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Warped Product Pointwise Hemi-Slant Submanifolds of Nearly Kaehler Manifolds

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Abstract: In this paper, we introduce the notion of pointwise hemi-slant submanifolds of nearly Kaehler manifolds. Further, we study their warped products and prove the necessary and sufficient condition that a pointwise hemi-slant submanifold to be a warped product manifold. Also, we prove that every pointwise hemi-slant warped product submanifold $M = M_\perp \times_f M_\theta$ which is mixed totally geodesic in an arbitrary nearly Kaehler manifold $\tilde{M}$ satisfies $\|h\|^2 \geq \frac{2p}{9}\cos^2\theta\|\nabla(\ln f)\|^2$, where $\|h\|$ is the length of the second fundamental form of M and $2p = \dim M_\theta$; while $\nabla(\ln f)$ is the gradient of $\ln f$ along $M_\perp$. The equality case of this inequality is also given.

Keywords: slant; pointwise slant; pointwise hemi-slant; warped products; nearly Kaehler manifolds

MSC: 53C15; 53C40; 53C42; 53B25

1. Introduction

The concept of warped products appeared in the mathematical and physical literature long ago introduced by Bishop and O'Niell [2]. The notion of warped products is one of the most fruitful generalizations of Riemannian products. Also, warped products play very important roles in differential geometry as well as in physics, especially in general relativity. Many basic solutions of the Einstein field equations are warped products. For instance, both Schwarzschild's and Robertson-Walker's models in general relativity are warped products. Basic properties on warped products can be found in [2] and [20]. A warped product $M_1 \times_f M_2$ of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2) with their metrics g_1 and g_2 is the product manifold $M = M_1 \times_f M_2$ equipped with the metric

$$g = g_1 + f^2 g_2 \tag{1}$$

where $f : M_1 \to \mathbb{R}^+$, a positive differentiable function on M_1. We call the function f, the *warping function* of the warped product [2]. If the warping function f is constant, then the manifold M is said to be trivial or simply a Riemannian product.

On the other hand, the study of warped product submanifolds was only initiated by B.-Y. Chen around the beginning of this century in his seminal papers [6,7]. Motivated by Chen's results on CR-warped product submanifolds, this problem became an active and fruitful field of research in differential geometry and many geometers extended this idea to CR-warped products [9,13,22], semi-slant warped products [1,21], hemi-slant warped products [23,26], CR-slant warped products [24,28], pointwise semi-slant warped products [25] in almost Hermitian manifolds.

In this paper, we study pointwise hemi-slant submanifolds of a more general class of almost Hermitian manifolds, namely; nearly Kaehler manifolds. First we give preparatory lemmas for pointwise hemi-slant submanifolds about the integrability of the involved distributions those are useful to develop the main results of this paper. A characterization theorem is proved about the

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necessary and sufficient conditions for a pointwise hemi-slant submanifold to be a warped product manifold. Furthermore, B.-Y. Chen's inequality is derived and its equality case is also discussed.

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with almost complex structure J and compatible Riemannian metric g such that

$$J^2 = -I, \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y) \tag{2}$$

for all vector fields X, Y on $\tilde{M}$, where I is the identity transformation on the tangent space $T\tilde{M}$ of $\tilde{M}$. Let $\tilde{\nabla}$ be the Levi-Civita connection on the tangent space of $\tilde{M}$ with respect to metric g. By A. Gray [15], if the almost complex structure J satisfies

$$(\tilde{\nabla}_X J)X = 0, \text{ or } (\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0, \tag{3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, then the manifold $\tilde{M}$ is called a nearly Kaehler manifold, where $\Gamma(T\tilde{M})$ is the set of vector fields on $\tilde{M}$.

