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Construction of Rank One Solvable Rigid Lie Algebras with Nilradicals of Decreasing Nilpotence Index

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Abstract: It is shown that for any integers $k \geq 2$, $q \geq 2k$ and $N \geq k+q+2$ there exists a real solvable Lie algebra of rank one with a maximal torus of derivations $\mathfrak{t}$ possessing the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1,2,\ldots,k,q,q+1\ldots,N)$ and a nilradical of nilpotence index $N-k$ and characteristic sequence $(N-k,1^k)$.

Keywords: Lie algebras; solvability; rigid; cohomology; Jacobi scheme

1. Introduction

In spite of the fact that the work of R. Carles on rigid Lie algebras presents a clear picture concerning their generic structural properties [1–4], unifying previous approaches [5,6] and establishing a subdivision of rigid algebras into six principal types [2], the problem of classifying and characterizing rigid Lie algebras is far from being solved in satisfactory manner. Although the cohomological tools have been shown to be an effective alternative [7], the existence of a purely geometrical notion of rigidity shows that other procedures, such as the Jacobi schemes [8], must be further developed and refined in order to obtain reliable classifications, even in comparatively low dimensions. Solvable Lie algebras are of special relevance among the rigid ones, as they correspond to a class of algebras that cannot be fully classified beyond low dimensions. In this context, the study of the weight systems of maximal tori of derivations [9–11] is a powerful technique to analyze the rigidity independently of cohomological tools, and several algorithmic procedures to determine rigid Lie algebras and construct them systematically from the eigenvalue spectra of maximal tori have been developed [12,13], eventually leading to a classification of low dimensional solvable rigid algebras [14–17] as well as the discovery of various rigid hierarchies in arbitrary dimension, in both the cohomological and geometrically rigid cases [18–20]. With the application of symbolic computer packages, further generalizations of some of the previous results have been made possible, as well as the determination of new series of geometrically rigid Lie algebras or the explicit computation of the integrability obstructions that appear in the cohomological approach [21,23]. In this context, recently various works have been devoted to the systematic analysis and classification of solvable rigid Lie algebras of rank one associated to various types of eigenvalue spectra ([24–26] and references therein), showing the possibility of a unified description of ample classes of spectra in dependence of one or more parameters by means of generating functions.

In this work we proceed with the study of eigenvalue spectra of one-dimensional tori, but focusing on the construction of rank one solvable cohomologically rigid Lie algebras such that the nilradical $\mathfrak{n}$ has a nilpotence index $\dim\mathfrak{n} - k$ for $k \geq 2$, hence enlarging to lower nilpotent indices results some of the constructions already known for the filiform case. In particular, we show that for arbitrary integers $k \geq 2$, $q \geq 2k$ and $N \geq k+q+2$ there exists a real solvable Lie algebra of rank one with a maximal torus of derivations $\mathfrak{t}$ possessing the eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1,2,\ldots,k,q,q+1\ldots,N+q-k-1)$ such that the nilradical has the nilpotence index $N-k$ and the characteristic sequence $(N-k,1^k)$.

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Unless otherwise stated, any Lie algebra in this work is finite-dimensional and defined over the field of real numbers $\mathbb{R}$.

1.1. General properties of nilpotent Lie algebras

Let $\mathfrak{n}$ be a nilpotent Lie algebra. For any $X \in \mathfrak{n} \setminus [\mathfrak{n}, \mathfrak{n}]$ we consider the decreasing sequence of dimensions of the Jordan blocks of the adjoint operator $\mathrm{ad}(X)$

$$c(X) = (c_1(X), c_2(X), \cdots, c_k(X), 1), \quad c_i(X) \geq c_{i+1}(X) \geq 1. \tag{1}$$

As $c(X)$ constitutes a similarity invariant, it determines an invariant $c(\mathfrak{n})$ defined as

$$c(\mathfrak{n}) = \sup \{c(X) \mid X \in \mathfrak{n} \setminus [\mathfrak{n}, \mathfrak{n}]\}. \tag{2}$$

and called the characteristic sequence of $\mathfrak{n}$ (see e.g. [18] and references therein). Another invariant is given by the dimensions of the central descending sequence, given recursively by

$$C^0(\mathfrak{n}) = \mathfrak{n}, \; C^k(\mathfrak{n}) = \left[\mathfrak{n}, C^{k-1}(\mathfrak{n})\right], \; k \geq 1. \tag{3}$$

