 0$ yields

$$w_I(x, y) = \frac{M_{x0}y}{2\pi D} \sum_m \frac{1}{m} e^{-\alpha_m y} \sin \alpha_m x_0 \sin \alpha_m x \quad (17)$$

This result can be found in Girkmann [5].

3.2 Plate of infinite length subjected at its end to a concentrated moment

The plate of infinite length is subjected at its end $y = 0$ to a concentrated moment, as represented in Figure 5.

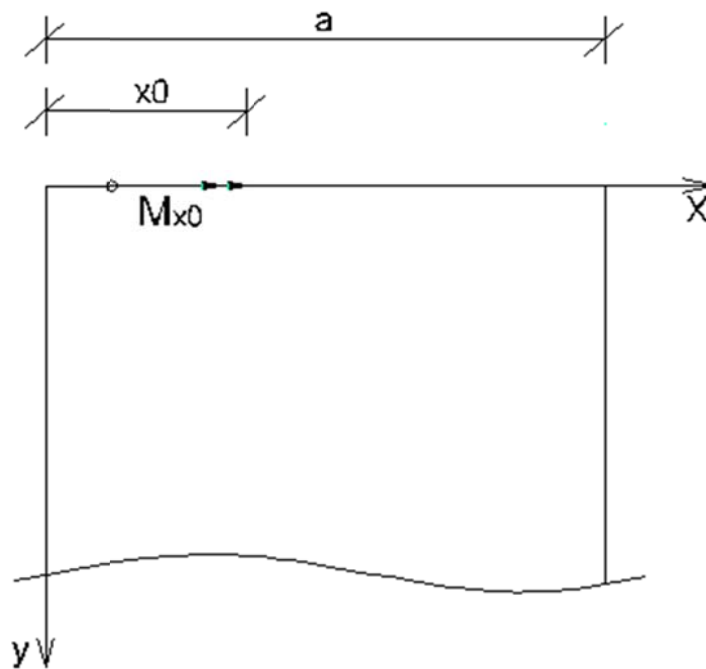


Figure 5 Rectangular plate of infinite length subjected at its end to a concentrated moment M_{x0}

Case 1: Edge $y = 0$ simply supported

The deflections are zero along the line $y = 0$ and the bending moment m_{yy} at $y = 0$ is equal to the distributed moment $m_{x0}(x)$ (the concentrated moment M_{x0} expanded into a Fourier series according to Equation (6)). Substituting these conditions into Equations (12) and (13b) yields

$$w(x, y) = \frac{M_{x0}y}{\pi D} \sum_m \frac{1}{m} e^{-\alpha_m y} \sin \alpha_m x_0 \sin \alpha_m x \quad (18)$$

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Case 2: Edge $y = 0$ free

The Kirchhoff shear force is zero along the line $y = 0$ and the bending moment m_{yy} at $y = 0$ is equal to the distributed moment (the concentrated moment M_{x0} expanded into a Fourier series according to Equation (6)). Substituting these conditions into Equations (13b) and (14) yields

$$w(x, y) = \frac{2M_{x0}}{D} \sum_m \left(\frac{-(1+\nu)a}{(1-\nu)(3+\nu)\pi^2 m^2} + \frac{y}{(3+\nu)\pi m} \right) e^{-\alpha_m y} \sin \alpha_m x_0 \sin \alpha_m x \quad (19)$$

4 Conclusion

In this paper, isotropic rectangular thin plates simply supported or clamped along two opposite edges and subjected to a concentrated bending moment perpendicular to these edges were analyzed. Bending moments parallel to the supported opposite edges are satisfactorily treated in the literature and were not developed in this paper. The standard approach to this problem is to replace the bending moment with a couple of forces infinitely close and to use the known expressions of efforts and deformations for the plate subjected to concentrated forces; the results are then related to the first derivatives of these efforts and deformations with respect to the position of application of the load. In this study the concentrated moment was expanded into a Fourier series, leading to a distributed external bending moment, and the boundary conditions and continuity equations were set. Plates of infinite length were also analyzed and the results obtained were identical to those in the literature.

