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Computation of Matrix Determinants by Cross-Multiplication: A Rethinking of Dodgson's Condensation and Reduction by Elementary Row Operations Methods

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Abstract: We formulate a more straightforward, symmetry-based techniques in manually computing the determinant of any $n \times n$ matrix by revisiting Dodgson's condensation method and strategically applying elementary row operations and the definition and properties of determinants. The result yields a more simplified mnemonical algorithm than the usual reduction by elementary row operations and co-factor expansion and with minimal terminal divisions performed compared to Dodgson's internal divisions.

Keywords: matrix determinant; Dodgson's condensation; cross-multiplication

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1. Introduction

The determinant is a unique number that is associated with a square matrix [1]. This determinant-square matrix one-to-one correspondence is a function $f(A) = |A|$ which assigns the number $|A|$ to a square matrix A, the value of which is determined from the entries of A. The methods of computing determinants usually presented in elementary linear algebra texts include the Leibniz's definition, co-factor expansion, reduction by elementary row or column operations, and sometimes Dodgson's condensation method.

Determinants by Definition

Definition 1. *The determinant of an* $n \times n$ *matrix A denoted by* $det\,[A]$ *or* $|A|$ *is defined as*

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} a_{3j_3} \dots a_{nj_n}$$

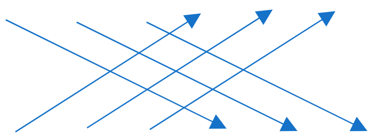
where $j_1 j_2 j_3 \dots j_n$ are column numbers derived from the permutations of the set $S = \{1, 2, 3, \dots, n\}$.

The sign is taken as + for even permutation and - for odd permutation [1].

Accordingly, the determinants of simple matrices can be readily computed. The determinant of a single-entry matrix equals the entry itself, $|a_{11}| = a_{11}$. The butterfly method for 2×2 matrices is so-called as the movement of computation suggests the butterfly wings, that is, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$. The Sarrus rule for 3×3 matrices provides the most efficient approach in computing the determinant [2]. By Definition 1, the determinant of a 3×3 is expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

By extending the first two columns of the matrix to the right, the downward arrows give the positive terms and the upward arrows give the negative terms in the expansion of the formula.



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$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Applying the definition in computing the determinant of $n \times n$ matrix which requires $n!$ terms with each containing n factors involving $n-1$ multiplications implies a lengthy process in manually computing the determinants as matrix size becomes larger. Several approaches were proposed to extend Sarrus rule to 4×4 and 5×5 matrices [2]. While these methods were shown to be efficient, the length of the process with added rules and number of factors in each term accumulate up as matrix size becomes larger.

Determinants by Cofactor Expansion

Two concepts defined below are essential in this method.

Definition 2. *The minor of an entry a_{ij} of an $n \times n$ matrix A is the determinant of the $n-1 \times n-1$ submatrix M_{ij} of A obtained by deleting the ith row and jth column of A. The co-factor of a_{ij} is a real number denoted by A_{ij} and is defined by*

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

After computing the minor and its associated co-factor, we compute the determinant by expanding about an ith row (the row containing all a_{ij}):

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \cdots + a_{in}A_{in}$$

or jth column:

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \cdots + a_{nj}A_{nj}$$

In co-factor expansion we "expand" the $n \times n$ matrix about a certain row or column into n submatrices M_{ij} each with $n-1$ rows and $n-1$ columns. This means at the first expansion, we compute n minors and n co-factors. For larger matrices, we may not readily compute the minors but have to perform second expansion of each of M_{ij} into $n-1$ submatrices each with $n-2$ rows and $n-2$ columns; that is, the process is repeated over the resulting submatrices until smaller submatrices with readily computed determinants are obtained. The determinant of the matrix is then computed by working backward.

Determinants by Row Operations

This alternative to computing the determinants employs elementary row operations to put the matrix into upper or lower triangular form in which the resulting entries in the main diagonal lead to the calculation of the determinant. Here, we revisit the three elementary row operations and their effect on the determinant of a given matrix [1][4].

Type 1. Multiplying a row through by a nonzero constant.

Theorem 1. *If $B = [b_{ij}]$ is the matrix that results when a single row of $A = [a_{ij}]$ is multiplied by a scalar c, then $\det(B) = c \det(A)$.*

Proof: Let $c \neq 0$ be a scalar multiplier of ith row, R_i, of A. The definition $\det(A) = \sum(\pm) a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{nj_n}$ denotes each entry of R_i is a factor in each term of $\sum(\pm) a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{nj_n}$, so that cR_i yields $\sum(\pm)\, c \cdot a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{nj_n} = c \cdot \sum(\pm) a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{nj_n} = c \det(A)$.

Type 2. Interchanging two rows.

Theorem 2. *If $B = [b_{ij}]$ is the matrix that results when two rows of $A = [a_{ij}]$ are interchanged, then $\det(B) = -\det(A)$.*

Proof: Suppose we obtain an $n \times n$ matrix $B = [b_{ij}]$ by interchanging rows r and s of $A = [a_{ij}]$ so that $b_{rj} = a_{sj}$ and $b_{sj} = a_{rj}$. By definition, $\det(B) = \sum(\pm) b_{1j_1}b_{2j_2}b_{3j_3} \dots b_{rj_r} \dots b_{sj_s} \dots b_{nj_n}$ in terms of the entries of B and $\det(B) = \sum(\pm) a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{sj_r} \dots a_{rj_s} \dots a_{nj_n}$ in terms of the entries of A.

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Interchanging to rows denotes a single change within the permutation and produces a change in the number of inversions as to being odd or even; hence,

$$\det(B) = \sum(\mp)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_s} \dots a_{sj_r} \dots a_{nj_n} = -\det(A)$$

Type 3. Adding a constant times one row to another.

Theorem 3. *If $B = [b_{ij}]$ is the matrix that results when a multiple of a row R_r of $A = [a_{ij}]$ is added to another row R_s, then $det(B) = det(A)$.*

Proof. Let $c \neq 0$ be a scalar multiplier R_r so that $a_{ij} = b_{ij}$; $i \neq s$ and $b_{sj} = a_{sj} + ca_{rj}$. By definition,

$$\det(B) = \sum(\pm)b_{1j_1}b_{2j_2}b_{3j_3} \dots b_{rj_r} \dots b_{sj_s} \dots b_{nj_n}$$

$$= \sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots (a_{sj_s} + ca_{rj_r}) \dots a_{nj_n}$$

Expanding the summation gives

$$\sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots a_{sj_s} \dots a_{nj_n} + \sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots ca_{rj_r} \dots a_{nj_n}$$

Note that the first term $\sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots a_{sj_s} \dots a_{nj_n} = \det(A)$ and $\sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots ca_{rj_r} \dots a_{nj_n} = c\sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots a_{rj_r} \dots a_{nj_n}$. The two equal factors denote two rows with same entries, thus $\sum(\pm)a_{1j_1}a_{2j_2}a_{3j_3} \dots a_{rj_r} \dots a_{rj_r} \dots a_{nj_n} = 0$. Finally, $\det(B) = \det(A) + c(0) = \det(A)$.

By reducing a matrix into upper or lower triangular form and applying co-factor expansion, the determinant of the matrix is determined by the product of the entries in the main diagonal as stated in the following theorem.