This sequence further determines the so-called associated graded Lie algebra $\mathfrak{gr}(\mathfrak{n}) = \mathfrak{g}_1(\mathfrak{n}) \oplus \cdots \oplus \mathfrak{g}_r(\mathfrak{n})$ with

$$\mathfrak{g}_k(\mathfrak{n}) = C^{k-1}(\mathfrak{n}) / C^k(\mathfrak{n}), \; k \geq 1. \tag{4}$$

The Lie algebra $\mathfrak{n}$ is called naturally graded if the isomorphism of Lie algebras $\mathfrak{n} \simeq \mathfrak{gr}(\mathfrak{n})$ holds.
We denote by $Der(\mathfrak{n})$ the Lie algebra of derivations of $\mathfrak{n}$, i.e., the space of linear maps $D : \mathfrak{n} \to \mathfrak{n}$ satisfying the condition

$$D[X, Y] = [D(X), Y] + [X, D(Y)], \quad X, Y \in \mathfrak{n}. \tag{5}$$

Definition 1. *Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus of derivations is any Abelian subalgebra of $Der(\mathfrak{g})$ the generators of which are semisimple.*

Elements in a (maximal) torus are simultaneously diagonalizable in the complex extension of the base field, i.e., $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})$ admit a diagonal matrix over $\mathbb{C}$ for some basis. As shown in [27], maximal tori of the complexified Lie algebra $\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C}$ are conjugate by an inner automorphism, which implies that their dimension is a scalar invariant of the Lie algebra, commonly referred to as the rank of $\mathfrak{n}$, and denoted by $\mathrm{r}(\mathfrak{n})$.

According to the general structure theory, a real or complex solvable Lie algebra $\mathfrak{r}$ admits the decomposition as semidirect sum

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}, \tag{6}$$

satisfying the relations

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \; [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \; [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \tag{7}$$

where $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{r}$ (the nilradical) and $\overrightarrow{\oplus}$ denotes the action of $\mathfrak{t}$ on $\mathfrak{n}$ by linearly nil-independent outer derivations. The dimension of $\mathfrak{t}$ is further upper bounded by the following inequality

$$\dim \mathfrak{n} - \dim [\mathfrak{n}, \mathfrak{n}] \geq \dim \mathfrak{t}. \tag{8}$$

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1.2. Solvable rigid Lie algebras

Let $\mathcal{L}^n$ denote the variety of n-dimensional Lie algebras $\mathfrak{g} = \left(\mathbb{K}^n, [\,,\,]_{\mathfrak{g}}\right)$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The general linear group $GL(n, \mathbb{K})$ acts naturally on $\mathcal{L}^n$ by changes of basis:

$$(f \star \mathfrak{g})(X, Y) = f^{-1}\left([f(X), f(Y)]_{\mathfrak{g}}\right), \; f \in GL(n, \mathbb{K}), \; X, Y \in \mathfrak{g}. \tag{9}$$

The orbit $\mathcal{O}(\mathfrak{g})$ of $\mathfrak{g}$ is therefore identified with the Lie algebras that are isomorphic to $\mathfrak{g}$.

Definition 2. *A Lie algebra $\mathfrak{g}$ is rigid if the orbit $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$ with respect to the Euclidean topology.*

This definition of rigidity, although mainly topological, admits various equivalent reformulations in analytical or algebraic terms (see e.g. [1,5,16]). In this context, using the adjoint cohomology of Lie algebras [7,28,29], several criteria to ensure rigidity have been proposed [15,30,31]:

Proposition 1. *Let $\mathfrak{g}$ be a Lie algebra. If the condition $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 0$ holds, then $\mathfrak{g}$ is rigid.*

According to this result, we say that a Lie algebra $\mathfrak{g}$ is cohomologically rigid if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$. This criterion, albeit not necessary for rigidity, has been extremely useful in the analysis of large classes of rigid Lie algebras, and has further allowed a detailed comparison with rigid algebras whose cohomology is not zero. Using the quadratic Rim map $\mathrm{Sq} : H^2(\mathfrak{g}, \mathfrak{g}) \rightarrow H^3(\mathfrak{g}, \mathfrak{g})$ defined by

$$\mathrm{Sq}(\psi)(X_i, X_j, X_k) := \psi\left(\psi(X_i, X_j), X_k\right) + \psi\left(\psi(X_j, X_k), X_i\right) + \psi\left(\psi(X_k, X_i), X_j\right), \tag{10}$$

another sufficiency criterion for rigidity was proved in [32,33]. This criterion states that if Sq is an injective map, then $\mathfrak{g}$ is a rigid Lie algebra.