The following aspect was not addressed in this study but could be analyzed in the future: Rectangular anisotropic plate

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ISOTROPIC RECTANGULAR PLATE SUBJECTED TO A CONCENTRATED BENDING MOMENT

Appendix A Coefficients A_{mI} , B_{mI} , C_{mI} , and D_{mI} and A_{mII} , B_{mII} , C_{mII} , and D_{mII} for various support conditions at $y = 0$ and $y = b$

Edges $y = 0$ and $y = b$ clamped

The plate is represented in Figure 2. We set

$$Y_0 = \alpha_m y_0, \quad Y_1 = \alpha_m y_1, \quad m_{x0m}^* = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

The boundary conditions and continuity equations are expressed in matrices form as follows, whereby the first two rows and the last two rows represent the boundary conditions at the edges $y = 0$ and $y = b$, respectively.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 & 0 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 0 & 0 & -1 & -1 \\ \cosh Y_0 & \frac{2 \cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2 \sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & -\frac{2}{1-\nu} & 0 & 0 \\ 0 & 2 \sinh Y_0 & 0 & 2 \cosh Y_0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & \cosh Y_1 & Y_1 \sinh Y_1 & \sinh Y_1 & Y_1 \cosh Y_1 \\ 0 & 0 & 0 & 0 & \sinh Y_1 & \sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \cosh Y_1 + Y_1 \sinh Y_1 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = m_{x0m}^* \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (A1)$$

ISOTROPIC RECTANGULAR PLATE SUBJECTED TO A CONCENTRATED BENDING MOMENT

Edge $y = 0$ simply supported and edge $y = b$ free

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & \frac{2}{1-\nu} & 0 & 0 & 0 & 0 & 0 & 0 \\
 \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 & 0 & 0 \\
 \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 0 & 0 & -1 & -1 \\
 \cosh Y_0 & \frac{2 \cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2 \sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & -\frac{2}{1-\nu} & 0 & 0 \\
 0 & 2 \sinh Y_0 & 0 & 2 \cosh Y_0 & 0 & 0 & 0 & -2 \\
 0 & 0 & 0 & 0 & \cosh Y_1 & \frac{2 \cosh Y_1}{1-\nu} + Y_1 \sinh Y_1 & \sinh Y_1 & \frac{2 \sinh Y_1}{1-\nu} + Y_1 \cosh Y_1 \\
 0 & 0 & 0 & 0 & \sinh Y_1 & \frac{\nu+1}{\nu-1} \sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \frac{\nu+1}{\nu-1} \cosh Y_1 + Y_1 \sinh Y_1
 \end{bmatrix}
 \times
 \begin{bmatrix}
 A_{ml} \\
 B_{ml} \\
 C_{ml} \\
 D_{ml} \\
 A_{mll} \\
 B_{mll} \\
 C_{mll} \\
 D_{mll}
 \end{bmatrix}
 =
 m_{x0m}^*
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}
 \quad (A2)$$

Plates with other combinations of support conditions at $y = 0$ and $y = b$ can be analyzed similarly. The first two rows and last two rows are modified accordingly.

ISOTROPIC RECTANGULAR PLATE SUBJECTED TO A CONCENTRATED BENDING MOMENT

Edge $y = 0$ simply supported and edge $y = b$ free: bending moment acting at $(x_0, y_0 = 0)$