Let M be a submanifold of a Riemannian manifold $\tilde{M}$ with the induced connection ∇ and the induced metric g. We denote g and $\tilde{g}$ by the same symbol g due to the equidistance. Then, the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{4}$$

where $X, Y \in \Gamma(TM)$, and ∇ is the induced Riemannian connection on M, ξ is a vector field normal to M, h is the second fundamental form of $M, \nabla^\perp$ is the connection in the normal bundle $T^\perp M$ and A_ξ is the shape operator of the second fundamental form. They are related by

$$g\left(A_\xi X, Y\right) = g(h(X, Y), \xi). \tag{5}$$

A submanifold M of an almost Hermitian manifold $\tilde{M}$ is said to be totally geodesic if the second fundamental form h of M vanishes identically. Moreover, M is called totally umbilical if h satisfies $h(X, Y) = g(X, Y)H$, for all $X, Y \in \Gamma(TM)$, where $H = \frac{1}{n} trace\, h,\ n = \dim M$ and M is minimal if $H = 0$.

For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, the transformations JX and $J\xi$ are decomposed into tangential and normal parts as follows

$$JX = TX + FX, \quad J\xi = t\xi + f\xi. \tag{6}$$

Now, $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ denote to the tangential and normal parts of $(\tilde{\nabla}_X J)Y$, respectively, i.e.,

$$(\tilde{\nabla}_X J)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y, \tag{7}$$

for all $X, Y \in \Gamma(TM)$. Using (6) and (4), we have the following relations

$$\mathcal{P}_X Y = (\tilde{\nabla}_X T)Y - A_{FY} X - th(X, Y),$$
$$\mathcal{Q}_X Y = (\tilde{\nabla}_X F)Y + h(X, TY) - fh(X, Y).$$

Similarly, for any $\xi \in \Gamma(T^\perp M)$, denote the tangential and normal parts of $(\tilde{\nabla}_X J)\xi$ by $\mathcal{P}_X \xi$ and $\mathcal{Q}_X \xi$, respectively. Then, we have

$$\mathcal{P}_X \xi = (\tilde{\nabla}_X t)\xi + TA_\xi X - A_{f\xi} X,$$
$$\mathcal{Q}_X \xi = (\tilde{\nabla}_X f)\xi + h(t\xi, X) + FA_\xi X.$$

And the covariant derivative of T, F, t and f are defined by

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \; (\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y,$$
$$(\tilde{\nabla}_X t)\xi = \nabla_X t\xi - t\nabla^{\perp}_X \xi, \; (\tilde{\nabla}_X f)\xi = \nabla^{\perp}_X f\xi - f\nabla^{\perp}_X \xi.$$

It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$

$$\begin{array}{ll} (p_1) & (i)\;\; \mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W, \qquad (ii)\;\; \mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W, \\ (p_2) & (i)\;\; \mathcal{P}_X(Y+W) = \mathcal{P}_X Y + \mathcal{P}_X W, \;\; (ii)\;\; \mathcal{Q}_X(Y+W) = \mathcal{Q}_X Y + \mathcal{Q}_X W, \\ (p_3) & (i)\;\; g(\mathcal{P}_X Y, W) = -g(Y, \mathcal{P}_X W), \quad (ii)\;\; g(\mathcal{Q}_X Y, \xi) = -g(Y, \mathcal{P}_X \xi), \\ (p_4) & \mathcal{P}_X JY + \mathcal{Q}_X JY = -J(\mathcal{P}_X Y + \mathcal{Q}_X Y). \end{array}$$

for all $X, Y, W \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$.

For a submanifold M of a nearly Kaehler manifold $\tilde{M}$, from (3) and (7), we have

$$\mathcal{P}_X Y + \mathcal{P}_Y X = 0, \;\; \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0, \;\; \forall\, X, Y \in \Gamma(TM). \tag{8}$$

3. Basic Definitions and Lemmas

A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called holomorphic (complex) if, for any $p \in M$, we have $J(T_pM) \subseteq T_pM$, where T_pM denotes the tangent space of M at p. It is called totally real if we have $J(T_pM) \subseteq T_p^{\perp}M$ for each $p \in M$, where $T_p^{\perp}M$ is the normal space of M in $\tilde{M}$ at p. A totally real submanifold is Lagrangian if $\dim M = \dim_{\mathbb{C}} \tilde{M}$.

For each non zero vector X tangent to M at $p \in M$, the angle $\theta(X)$ between JX and T_PM is called the Wirtinger angle of X. Thus, M is said to be a slant submanifold if the angle $\theta(X)$ is constant, which is independent of the choice of $p \in M$ and $X \in T_PM$.