We also recall briefly the Hochschild-Serre factorization theorem [28,34], that provides a practical procedure for explicitly computing the cohomology spaces of semidirect sums of Lie algebras. Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ denote a solvable Lie algebra such that $\mathfrak{t}$ is Abelian and the operators $\mathrm{ad}_{\mathfrak{r}} T$ $(T \in \mathfrak{t})$ are diagonal. Then the adjoint cohomology $H^p(\mathfrak{r}, \mathfrak{r})$ satisfies the following isomorphism

$$H^p(\mathfrak{r}, \mathfrak{r}) \simeq \sum_{a+b=p} H^a(\mathfrak{t}, \mathbb{R}) \otimes H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}}, \tag{11}$$

where

$$H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}} = \left\{[\varphi] \in H^b(\mathfrak{n}, \mathfrak{r}) \mid (T.\varphi) = 0, \; T \in \mathfrak{t}\right\} \tag{12}$$

is the space of $\mathfrak{t}$-invariant cocycle classes of $\mathfrak{n}$ with values in $\mathfrak{r}$. Invariance of cocycles is determined by the condition

$$(T.\varphi)(Z_1, \cdots, Z_b) = [T, \varphi(Z_1, \cdots, Z_b)] - \sum_{s=1}^{b} \varphi(Z_1, \cdots, [T, Z_s], \cdots, Z_b). \tag{13}$$

Observing that $H^p(\mathfrak{t}, \mathbb{R}) = \bigwedge^p \mathfrak{t}$, it can be easily justified that $H^p(\mathfrak{r}, \mathfrak{r}) = 0$ is equivalent to the identities $H^b(\mathfrak{n}, \mathfrak{r}) = 0$ for $0 \leq b \leq p$. If in addition $\mathfrak{r}$ is a complex solvable rigid Lie algebras, the decomposition theorem of Carles implies that the torus $\mathfrak{t}$ is indeed a maximal external torus of derivations of the nilradical $\mathfrak{n}$ [2].

2. Structural properties of the nilpotent Lie algebra $\mathfrak{n}_{N,k}^{0}$

For any $k \geq 1$ and $N \geq 2k+1$ let $\mathfrak{n}_{N,k}^{0}$ be the Lie algebra with nonvanishing commutators

$$\begin{aligned} \left[X_1, X_j\right] &= X_{j+1}, \quad k+1 \leq j \leq N-1, \\ \left[X_2, X_j\right] &= X_{j+2}, \quad k+1 \leq j \leq N-2, \\ &\dots \\ \left[X_k, X_j\right] &= X_{j+k}, \quad k+1 \leq j \leq N-k, \end{aligned} \tag{14}$$

over the basis $\mathcal{B} = \{X_1, \dots, X_N\}$. The central descending sequence is given by

$$C^s(\mathfrak{n}_{N,k}^{0}) = \langle X_{k+s}, \dots, X_N \rangle, \quad 2 \leq s \leq N-k; \quad C^{N+1-k}(\mathfrak{n}_{N,k}^{0}) = 0,$$

showing that $\mathfrak{n}_{N,k}^{0}$ is nilpotent with nilpotence index $N-k$. It is straightforward to verify that the characteristic sequence of the Lie algebra is given by $c(\mathfrak{n}_{N,k}^{0}) = (N-k, 1^k)$. We further observe that $\mathfrak{n}_{N,k}^{0}$ is naturally graded only for $k=1$, in which case $\mathfrak{n}_{N,k}^{0}$ is isomorphic to the model filiform Lie algebra L_N [14]. In a certain sense, the algebras (14) constitute an extension of the models of Bratzlavsky type [10,23] to lower characteristic sequences.