Theorem 4. *If A = $[a_{ij}]$ is upper (lower) triangular, then det(A) = $a_{11}a_{22}a_{33} \dots a_{nn}$; that is, the determinant of a triangular matrix is the product of the elements on the main diagonal.*

Some methods along this approach include Chio's condensation, triangle's rule, Gaussian elimination procedure, LU decomposition, QR decomposition, and Cholesky decomposition (Sowabomo, 2016)

Dodgson's Condensation Method

The English clergyman Rev. Charles Lutwidge Dodgson (1832-1898), famously known as Lewis Caroll for his literary works *Alice in Wonderland* and *Through the Looking Glass*, invented an algorithm for computing the determinant of a square matrix that is more efficient than Leibniz's definition especially for larger matrices [4][5]. Dodgson's method aims to 'condense' the determinant by producing an $(n-1) \times (n-1)$ matrix from an $n \times n$ matrix, then $(n-2) \times (n-2)$ until a 1×1 matrix is obtained. The condensation method is founded on Jacobi's theorem [5].

Theorem 5. *Jacobi's Theorem. Let A be an $n \times n$ matrix, let $A_{ij}]$ be an $m \times m$ minor of A, where $m < n$, let $[A'_{ij}]$ be the corresponding $m \times m$ minor of A', and let $[A^*_{ij}]$ be the complementary $(n-m) \times (n-m)$ minor of A. Then:*

$$\det\ [A'_{ij}] = \det(A)^{m-1} \cdot \det[A^*_{ij}].$$

With $m = 2$, Dodgson realized that the determinant of A can be readily computed with

$$\det[A] = \frac{\det[A'_{ij}]}{\det[A^*_{ij}].}.$$

His algorithm consists of the following steps.

1. Check the interior of A for a zero entry. The interior of A is an $(n-2) \times (n-2)$ matrix that remains when the first row, last row, first column, and last column of A are deleted. We perform elementary row operations to remove all zeros from the interior of A.

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2. Compute the determinant of every four adjacent terms to form a new $(n-1)\times(n-1)$ matrix B.

3. Repeat Step 2 to produce an $(n-2)\times(n-2)$ matrix. We then divide each term by the corresponding entry in the interior of the original matrix A, to obtain matrix C.

4. We continue the process of condensation with the succeeding matrices a 1×1 matrix obtained which gives det A.

Note that Dodgson's method employs division of entries in succeeding matrices beginning with $(n-2)\times(n-2)$ matrix, a step which may magnify computational error especially for matrices with non-integral entries.

Some technology-based approaches are proposed to efficiently compute the determinant. Matrices with numerically very large or small entries are scaled down to optimize computing efficiency and space; which, however may affect accuracy. Hence, some methods are proposed such as computation of determinant of square matrices without division[6]. Some matrices naturally take specialized forms which facilitate the development of formulas to compute the determinant. These include division-free algorithm to compute the determinant of quasi-tridiagonal matrices [7], development of determinant formula for special matrices involving symmetry such as block matrices [8]; break-down free algorithm for computing determinants of periodic tridiagonal matrices [9]; and block diagonalization based algorithm of block k-tridiagonal matrices [10].

The goal of this paper is to propose a more straightforward and mnemonical algorithm that can be applied to any $n \times n$ numerical matrices.

2. Development of Proposed Algorithm

We start with an $n\times n$ matrix denoted by A_n with real entries

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where the first entries are non-zero real numbers. Our goal is to produce a matrix in upper triangular form so as to readily compute the determinant by Theorem 4.

Reduction by Cross-Multiplication

In this paper, we define reduction as the process of performing elementary row operations on a matrix to zero out entries under the first non-zero entry of the first row. A more straightforward way of zeroing out entries under a_{11} in A starting from the bottom row R_n is through $a_{n-1\,1}R_n - a_{n1}R_{n-1}$. By Theorem 3, the process introduces the factor $a_{n-1\,1}$ into the determinant. Working our way up, we zero out the first entry of R_{n-1} through $a_{n-2\,1}R_{n-1} - a_{n-1\,1}R_{n-2}$ which then introduces the factor $a_{n-2\,1}$ into the determinant. By repeating the process upward until R_2, we have introduced the following factors into the determinant of A: $a_{11}, a_{21}, \ldots, a_{n-2\,1}, a_{n-1\,1}$. At this stage, we obtain a matrix A_{n-1} with $n-1$ rows and $n-1$ columns by excluding the first column at the left with zero entries. We specified the entries as follows:

$$A_{n-1} \quad \begin{matrix} a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} & \cdots & a_{11}a_{2n} - a_{21}a_{1n} \\ a_{21}a_{32} - a_{31}a_{22} & a_{21}a_{33} - a_{31}a_{23} & \cdots & a_{21}a_{3n} - a_{31}a_{2n} \\ . & . & . & . \\ a_{n-11}a_{n2} - a_{n1}a_{n-12} & a_{n-11}a_{n3} - a_{n1}a_{n-13} & \cdots & a_{n-11}a_{nn} - a_{n1}a_{n-1n} \end{matrix}$$

For the purpose of this algorithm, we do away with the usual matrix notation and we put the rows of A_{n-1} under the given matrix.

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	a_{11}	a_{12}		$\cdots$	a_{1n}
	a_{21}	a_{22}		$\cdots$	a_{2n}
A	.	.		$\cdots$	.
	.	.		$\cdots$	.
	$a_{n-1\,1}$	$a_{n-1\,2}$			$a_{n-1\,n}$
	a_{n1}	a_{n2}		$\cdots$	a_{nn}
		$a_{11}^{(1)}$	$a_{12}^{(1)}$		$a_{1n-1}^{(1)}$
		$a_{21}^{(1)}$	$a_{22}^{(1)}$		$a_{2\,n-1}^{(1)}$
A_{n-1}		.		$\cdots$	.
		$a_{n-2\,1}^{(1)}$	$a_{n-2\,2}^{(1)}$	$\cdots$	$a_{n-2\,n-1}^{(1)}$
		$a_{n-1\,1}^{(1)}$	$a_{n-1\,2}^{(1)}$	$\cdots$	$a_{n-1\,n-1}^{(1)}$

The next stage is obtaining the rows for A_{n-2} which is equivalent to zeroing out entries under $a_{11}^{(1)}$ by following similar process done in obtaining the rows for A_{n-1}. The process also introduces the following factors into the determinant: $a_{11}^{(2)}, a_{21}^{(2)}, \dots, a_{n-3\,1}^{(2)}, a_{n-2\,1}^{(2)}$.

