Article

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SU(3) Brachistochrone and the Isotropic Oscillator

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Abstract

This paper discusses two major applications of SU(3) theory, the first being the question of quantum brachistochrones and problems in time optimal control theory, the second being the use of spherical tensor decompositions of the special functions which are associated to SU(3) via the isotropic oscillator. We discuss the derivation of the quantum brachistochrone problem more generally from the perspective of von Neumann equations and matrix mechanics, arriving at an equivalent formulation to the Lagrangian method. Discussion is given to the application of such advanced methods in quantum computation, and some future directions that may be amenable to a similar sort of analysis.

Keywords: Gell-Mann, calculus, quantum, control, brachistochrone, optimal, time, operator, eigenstate, eigenmatrix

1 Introduction

The time optimal control of quantum systems is a rapidly expanding area of research, with many new results emerging as a result of this interesting aspect of variational calculus. The application of open-loop control methods is desirable in quantum systems, as the delicate nature of measurement has difficulties associated with the use of feedback systems that operate in the same way as in a classically controlled system. This paper discusses several highly technical aspects of this emerging science of time optimal control.

With many resources in quantum information science devoted towards the understanding of single-qubit systems and their composition via tensor products, it is important to consider alternative methodologies. We take the view that qutrits offer a broader description of the wider variety of quantum phenomena we can expect to

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observe in qudit systems of arbitrary dimension, not necessarily reducible to a decomposition over qubits. This paper shows how one can arrive at similar results to those found in Gell-Mann's papers [8] and the collective motion models of the isotropic oscillator through the use of the quantum brachistochrone equation and time optimal control theory.

2 Quantum Fermat Principle

We shall briefly outline the context of the brachistochrone problem, and its analogous counterpart in quantum mechanics. In classical theory, the brachistochrone problem solves for the shortest time path between any two points, given constraints. This question, originally formulated by Bernoulli (1696) may be stated classically in the form of such questions as "Given two points, A & B, in a vertical plane, what is the curve traced out by a point acted on by gravity, which starts at A and reaches B in the shortest time?". The answer, as was found by Bernoulli, is the cycloid that connects the two points A and B.

Let us now discuss the quantum generalisation of this question. The simplest way in which to pose a similar problem occurred in [2, 3], where the authors determined for any two reachable quantum states the Hamiltonian operator that drives the state from one configuration to another in least time, given constraints. Other variants on this problem include the construction of the unitary operator associated to this Hamiltonian. In either sense, we can think of both these problems as being somewhat equivalent to Fermat's principle of optics, which states that the physical path of light through a medium is the path which takes the least time, and also Huygen's principle, where it has been known classically that the two principles are equivalent in every nature. Young (1809) was of the opinion that "The principle of Fermat, although it was assumed by that mathematician on hypothetical... grounds, is in fact a fundamental law with respect to undulatory motion, and is explicitly the basis of every determination in the Huygenian theory." [18]. From such considerations, one may go on to develop such formulations of ray optics as Snell's law of refraction. Of course, it was known to Hamilton, Laplace and others that the eikonal equation provides the connection between the two seemingly disparate perspectives. In quantum terms, we may coin a quantum Fermat principle, and state that the quantum state travels on the complex projective space such that the time between any two points is a minimum, given constraints on our objective which are represented as Lagrange multipliers. The question of the correct formulation of this yields naturally to variational calculus, as initially explored in [2, 3]. We shall not pursue their Lagrangian method in this work, as we shall explore the complementary Hamiltonian mechanics from a different perspective. Our originating principles shall overlap, but generally we shall use the von Neumann approach to quantum mechanics and some insights from projective geometry in order to analyse time optimal control problems in the quantum realm.

3 Review

The following paper is divided into two major sections. We shall review the relevant literature for each section in turn below.

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3.1 Matrix Mechanics and the Quantum Brachistochrone

We take the view that the results which naturally follow from such a time optimisation as implied by the quantum Fermat principle may be derived from application of the Floquet theorem [1] and the von Neumann equation [17]. The quantum brachistochrone equation, originally defined and calculated for qubit systems in [2, 3], is an interesting question in open-loop quantum control, which has been extended to problems of time optimal control of qutrits in [9, 13]. Much effort especially in the field of quantum information science has been focused on the use of qubits as a fundamental data carrier; this paper further investigates the use of qutrits as an alternative type of state logic. Note that this is underpinned by the theory of the eightfold way [8], and indeed we shall show that a certain concept of symmetry as given by what amounts to a Gram decomposition of the unitary matrices is an essential part of the solution of the SU(3) problems considered in these calculations. We note the extension of the system of time optimal control has been carried out for spinor type systems in [9, 12], where the author was able to demonstrate a number of useful known properties related to physical quantities such as the electronic rest mass and scattering theory.

3.2 Cartesian Tensors and the Isotropic Oscillator

The second structure we shall consider in this paper is given by the theory of SU(3) as a model of collective nuclear motion. The isotropic oscillator formulation and solution using SU(3) can be seen as a nuclear shell model of the atom, expounded upon originally in the papers of Elliott [15, 16]. More recent reviews of the topic may be found in the papers found at [5–7]. We note the use of Dyson models in self-consistent collective coordinates in [4], and the application to quadrupole-quadrupole interactions as given in [11]. Most importantly, it has been determined that the Cartesian tensor decomposition of the isotropic oscillator, originally given by Fradkin in [14], is of particular interest to the development of the model of SU(3) as given by creation and annihilation operator theory. The second part of our proof shall focus on the application of this concept; the paper given in [14] has differing goals in that our approach shall borrow from the theory of special functions in order to analyse separable solutions for the state equations of a similar oscillatory system.

Our primary goal herein is to show the contrast between the two methods of identification of the behaviour of the quantum system. On the one hand, we have a system of matrix equations, the other a system of continuous differential operators. As we shall see, the nature of the qutrit system is such that both of these perspectives of quantum mechanics are not mutually inconsistent. In fact, the deeper implication is one of representation theory of groups, which we shall discuss in the conclusion of this paper.