Clearly, holomorphic and totally real submanifolds are slant submanifolds with slant angles 0 and $\pi/2$, respectively [4,5].

On the other hand, a submanifold M of an almost Hermitian manifold *is called pointwise slant* [12] if, at each point $p \in M$, the Wirtinger angle $\theta(X)$ between JX and T_pM is independent of the choice of vector $0 \neq X \in T_pM$. In this case, the Wirtinger angle gives rise to a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$ which is called the *slant function* of the pointwise slant submanifold. A pointwise slant submanifold is a *slant submanifold* in the sense of [4,5] if its slant function is globally constant on M. Note that every slant submanifold is a pointwise slant submanifold. A pointwise slant submanifold M is called *proper pointwise slant* if it is neither holomorphic nor totally real or slant.

It follows from ([12] [Lemma 2.1]) that a submanifold M of an almost Hermitian manifold is pointwise slant if and only if

$$T^2X = -\cos^2\theta(p)X \tag{9}$$

holds for some real-valued function θ on M, at $p \in M$. Note that the next two relations are consequences of (9):

$$g(TX, TY) = \cos^2\theta(p)\, g(X, Y), \tag{10}$$

$$g(FX, FY) = \sin^2\theta(p)\, g(X, Y) \tag{11}$$

for any $X, Y \in \Gamma(TM)$. The following relations for the pointwise slant submanifolds of an almost Hermitian manifold also follow easily from (2) and (9):

$$tFX = -\sin^2\theta(p)\, X, \;\; fFX = -FTX, \;\; \forall X \in \Gamma(TM). \tag{12}$$

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Definition 1. *A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called a pointwise hemi-slant submanifold if there exist a pair of orthogonal distributions $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ on M such that*

(i) *TM admits the orthogonal direct decomposition $TM = \mathfrak{D}^{\perp} \oplus \mathfrak{D}^{\theta}$.*
(ii) *The distribution $\mathfrak{D}^{\perp}$ is a totally, i.e., $J\mathfrak{D}^{\perp} \subseteq T^{\perp}M$.*
(iii) *The distribution $\mathfrak{D}^{\theta}$ is pointwise slant with slant function θ.*

The normal bundle of a pointwise hemi-slant submanifold M is decomposed as

$$T^{\perp}M = J\mathfrak{D}^{\perp} \oplus F\mathfrak{D}^{\theta} \oplus \mu, \tag{13}$$

where μ is the j-invariant normal subbundle of $T^{\perp}M$.

A pointwise hemi-slant submanifold is hemi-slant if $\mathfrak{D}^{\theta}$ is a slant distribution on M, i.e., θ is globally constant on M. Furthermore, it is proper pointwise hemi-slant if $\mathfrak{D}^{\theta}$ is neither slant nor holomorphic on M.

Now, we have the following preparatory results for later use.

Lemma 1. *Let M be a proper pointwise hemi-slant submanifold of a nearly Kaehler manifold $\tilde{M}$. Then, we have*

$$g(\nabla_Z W, X) = \sec^2\theta[g(h(Z,W), FTX) - g(h(TX,Z), JW) + g(\mathcal{Q}_Z X, JW) + g(\mathcal{Q}_Z W, FX)],$$

for any $X \in \Gamma(\mathfrak{D}^{\theta})$ and $Z, W \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. From (4), we have

$$g(\nabla_Z W, X) = -g(W, \tilde{\nabla}_Z X) = -g(J\tilde{\nabla}_Z X, JW) = g((\tilde{\nabla}_Z J)X, JW) - g(\tilde{\nabla}_Z JX, JW).$$

Then using (6), (4) and (7), we derive

$$\begin{aligned} g(\nabla_Z W, X) &= g((\tilde{\nabla}_Z J)X, JW) - g(\tilde{\nabla}_Z TX, JW) - g(\tilde{\nabla}_Z FX, JW) \\ &= g(\mathcal{Q}_Z X, JW) - g(h(TX,Z), JW) + g(J\tilde{\nabla} FX, W). \end{aligned} \tag{14}$$