For later use, it is convenient to consider the Maurer-Cartan equations of $\mathfrak{n}_{N,k}^{0}$. If $\{\omega^1, \dots, \omega^N\}$ denotes the dual basis of $\mathcal{B}$, these are given by

$$\begin{aligned} d\omega^s &= 0, \quad 1 \leq s \leq k+1, \\ d\omega^p &= \sum_{a=1}^{p-k-1} \omega^a \wedge \omega^{p-a}, \quad k+2 \leq p \leq 2k \\ d\omega^p &= \sum_{a=1}^{k} \omega^a \wedge \omega^{p-a}, \quad 2k+1 \leq p \leq N. \end{aligned} \tag{15}$$

If now $\theta = \sum_{\ell=1}^{N} a_\ell d\omega^\ell \in \mathcal{L}(\mathfrak{n}_{N,k}^{0}) = \mathbb{R}\left\{d\omega_i\right\}_{1 \leq i \leq N}$ is a generic linear combination of the 2-forms in (15), it is straightforward to verify that

$$\bigwedge^{k} \theta \equiv 0, \quad \bigwedge^{k-1} \theta \neq 0.$$

The quantity

$$j_0\left(\mathfrak{n}_{N,k}^{0}\right) = \max\left\{j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{n}_{N,k}^{0})\right\} = k \tag{16}$$

depends only on the structure of $\mathfrak{n}_{N,k}^{0}$, and constitutes a numerical invariant of the Lie algebra [35].[1]

Lemma 1. *For any $k \geq 1$ and $N \geq 2k+1$, the rank of $\mathfrak{n}_{N,k}^{0}$ is two.*

Let $f(X_\ell) = \sum_{s=1}^{N} f_\ell^s X_s$ be the expression of a derivation of $\mathfrak{n}_{N,k}^{0}$. As the centre is generated by X_N, it follows immediately that $f(X_N) = f_N^N X_N$. Evaluation of the derivation condition (5) for $X = X_1, Y = X_{N-1}$ shows in particular that $f(X_{N-1}) = \sum_{s=1}^{k} f_{N-1}^s X_s + f_{N-1}^{N-1} X_{N-1} + f_{N-1}^{N} X_N$. Now, computation for the pair $X = X_{k+1}, Y = X_{N-1}$ implies that

$$f_{N-1}^s = 0, \quad 1 \leq s \leq k; \quad f_{k+1}^1 = 0.$$

[1] This actually means that $\mathfrak{n}_{N,k}^{0}$ possesses $N-2k$ functionally independent invariants for the coadjoint representation.

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Iterating the computation for the pair $X = X_1$, $Y = X_{N-p}$ (for $N-p \geq k+1$) first shows that $f(X_{N-p}) = \sum_{s=1}^{k} f_{N-p}^{s} X_s + \sum_{q=N-p}^{N} f_{N-p}^{q} X_q$, while evaluation of (5) for $X = X_{k+1}$, $Y = X_{N-p}$ successively leads to the conditions

$$f_{N-p}^{s} = 0, \quad 1 \leq s \leq k; \quad f_{k+1}^{p} = 0.$$

From these identities we conclude that $f(X_q) = \sum_{s=q}^{N} f_q^s X_s$ for $q \geq k+1$. Considering now the pair $X = X_m$, $Y = X_{k+1}$ for $m \leq k$, we obtain

$$\sum_{s=k+1}^{N-m} f_{k+1}^{s} X_s - \sum_{s=1}^{m} f_m^s X_s = \sum_{s=m+k+1}^{N} f_{m+k+1}^{s} X_s,$$

from which it follows by iteration on the value of m that $f(X_m) = \sum_{s=m}^{N} f_m^s X_s$, showing that the matrix of f is triangular. In order to compute the semisimple derivations, it therefore suffices to consider a generic diagonal derivation $\Phi(X_i) = \lambda_i X_i$. From the commutators in (14) the following relations are easily obtained:

$$\lambda_i + \lambda_j = \lambda_{i+j}, \quad 1 \leq i \leq k,\ k+1 \leq j \leq N-i. \tag{17}$$

Considering $i = 1$, it follows for $s \geq 2$ that

$$\lambda_{k+s} = \lambda_1 + \lambda_{k+s-1} = 2\lambda_1 + \lambda_{k+s-2} \cdots = (s-1)\lambda_1 + \lambda_{k+1}.$$

On the other hand, for $1 < i \leq k$ the relation

$$\lambda_i + \lambda_{k+1} = \lambda_{i+k+1} = i\lambda_1 + \lambda_{k+1} \tag{18}$$

implies that $\lambda_i = i\lambda_1$. It follows that there exist two diagonalizable derivations F_1 and F_2 with eigenvalues

$$\begin{aligned} \mathrm{spec}(F_1) &= (1,2,\ldots,k,0,1,2,\ldots,(N-k-1))\,, \\ \mathrm{spec}(F_2) &= (0,0,\ldots,0,1,1,1,\ldots,1)\,, \end{aligned} \tag{19}$$

from which we conclude that the rank of $\mathfrak{n}_{N,k}^0$ is two. We denote a maximal torus of $\mathfrak{n}_{N,k}^0$ by $\mathfrak{t}_0$.