4 Structure of the Paper

The sections are as follows; firstly, we outline the basic machinery required from matrix mechanics required in this proof, including the basic brachistochrone equations, constraints and definition of the quantum Fermat principle. The following proof shows

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how the von Neumann perspective of quantum mechanics has particular applicability to this problem, and demonstrates how one may set up the problem of time optimal control using some simple constraints and relationships from matrix operator theory. We then apply this technique to the simple toy problem for a qubit as given in [2, 3] by a more complicated method, following with a brief review of known advanced results from qutrit operator theory as given in [10]. The final section of the paper is devoted to an exposition of spherical operator theory as it is applied to the problem of an isotropic oscillator, including its relation to SU(3), the analysis of group relations related to spherical harmonics, and radial Whittaker functions.

5 Machinery of Finite Hilbert Spaces

Let us now define the basic structures we shall encounter in this proof. In general, we shall be dealing with quantum states that exist on a complex vector space, normalised to probability 1 and of finite dimension. The state vector shall be written as a column vector of possibly complex functions of time:

$$|\Psi\rangle = \sum_j c_j |j\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{1}$$

with Hermitian conjugate defined by the adjoint:

$$\langle\Psi| = (|\Psi\rangle)^\dagger = \left[\, c_1^*,\ c_2^*,\ \cdots\ ,c_n^* \,\right] \tag{2}$$

The inner product is such that we are constrained to the surface of a sphere via:

$$\langle\Psi|\ \Psi\rangle = |c_1|^2 + |c_2|^2 + ... + |c_n|^2 = 1 \tag{3}$$

with outer product given by the density matrix:

$$|\Psi\rangle\langle\Psi| = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \left[\, c_1^*\ c_2^*\ \cdots\ c_n^* \,\right] = \begin{bmatrix} |c_1|^2 & c_1c_2^* & \cdots & \\ c_2c_1^* & |c_2|^2 & \ddots & \\ \vdots & \ddots & \ddots & \\ & & & |c_n|^2 \end{bmatrix} \tag{4}$$

Finally, the expectation value and trace of an operator are defined through:

$$\langle\Psi|\,\tilde{H}\,|\Psi\rangle = \left\langle \tilde{H} \right\rangle \tag{5}$$

$$\mathbf{Tr}(\tilde{A}) = \sum_k A_{kk} \tag{6}$$

These are the basic vector and matrix operators that we shall require in this proof. The Dirac notation in this case is abused to merely mean finite state vectors. However, as we shall show, it has particular utility in allowing a compact formulation of this theory of time optimality.

5.1 Quantum Brachistochrone

We shall now demonstrate a simple way in which the quantum brachistochrone equation and Fermat principle may be understood. If we take the standard expression for the energy variance, we may write this as the expectation value:

$$\Delta E^2 = \left\langle \tilde{H}^2 \right\rangle - \left\langle \tilde{H} \right\rangle^2 = \langle\Psi| (\tilde{H} - \left\langle \tilde{H} \right\rangle \mathbf{1})^2 |\Psi\rangle \tag{7}$$

The Fubini-Study metric in this instance is defined through the simple projective equation:

$$ds^2 = 1 - |\langle\Psi + d\Psi| \Psi\rangle|^2 \tag{8}$$
$$= \langle d\Psi| (\mathbf{1} - \hat{P}) |d\Psi\rangle \tag{9}$$

where $\hat{P} = |\Psi\rangle\langle\Psi|$ is the projection matrix defined by the outer product of the state with itself. This is the natural metric associated with complex projective state space. The velocity of the state may be specified through $\frac{ds}{dt} = \Delta E$. Following [2, 3], the action principle or Lagrangian can be written as:

$$S = \int_{t_0}^{t_f} (\mathcal{L}_T + \mathcal{L}_S + \mathcal{L}_C)\, dt \tag{10}$$

where

$$\mathcal{L}_T = \frac{1}{\Delta E}\frac{ds}{dt} = \frac{\sqrt{\left\langle \dot{\Psi} \right| (\mathbf{1} - \hat{P}) \left| \dot{\Psi} \right\rangle}}{\Delta E} = 1 \tag{11}$$
$$\mathcal{L}_S = i\left(\left\langle \dot{\phi} \right| \Psi\rangle - \langle\Psi| \dot{\phi}\rangle\right) + \left(\langle\Psi| \tilde{H} |\phi\rangle + \langle\phi| \tilde{H} |\Psi\rangle\right) \tag{12}$$
$$\mathcal{L}_C = \sum_i \lambda_j f_j(\tilde{H}) \tag{13}$$

where the Lagrangian components have meaning $\int \mathcal{L}_T dt = \int 1 dt$, i.e. least time, $\mathcal{L}_S$ constrains the wave vector to follow the quantum equation, $\mathcal{L}_C$ is a set of constraints on the Hamiltonian operator. The variables $|\phi\rangle$, $|\Psi\rangle$, $\tilde{H}$ and λ_j then form the system of variables for the Euler-Lagrange equations for which the functional is to be minimised. We need more information about the constraints to proceed.

6 Constraints and the von Neumann perspective

The Euler-Lagrange equation in the variable $|\phi\rangle$ for the action principle gives

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}_S}{\partial \left\langle \dot{\phi} \right|}\right) = \frac{\partial \mathcal{L}_S}{\partial \langle\phi|} \tag{14}$$
$$i\frac{\partial}{\partial t}|\Psi\rangle = \tilde{H}(t)|\Psi\rangle \tag{15}$$

To define the state, we need access to the form of the Hamiltonian operator as a function of time. One simple system where a solution exists is the case of finite isotropic energy, and a linear constraint on the accessible degrees of freedom. In this situation we may write the isotropic condition as:

$$f_1(\tilde{H}) = \mathbf{Tr}\left(\frac{\tilde{H}^2}{2}\right) - k < \infty \tag{16}$$

and the linear constraint on the Hamiltonian operator in the form:

$$f_2(\tilde{H}) = \mathbf{Tr}\left(\tilde{H}\tilde{F}\right) = 0 \tag{17}$$

A lengthy calculation following the prescription in [2, 3] yields the resulting differential equations for the Hamiltonian operator, but there is a direct method that hinges on the application of the von Neumann perspective. In this case, we take as our basis axiom the von Neumann equation for a matrix operator:

$$i\frac{d\hat{A}}{dt} = [\tilde{H}(t), \hat{A}] = \tilde{H}\hat{A} - \hat{A}\tilde{H} \tag{18}$$

for any Hermitian matrix $\hat{A} = \hat{A}^{\dagger}$. Choose a particular $\hat{A} = \tilde{H} + \tilde{F}$, substituting above, obtain

$$i\frac{d}{dt}\left(\tilde{H} + \tilde{F}\right) = [\tilde{H}, \tilde{H} + \tilde{F}] = [\tilde{H}, \tilde{F}] \tag{19}$$