Then, the third term of (14), can be evaluated as by using (6) and (7) as follows

$$\begin{aligned} g(J\tilde{\nabla} FX, W) &= g(\tilde{\nabla} JFX, W) - g((\tilde{\nabla}_Z J)FX, W) \\ &= g(\tilde{\nabla}_Z tFX, W) + g(\tilde{\nabla}_Z fFX, W) - g(\mathcal{P}_Z FX, W). \end{aligned}$$

Using (12) and property p_3 (ii), we derive

$$\begin{aligned} g(J\tilde{\nabla} FX, W) &= -\sin^2\theta g(\tilde{\nabla}_Z X, W) - \sin 2\theta Z(\theta) g(X, W) + g(A_{FTX} Z, W) + g(\mathcal{Q}_Z W, FX) \\ &= \sin^2\theta g(\tilde{\nabla}_Z W, X) + g(h(Z,W), FTX) + g(\mathcal{Q}_Z W, FX). \end{aligned} \tag{15}$$

Then, from (14) and (15), we find

$$\cos^2\theta g(\nabla_Z W, X) = g(\mathcal{Q}_Z X, JW) + g(\mathcal{Q}_Z W, FX) + g(h(Z,W), FTX) - g(h(TX,Z), JW),$$

which proves our assertion. □

The following corollary is an immediate consequence of above lemma.

Corollary 1. *The leaves of the totally real distribution $\mathfrak{D}^{\perp}$ of a pointwise hemi-slant submanifold M in a nearly Kaehler manifold $\tilde{M}$ are totally geodesic in M if and only if*

$$g(A_{JW}TX - A_{FTX}W, Z) = g(\mathcal{Q}_Z X, JW) + g(\mathcal{Q}_Z W, FX),$$

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for any $X \in \Gamma(\mathfrak{D}^\theta)$ *and* $Z, W \in \Gamma(\mathfrak{D}^\perp)$.

In a similar way of Lemma 1, we also have

Lemma 2. *Let* M *be a proper pointwise hemi-slant submanifold of a nearly Kaehler manifold* $\tilde{M}$. *Then, the following holds*

$$\cos^2\theta g(\nabla_X Y, Z) = g(h(X, TY), JZ) - g(h(X, Z), FTY) - g(\mathcal{Q}_X Y, JZ) - g(\mathcal{Q}_X Z, FY)$$

for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ *and* $Z \in \Gamma(\mathfrak{D}^\perp)$.

We skip the proof of this lemma because of the similar procedure used in the proof of Lemma 1. The following two results derived from Lemma 2 are also useful.

Proposition 1. *Let* M *be a proper pointwise hemi-slant submanifold of a nearly Kaehler manifold* $\tilde{M}$. *Then, we have*

$$\begin{aligned}\cos^2\theta g([X, Y], Z) = {} & g(h(X, TY) - h(TX, Y), JZ) - g(h(X, Z), FTY) + g(h(Y, Z), FTX) \\ & - 2g(\mathcal{Q}_X Y, JZ) - g(\mathcal{Q}_X Z, FY) + g(\mathcal{Q}_Y Z, FX)\end{aligned}$$

for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ *and* $Z \in \Gamma(\mathfrak{D}^\perp)$.

Corollary 2. *The leaves of a pointwise slant distribution* $\mathfrak{D}^\theta$ *in a pointwise hemi-slant submanifold* M *of a nearly Kaehler manifold* $\tilde{M}$ *are totally geodesic in* M *if and only if*

$$g(A_{JZ}TX - A_{FTX}Z, Y) = g(\mathcal{Q}_Y Z, FX) - g(\mathcal{Q}_X Y, JZ),$$

for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ *and* $Z \in \Gamma(\mathfrak{D}^\perp)$.

4. Pointwise Hemi-Slant Warped Products

In [27], Uddin and Chi investigated warped product hemi-slant submanifolds of nearly Kaehler manifolds under the name of pseudo-slant warped products and they proved the non-existence of the warped products $M = M_\perp \times_f M_\theta$ under the condition that $\mathcal{P}_X TX \in \Gamma(TM_\theta)$, $\forall\, X \in \Gamma(TM_\theta)$ in an arbitrary nearly Kaehler manifold $\tilde{M}$, where $M_\perp$ is a totally real submanifold and M_θ is a proper slant submanifold of $\tilde{M}$. In fact, these warped product exist in nearly Kaehler manifolds without imposing any condition.