		$a_{11}^{(1)}$	$a_{12}^{(1)}$			$a_{1n-1}^{(1)}$
		$a_{21}^{(1)}$	$a_{22}^{(1)}$			$a_{2\,n-1}^{(1)}$
A_{n-1}		.		$\cdots$		.
		$a_{n-2\,1}^{(1)}$	$a_{n-2\,2}^{(1)}$	$\cdots$		$a_{n-2\,n-1}^{(1)}$
		$a_{n-1\,1}^{(1)}$	$a_{n-1\,2}^{(1)}$	.	$\cdots$	$a_{n-1\,n-1}^{(1)}$
			$a_{11}^{(2)}$	$a_{12}^{(2)}$		$a_{1\,n-2}^{(2)}$
			$a_{21}^{(2)}$	$a_{22}^{(2)}$		$a_{2\,n-2}^{(2)}$
A_{n-2}			.	.	$\cdots$	.
			$a_{n-3\,1}^{(2)}$	$a_{n-3\,2}^{(2)}$		$a_{n-3\,n-2}^{(2)}$

			$a^{(2)}_{n-2\,1}$	$a^{(2)}_{n-2\,2}$		$a^{(1)}_{n-2\,n-2}$

Assuming no first entries of each A_i is zero, continuing the process of reduction leads to 2×2 submatrix A_2 with an introduced factor of $a^{(n-2)}_{1\,1}$ and eventually 1×1 submatrix A_1.

A_2	$a^{(n-2)}_{1\,1}$	$a^{(n-2)}_{12}$
	$a^{(n-2)}_{21}$	$a^{(n-2)}_{22}$
A_1		$a^{(n-1)}_{1\,1}$

By taking the first row from each submatrix, we can reconstruct an $n\times n$ matrix B in upper triangular form that is row equivalent to the given $n\times n$ matrix A.

a_{11}	a_{12}		$\cdots$				a_{1n}
0	$a^{(1)}_{11}$	$a^{(1)}_{13}$					$a^{(1)}_{1n-1}$
0	0	$a^{(2)}_{11}$	$a^{(2)}_{14}$				$a^{(2)}_{1\,n-2}$
0	0	0	.	.	$\cdots$		
0	0	0		.	$\cdots$		
0	0	0		.	$\cdots$		
0	0	0		.	$\cdots$	$a^{(n-2)}_{11}$	$a^{(n-2)}_{12}$
0	0	0	0	0	$\cdots$	0	$a^{(n-1)}_{1\,1}$

By Theorem 4

$$\det(B) = a_{11}\cdot\ a^{(1)}_{11}\cdot\ a^{(2)}_{11}\cdot \ldots \cdot\ a^{(n-2)}_{1\,1}\cdot a^{(n-1)}_{11}.$$

Since A is row equivalent to B and these factors

$$a_{11}a_{21}a_{31}\cdots a_{n-1\,1}\cdot\ a^{1}_{11}a^{(1)}_{21}\cdots a^{(1)}_{n-2\,1}a^{(1)}_{n-1\,1}\cdot \ldots \cdot a^{(n-2)}_{1\,1}$$

are introduced into the determinant of A in the process of forming B, then

$$\det\,(\mathrm{B}) = a_{11}a_{21}a_{31}\cdots a_{n-1\,1}\cdot\ a^{1}_{11}a^{(1)}_{21}\cdots a^{(1)}_{n-2\,1}a^{(1)}_{n-1\,1}\cdot \ldots \cdot a^{(n-2)}_{11}\det(A)$$

from which,

$$\det(A) = \frac{\det\,(\mathrm{B})}{a_{11}a_{21}a_{31}\cdots a_{n-1\,1}\cdot\ a^{1}_{11}a^{(1)}_{21}\cdots a^{(1)}_{n-2\,1}a^{(1)}_{n-1\,1}\cdot \ldots \cdot a^{(n-2)}_{11}}$$

$$\det(A) = \frac{a_{11}\cdot a^{(1)}_{12}\cdot a^{(2)}_{13}\cdot\ldots\cdot a^{(n-2)}_{11}\cdot a^{(n-1)}_{11}}{a_{11}a_{21}a_{31}\cdots a_{n-1\,1}\cdot a^{1}_{11}a^{(1)}_{21}\cdots a^{(1)}_{n-2\,1}a^{(1)}_{n-1\,1}\cdot\ldots\cdot a^{(n-2)}_{11}} \qquad (1)$$

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Finally,

$$\det(A) = \frac{a_{11}^{(n-1)}}{a_{21}a_{31}\cdots a_{n-1\,1}a_{21}^{(1)}\cdots a_{n-2\,1}^{(1)}a_{n-1\,1}^{(1)}\cdot a_{21}^{(2)}a_{31}^{(2)}\cdot a_{n-21}^{(2)}\ldots a_{21}^{(n-4)}a_{31}^{(n-4)}\cdot a_{21}^{(n-3)}}. \qquad (2)$$

Examining the table below, the denominators are the in-between first entries in the submatrices.

	a_{11}	a_{12}		$\ldots$			a_{1n}
	a_{21}	a_{22}		$\ldots$			a_{2n}
	.	.		$\ldots$			.
A	.	.		$\ldots$			.
	$a_{n-1\,1}$	$a_{n-1\,2}$					$a_{n-1\,n}$
	a_{n1}	a_{n2}		$\ldots$			a_{nn}
		$a_{11}^{(1)}$	$a_{12}^{(1)}$				$a_{1n-1}^{(1)}$
		$a_{21}^{(1)}$	$a_{22}^{(1)}$				$a_{2\,n-1}^{(1)}$
A_{n-1}		.		$\ldots$			.
		$a_{n-2\,1}^{(1)}$	$a_{n-2\,2}^{(1)}$	$\ldots$			$a_{n-2\,n-1}^{(1)}$
		$a_{n-1\,1}^{(1)}$	$a_{n-1\,2}^{(1)}$				$a_{n-1\,n-1}^{(1)}$
			$a_{11}^{(2)}$	$a_{12}^{(2)}$			$a_{1\,n-2}^{(2)}$
			$a_{21}^{(2)}$	$a_{22}^{(2)}$			$a_{2\,n-2}^{(2)}$
A_{n-2}			.	.			.
			$a_{n-3\,1}^{(2)}$	$a_{n-3\,2}^{(2)}$			$a_{n-3\,n-2}^{(2)}$
			$a_{n-2\,1}^{(2)}$	$a_{n-2\,2}^{(2)}$			$a_{n-2\,n-2}^{(1)}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$		$\ldots$	$\ldots$
					$a_{1\,1}^{(n-3)}$	$a_{1\,2}^{(n-3)}$	$a_{1\,3}^{(n-3)}$
A_3					$a_{2\,1}^{(n-3)}$	$a_{22}^{(n-3)}$	$a_{23}^{(n-3)}$
					$a_{3\,1}^{(n-3)}$	$a_{3\,2}^{(n-3)}$	$a_{3\,3}^{(n-3)}$

A_2						$a_{1\,1}^{(n-2)}$	$a_{12}^{(n-2)}$
						$a_{21}^{(n-2)}$	$a_{22}^{(n-2)}$
A_1							$a_{1\,1}^{(n-1)}$

Remarks. The expression for each entry suggests a cross-multiplication pattern involving the first entries of two adjacent rows of A, a_{i1} and $a_{i+1\,1}$, and a pair of entries in the jth column, a_{ij} and $a_{i+1\,j}$ such that a resulting entry denoted by $a_{ij-1}^{(m)}$ in the submatrix A_i is determined by $a_{ij-1}^{(m)} = a_{i1}a_{i+1\,j} - a_{i+1\,1}a_{i\,j}$ where (m) is the number of reductions performed. Hence, we name this method cross-multiplication. In A_{n-1} for example, $a_{11}^{(1)} = a_{11}a_{22} - a_{21}a_{12}$, $a_{12}^{(1)} = a_{11}a_{22} - a_{21}a_{12}$, …, $a_{1n-1}^{(1)} = a_{11}a_{2n} - a_{21}a_{1n}$, and so on until $a_{n-1\,n-1}^{(1)} = a_{n-1\,1}a_{nn} - a_{n1}a_{n-1\,n}$.