This is now a set of coupled differential equations in the matrix operators. The Euler-Lagrange equation in the variable $\tilde{H}$ will contain a term

$$\frac{\partial \mathcal{L}_C}{\partial \tilde{H}} = \sum_j \lambda_j \frac{\partial f_j(\tilde{H})}{\partial \tilde{H}} = \lambda_1 \tilde{H} + \lambda_2 \tilde{F} \tag{20}$$

and with some effort it is possible to show that the dynamical equations can be re-arranged to give the same result as the ansatz obtained from the von Neumann equation. We call this the quantum brachistochrone equation. The system of equations is then completely specified in this situation by the following set of matrix differential equations:

Linear Constraint

$$\mathbf{Tr}\left(\tilde{H}\tilde{F}\right) = 0 \tag{21}$$

which is a Lagrange multiplier to hold the system degrees of freedom to a hyperplane.

Isotropic Condition

$$\mathbf{Tr}\left(\frac{\tilde{H}^2}{2}\right) = k \tag{22}$$

i.e. the net length of the vector represented through the matrix $\tilde{H}$ is finite, which holds the net overhead energy to some finite value.

Quantum Brachistochrone Equation

$$i\frac{d}{dt}\left(\tilde{H}+\tilde{F}\right)=[\tilde{H},\tilde{F}] \tag{23}$$

which is the Euler-Lagrange equation for constraints plus least time in the Hamiltonian variable.

Schrödinger Equation

$$i\frac{\partial}{\partial t}\left|\Psi\right\rangle=\tilde{H}(t)\left|\Psi\right\rangle \tag{24}$$

i.e. the Euler-Lagrange equation in the state variable.

Unitary Evolution

$$\hat{U}(t,s)\tilde{H}(s)\hat{U}^{\dagger}(t,s)=\tilde{H}(t) \tag{25}$$

which gives the unitary evolution in terms of the Hamiltonian operator via

$$\hat{U}(t,s)=\exp\left(-i\int_{s}^{t}\tilde{H}(\tau)d\tau\right) \tag{26}$$

The quantum brachistochrone also takes into account boundary conditions, but we shall see that by identifying unitary evolutions on the state space we can avoid this complication. We also have not discussed the gauge invariance of the action principle, which is an important point that allows us to use the traceless component of the Hamiltonian matrix.

7 Simple Problem

Consider the smallest non-trivial problem in time optimal quantum control. If we take the spinor space defined over the Pauli matrices $\{\hat{\sigma}_x,\hat{\sigma}_y,\hat{\sigma}_z\}$, we have:

$$\hat{\sigma}_x=\begin{bmatrix}0&1\\1&0\end{bmatrix},\hat{\sigma}_y=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\hat{\sigma}_z=\begin{bmatrix}1&0\\0&-1\end{bmatrix} \tag{27}$$

If we consider the geometric algebra, we have spinor geometry defined by product formula:

$$(\mathbf{a}\cdot\sigma)(\mathbf{b}\cdot\sigma)=(\mathbf{a}\cdot\mathbf{b})\mathbf{1}+i(\mathbf{a}\times\mathbf{b})\cdot\sigma \tag{28}$$

$$\mathbf{n}\cdot\sigma=\sum_j n_j\hat{\sigma}_j=\begin{bmatrix}n_z&n_x-in_y\\n_x+in_y&-n_z\end{bmatrix} \tag{29}$$

Choosing then a Hamiltonian matrix through $\tilde{H}(t)=\lambda_x(t)\hat{\sigma}_x+\lambda_y(t)\hat{\sigma}_y=\mathbf{a}\cdot\sigma$, we are able to specify the time optimal control problem. In terms of the isotropic constraint, we have

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$$\mathbf{Tr}\left(\frac{\tilde{H}^2}{2}\right) = \frac{1}{2}\mathbf{Tr}\left((\mathbf{a}\cdot\sigma)(\mathbf{a}\cdot\sigma)\right) = \frac{1}{2}\mathbf{Tr}\left((\mathbf{a}\cdot\mathbf{a})\mathbf{1} + i(\mathbf{a}\times\mathbf{a})\cdot\sigma\right) \tag{30}$$

$$= \frac{(\mathbf{a}\cdot\mathbf{a})}{2}\mathbf{Tr}\left(\mathbf{1}\right) = |\mathbf{a}|^2 = k < \infty \tag{31}$$

We can see that this forces the vector which defines the Hamiltonian matrix to be of finite length. The linear constraint gives

$$\mathbf{Tr}\left(\tilde{H}\tilde{F}\right) = \mathbf{Tr}\left((\mathbf{a}\cdot\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\cdot\sigma\right) = 0 \tag{32}$$

The second term $\mathbf{Tr}\left(i(\mathbf{a}\times\mathbf{b})\cdot\sigma\right) = i\mathbf{Tr}\left(\mathbf{c}\cdot\sigma\right) = 0$, so we must have $\mathbf{a}\cdot\mathbf{b} = 0$, meaning that we have $\tilde{F} = \Omega(t)\hat{\sigma}_z$. The matrix operators with functions to be determined are then

$$\tilde{H}(t) = \lambda_x(t)\hat{\sigma}_x + \lambda_y(t)\hat{\sigma}_y = \begin{bmatrix} 0 & \varepsilon(t) \\ \varepsilon^*(t) & 0 \end{bmatrix} \tag{33}$$

with $\varepsilon(t) = \lambda_x(t) - i\lambda_y(t)$, and

$$\tilde{F} = \Omega(t)\hat{\sigma}_z = \begin{bmatrix} \Omega(t) & 0 \\ 0 & -\Omega(t) \end{bmatrix} \tag{34}$$