In this section we study the warped product submanifold $M = M_\perp \times_f M_\theta$, when M_θ is proper pointwise slant submanifolds of a nearly Kaehler manifold $\tilde{M}$ and we called them pointwise hemi-slant warped products.

The following given results are useful to proves the main theorems.

Proposition 2. *[27] Let* $M = M_\perp \times_f M_\theta$ *be a warped product submanifold of a nearly Kaehler manifold* $\tilde{M}$, *where* $M_\perp$ *and* M_θ *are totally real and proper slant submanifolds of* $\tilde{M}$, *respectively. Then*

$$2g(h(Z, W), FX) = g(h(X, Z), JW) + g(h(X, W), JZ),$$

for any $X, Y \in \Gamma(TM_\theta)$ *and* $Z, W \in \Gamma(TM_\perp)$.

Remark 1. *Notice that above result is also true for warped product pointwise hemi-slant submanifolds.*

Next, we prove the following:

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Lemma 3. *Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise hemi-slant submanifold of a nearly Kaehler manifold $\tilde{M}$. Then*

$$g(h(X,W),FY) = g(h(X,Y),JW) + \frac{1}{3}W(\ln f)g(TX,Y)$$

for any $X,Y \in \Gamma(TM_\theta)$ and $W \in \Gamma(TM_\perp)$.

Proof. For any $W \in \Gamma(TM_\perp)$ and $X,Y \in \Gamma(TM_\theta)$, we have

$$\begin{aligned} g(h(X,W),FY) &= g(\tilde{\nabla}_X W, JY) - g(\tilde{\nabla}_X W, TY) \\ &= g((\tilde{\nabla}_X J)W, Y) - g(\tilde{\nabla}_X JW, Y) - W(\ln f)g(X,TY) \\ &= g((\tilde{\nabla}_X J)W, Y) + g(A_{JW}X, Y) + W(\ln f)g(TX,Y). \end{aligned} \tag{16}$$

On the other hand, we also have

$$\begin{aligned} g(h(X,W),FY) &= g(\tilde{\nabla}_W X, JY) - g(\tilde{\nabla}_W X, TY) \\ &= g((\tilde{\nabla}_W J)X, Y) - g(\tilde{\nabla}_W JX, Y) - W(\ln f)g(X,TY) \\ &= g((\tilde{\nabla}_W J)X, Y) - g(\tilde{\nabla}_W TX, Y) - g(\tilde{\nabla}_W FX, Y) + W(\ln f)g(TX,Y) \\ &= g((\tilde{\nabla}_W J)X, Y) - W(\ln f)g(TX,Y) + g(A_{FX}W, Y) + W(\ln f)g(TX,Y) \\ &= g((\tilde{\nabla}_W J)X, Y) + g(h(Y,W),FX). \end{aligned} \tag{17}$$

Then, from (16) and (17) with the help of (3), we derive

$$2g(h(X,W),FY) = g(h(X,Y),JW) + g(h(Y,W),FX) + W(\ln f)g(TX,Y). \tag{18}$$

Interchanging X with Y in (18), we obtain

$$2g(h(Y,W),FX) = g(h(X,Y),JW) + g(h(X,W),FY) - W(\ln f)g(TX,Y). \tag{19}$$

Hence, the result follows from (18) and (19). □

Following relations are easily obtained by using (9), (10) in Lemma 3.

$$g(h(TX,Y),JW) = g(h(TX,W),FY) + \frac{1}{3}\cos^2\theta W(\ln f)g(X,Y), \tag{20}$$

$$g(h(X,TY),JW) = g(h(X,W),FTY) - \frac{1}{3}\cos^2\theta W(\ln f)g(X,Y), \tag{21}$$

$$g(h(TX,TY),JW) = g(h(TX,W),FTY) + \frac{1}{3}\cos^2\theta W(\ln f)g(X,TY). \tag{22}$$

5. A Characterization Theorem: MAIN Result 1

Now, we have the following theorem which provide the necessary and sufficient conditions that a pointwise hemi-slant submanifold to be a warped product.