3. Cross-Multiplication and Dodgson's Condensation Method in 3 x 3 Matrices

We now show the equivalence of Dodgson's condensation method and cross-multiplication which as shown above is derived from elementary row operations for 3×3 matrix.

Given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ where $a_{11}, a_{21}, a_{31}\ and\ a_{22}$ are nonzero. By Dodgson's condensation method, we take four adjacent entries at a time, compute their determinants to form a 2×2 matrix.

$$\begin{bmatrix} a_{11}a_{22} - a_{21}a_{12} & a_{12}a_{23} - a_{22}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & a_{22}a_{33} - a_{32}a_{23} \end{bmatrix}.$$

We then compute the determinant of the resulting matrix to obtain a 1×1 matrix and divide the resulting entry by the corresponding entry of the interior of A which is a_{22}.

$$[(a_{11}a_{22}a_{22}a_{33} - a_{11}a_{22}a_{32}a_{23} - a_{21}a_{12}a_{22}a_{33} + a_{21}a_{12}a_{32}a_{23}) - (a_{21}a_{32}a_{12}a_{23} - a_{21}a_{32}a_{22}a_{13} - a_{31}a_{22}a_{12}a_{23} + a_{31}a_{22}a_{22}a_{13}].$$

With $a_{21}a_{12}a_{32}a_{23}$ cancelled, this gives

$$B = [a_{11}a_{22}a_{22}a_{33} - a_{11}a_{22}a_{32}a_{23} - a_{21}a_{12}a_{22}a_{33} + a_{21}a_{32}a_{22}a_{13} + a_{31}a_{22}a_{12}a_{23} - a_{31}a_{22}a_{22}a_{13}]$$

Since a_{22} is a common factor, then $B = a_{22}[a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}]$. By Definition 1, $|A| = |a_{11}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}|$. Thus, $|A| = \frac{|B|}{a_{22}}$.

By exchanging columns 1 and 2 of A, we have $A' = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$ and it can also be shown that $|A| = -\frac{|B'|}{a_{21}}$.

By cross-multiplication

A_3	a_{11}	a_{12}	a_{13}
	a_{21}	a_{22}	a_{23}
	a_{31}	a_{32}	a_{33}
A_2		$a_{11}a_{22} - a_{21}a_{12}$	$a_{11}a_{23} - a_{21}a_{13}$
		$a_{21}a_{32} - a_{31}a_{22}$	$a_{21}a_{33} - a_{31}a_{23}$

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A_1			$a_{11}a_{22}a_{21}a_{33} - a_{11}a_{22}a_{31}a_{23}$ $- a_{21}a_{12}a_{21}a_{33}$ $+ a_{21}a_{12}a_{31}a_{23}$ $- (a_{21}a_{32}a_{11}a_{23}$ $- a_{21}a_{32}a_{21}a_{13}$ $- a_{31}a_{22}a_{11}a_{23}$ $+ a_{31}a_{22}a_{21}a_{13})$

Here, a_{21} is the common factor hence, $|A| = \frac{|A_1|}{a_{21}}$ which is consistent with Equation 2.

Similar to Dodgson's approach, $a_{ij-1}^{(m)}$ can be thought of as the determinant of 2×2 matrix formed by the first entries of two adjacent rows and the pair of corresponding entries in the jth column; that is, $a_{ij-1}^{(m)} = \begin{vmatrix} a_{i1} & a_{ij} \\ a_{i+1\,1} & a_{i+1\,j} \end{vmatrix}$. Here, we fix the first entries of two adjacent rows as the pivot in computing the determinants of consecutive 2×2 matrices. By doing so, $a_{i+1\,1}$ is the factor introduced into the original determinant, analogous to the corresponding entry of the interior matrix in the condensation method. By fixing the pivot at the first two entries, we can limit the number of divisions to be performed in preserving the determinant. With $n \geq 3$ Dodgson's method employs $n^2 - 3n + 1$ internal divisions while cross-multiplications only employs $\frac{n^2-3n+2}{2}$ divisions at the final stage of calculations.

4. Alternate Proof of Cross-Multiplication Method

Suppose that B is an $n \times n$ matrix where all entries under the first element of the first row are zeros,

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & b_{n-1\,2} & \cdots & b_{n-1\,n} \\ 0 & b_{n\,2} & \cdots & b_{n\,n} \end{bmatrix}.$$

By co-factor expansion $\det(B) = b_{11}(-1)^{1+1}\begin{bmatrix} b_{22} & \cdots & b_{2n} \\ \cdot & \cdots & \cdot \\ b_{n-1\,2} & \cdots & b_{n-1\,n} \\ b_{n2} & \cdots & b_{nn} \end{bmatrix}$. If $n > 4$, then to compute the determinant of the resulting $n-1 \times n-1$ submatrix requires a full co-factor expansion or elementary row operations. In deriving the proposed method, we first transform a given matrix A into the form of B through elementary row operations then apply the definition of co-factor expansion by expanding about the first column. We then repeat the process with the succeeding submatrices until a relatively smaller matrix is obtained where we can readily compute the determinant. By introducing the cross-multiplication, we only form one $n-1 \times n-1$ submatrix instead of n $n-1 \times n-1$ submatrices in co-factor expansion.

Let A be an $n \times n$ matrix with real entries where $a_{11}, a_{21}, a_{n-1\,1}, a_{n1} \neq 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We zero out entries under a_{11} through the process illustrated in this paper: With two successive rows R_i and R_{i+1} with first entries a_{i1} and a_{i+11}, respectively, and by $a_{i1}R_{i+1} - a_{i+1\,1}R_i$, a matrix B_1 is produced.

$$B_1 = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ 0 & a_{11}^{(1)} & \cdots & a_{1n-1}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ . & . & \cdots & . \\ 0 & a_{n-1\,1}^{(1)} & \cdots & a_{n-1\,n-1}^{(1)} \end{bmatrix}$$

In the process, the factors $a_{11}, a_{21}, a_{31}, \dots, a_{n-1\,1}$ are introduced into $\det[A]$. Thus by Theorem 1,

$$\det[B_1] = a_{11}, a_{21}, a_{31}, \dots, a_{n-1\,1} \det(A)$$

By cofactor expansion,

$$\det[B_1] = a_{11} \begin{vmatrix} a_{11}^{(1)} & \cdots & a_{1n-1}^{(1)} \\ \cdots & \cdots & \cdots \\ a_{n-1\,1}^{(1)} & \cdots & a_{n-1\,n-1}^{(1)} \end{vmatrix} = a_{11} \det[A_{n-1}]$$

where $A_{n-1} = \begin{bmatrix} a_{11}^{(1)} & \cdots & a_{1n-1}^{(1)} \\ \cdots & \cdots & \cdots \\ a_{n-1\,1}^{(1)} & \cdots & a_{n-1\,n-1}^{(1)} \end{bmatrix}$.