We can show that we satisfy the linear constraint using elementary matrix operations:

$$\mathbf{Tr}\left(\tilde{H}\tilde{F}\right) = \mathbf{Tr}\left(\begin{bmatrix} 0 & \varepsilon(t) \\ \varepsilon^*(t) & 0 \end{bmatrix}\begin{bmatrix} \Omega(t) & 0 \\ 0 & -\Omega(t) \end{bmatrix}\right) = 0 \tag{35}$$

whereas the isotropic condition reads as

$$\mathbf{Tr}\left(\frac{\tilde{H}^2}{2}\right) = \frac{1}{2}\mathbf{Tr}\left(\begin{bmatrix} 0 & \varepsilon(t) \\ \varepsilon^*(t) & 0 \end{bmatrix}\begin{bmatrix} 0 & \varepsilon(t) \\ \varepsilon^*(t) & 0 \end{bmatrix}\right) = |\varepsilon(t)|^2 = k \tag{36}$$

The quantum brachistochrone equation is then defined by $i\frac{d}{dt}\left(\tilde{H} + \tilde{F}\right) = [\tilde{H}, \tilde{F}]$, plugging in the form of the matrices as from above, we find:

$$i\frac{d}{dt}\begin{bmatrix} \Omega & \varepsilon \\ \varepsilon^* & -\Omega \end{bmatrix} = 2\Omega\begin{bmatrix} 0 & -\varepsilon \\ \varepsilon^* & 0 \end{bmatrix} \tag{37}$$

i.e. $\dot{\Omega} = 0$, which implies $\Omega(t) = \Omega$ (a constant). The remaining expressions in vector-matrix form

$$i\frac{d}{dt}\begin{bmatrix} \varepsilon \\ \varepsilon^* \end{bmatrix} = 2\Omega\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varepsilon \\ \varepsilon^* \end{bmatrix} \tag{38}$$

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$$\tilde{H}(t) = \begin{bmatrix} 0 & \varepsilon(t) \\ \varepsilon^*(t) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon(0)e^{+2i\Omega t} \\ \varepsilon^*(0)e^{-2i\Omega t} & 0 \end{bmatrix} \tag{39}$$

Using $\varepsilon(0) = re^{i\phi}$:

$$\tilde{H}(t) = r \begin{bmatrix} 0 & e^{+i(2\Omega t+\phi)} \\ e^{-i(2\Omega t+\phi)} & 0 \end{bmatrix} \tag{40}$$

7.1 Unitary Evolution

We now have to evaluate the exponential operator of the Hamiltonian $\hat{U}(t,s) = \exp\left(-i\int_s^t \tilde{H}(\tau)d\tau\right)$. In general this is a very difficult problem, especially for time dependent operators as we are working with. Unitary evolution is defined by

$$\hat{U}(t,s)\tilde{H}(s)\hat{U}^\dagger(t,s) = \tilde{H}(t) \tag{41}$$

The Hamiltonian may be diagonalised at any point in time using

$$\tilde{H}(t) = \hat{W}(t)\hat{L}\hat{W}^{-1}(t) \tag{42}$$

Consider the initial time $\tilde{H}(0) = \hat{W}(0)\hat{L}\hat{W}^{-1}(0)$, then we can write $\hat{L} = \hat{W}^{-1}(0)\tilde{H}(0)\hat{W}(0)$ implying

$$\tilde{H}(t) = \hat{W}(t)\hat{L}\hat{W}^{-1}(t) = \hat{W}(t)\hat{W}^{-1}(0)\tilde{H}(0)\hat{W}(0)\hat{W}^{-1}(t) \tag{43}$$

$$= \hat{U}(t,0)\tilde{H}(0)\hat{U}^\dagger(t,0) \tag{44}$$

and

$$\hat{U}(t,0) = \hat{W}(t)\hat{W}^{-1}(0), \hat{U}^\dagger(t,0) = \hat{W}(0)\hat{W}^{-1}(t) \tag{45}$$

Further, we can write $\hat{U}(t,s) = \hat{U}(t,0)\hat{U}^\dagger(s,0)$, $\hat{U}^{-1}(t,0) = \hat{U}^\dagger(t,0)$, which gives

$$\hat{U}(t,s) = \hat{W}(t)\hat{W}^{-1}(s) \tag{46}$$

Applying this formula for the Hamiltonian matrix

$$\tilde{H}(t) = r \begin{bmatrix} 0 & e^{+i(2\Omega t+\phi)} \\ e^{-i(2\Omega t+\phi)} & 0 \end{bmatrix} \tag{47}$$

we find:

$$\tilde{H}(t) = \hat{W}(t) \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix} \hat{W}^{-1}(t) \tag{48}$$

with eigenmatrix

$$\hat{W}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i(2\Omega t+\phi)} & -e^{i(2\Omega t+\phi)} \\ 1 & 1 \end{bmatrix} \tag{49}$$

and inverse given by:

$$\hat{W}^{-1}(t) = \hat{W}^\dagger(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i(2\Omega t+\phi)} & 1 \\ -e^{-i(2\Omega t+\phi)} & 1 \end{bmatrix} \tag{50}$$

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The unitary operator is then calculated using $\hat{U}(t,s) = \hat{W}(t)\hat{W}^{\dagger}(s)$ with matrix:

$$\hat{U}(t,s) = \begin{bmatrix} e^{2i\Omega(t-s)} & 0 \\ 0 & 1 \end{bmatrix} \tag{51}$$

This is the Cartan sub-group of the SU(2) Lie group. Further results have shown that through a process of isomorphic transformation one may map this to a number of other solutions which exist by permutation of the space indices of the basis matrices.