Theorem 1. *Let M be a proper pointwise hemi-slant submanifold of nearly Kaehler manifold $\tilde{M}$ such that the normal component of $(\tilde{\nabla}_U J)V$ lies in the invariant normal subbundle of M for any $U,V \in \Gamma(TM)$. Then M is locally a warped product submanifold of the form $M = M_\perp \times_f M_\theta$ if and only if*

$$A_{JW}TX - A_{FTX}W = -\frac{1}{3}\cos^2\theta W(\lambda)X \tag{23}$$

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for any $W \in \Gamma(\mathfrak{D}^{\perp})$ *and* $X \in \Gamma(\mathfrak{D}^{\theta})$ *and for a differentiable function* λ *on* M *satisfying* $Y(\lambda) = 0, \ \forall\, Y \in \Gamma(\mathfrak{D}^{\theta})$.

Proof. If M be warped product submanifold, then (23) directly follows from Lemma 3.

Conversely, if M is a pointwise hemi-slant submanifold with the assumptions, then from Lemma 1, we have

$$\cos^2\theta g(\nabla_Z W, X) = -g(A_{JW}TX - A_{FTX}W, Z) = \frac{1}{3}\cos^2\theta W(\lambda) g(X, Z) = 0,$$

for any $X \in \Gamma(\mathfrak{D}^{\theta})$ and $Z, W \in \Gamma(\mathfrak{D}^{\perp})$. Since, M is proper pointwise hemi-slant, then we find from above equality that the leaves of the distribution $\mathfrak{D}^{\perp}$ are totally geodesic in M. Furthermore, from Proposition 1 with the hypothesis of theorem, we have

$$\cos^2\theta g([X, Y], Z) = g(A_{JZ}TY - A_{FTY}Z, X) - g(A_{JZ}TX - A_{FTX}Z, Y).$$

Then, using (23), we get $\cos^2\theta g([X, Y], Z) = 0$, which implies that $\mathfrak{D}^{\theta}$ is integrable and also from Lemma 2 with (23), we find its leaves are also totally geodesic in M. If we denote M_θ is a leaf of $\mathfrak{D}^{\theta}$ in M and h^θ is the second fundamental form of M_θ in M. Then, we have

$$g(h^\theta(X, Y), Z) = g(\nabla_X Y, Z) = g(J\tilde{\nabla}_X Y, JZ) = g(\tilde{\nabla}_X JY, JZ) - g((\tilde{\nabla}_X J)Y, JZ).$$

Then, using (6) and (7), we derive

$$g(h^\theta(X, Y), Z) = g(\tilde{\nabla}_X TY, JZ) + g(\tilde{\nabla}_X FY, JZ) - g(\mathcal{Q}_X Y, JZ).$$

By the hypothesis of the theorem that $\mathcal{Q}_U V \in \Gamma(\mu)$, we find

$$\begin{aligned} g(h^\theta(X, Y), Z) &= g(h(X, TY), JZ) + g((\tilde{\nabla}_X J)FY, Z) - g(\tilde{\nabla}_X JFY, Z) \\ &= g(A_{JZ}TY, X) + g(\mathcal{P}_X FY, Z) - g(\tilde{\nabla}_X tFY, Z) - g(\tilde{\nabla}_X fFY, Z). \end{aligned}$$

Using the property p_3 (ii) and (12), we obtain

$$\begin{aligned} g(h^\theta(X, Y), Z) = g(A_{JZ}TY, X) - g(\mathcal{Q}_X Z, FY) + \sin^2\theta g(\tilde{\nabla}_X Y, Z) \\ + \sin 2\theta X(\theta) g(Y, Z) + g(\tilde{\nabla}_X FTY, Z). \end{aligned}$$