We repeat the process of reduction on A_{n-1} to form the matrix

$$B_2 = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & & a_{1n-1}^{(1)} \\ 0 & a_{11}^{(2)} & \cdots & a_{1n-2}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ . & . & \cdots & . \\ 0 & a_{n-2\,1}^{(2)} & \cdots & a_{n-2\,n-2}^{(1)} \end{bmatrix}$$

and which

$$\det[B_2] = a_{11}^{(1)} a_{21}^{(1)} a_{31}^{(1)} \dots a_{n-21}^{(1)} \det[A_{n-1}]$$

$$\det[B_2] = a_{11}^{(1)} \det(A_{n-2}) \text{ where } A_{n-2} = \begin{bmatrix} a_{11}^{(2)} & \cdots & a_{1n-2}^{(2)} \\ \cdots & \cdots & \cdots \\ a_{n-2\,1}^{(2)} & \cdots & a_{n-2\,n-2}^{(2)} \end{bmatrix}$$

Continuing the reduction on subsequent submatrices we summarize the following results

$$\det[B_1] = a_{11}, a_{21}, a_{31}, \dots, a_{n-1\,1} \det[A] = a_{11} \det[A_{n-1}]$$

$$\det[B_2] = a_{11}^{(1)} a_{21}^{(1)} a_{31}^{(1)} \dots a_{n-21}^{(1)} \det[A_{n-1}] = a_{11}^{(1)} \det[A_{n-2}]$$

$$\det[B_3] = a_{11}^{(2)} a_{21}^{(2)} a_{31}^{(2)} \dots a_{n-31}^{(2)} \det[A_{n-2}] = a_{11}^{(2)} \det[A_{n-3}]$$

$$\dots$$

$$\det[B_{n-2}] = a_{11}^{(n-3)} a_{21}^{(n-3)} \det[A_2] = a_{11}^{(n-3)} \det[A_1]$$

$$\det[B_{n-1}] = a_{11}^{(n-2)} \det[A_2] = a_{11}^{(n-2)} \det[A_1] = a_{11}^{(n-1)}$$

Working backwards leads to

$$\det[A] = \frac{a_{11}}{a_{11}, a_{21}, a_{31}, \dots, a_{n-1\,1}} \cdot \frac{a_{11}^{(1)}}{a_{11}^{(1)} a_{21}^{(1)} a_{31}^{(1)} \dots a_{n-21}^{(1)}} \cdots \frac{a_{11}^{(n-3)}}{a_{11}^{(n-3)} a_{21}^{(n-3)}} \cdot a_{11}^{(n-1)}$$

Hence,

$$\det[A] = \frac{a_{11}^{(n-1)}}{a_{21}a_{31}\cdots a_{n-1\,1}a_{21}^{(1)}\cdots a_{n-2\,1}^{(1)}a_{n-1\,1}^{(1)}\cdot a_{21}^{(2)}a_{31}^{(2)}\cdot a_{n-21}^{(2)}\ldots a_{21}^{(n-4)}a_{31}^{(n-4)}\cdot\; a_{21}^{(n-3)}}.$$

5. Some Special Cases

The equation in computing the determinant with the developed method holds if none of the first entries in any row is zero. Here we consider cases where there are zero first entries in any of the submatrices A_{n-i}.

Zero First Entries of First or Last Row of A_{n-i}; $n-i>1$

If the first entry of either first or last row of A_{n-i} except the last submatrix A_1 is zero, then Equation 2 still holds. These rows with zero first entries are considered standby rows and do not participate in row reduction. They are then transferred to the next submatrix retaining their placement as first or last row.

A_{n-i}	0	$a_{12}^{(i)}$		
	$a_{21}^{(i)}$	$a_{22}^{(i)}$		
	$a_{21}^{(i)}$	$a_{22}^{(i)}$		
	…			
	$a_{n-i+1\,1}^{(i)}$			
	0	$a_{n-i\,2}^{(i)}$		
A_{n-i-1}		$a_{12}^{(i)}$		
		$a_{11}^{(i+1)}$		
		…		
		$a_{n-i\,1}^{(i)}$		

By Theorem 3, we can also add any row with nonzero first entry to a row with zero first entry and go on with the usual process of reduction.

Some In-Between First Entries are Zeros

The general approach is to apply Theorem 3 by adding any row with nonzero first entry to a row with zero first entry. We can also apply Theorem 2 to lessen the number of operations especially when there are more rows with zero first entries than rows with nonzero first entries. If an mth row of A_{n-i} has $a_{m1}^{(i)}=0$, then the reduction $a_{m1}^{(i)}R_{m+1}-a_{m+1\,1}^{(i)}R_m$ is not possible since this will introduce a zero factor in the divisor of Equation 2. We reduce the entire submatrix by first moving the rows with zero first entries to the bottom part then perform reduction to rows with nonzero first entries. The rows with zero first entries are treated as "standby rows" and are then moved to the next submatrix in the same placement. Since $\det(A_{n-i}) = a_{11}^{(k)}\det(A_{n-i-1})$, by Theorem 2, interchanging R_m and the last row will lead to $\det(A_{n-i}) = -a_{11}^{(k)}\det(A_{n-i-1})$. We introduce the factor $(-1)^k$ to Equation 2 where k indicates the number of row exchanges performed.

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$$\det(\mathrm{A}) = \frac{a_{11}^{(n-1)}}{a_{21}a_{31}\cdots a_{n-1\,1}a_{21}^{(1)}\cdots a_{n-2\,1}^{(1)}a_{n-1\,1}^{(1)}\cdot a_{21}^{(2)}a_{31}^{(2)}\cdot a_{n-21}^{(2)}\ldots a_{21}^{(n-4)}a_{31}^{(n-4)}\cdot a_{21}^{(n-3)}} \cdot (-1)^k\,;$$

A_{n-i}	$a_{11}^{(i)}$	$a_{11}^{(i)}$		
	$a_{21}^{(i)}$	$a_{22}^{(i)}$		
	…			
	0	$a_{m-1\,2}^{(i)}$		
	0	a_{m2}^{i}		
	$a_{m+1\,1}^{(i)}$	$a_{m+1\,2}^{(i)}$		
	…			
	$a_{n-i\,1}^{(i)}$	$a_{n-i\,1}^{(i)}$		
A_{n-i}	$a_{11}^{(i)}$	$a_{11}^{(i)}$		
	$a_{21}^{(i)}$	$a_{22}^{(i)}$		
	…			
	$a_{m+1\,1}^{(i)}$	$a_{m+1\,2}^{(i)}$		
	$a_{n-i\,1}^{(i)}$	$a_{n-i\,1}^{(i)}$		
	0	$a_{m-1\,2}^{(i)}$		
	0	$a_{m2}^{(i)}$		
A_{n-i-1}		$a_{11}^{(i+1)}$	$a_{12}^{(i+1)}$	
		…		
		$a_{m-1\,2}^{(i)}$		
		a_{m2}^{i}		

Only One Row Has Non-zero First Entry

If only one row of A_{n-i} has non-zero first entry, we place this row as R_1 with first entry $a_{11}^{(n-i)}$. No reduction is performed since the A_{n-i} already takes the reduced form. By Theorem 4, $a_{11}^{(n-i)}$ is a diagonal entry thus this leads to the modified formula:

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$$\det(\mathrm{A}) = \frac{a_{11}^{(n-i)} \cdot a_{11}^{(n-1)}}{a_{21}a_{31}\cdots a_{n-1\,1}a_{21}^{(1)}\cdots a_{n-2\,1}^{(1)}a_{n-1\,1}^{(1)}\cdot a_{21}^{(2)}a_{31}^{(2)}\cdot a_{n-21}^{(2)}\ldots a_{21}^{(n-4)}a_{31}^{(n-4)}\cdot\; a_{21}^{(n-3)}}\cdot(-1)^k$$

All Rows Have Zero First Entries

If all rows of A_{n-i} have zero first entries, then no reduction is performed and since $a_{11}^{(n-i)} = 0$, then by Equation 2, $\det(A) = 0$.