Let $\mathfrak{r}_0 = \mathfrak{t}_0 \overrightarrow{\oplus} \mathfrak{n}$ be a solvable Lie algebra such that the torus $\mathfrak{t}_0$ is generated by two diagonalizable derivations T_1, T_2 with eigenvalues as given in (19). Although it is not essential for the following, using the properties of the root system associated to solvable Lie algebras [13], the rigidity of $\mathfrak{r}_0$ can be shown directly without applying cohomological methods. Thus $\mathfrak{r}_0$ defines a series of rank two solvable rigid Lie algebras for any $k \geq 2$ and $q \geq 2k$. In particular, for $k = 1$ we recover the rigid Lie algebra associated to the model filiform Lie algebra L_n [36]. For higher values of k, the algebra can be seen as the counterpart of the model algebra for characteristic sequences $c(\mathfrak{n}_{N,k}^0) = (N-k, 1^k)$.[2]

2.1. Generation of rank one solvable Lie algebras

In this section, we analyze how to derive nilpotent Lie algebras that have rank one using the Lie algebra $\mathfrak{n}_{N,k}^0$, and such that the eigenvalues of a maximal torus are given in terms of (19). Considering a linear combination $F_1 + qF_2$, we get a diagonal derivation with eigenvalues

$$\mathrm{spec}(F_1 + qF_2) = (1,2,\ldots,k,q,q+1,q+2,\ldots,(N+q-k-1))\,, \quad q \neq 0. \tag{20}$$

[2] Incidentally, the algebras $\mathfrak{r}_0$ are actually cohomologically rigid.

In this context, it can be asked whether, starting from the nilpotent Lie algebra $\mathfrak{n}_{N,k}^0$, we can get another nilpotent Lie algebra that is isomorphic to a nontrivial deformation of $\mathfrak{n}_{N,k}^0$ and such that it has rank one, with a torus $\mathfrak{t}$ whose eigenvalues are given by (20). A first example in this direction was already given in [18] for fixed dimension, where the cohomological rigidity of the solvable Lie algebra $\mathfrak{r}_{q+4,q}$ with commutators

$$\begin{aligned} [T, X_j] &= \mu_j X_j, \\ [X_1, X_j] &= X_{j+1}, \quad 3 \leq j \leq q+3, \\ [X_2, X_j] &= X_{j+2}, \quad 3 \leq j \leq q+2, \\ [X_3, X_4] &= X_{q+4}, \end{aligned} \tag{21}$$

with $q \geq 4$, $\mu_1 = 1, \mu_2 = 2$ and $\mu_s = q + s - 3$ for $3 \leq s \leq q+4$, was proved. In this case, the spectrum of the torus generated by T is given by

$$\mathrm{spec}(T) = (1, 2, q, q+1, q+2, \ldots, 2q+1), \quad q \geq 4, \tag{22}$$

thus belongs to type (20) with $k = 2$ and $N = q + 4$. We further observe that the nilradical is isomorphic to the deformation $\mathfrak{n}_{q+4,2}^0 + \varphi$, where $\varphi(X_3, X_4) = X_{q+4}$ defines a nontrivial cocycle. The addition of this cocycle implies in particular that F_2 cannot be a derivation of the deformed algebra, from which the rank reduction follows.

The Lie algebra (21) can actually be seen as the first element in a series of solvable Lie algebras of rank one with vanishing cohomology. To this extent, consider $N \geq q + 4$ and the skew-symmetric 2-form φ on $\mathfrak{n}_{N,2}^0$ defined by

$$\varphi\left(X_3, X_j\right) = X_{q+j}, \quad 4 \leq j \leq N - q. \tag{23}$$

It is immediate to verify that φ is a 2-cocycle of $\mathfrak{n}_{N,2}^0$. In order to prove that the cohomology class of φ is nonzero, we consider the following 2-form on the (linearly) deformed Lie algebra $\mathfrak{n}_{N,2}^0 + \varepsilon\varphi$:

$$\theta = d\omega^N = \omega^1 \wedge \omega^{N-1} + \omega^2 \wedge \omega^{N-2} + \varepsilon\, \omega^3 \wedge \omega^{N-3}.$$

For any $\varepsilon \neq 0$ we have $\theta \wedge \theta \wedge \theta \neq 0$, while for $\varepsilon = 0$ the index of a generic 2-form over $\mathfrak{n}_{N,2}^0$ is 2 (see Equation (16)), showing that both algebras are not isomorphic, hence implying that $[\varphi] \neq 0$.