The plate is represented in Figure 2 with $y_0 = 0$. We set

$$Y_1 = \alpha_m b, \quad m_{x0m}^* = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{2}{1-\nu} & 0 & 0 \\ \cosh Y_1 & \frac{2 \cosh Y_1}{1-\nu} + \frac{\sinh Y_1}{Y_1 \sinh Y_1} & \sinh Y_1 & \frac{2 \sinh Y_1}{1-\nu} + \frac{\cosh Y_1}{Y_1 \cosh Y_1} \\ \sinh Y_1 & \frac{\nu+1}{\nu-1} \sinh Y_1 + \frac{\cosh Y_1}{Y_1 \cosh Y_1} & \cosh Y_1 & \frac{\nu+1}{\nu-1} \cosh Y_1 + \frac{\sinh Y_1}{Y_1 \sinh Y_1} \end{bmatrix} \times \begin{bmatrix} A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ -m_{x0m}^* \\ 0 \\ 0 \end{bmatrix} \quad (\text{A3})$$

Edge $y = 0$ free and edge $y = b$ clamped: bending moment acting at $(x_0, y_0 = 0)$

$$\begin{bmatrix} 0 & 0 & 1 & \frac{\nu+1}{\nu-1} \\ 1 & \frac{2}{1-\nu} & 0 & 0 \\ \cosh Y_1 & Y_1 \sinh Y_1 & \sinh Y_1 & Y_1 \cosh Y_1 \\ \sinh Y_1 & \sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \cosh Y_1 + Y_1 \sinh Y_1 \end{bmatrix} \times \begin{bmatrix} A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ -m_{x0m}^* \\ 0 \\ 0 \end{bmatrix} \quad (\text{A4})$$

ISOTROPIC RECTANGULAR PLATE SUBJECTED TO A CONCENTRATED BENDING MOMENT

Plates with other support conditions at $y = b$ can be analyzed similarly, the last two rows being modified accordingly. Then, the bending moments m_{yy} are calculated using Equation (8c), and the bending moments m_{xx} and twisting moments m_{xy} are calculated using Equations (11a-b).

Appendix B Plate of infinite length: coefficients A_{mI} , B_{mI} , C_{mI} , D_{mI} , A_{mII} , and B_{mII} for various support conditions at $y = 0$

Edge $y = 0$ simply supported

The plate is represented in Figure 2. We set

$$Y_0 = \alpha_m y_0, \quad m_{x0m}^* = \frac{1}{\alpha_m^2(1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

The boundary conditions and continuity equations are expressed in matrices form as follows, whereby the first two rows represent the boundary conditions at the edge $y = 0$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{2}{1-\nu} & 0 & 0 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 1 & -1 \\ \cosh Y_0 & \frac{2 \cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2 \sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & \frac{2}{1-\nu} \\ 0 & 2 \sinh Y_0 & 0 & 2 \cosh Y_0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m_{x0m}^* \\ 0 \end{bmatrix} \quad (B1)$$

ISOTROPIC RECTANGULAR PLATE SUBJECTED TO A CONCENTRATED BENDING MOMENT

Edge $y = 0$ clamped

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 \\
 \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 \\
 \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 1 & -1 \\
 \cosh Y_0 & \frac{2 \cosh Y_0}{1 - \nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2 \sinh Y_0}{1 - \nu} + Y_0 \cosh Y_0 & -1 & \frac{2}{1 - \nu} \\
 0 & 2 \sinh Y_0 & 0 & 2 \cosh Y_0 & 0 & -2
 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m_{x0m}^* \\ 0 \end{bmatrix} \quad (B2)$$

Edge $y = 0$ free

$$\begin{bmatrix}
 0 & 0 & 1 & \frac{\nu + 1}{\nu - 1} & 0 & 0 \\
 1 & \frac{2}{1 - \nu} & 0 & 0 & 0 & 0 \\
 \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 \\
 \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 1 & -1 \\
 \cosh Y_0 & \frac{2 \cosh Y_0}{1 - \nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2 \sinh Y_0}{1 - \nu} + Y_0 \cosh Y_0 & -1 & \frac{2}{1 - \nu} \\
 0 & 2 \sinh Y_0 & 0 & 2 \cosh Y_0 & 0 & -2
 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m_{x0m}^* \\ 0 \end{bmatrix} \quad (B3)$$

ISOTROPIC RECTANGULAR PLATE SUBJECTED TO A CONCENTRATED BENDING MOMENT

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