6. Conclusions

The cross-multiplication method of computing the determinants is a mnemonical use of elementary row operations in reducing a matrix to lower triangular form by using the first entries of adjacent rows as scalar multipliers. Doing so retains the butterfly movement in computing the determinant of 2×2 matrix as each entry is easily computed through $a_{ij-1}^{(m)} = a_{i1}a_{i+1\,j} - a_{i+1\,1}a_{i\,j}$ or $a_{ij-1}^{(m)} = \begin{vmatrix} a_{i1} & a_{ij} \\ a_{i+1\,1} & a_{i+1\,j}\end{vmatrix}$ such that a_{i1} and $a_{i+1\,1}$ are the pivot entries. The algorithm proceeds by computing each entry to produce a submatrix that is one row and one column less than the preceding submatrix until a 1×1 matrix, $[a_{11}^{(n-1)}]$, is obtained. The determinant is then computed as to the following cases.

1. When all first entries in the rows of submatrices are nonzero, the determinant is solved by

$$\det(\mathrm{A}) = \frac{a_{11}^{(n-1)}}{a_{21}a_{31}\cdots a_{n-1\,1}a_{21}^{(1)}\cdots a_{n-2\,1}^{(1)}a_{n-1\,1}^{(1)}\cdot a_{21}^{(2)}a_{31}^{(2)}\cdot a_{n-21}^{(2)}\ldots a_{21}^{(n-4)}a_{31}^{(n-4)}\cdot\; a_{21}^{(n-3)}}$$

where the factors in the denominator are the nonzero in-between first entries of submatrices, representing the factors introduced into the original determinant in the reduction process.

2. When some first entries in the rows are zero, the rows are transferred to the bottom of the submatrix and then transferred to the next submatrix. The determinant is solved by

$$\det(\mathrm{A}) = \frac{a_{11}^{(n-1)}}{product\ of\ nonzero\ in\text{–}between\ first\ entries}\cdot(-1)^k$$

where k is the number of row exchanges.

3. When rows with zero first entries are added by rows with nonzero first entries, the formula in (1) holds.

4. When each of the submatrices $A_{n-i}, A_{n-j}, A_{n-k}, \ldots$ has exactly one row with nonzero first entry, we set these rows as first rows where necessary and compute the determinant by

$$\det(\mathrm{A}) = \frac{a_{11}^{(n-i)}\cdot a_{11}^{(n-j)}\cdot a_{11}^{(n-k)}\cdot\ldots\cdot a_{11}^{(n-1)}}{product\ of\ nonzero\ in - between\ first\ entries}\cdot(-1)^k$$

where k is the number of row exchanges.

4. When all rows of a submatrix A_{n-i} have zero first entries, then $\det[A] = 0$.

For $n = 3$, cross-multiplication and Dodgson's condensation method are shown to be equivalent. Unlike the condensation method which employs $n^2 - 3n + 1$ internal divisions, the proposed method employs much fewer, $\frac{n^2-3n+2}{2}$,terminal divisions thus minimizing the propagation of computational errors. Both methods require row adjustments whenever there are zero pivot entries.

Compared to co-factor expansion which generates n number of $(n-1)\times(n-1)$ submatrices, then $n(n-1)$ number of $(n-2)\times(n-2)$ submatrices, and so on, the cross-multiplication method generates only one of each of $(n-1)\times(n-1), (n-2)\times(n-2)$, … (2×2), and (1×1) submatrices.

The simplicity and efficiency of the proposed methods lies in the consistency and symmetry of of its iterative process. No procedural adjustments are introduced as the matrix size increases compared to other methods such as extension of Sarrus rule to matrices with $n \geq 4$.

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Appendix A

Experiments with Proposed Methods

We provide illustrative examples for the proposed methods and we verify the results using co-factor expansion and Matlab (See Appendix B).

Illustration 1. 4×4 matrix All First Entries Are Non-zeros

$$A = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 2 & 3 & 2 & 3 \\ 1 & -1 & 4 & 2 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

Solution by cross-multiplication method.

A_4	2	1	5	2
	2	3	2	3
	1	-1	4	2
	1	2	4	1
A_3		4	-6	2
		-5	6	1
		3	0	-1
A_2			-6	14
			-18	2
A_1				240
	$\det[A] = -\frac{240}{(2)(1)(-5)} = -24$			

Solution by co-factor expansion

We expand at the first column to get

$$\begin{vmatrix} 2 & 1 & 5 & 2 \\ 2 & 3 & 2 & 3 \\ 1 & -1 & 4 & 2 \\ 1 & 2 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 & 3 \\ -1 & 4 & 2 \\ 2 & 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 & 2 \\ -1 & 4 & 2 \\ 2 & 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ 3 & 2 & 3 \\ 2 & 4 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 5 & 2 \\ 3 & 2 & 3 \\ -1 & 4 & 2 \end{vmatrix}$$

$$= 2(12 - 12 + 8 - 24 + 2 - 24) - 2(4 - 8 + 20 - 16 + 5 - 8) + (2 + 24 + 30 - 8 - 15 - 12) - (4 + 24 - 15 + 4 - 30 - 12)$$

$$= 2(-38) - 2(-3) + 21 - (-25)$$

$= -24$

Illustration 2. 5×5 matrix All First Entries Are Non-zeros

$$A = \begin{bmatrix} 2 & 2 & 1 & 3 & 1 \\ 1 & 3 & -1 & 1 & 2 \\ 1 & 2 & 4 & -2 & 3 \\ 2 & 2 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 & 5 \end{bmatrix}$$

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Computation of determinants by cross-multiplication method.