Then, from (4) and the assumption that $\mathcal{Q}_U V \in \Gamma(\mu)$, we derive

$$\cos^2\theta g(h^\theta(X, Y), Z) = g(A_{JZ}TY - A_{FTY}Z, X) = -\frac{1}{3}\cos^2\theta Z(\lambda) g(X, Y),$$

or equivalently

$$h^\theta(X, Y) = -\frac{1}{3}\nabla(\lambda) g(X, Y),$$

where $\nabla(\lambda)$ is the gradient of λ. Thus, M_θ is a totally umbilical submanifold of M with mean curvature $H^\theta = -\frac{1}{3}\nabla(\lambda)$. Also, since $Y(\lambda) = 0$, for all $Y \in \Gamma(\mathfrak{D}^{\theta})$, we can prove that H^θ is parallel corresponding to the normal connection ∇^θ of M_θ in M (see, for more detail [1]). Thus, M_θ is an extrinsic sphere in M. Hence, by a result of Hiepko [17], we conclude that M is a warped product submanifold. Hence, the proof is complete. □

6. A General Inequality: Main Result 2

In this section, we develop the B.-Y. Chen's inequality for pointwise hemi-slant warped products in nearly Kaehler manifolds. For this, we assume the frame fields of $M = M_\perp \times {}_f M_\theta$ as follows:

Let $2n = \dim_{\mathbf{R}} \tilde{M}$, $q = \dim M_\perp$ and $2p = \dim M_\theta$ such that $\dim M = n = 2p + q$. If we denote $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, the tangent bundles on $M_\perp$ and M_θ, respectively and let $\{e_1, \ldots, e_q\}$ and $\{e_{q+1} = e_1^*, \cdots, e_{p+q} = e_p^*, e_{p+q+1} = \sec\theta Te_1^*, \cdots, e_n = e_{2p}^* = \sec\theta Te_p^*\}$ be the local orthonormal frames of $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively. Then, the orthonormal frames of $J\mathfrak{D}^\perp$ and $F\mathfrak{D}^\theta$ are $\{e_{n+1} = \tilde{e}_1 = Je_1, \cdots, \tilde{e}_q = Je_q\}$ and $\{e_{n+q+1} = \tilde{e}_1^* = \csc\theta Fe_1^*, \cdots, e_{n+q+p} = \tilde{e}_p^* = \csc\theta Fe_p^*, e_{n+q+p+1} = \tilde{e}_{p+1}^* = \csc\theta\sec\theta FTe_1^*, \cdots, e_{n+q+2p} = \tilde{e}_{2p} = \csc\theta\sec\theta FTe_p^*\}$, respectively. Notice, that we assume here $\dim\tilde{M} = 2\dim M$, i.e., the normal invariant subbundle $\mu = \{0\}$.

Theorem 2. *Let $M^n = M_\perp \times_f M_\theta$ be a warped product pointwise hemi-slant submanifold of a nearly Kaehler manifold $\tilde{M}^{2n}$ such that M is mixed totally geodesic, where $M_\perp$ and M_θ are totally real and proper poinwise slant submanifolds of $\tilde{M}$, respectively. Then, the second fundamental form h of M satisfies*

$$\|h\|^2 \geq \frac{2p}{9}\cos^2\theta\|\nabla\ln f\|^2, \tag{24}$$

where $\nabla\ln f$ is the gradient of $\ln f$ along $M_\perp$ and $2p = \dim M_\theta$.
If the equality sign in (24) holds identically, then $M_\perp$ is totally geodesic and M_θ is totally umbilical in $\tilde{M}$.

Proof. From the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right) = \sum_{k=n+1}^{2n}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), e_k\right)^2.$$

For the frame fields of $J\mathfrak{D}^\perp$ and $F\mathfrak{D}^\theta$ the above relation spilt as

$$\|h\|^2 = \sum_{k=n+1}^{q}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), \tilde{e}_k\right)^2 + \sum_{k=1}^{2p}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), \tilde{e}_k^*\right)^2. \tag{25}$$