Let $\mathfrak{n}_{2,q,N} = \mathfrak{n}_{N,2}^0 + \varphi$. Repeating the argumentation of Lemma 1, it follows at once that any derivation f of $\mathfrak{n}_{2,q,N}$ is triangular. Assuming that f is a diagonal derivation, it satisfies in particular the conditions (17) and (18) for $k = 2$, so that

$$f(X_1) = \lambda_1 X_1,\ f(X_2) = 2\lambda_1 X_2,\ f(X_3) = \lambda_3 X_3,\ f(X_j) = \left((j-3)\lambda_1 + \lambda_3\right) X_j,\ 4 \leq j \leq N.$$

In addition to these constraints, the condition $f\left([X_3, X_j]\right) = [f(X_3), X_j] + [X_3, f(X_j)]$ must be fulfilled, leading to the eigenvalue identities

$$\lambda_3 + (j-3)\lambda_1 + \lambda_3 = (q + j - 3)\lambda_1 + \lambda_3, \quad j \geq 4, \tag{24}$$

from which $\lambda_3 = q$ follows at once. We conclude that $\mathfrak{n}_{2,q,N}$ has rank one with a maximal torus $\mathfrak{t}$ having eigenvalues as given in (20) for $k = 2$.

Proposition 2. *For any $q \geq 4$ and $N \geq q + 4$ the solvable Lie algebra $\mathfrak{r}_{2,q,N} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}_{2,q,N}$ is rigid with vanishing cohomology $H^2\left(\mathfrak{r}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)$.*

The proof follows by application of the Hochschild-Serre factorization theorem [28]. It is straightforward to verify that any invariant 1-cochain $\varphi \in C^1\left(\mathfrak{r}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)$ has the form

$$\varphi(X_i) = a_i^i X_i, \quad 1 \leq i \leq N.$$

For the coboundary operator we have the nonvanishing entries

$$\begin{aligned}
d\varphi(X_1, X_j) &= (a_1^1 + a_j^j - a_{1+j}^{j+1}) X_{1+j}, \quad j \geq 3,\\
d\varphi(X_2, X_j) &= (a_2^2 + a_j^j - a_{2+j}^{j+1}) X_{2+j}, \quad j \geq 3,\\
d\varphi(X_3, X_j) &= (a_3^3 + a_j^j - a_{q+j}^{q+j}) X_{q+j}, \quad j \geq 4,
\end{aligned}$$

from which it follows at once that $d\varphi = 0$ only if $a_2 = 2a_1$, $a_j = (q+j-3)a_1$ for $j \geq 3$, further showing that $\dim B^2\left(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)^{\mathfrak{t}} = N-1$. On the other hand, a $\mathfrak{t}$-invariant 2-form has the shape

$$\begin{aligned}
d\varphi(X_1, X_j) =& b_{1,j}^{j+1} X_{1+j}, \quad j \geq 3,\\
d\varphi(X_2, X_j) =& b_{2,j}^{2+j} X_{2+j}, \quad j \geq 3,\\
d\varphi(X_i, X_j) =& b_{i,j}^{i+j+q-3} X_{i+j+q-3}, \quad j \geq 4,
\end{aligned}$$

Imposing the condition $d\varphi = 0$ leads to the system of coefficients

$$\begin{aligned}
b_{2,j}^{j+2} - b_{1,j}^{1+j} - b_{2,j+1}^{j+3} + b_{1,j+2}^{j+3} &= 0, \quad j \geq 4\\
b_{3,j}^{j+q} - b_{2,j}^{j+2} - b_{3,j+2}^{j+2+q} + b_{2,j+q}^{j+q+2} &= 0, \quad j \geq 4\\
b_{3,j}^{j+q} - b_{1,j}^{j+1} &= 0, \quad j \geq 5.
\end{aligned}$$

A routine computation shows that, as a basis of independent coefficients, we can choose $b_{3,4}^{4+q}$, $b_{2,3}^5$ and $b_{1,j}^{j+1}$ for $3 \leq j \leq N-1$, implying that $\dim Z^2\left(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)^{\mathfrak{t}} = N-1$. It follows at once that $\dim H^2\left(\mathfrak{n}_{2,q,N}, \mathfrak{r}_{2,q,N}\right)^{\mathfrak{t}} = 0$, showing that the algebra is cohomologically rigid.