7.2 A Harder Example- SU(3)

If we take an identical problem, but increase the dimensions, various subgroups and implied subalgebras present themselves naturally on SU(3). Due to the higher dimensionality, it is possible to define such problems as:

$$\tilde{H}_Q(t) = \begin{bmatrix} 0 & \varepsilon_1 & 0 \\ \varepsilon_1^* & 0 & \varepsilon_2 \\ 0 & \varepsilon_2^* & 0 \end{bmatrix} \tag{52}$$

$$\tilde{H}_J(t) = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & -i\beta \\ 0 & i\beta & 0 \end{bmatrix} \tag{53}$$

$$\tilde{H}_D(t) = i\begin{bmatrix} 0 & -\Lambda & 0 \\ \Lambda & 0 & -\Xi \\ 0 & \Xi & 0 \end{bmatrix} \tag{54}$$

The quantum brachistochrone problem is now on SU(3)- the space of unitary transformations of the qutrit state. As before, we have the set of matrix equations

Linear Constraint

$$\mathbf{Tr}\left(\tilde{H}\tilde{F}\right) = 0 \tag{55}$$

Isotropic Condition

$$\mathbf{Tr}\left(\frac{\tilde{H}^2}{2}\right) = k \tag{56}$$

Quantum Brachistochrone

$$i\frac{d}{dt}\left(\tilde{H} + \tilde{F}\right) = [\tilde{H}, \tilde{F}] \tag{57}$$

so each Hamiltonian matrix will have an associated isotropic condition and linear constraint. Solving the quantum brachistochrone for each of the systems and associated constraint, and normalising each Hamiltonian to unity, we find:

$$\tilde{H}_Q(t) = \begin{bmatrix} 0 & \cos t & 0 \\ \cos t & 0 & -ie^{-i\theta}\sin t \\ 0 & ie^{+i\theta}\sin t & 0 \end{bmatrix} \tag{58}$$

$$\tilde{H}_J(t) = \begin{bmatrix} 0 & \cos t & 0 \\ \cos t & 0 & -i\sin t \\ 0 & i\sin t & 0 \end{bmatrix} \tag{59}$$

$$\tilde{H}_D(t) = i\begin{bmatrix} 0 & -\cos t & 0 \\ \cos t & 0 & -\sin t \\ 0 & \sin t & 0 \end{bmatrix} \tag{60}$$

Consider the first Hamiltonian matrix, the others are solved in a similar fashion. There exists a diagonalisation such that

$$\tilde{H}_Q(t) = \hat{Q}(t)\hat{L}\hat{Q}^\dagger(t) \tag{61}$$

and likewise for $\tilde{H}_J(t)$, $\tilde{H}_D(t)$, which map to the same $\hat{L}$ via different unitary operators. The operator $\hat{Q}(t)$ has matrix form

$$\hat{Q}(t) = \begin{bmatrix} \frac{\cos t}{\sqrt{2}} & -\frac{\cos t}{\sqrt{2}} & ie^{-i\theta}\sin t \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{ie^{i\theta}\sin t}{\sqrt{2}} & -\frac{ie^{i\theta}\sin t}{\sqrt{2}} & \cos t \end{bmatrix} \tag{62}$$

with properties $\hat{Q}(t)\hat{Q}^\dagger(t) = \hat{Q}^\dagger(t)\hat{Q}(t) = \mathbf{1}$, also $\det\hat{Q}(t) = 1$ so it is a unitary rotation. Once again we can use the unitary evolution equation

$$\tilde{H}_Q(t) = \hat{U}(t,s)\tilde{H}_Q(s)\hat{U}^\dagger(t,s) \tag{63}$$

and hence $\hat{U}(t,s) = \hat{Q}(t)\hat{Q}^\dagger(s)$ as before, which may be easily computed as

$$\hat{U}(t,s) = \hat{Q}(t)\hat{Q}^\dagger(s) = \begin{bmatrix} \cos(t-s) & 0 & ie^{-i\theta}\sin(t-s) \\ 0 & 1 & 0 \\ ie^{i\theta}\sin(t-s) & 0 & \cos(t-s) \end{bmatrix} \tag{64}$$

This is an element of the set of unitary matrices which operate on the state vector via

$$\hat{U}(t,s)\,|\Psi(s)\rangle = |\Psi(t)\rangle \tag{65}$$

By observation, we plainly have time translation invariance $\hat{U}(t,s) = \hat{U}(t-s,0)$. This method of constructing the time optimal Hamiltonian is particularly efficient, and may be applied to any problem of finite dimension. We note the deep connection between such observations, the differential geometry as given by the Gauss fundamental forms, and the Gram and Cholesky decompositions of matrix operators. This result and method is startlingly straightforward when compared to the techniques used in [2, 3] when originally evaluating the quantum brachistochrone. All extraneous complication resulting from the application of boundary conditions, anti-commutative operators and variational calculus are essentially short-circuited

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from this perspective, as the matrix mechanics perspective of von Neumann coupled with these matrix decompositions allows ready analysis free of these difficulties.

8 Constructing SU(3) via Isotropic Oscillator

We shall now outline a simple method where one may construct a set of states with analogous properties to the finite matrix systems considered above. Working now in the representation theory, our states will be given by basis functions over the Hilbert space. The Schrödinger equation is defined by:

$$i\frac{\partial\Phi}{\partial t} = \hat{H}\Phi \tag{66}$$

where $\Phi = \Phi(\mathbf{x}, t)$ and $\hat{H}$ is some Hermitian differential operator. We shall be concerned with the differential equation given by:

$$E\psi = -\frac{1}{2}\nabla^2\psi + \frac{1}{2}\omega^2 r^2\psi \tag{67}$$

where $\psi = \psi(\mathbf{x})$, and $\Phi(\mathbf{x}, t) = \varphi(t)\psi(\mathbf{x})$ is a separable solution to the differential equation, which is the form a time independent solution will obey. In terms of the potential energy, we can write $V(r) = \frac{1}{2}\omega^2 r^2$, which is isotropic as it contains no angular dependence. If we think back to Hooke's law for a spring, in this situation we will have $\ddot{x} = -kx$, hence $V(x) = \frac{1}{2}kx^2$, so in this sense the system is an isotropic oscillator. Indeed, the fundamental eigenfunction equation may be written:

$$E\psi = \frac{1}{2}\left(-\nabla^2 + \omega^2 r^2\right)\psi \tag{68}$$

and following the analysis in [11, 14, 16], the basic identities for the angular momentum are readily evaluated as:

$$\hat{L}_i = \epsilon_{ijk} x_j \hat{p}_k \tag{69}$$

where ϵ_{ijk} is the Levi-Civita symbol and $\hat{p}_k = -i\partial_k = -i\frac{\partial}{\partial x_k}$, e.g. $\hat{L}_x = i(z\partial_y - y\partial_z)$ and so on, the commutation brackets being given by identities such as:

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hat{L}_z \tag{70}$$

& cyclic permutations thereof, the Laplacian given by:

$$\nabla^2\psi = -\frac{1}{r^2}\hat{L}^2\psi + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \tag{71}$$

where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. Finally, the constants of motion and angular momentum tensor may be found through:

$$\left[\hat{H}, \hat{L}^2\right]\psi = \left[\hat{H}, \hat{L}_z\right]\psi = 0 \tag{72}$$

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$$\hat{l}_{ij} = \left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{ijk}\hat{L}_k \tag{73}$$

$$\left[\hat{l}_{ij}, \hat{l}_{km}\right] = \delta_{jm}\hat{l}_{ki} - \delta_{jn}\hat{l}_{mi} - \delta_{im}\hat{l}_{nj} + \delta_{in}\hat{l}_{mj} \tag{74}$$

with the raising and lowering operators given by the standardised expressions:

$$\hat{a}_j = \frac{1}{\sqrt{2\omega}}\left(\omega x_j + i\hat{p}_j\right) \tag{75}$$

with $\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}$ and $\left[\hat{a}_j, \hat{a}_k\right] = \left[\hat{a}_j^\dagger, \hat{a}_k^\dagger\right] = 0$. The Hamiltonian operator can be written as

$$\hat{H} = \frac{1}{2}\left(-\nabla^2 + \omega^2 r^2\right) = \omega\sum_j(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2}) \tag{76}$$

where the sum over the latin indices represents the coordinate degrees of freedom of the system. If we take a particular coordinate, the creation and annihilation operators may be written:

$$\hat{a}_x = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + i\hat{p}_x) \tag{77}$$

$$\hat{a}_x^\dagger = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} - i\hat{p}_x) \tag{78}$$

Calculating commutator $\left[\hat{g}, \hat{h}\right] = \hat{g}\hat{h} - \hat{h}\hat{g}$ for these operators we find:

$$\left[\hat{a}_x, \hat{a}_x^\dagger\right]\psi = \frac{1}{2\omega}\left[(\omega\hat{x} + i\hat{p}_x)(\omega\hat{x} - i\hat{p}_x) - (\omega\hat{x} - i\hat{p}_x)(\omega\hat{x} + i\hat{p}_x)\right]\psi \tag{79}$$

$$= \frac{1}{2\omega}\left[2i\omega\left(\hat{p}_x\hat{x} - \hat{x}\hat{p}_x\right)\right]\psi = i\left[\hat{p}_x, \hat{x}\right]\psi \tag{80}$$

To calculate $[\hat{p}_x, \hat{x}]$ use $\hat{p}_x = -i\frac{\partial}{\partial x}$, we find the standard commutation relations:

$$\left[\hat{p}_x, \hat{x}\right]\psi = -i\frac{\partial}{\partial x}(x\psi) + ix\frac{\partial\psi}{\partial x} = -i\psi \tag{81}$$

which means that we can write $[\hat{p}_x, \hat{x}] = -i$. Therefore the commutator of the creation and annihilation operators is given by $\left[\hat{a}_x, \hat{a}_x^\dagger\right] = 1$. The Hamiltonian operator for the spherical isotropic oscillator is:

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \frac{1}{2}\omega^2(x^2 + y^2 + z^2) = -\frac{1}{2}\nabla^2 + \frac{1}{2}\omega^2 r^2 \tag{82}$$

Consider the operator $\hat{a}_x^\dagger\hat{a}_x$. Writing this out in terms of the momentum and position operators

$$\hat{a}_x^\dagger\hat{a}_x = \frac{1}{2\omega}(\omega\hat{x} - i\hat{p}_x)(\omega\hat{x} + i\hat{p}_x) \tag{83}$$

$$= \frac{1}{2\omega}\left(\omega^2 x^2 + \hat{p}_x^2 - i\left[\hat{p}_x, \hat{x}\right]\right) \tag{84}$$

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$$= \frac{1}{2\omega}\left(\omega^2 x^2 + \hat{p}_x^2 - 1\right) \tag{85}$$

Rearranging, we therefore have $\omega\left(\hat{a}_x^\dagger \hat{a}_x + \frac{1}{2}\right) = \frac{1}{2}\left(\omega^2 x^2 + \hat{p}_x^2\right)$. Summing over the other components, we find:

$$\hat{H} = \omega \sum_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) \tag{86}$$

as required. We have the formula for the shift operators given as $\hat{G}_{ij} = \frac{1}{2}(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_j \hat{a}_i^\dagger)$. Take the component $\hat{G}_{xx}$, the other components will follow similarly.

$$\hat{G}_{xx} = \frac{1}{2}(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_x \hat{a}_x^\dagger) \tag{87}$$

We have already evaluated the commutator, i.e. $\left[\hat{a}_x, \hat{a}_x^\dagger\right] = \hat{a}_x \hat{a}_x^\dagger - \hat{a}_x^\dagger \hat{a}_x = 1$. Rearranging, $\hat{a}_x \hat{a}_x^\dagger = 1 + \hat{a}_x^\dagger \hat{a}_x$. Substituting this into the expression for $\hat{G}_{xx}$ we find

$$\hat{G}_{xx} = \frac{1}{2}(\hat{a}_x^\dagger \hat{a}_x + 1 + \hat{a}_x^\dagger \hat{a}_x) = \hat{a}_x^\dagger \hat{a}_x + \frac{1}{2} \tag{88}$$

and similar for $\hat{G}_{yy}$ and $\hat{G}_{zz}$, hence the expression for the Hamiltonian operator is given by

$$\hat{H}/\omega = \hat{G}_{xx} + \hat{G}_{yy} + \hat{G}_{zz} \tag{89}$$

From this perspective, we can see how the isotropic oscillator problem is related to the tensor defined through the creation and annihilation operators. We shall now discuss some basic properties of these matrix operators as related to SU(3).