A_5	2	2	1	3	1
	1	3	-1	1	2
	1	2	4	-2	3
	2	2	3	2	1
	1	3	2	1	5
A_4		4	-3	-1	3
		-1	5	-3	1
		-2	-5	6	-5
		4	1	0	9
A_3			17	-13	7
			15	-12	7
			18	-24	2
A_2				-9	14
				-144	-96
A_1					2880
					48

Verification of result by co-factor expansion.

$$\det[A] = \begin{vmatrix} 2 & 2 & 1 & 3 & 1 \\ 1 & 3 & -1 & 1 & 2 \\ 1 & 2 & 4 & -2 & 3 \\ 2 & 2 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 & 5 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & -1 & 1 & 2 \\ 2 & 4 & -2 & 3 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 3 & 1 \\ 2 & 4 & -2 & 3 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 & 1 \\ 3 & -1 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{vmatrix}$$

$$- 2\begin{vmatrix} 2 & 1 & 3 & 1 \\ 3 & -1 & 1 & 2 \\ 2 & 4 & -2 & 3 \\ 3 & 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 & 1 \\ 3 & -1 & 1 & 2 \\ 2 & 4 & -2 & 3 \\ 2 & 3 & 2 & 1 \end{vmatrix}$$

We compute the determinant of each submatrix separately.

$$\begin{vmatrix} 3 & -1 & 1 & 2 \\ 2 & 4 & -2 & 3 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 5 \end{vmatrix}$$

$$= 3\begin{vmatrix} 4 & -2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 & 2 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 & 2 \\ 4 & -2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 3(40 - 4 + 9 - 12 + 30 - 4) - 2(-10 + 2 + 6 - 8 - 15 + 1)$$
$$+2(10 + 8 + 6 + 8 - 20 + 3) - 3(2 + 16 + 9 + 12 - 4 + 6)$$

$= 132$

$$\begin{vmatrix}2&1&3&1\\2&4&-2&3\\2&3&2&1\\3&2&1&5\end{vmatrix} = 2\begin{vmatrix}4&-2&3\\3&2&1\\2&1&5\end{vmatrix} - 2\begin{vmatrix}1&3&1\\3&2&1\\2&1&5\end{vmatrix} + 2\begin{vmatrix}1&3&1\\4&-2&3\\2&1&5\end{vmatrix} - 3\begin{vmatrix}1&3&1\\4&-2&3\\3&2&1\end{vmatrix}$$

$$= 2(20-4+9-12+30-4) - 2(10+3+6+4-60-9) + 2(-10+18+4+4-60-3) - 3(-2+27+8+6-12-6)$$

$$= 23$$

$$\begin{vmatrix}2&1&3&1\\3&-1&1&2\\2&3&2&1\\3&2&1&5\end{vmatrix} = 2\begin{vmatrix}-1&1&2\\3&2&1\\2&1&5\end{vmatrix} - 3\begin{vmatrix}1&3&1\\3&2&1\\2&1&5\end{vmatrix} + 2\begin{vmatrix}1&3&1\\-1&1&2\\2&1&5\end{vmatrix} - 3\begin{vmatrix}1&3&1\\-1&1&2\\2&1&5\end{vmatrix}$$

$$= 2(-10+6+2-8+1-15) - 3(10+3+6-4-45-2) + 2(5-1+12-2+15-2) - 3(5-1+12-2+15-2)$$

$$= 60$$

$$\begin{vmatrix}2&1&3&1\\3&-1&1&2\\2&4&-2&3\\3&2&1&5\end{vmatrix}$$

$$= 2\begin{vmatrix}-1&1&2\\4&-2&3\\2&1&5\end{vmatrix} - 3\begin{vmatrix}1&3&1\\4&-2&3\\2&1&5\end{vmatrix} + 2\begin{vmatrix}1&3&1\\-1&1&2\\2&1&5\end{vmatrix} - 3\begin{vmatrix}1&3&1\\-1&1&2\\4&-2&3\end{vmatrix}$$

$$= 2(10+8+6+8-20+3) - 3(-10+18+4+4-60-3) + 2(5-1+12-2+15-2) - 3(3+2+24-4+9+4)$$

$$= 111$$

$$\begin{vmatrix}2&1&3&1\\3&-1&1&2\\2&4&-2&3\\2&3&2&1\end{vmatrix}$$

$$= 2\begin{vmatrix}-1&1&2\\4&-2&3\\3&2&1\end{vmatrix} - 3\begin{vmatrix}1&3&1\\4&-2&3\\3&2&1\end{vmatrix} + 2\begin{vmatrix}1&3&1\\-1&1&2\\3&2&1\end{vmatrix} - 2\begin{vmatrix}1&3&1\\-1&1&2\\4&-2&3\end{vmatrix}$$

$$= 2(2+16+9+12-4+6) - 3(-2+27+8+6-12-6) + 2(1-2+18-3+3-4) - 2(3+2+24-4+9+4)$$

$$= -31$$

Thus, $\det[A] = 2(132) - 23 + 60 - 2(111) - 31 = 48$

Illustration 3. First Entry of First or Last Row is Zero

$$C = \begin{bmatrix}0&-2&1&1\\1&2&3&1\\2&5&2&1\\3&2&2&5\end{bmatrix}$$

A_4	0	-2	1	1	Standby row

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	1	2	3	1	
	2	5	2	1	
	3	2	2	5	
A_3		-2	1	1	Transferred row
		1	-4	-1	
		-11	-2	7	
A_2			7	1	
			-46	-4	
A_1				-18	
				-9	

The other approach is to add the row with zero first entry by another row with nonzero first entry.

A_4	1	0	4	2	Row 2 added to Row 1
	1	2	3	1	
	2	5	2	1	
	3	2	2	5	
A_3		2	-1	-1	
		1	-4	-1	
		-11	-2	7	
A_2			-7	-1	
			-46	-4	
A_1				-18	
				-9	

Illustration 4. Some In-between First Entries are Zeros

$$D = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 0 & 3 & 2 & 3 \\ 1 & -1 & 4 & 2 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

A_4	2	1	5	2			
	0	3	2	3			
	1	-1	4	2			
	0	2	4	1			
A_4	2	1	5	2			
	1	-1	4	2			
m=1	0	3	2	3	Standby row		
	0	2	4	1	Standby row		
A_3		-3	3	2	2(-1)-1(1)=-3	2(4)-1(5)=3	2(2)-1(2)=2
		3	2	3	Transferred standby row		

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		2	4	1	Transferred standby row		
A_2			-15	-15		-3(2)-3(3)=-15	-3(3)-3(2)=-15
			8	-3		3(4)-2(2)=8	3(1)-2(3)=-3
A_1				165			-15(-3)-8(-15)=165
				-55			156/(3)(-1)

We apply similar technique as in Illustration 2.

A_4	2	1	5	2	
	0	3	2	3	
	1	-1	4	2	
	0	2	4	1	
A_4	2	1	5	2	
	2	4	7	5	Row 1 added to row 2
	1	-1	4	2	
	1	1	8	3	Row 3 added to row 4
A_3		6	4	6	
		-6	1	-1	
		2	4	1	
A_2			30	30	
			-26	-4	
A_1				780	
				-55	

Illustration 5. Only One Row Has Non-zero First Entry

$$E = \begin{bmatrix} 0 & 2 & 1 & 3 & 1 \\ 0 & 0 & -2 & 1 & 1 \\ 3 & 3 & 4 & 1 & 5 \\ 0 & 2 & 5 & 2 & 1 \\ 0 & 3 & 2 & 2 & 5 \end{bmatrix}$$

A_5	0	2	1	3	1			Exchange with R3
	0	0	-2	1	1			Standby
	3	3	4	1	5			Exchange with R1
	0	2	5	2	1			Standby
	0	3	2	2	5			Standby
A_5	3	3	4	1	5	Exchanged row		