3. The solvable Lie algebras $\mathfrak{r}_{k,q,N}$

As the preceding proof does not essentially depend on the value of k, it is naturally suggested that the result can be easily generalized to nilradicals with characteristic sequence $c(\mathfrak{n}_{N,k}^0) = (N-k, 1^k)$ for arbitrary $k \geq 3$, by considering the 2-cocycle class of $\mathfrak{n}_{N,k}^0$ defined by

$$\varphi\left(X_{k+1}, X_j\right) = X_{k+q}, \quad k+1 \leq j \leq N-q. \tag{25}$$

Consider the Maurer-Cartan equations of $\mathfrak{n}_{N,k}^0 + \varepsilon\varphi$. It is immediate to verify that

$$\theta = d\omega^N = \sum_{p=1}^{k} \omega^p \wedge \omega^{N-p} + \varepsilon\, \omega^{k+1} \wedge \omega^{N-k-1}$$

satisfies the identity $\bigwedge^{k+1}\theta \neq 0$ for $\varepsilon \neq 0$, showing that it is not isomorphic to $\mathfrak{n}_{N,k}^0$. Using Lemma 1, the same reasoning as that used in Equation (24) shows that $\mathfrak{n}_{k,q,N} = \mathfrak{n}_{N,k}^0 + \varphi$ has rank one with a maximal torus $\mathfrak{t}$ possessing the eigenvalues

$$\mathrm{spec}(\mathfrak{t}) = (1, 2, \ldots, k, q, q+1, q+2, \ldots, q+N-k-1), \quad q \geq 2k. \tag{26}$$

In analogy with the previous case, we define the solvable real Lie algebra of rank one $\mathfrak{r}_{k,q,N} = \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}_{k,q,N}$. Over a basis $\{T, X_1, \ldots, X_N\}$ with $N \geq 2q+1$, the precise brackets are given by

$$\begin{aligned} [T, X_i] &= i\, X_i, \quad 1 \leq i \leq k \\ [T, X_j] &= (q+j-k-1)X_j, \quad k+1 \leq j \leq N \\ [X_a, X_j] &= X_{j+a}, \quad 1 \leq a \leq k,\ k+1 \leq j \leq N-a, \\ [X_{k+1}, X_j] &= X_{q+j}, \quad k+2 \leq j \leq N-k-1. \end{aligned} \tag{27}$$

Proposition 3. *For any $k \geq 2$, $q \geq 2k$ and $N \geq 2q+1$ the solvable Lie algebra $\mathfrak{r}_{k,q,N}$ is cohomologically rigid.*

The proof is completely analogous to that of Proposition 1, for which reason we omit the detailed computations. The results above show that for any $k \geq 2$ and any dimension $N \geq 4k+1$ such Lie algebras exist, with $N = 9$ (for $k = 2$) being the lowest dimension for which an eigenvalue spectrum as given in (26) appears. The series $\mathfrak{r}_{k,q,N}$ hence gives a partial answer to a question formulated in [19], namely finding conditions for the existence of rank one rigid Lie algebras such that the nilradical has a given characteristic sequence.

4. Conclusions

In this work certain results of [10] and [18] concerning rank one solvable rigid Lie algebras have been extended to the case of nilradicals having characteristic sequence $c(\mathfrak{n}) = (n-k, 1^k)$ for arbitrary $k \geq 2$, a one dimensional torus of derivations with eigenvalues (26) and dimensions $N \geq k+q+2$. It also solves a subsidiary question formulated in [19], providing minimal dimensions for which rank one rigid Lie algebra with a certain characteristic sequence can appear. The guiding principle has been to consider certain deformations of the nilpotent Lie algebra $\mathfrak{n}^0_{N,k}$ that imply the existence of a unique diagonal derivation, hence guaranteeing that the rank is one. However, this approach merely constitutes one of the multiple possibilities that are conceivable. Rigid algebras structurally analogous but not related to $\mathfrak{n}^0_{N,k}$ can also be constructed along similar lines. Considering for example the eigenvalue sequence $\Lambda = (1, 2, 4, \ldots, 9, 18, 19, \ldots 37)$, a routine computation shows that the 28-dimensional nilpotent algebra defined by