8.1 Generators of SU(3)

In terms of the basic spin subalgebra, using the isotropic tensor, one can find the Gell-Mann matrices [8] as given by e.g.:

$$\hat{\lambda}_1 = \hat{G}_{xy} + \hat{G}_{yx} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \hat{\lambda}_2 = i(\hat{G}_{yx} - \hat{G}_{xy}) \to \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{90}$$

where the centralising element is:

$$\hat{\lambda}_3 = \hat{G}_{xx} - \hat{G}_{yy} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{91}$$

The other subalgebras may be readily written down in an analogous fashion, viz.:

$$\hat{\lambda}_4 = \hat{G}_{xz} + \hat{G}_{zx} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \hat{\lambda}_5 = i(\hat{G}_{zx} - \hat{G}_{xz}) \to \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \tag{92}$$

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$$\hat{\lambda}_6 = \hat{G}_{yz} + \hat{G}_{zy} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \hat{\lambda}_7 = i(\hat{G}_{zy} - \hat{G}_{yz}) \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad (93)$$

Finally, the commuting element of the group may be written as:

$$\hat{\lambda}_3 = \frac{1}{\sqrt{3}}\left(\hat{G}_{xx} + \hat{G}_{yy} - 2\hat{G}_{zz}\right) \rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad (94)$$

We can see how one may easily generate all the group elements associated with SU(3) by using the correct types of creation and annihilation operators for the modes of the isotropic oscillator. It is possible that such descriptions of collective motion may prove to be of use in qutrit computation and control.

8.2 Spherical Operators

We shall now show how one may derive the quadrupole operator for the isotropic oscillator. The shift operator may be written in Cartesian form as the dyadic product:

$$\hat{G}_{ij} = \frac{1}{2\omega}\hat{p}_i\hat{p}_j + \frac{\omega}{2}\hat{x}_i\hat{x}_j + \frac{i}{2}\epsilon_{ijk}\hat{J}_k \quad (95)$$

In terms of the spherical representation for a tensor, we know that any tensor is defined by the sum of the symmetric and antisymmetric parts. Indeed, generalising this, one may simply write that any tensor is given by the sum of the trace, the antisymmetric part, and the symmetric part minus the trace, resulting in the spherical representation formula:

$$\hat{G}_{ij} = \frac{1}{3}\delta_{ij}\hat{G}_{kk} + \frac{1}{2}\left(\hat{G}_{ij} - \hat{G}_{ji}\right) + \left(\frac{1}{2}\left(\hat{G}_{ij} + \hat{G}_{ji}\right) - \frac{1}{3}\delta_{ij}\hat{G}_{kk}\right) \quad (96)$$

where the Hamiltonian given by $\hat{H} = \hat{G}_{kk} = \hat{G}_{xx} + \hat{G}_{yy} + \hat{G}_{zz}$ is the trace part. The antisymmetric part is then:

$$\frac{1}{2}\left(\hat{G}_{ij} - \hat{G}_{ji}\right) = i\epsilon_{ijk}\hat{J}_k \quad (97)$$

which leaves us with the last term (called quadrupole):

$$\frac{1}{3}\hat{Q}_{ij} = \frac{1}{2}\left(\hat{G}_{ij} + \hat{G}_{ji}\right) - \frac{1}{3}\delta_{ij}\hat{G}_{kk} \quad (98)$$

This tensor is traceless, and we therefore have 5 independent components. Writing $\hat{H}_{ij} = \frac{1}{2}(\hat{p}_i\hat{p}_j + \omega^2\hat{x}_i\hat{x}_j)$, we have $\hat{G}_{ij} = \frac{1}{\omega}\hat{H}_{ij} + \frac{i}{2}\epsilon_{ijk}\hat{J}_k$. The quadrupole tensor is then

$$\frac{1}{3}\hat{Q}_{ij} = \frac{1}{\omega}\hat{H}_{ij} - \frac{1}{3\omega}\delta_{ij}\hat{H}_{kk} \quad (99)$$

where we used $\epsilon_{ijk} = -\epsilon_{jik}$, also $\hat{H}_{ij} = \hat{H}_{ji}$. The z, z component of this tensor is then given by

$$\frac{1}{3}\hat{Q}_{zz} = \frac{1}{\omega}\left(\hat{H}_{zz} - \frac{1}{3}\hat{H}_{kk}\right) \tag{100}$$

$$= \frac{1}{2\omega}(2\hat{p}_z^2 - \hat{p}_x^2 - \hat{p}_y^2) + \omega(2\hat{z}^2 - \hat{x}^2 - \hat{y}^2) \tag{101}$$

$$= \sqrt{\frac{16\pi}{5}}r^2\left(\frac{1}{2\omega}Y_0^2(p,p) + \frac{\omega}{2}Y_0^2(x,x)\right) \tag{102}$$

where $Y_0^2(p,p) = \sqrt{\frac{5}{16\pi}}\frac{1}{r^2}(2\hat{p}_z^2 - \hat{p}_x^2 - \hat{p}_y^2)$ is a spherical harmonic function. We can see from this analysis that the spherical harmonics are deeply related to the structure of these types of spherical tensors on the space of states.

8.3 Spherical Harmonics

We shall now show how one may derive the whole solution for the isotropic oscillator using some simple results from the theory of partial differential equations. For the DE

$$E\psi = \frac{1}{2}(-\nabla^2 + \omega^2 r^2)\psi \tag{103}$$

we can write a separable solution in spherical polar co-ordinates as $\psi(\mathbf{x}) = \psi(r,\theta,\phi) = R(r)f(\theta,\phi)$. We have the expression for the Laplacian:

$$\nabla^2\psi = -\frac{1}{r^2}\hat{L}^2\psi + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \tag{104}$$

We have that $\nabla^2(Rf) = (\omega^2r^2 - 2E)Rf$, substituting on the left hand side above and dividing through by Rf, we get

$$\frac{\hat{L}^2(f)}{fr^2} = \frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + (2E - \omega^2r^2) \tag{105}$$

which can be rewritten

$$\frac{\hat{L}^2(f)}{f} = \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + (2Er^2 - \omega^2r^4) \tag{106}$$

The left hand side is solely a function of the angular variables, and the right hand side the radial terms. We must therefore have each side equal to a constant

$$\frac{\hat{L}^2(f)}{f} = E_l = \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + (2Er^2 - \omega^2r^4) \tag{107}$$

Simplifying, we have the two differential equations:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + (2E - \frac{E_l}{r^2} - \omega^2r^2)R = 0 \tag{108}$$

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$$E_l f = \hat{L}^2(f) \tag{109}$$

Again, separating the angular variables, we write $f = \eta(\phi)\Lambda(\theta)$, and we find

$$\frac{d^2\eta}{d\phi^2} = -m^2\eta \tag{110}$$

$$\frac{d^2\Lambda}{d\theta^2} + \cot\theta\frac{d\Lambda}{d\theta} + (E_l - \frac{m^2}{\sin^2\theta})\Lambda = 0 \tag{111}$$