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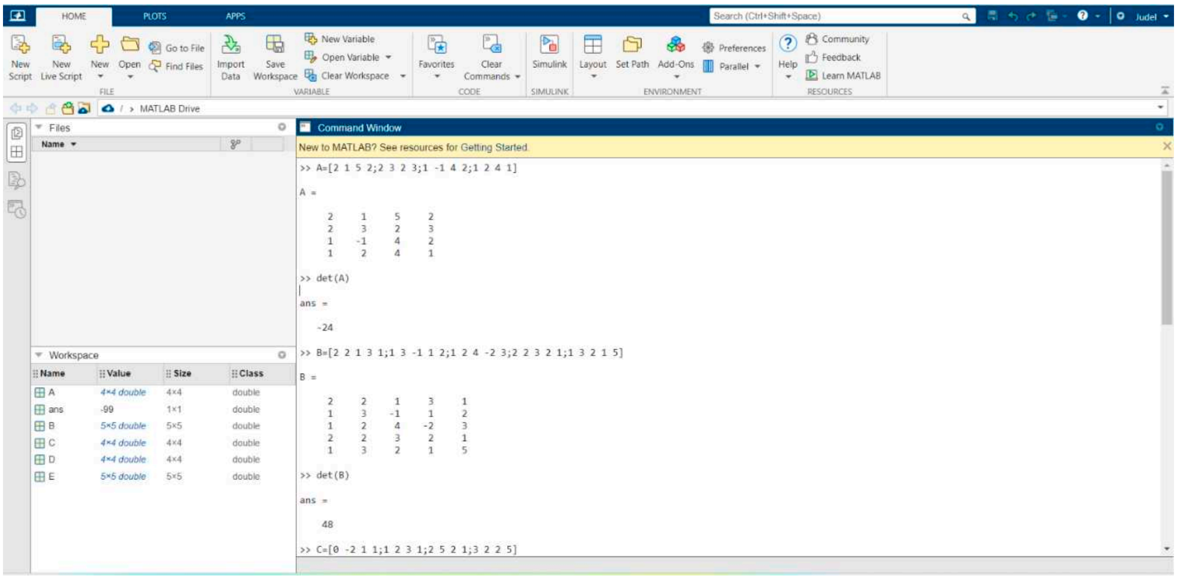
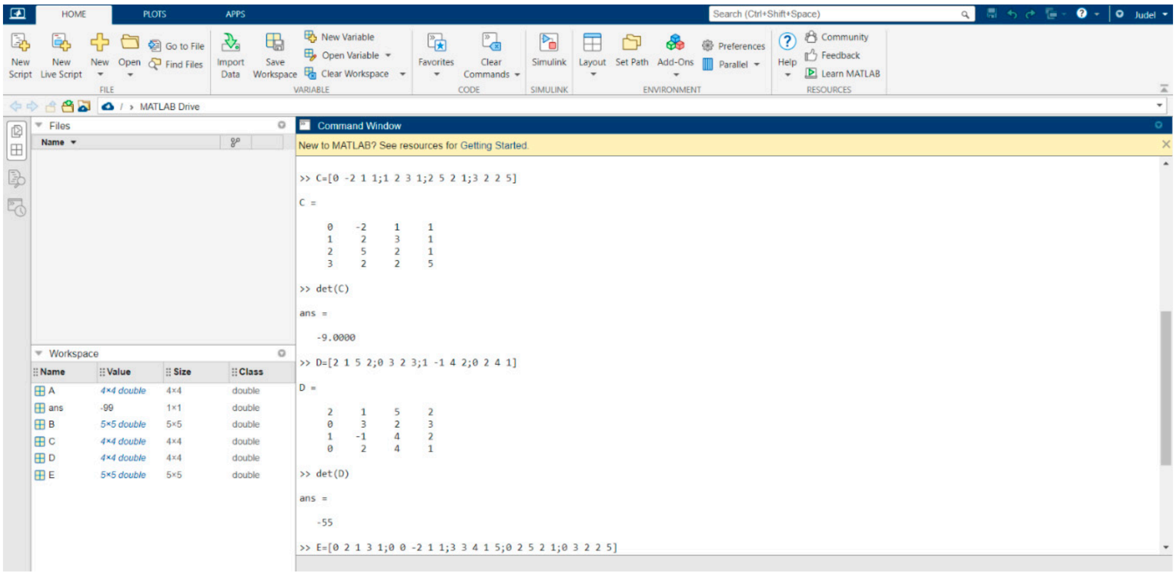
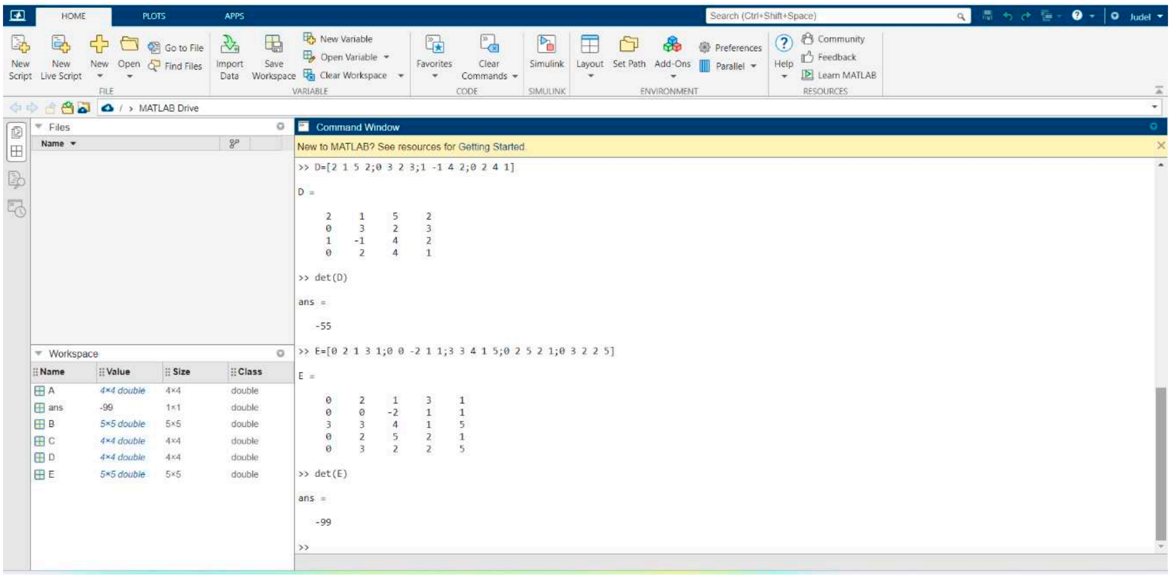
m = 1	0	0	-2	1	1			Standby row
	0	2	1	3	1			Exchanged row
	0	2	5	2	1			Standby row
	0	3	2	2	5			Standby row
A_4		0	-2	1	1			Transferred/Standby row
		2	1	3	1			Transferred row
		2	5	2	1			Transferred row
		3	2	2	5			Transferred row
A_4			-2	1	1			Transferred row
			8	-2	0	2(5)-2(1)=8	2(2)-2(3) = -2	2(1)-2(1) = 0
			-11	-2	7	2(2)-3(5) = -11	2(2)-3(2) = -2	2(5)-3(1) = 7
A_3				-4	-8		-2(-2)-8(1) = -4	-2(0)-8(1)= - 8
				-38	56		8(-2) – (-11)(-2) = -38	8(7)-(-11)(0) = 56
A_2					528			-4(56)-(-38)(-8) = 528
A_1					-99			(3)528/(2)(8)(-1) = -99

Appendix B

Matlab Generated Results







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