$$\begin{aligned} [X_a, X_j] &= X_{j+a},\ 1 \leq a \leq 8; \quad 9 \leq j \leq 28-a, \\ [X_9, X_{10}] &= X_{28} \end{aligned}$$

has rank one, with a maximal torus having the eigenvalues Λ. The corresponding extension of the nilradical by the torus determines a rank one solvable rigid Lie algebra with vanishing cohomology. In contrast to the series derived from $\mathfrak{n}^0_{N,k}$, the eigenvalues of Λ are not obtainable as a linear combination of the elements in (19).

On the other hand, from the Jacobi scheme associated to the eigenvalue spectrum (26), it follows that a decreasing nilpotence index allows the existence of different characteristic sequences, with the rigidity type (cohomological or geometrical) being deeply related to the particular structure of the characteristic sequence.[3] In other words, the eigenvalue spectrum (26) does not uniquely determine the nilradical. A systematic analysis of these additional solutions, as well as their potential rigidity (either cohomological or geometrical), constitutes a problem worthy to be inspected more into the detail. In order to illustrate how examples of geometrically rigid Lie algebras arise in this context,

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[3] This phenomenon cannot occur for filiform algebras, as these correspond to the maximal possible nilpotence index.

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consider $k = 3$, $q = 8$, $N = 13$ and the torus $\mathfrak{t}$ with eigenvalue spectrum $(1, 2, 3, 8, 9, \ldots, 17)$. The nilpotent Lie algebra $\mathfrak{m}$ given by

$$\begin{aligned} [X_1, X_2] &= X_3, \\ [X_1, X_j] &= X_{j+1}, \quad 4 \leq j \leq 12, \\ [X_2, X_j] &= X_{j+2}, \quad 4 \leq j \leq 11, \\ [X_4, X_5] &= X_{13} \end{aligned} \tag{28}$$

admits $\mathfrak{t}$ as a maximal torus of derivations. The corresponding solvable extension $\mathfrak{R} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{m}$ has a one-dimensional adjoint cohomology space, generated by the cocycle class ψ defined by

$$\begin{aligned} \psi\left(X_2, X_j\right) &= (j-4)X_{j+2}, \quad 5 \leq j \leq 11, \\ \psi\left(X_3, X_j\right) &= -X_{j+3}, \quad 4 \leq j \leq 10. \end{aligned} \tag{29}$$

Although this cocycle is not integrable, using the Rim map (10) it can be easily verified that

$$\mathrm{Sq}(\psi)(X_2, X_3, X_5) = 3X_{10} \neq 0,$$

from which we deduce that $\mathrm{Sq} : H^2(\mathfrak{R}, \mathfrak{R}) \to H^3(\mathfrak{R}, \mathfrak{R})$ is injective. Following the criterion in [33], $\mathfrak{R}$ is rigid with nonvanishing cohomology. It is worthy to be observed that, as happened for the filiform case, a same eigenvalue spectrum can lead to either cohomologically or geometrically rigid Lie algebras depending on the dimension of the nilradical (see e.g. [4,12,23,26]). The interesting fact that distinguishes this type of eigenvalue spectrum from those associated to filiform algebras is that $\mathfrak{m}$ has characteristic sequence $c(\mathfrak{m}) = (10, 2, 1)$, and the natural question that arises is whether it is the lowest dimensional hierarchy of a series that generalizes recent constructions of geometrically rigid algebras (as that e.g. proposed in [23]) to characteristic sequences of the type $c(\mathfrak{m}) = (c_1, c_2, 1^{c_1+c+2})$. In a more wide context, it can be asked what conditions must be satisfied by the elements of a sequence of integers $\{c_1, \ldots, c_s\}$ in order to imply the existence of a nilradical with characteristic sequence $(c_1, \ldots, c_s, 1^{s+1})$ associated to a rigid Lie algebra of rank one. A complete answer to this question will probably require the use of symbolic computer packages, due to the relatively high dimensions and the number of solutions of the Jacobi equations involved. Work in this direction is currently in progress.

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