Noting that the angular functions are eigenfunctions of the operator $\hat{L}^2$ with eigenvalues $E_l = l(l+1)$, we arrive at the system of differential equations:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + (2E - \frac{l(l+1)}{r^2} - \omega^2r^2)R = 0 \tag{112}$$

$$\frac{d^2\Lambda}{d\theta^2} + \cot\theta\frac{d\Lambda}{d\theta} + (l(l+1) - \frac{m^2}{\sin^2\theta})\Lambda = 0 \tag{113}$$

$$\frac{d^2\eta}{d\phi^2} = -m^2\eta \tag{114}$$

For the angular part, we may immediately solve the equation, yielding:

$$f(\theta,\phi) = Y_l^m(\theta,\phi) = P_l^m(\cos\theta)e^{\pm im\phi} \tag{115}$$

termed the spherical harmonics, with $P_l^m(\cos\theta)$ the associated Legendre polynomial. The radial part of the solution is solved by

$$R(r) = C\frac{W_{\kappa,\mu}(\omega r^2)}{r^{3/2}} \tag{116}$$

where $\mu = \frac{1}{2}(l + \frac{1}{2})$, $\kappa = \frac{E}{2\omega}$ and $W_{\kappa,\mu}(\omega r^2)$ is the Whittaker function. The first few spherical harmonics in Cartesian form are defined by:

$$Y_0^2(\mathbf{x}) = \sqrt{\frac{5}{16\pi}}\frac{1}{r^2}(2\hat{z}^2 - \hat{x}^2 - \hat{y}^2) \tag{117}$$

$$Y_{\pm1}^2(\mathbf{x}) = \mp\sqrt{\frac{15}{8\pi}}\frac{\hat{z}}{r^2}(\hat{x} \pm i\hat{y}) \tag{118}$$

$$Y_{\pm2}^2(\mathbf{x}) = \sqrt{\frac{15}{32\pi}}\frac{1}{r^2}(\hat{x} \pm i\hat{y})^2 \tag{119}$$

Noting that in the quadrupole tensor we have terms $Y_0^2(\mathbf{x})$ and $Y_0^2(\mathbf{p})$, which we write as $Y_0^2(\mathbf{x}) = Y_0^2(x,x)$ and $Y_0^2(\mathbf{p}) = Y_0^2(p,p)$, one may define mixed terms such as $Y_0^2(x,p)$ and $Y_0^2(p,x)$ via the formulae

$$Y_0^2(x,p) = \sqrt{\frac{5}{16\pi}}\frac{1}{r^2}(2\hat{z}\hat{p}_z - \hat{x}\hat{p}_x - \hat{y}\hat{p}_y) \tag{120}$$

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$$Y_0^2(p,x) = \sqrt{\frac{5}{16\pi}}\frac{1}{r^2}(2\hat{p}_z\hat{z} - \hat{p}_x\hat{x} - \hat{p}_y\hat{y}) \tag{121}$$

where we remember that these are now operators which operate on a function from the right. Writing out the commutation relationships for these operators, we find

$$\left[Y_0^2(p,p), r^2\right] = 4iY_0^2(x,p) \tag{122}$$

$$\left[Y_{\pm 2}^2(p,p), r^2\right] = -4iY_{\pm 2}^2(x,p) \tag{123}$$

$$\left[Y_{\pm 1}^2(p,p), r^2\right] = -2i\left(Y_{\pm 1}^2(x,p) + Y_{\pm 1}^2(p,x)\right) \tag{124}$$

$$\left[Y_{\pm l}^2(x,p), r^2\right] = \left[Y_{\pm l}^2(p,x), r^2\right] = -2iY_{\pm l}^2(x,x) \tag{125}$$

and finally, the Abelian subgroup:

$$\left[Y_{\pm l}^2(x,x), r^2\right] = \left[Y_0^2(x,x), r^2\right] = 0 \tag{126}$$

where $l = 1, 2$. We can see that under the commutation bracket with the potential energy $\left[(.), r^2\right]$ this set of operators forms a closed group. This completes our calculation of the basics of the isotropic oscillator as related to the theory of SU(3), and our calculation of various brachistochrones. We shall now discuss some more general conclusions which can be drawn from this primitive calculation.

9 Discussion and Conclusions

We have shown in this paper how one may arrive at a description of models of SU(3) in quantum systems using two seemingly different methods; the first was a direct technique using insights from matrix mechanics, the second being a PDE method based on analysis of a quantum isotropic oscillator. In both aspects we have been successful in illustrating the contrasts and similarities between these two perspectives of quantum mechanics. In particular, the first part of this paper can be considered a concrete application of von Neumann matrix mechanics and Hamiltonian principles to the question of time optimisation and control. We have shown explicitly how even complicated problems that require lengthy calculations using methods of bounded operators and variational calculus can be reduced to a Gram or Cholesky type decomposition of the unitary operator. This considerably reduces the leg-work involved in such analysis and is of definite utility in this work. We have demonstrated in a clear and concise fashion how the model of nuclear motion as given by the isotropic oscillator can be directly related to the Cartesian operator representation of spherical tensors, as originally expounded in [14]. This opens up the prospect of using such models as an advanced area of experimentation in which to problem the properties of qutrit states and unitary symmetries as implied through the brachistochrone problem. Indeed, the question of time optimality should be at the forefront of any proper analysis of quantum computation and control, and it is hoped that this work assists in emphasising the interesting results that may be obtained through use of this model of quantum mechanics.

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10 Future Directions

This paper poses an interesting question as to the exact link between the model of unitary operators and those of continuous differential operators. The answer to this question lies deep in group theory, specifically the theory of group representations. Although we have not covered it in this paper, a meta-analysis of this question and development of the science of time optimal control may be understood through use of the Fubini-Study metric and the Laplace operator. An upcoming series of papers shall discuss the application of this concept to a number of different problems related to qutrit state control, hyperbolic systems theory and the analysis of special functions.

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12 Conflicts of Interest and Data Sharing

The author indicates that no data was produced in the construction of this paper and its results. The author has no conflicts of interest to declare that are relevant to the content of this article.

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