Then for the orthonormal frames of $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, the above equality takes the form

$$\begin{aligned}\|h\|^2 = &\sum_{k=1}^{q}\sum_{i,j=1}^{q} g\left(h\left(e_i, e_j\right), Je_k\right)^2 + 2\sum_{k=1}^{q}\sum_{i=1}^{q}\sum_{j=1}^{2p} g\left(h\left(e_i, e_j^*\right), Je_k\right)^2 \\ &+ \sum_{k=1}^{q}\sum_{i,j=1}^{2p} g\left(h\left(e_i^*, e_j^*\right), Je_k\right)^2 + \sum_{k=1}^{2p}\sum_{i,j=1}^{q} g\left(h\left(e_i, e_j\right), \tilde{e}_k^*\right)^2 \\ &+ 2\sum_{k=1}^{2p}\sum_{i=1}^{q}\sum_{j=1}^{2p} g\left(h\left(e_i, e_j^*\right), \tilde{e}_k^*\right)^2 + \sum_{k=1}^{2p}\sum_{i,j=1}^{2p} g\left(h\left(e_i^*, e_j^*\right), e_k^*\right)^2.\end{aligned} \tag{26}$$

There is no relation in terms of the warped products for the first and last terms in (26), so leave these positive terms. Then, using the fact that M is mixed totally geodesic submanifold, the second and fifth terms are identically zero. Furthermore, from Proposition 2 with mixed totally geodesic condition, forth term vanishes identically. Thus evaluated third term can be expanded as

$$\begin{aligned}\|h\|^2 \geq &\sum_{k=1}^{q}\sum_{i,j=1}^{p} g\left(h\left(e_i^*, e_j^*\right), Je_k\right)^2 + \sec^2\theta\sum_{k=1}^{q}\sum_{i,j=1}^{p} g\left(h\left(Te_i^*, e_j^*\right), Je_k\right)^2 \\ &+ \sec^2\theta\sum_{k=1}^{q}\sum_{i,j=1}^{p} g\left(h\left(e_i^*, Te_j^*\right), Je_k\right)^2 + \sec^4\theta\sum_{k=1}^{q}\sum_{i,j=1}^{p} g\left(h\left(Te_i^*, Te_j^*\right), Je_k\right)^2.\end{aligned}$$

Using Lemma 3 and (20)-(22), we arrive at

$$\begin{aligned}\|h\|^2 &\geq \frac{2}{9}\sum_{k=1}^{q}(e_k \ln f)^2 \sum_{i,j=1}^{p}\Big(g(Te_i^*, e_j^*)\Big)^2 + \frac{2}{9}\cos^2\theta \sum_{k=1}^{q}(e_k \ln f)^2 \sum_{i,j=1}^{p}\Big(g(e_i^*, e_j^*)\Big)^2 \\ &= \frac{2p}{9}\cos^2\theta \|\nabla(\ln f)\|^2,\end{aligned}$$

which is required inequality. Since, M is mixed totally geodesic, i.e.,

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\theta}) = \{0\}. \tag{27}$$

For the leaving first term in (26), we have

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) \perp J\mathfrak{D}^{\perp}. \tag{28}$$

Also, from the leaving sixth term in (26), we observe that

$$h(\mathfrak{D}^{\theta}, \mathfrak{D}^{\theta}) \perp F\mathfrak{D}^{\theta}. \tag{29}$$

Furthermore, from Proposition 2 with mixed totally geodesic condition, we find

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) \perp F\mathfrak{D}^{\theta}. \tag{30}$$

Then, from (28) and (30), we obtain

$$h(\mathfrak{D}^{\perp}, \mathfrak{D}^{\perp}) = \{0\}. \tag{31}$$

Since, $M_{\perp}$ is totally geodesic in M [2,6], using this fact with (27) and (31), we conclude that $M_{\perp}$ is totally geodesic in $\tilde{M}$. Moreover, Since M_{θ} is totally umbilical in M [2,6], then with this fact (27) and (30) imply that M_{θ} is totally umbilical in $\tilde{M}$. Thus, the theorem is proved completely. □

Remark 2. *Notice that the above inequality (24) is true only for proper pointwise hemi-slant submanifolds. For example, on a pointwise slant submanifold, if every point is a complex point i.e., M_{θ} becomes a holomorphic submanifold with slant function $\theta = 0$, then this is the case of non-existence of such warped products.*

Remark 3. *Furthermore the inequality (24) is valid also for proper hemi-slant warped products, i.e., if θ is globally constant on M_{θ}, then (24) can be generalised for warped product hemi-slant submanifolds studied in [23,26].*

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