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Posted Date: 22 November 2023

doi: 10.20944/preprints202306.1472.v4

Keywords: Quantum Gravity; M-theory; Dark Energy; Dark Matter

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General Quantum Gravity

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Abstract: Quantum Gravity is not sufficient to predict the properties of particles (baryonic and non-baryonic) to a great extent. So, we develop General Quantum Gravity (GQG), which is a renormalizable formalization of quantized gravity that emerges from General Relativity through a new scenario of Quantum Mechanics. Inside every observable $(3+1)D$ spacetime in this new scenario of Quantum Mechanics, there must contain an internal hidden $(n+1)D$ spacetime, which immediately yields extra hidden dimensions by a closed continuous mapping, so that the overall spacetime must acquire Supersymmetry. We have also found that Superstrings are inevitably natural and universal but hidden inside every $(3+1)D$ observable spacetime. Superstrings are found eleven-dimensional (rather than lower dimensional) within this quantum spacetime. Consequently, eleven-dimensional Supergravity is necessarily a natural phenomenon within the GQG spacetime. GQG is formalized here in two different aspects. The first one gives an Einstein field equation in a semi-quantum Minkowski spacetime with semi-quantum Lorentzian metric tensor, whereas, the second one yields a purely Quantum Mechanical Einstein field equation in a quantum non-Minkowski spacetime with non-Lorentzian metric tensor. Dark Energy (as well as Dark Matter) appears from GQG quite naturally. We develop here Gravitational Electroweak Dark Energy interactions, where gravity and Dark Energy are allowed to combine with electroweak symmetry. Likewise, in Gravitational Chromodynamic Dark Energy interactions, we combine QCD with gravitational and Dark Energy symmetries. In a Dark Matter gauge symmetry model, we combine Dark matter with all possible gauge symmetries. A Universal Model $SU(7)_{UM}$ combines entire baryonic and non-baryonic particle interactions all together.

Keywords: quantum gravity; M-theory; dark energy; dark matter

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1. Introduction

This work is dedicated to those innocent children, women and infants, who were molested and brutally killed by the invading militants during and the aftermath week of 7^{th} October, 2023.

Present mathematical physics is unable to provide us a more acceptable scenario of Einstein field equation which may be developed in a quantum spacetime. Even cosmological observations are inconsistent with Einstein's equations of General Relativity in the absence of Dark Energy and Dark Matter. Additionally, Einstein field equation is not sufficient to predict the properties of particles (baryonic and non-baryonic) to a great extent.

Fortunately, if we can able to develop a formalism of generalized Quantum Gravity, it is possible that the cosmological constant problem could be resolved by replacing classical General Relativity with an alternative theory of gravity, with no dark components being imposed separately but comprised within the explanation of this alternative theory of gravity. This is the fundamental reason why we develop a formalism of General Quantum Gravity (GQG) in our present investigation. GQG is a formalization of quantized gravity that emerges from General Relativity through a new scenario of Quantum Mechanics. This new scenario is analogous, but not exactly similar to the Classical Quantum Mechanics. The formalism of GQG yields the Einstein field equation either in Semi-Quantum Minkowski Spacetime, or in Non-Minkowskian Spacetime, quite unfamiliar with the previous attempts of Quantum Gravity which have been published yet.

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In GQG, we describe gravity through Quantum Mechanics without considering Planck scales in general. But, if we consider Planck scales in this quantum gravity formalism, our proposed scenarios immediately develop a set of bosonic and fermionic fields for both Dark Energy and Dark Matter quite naturally without presuming any additional conditions, such as supersymmetry, superstrings, etc.

During the development of GQG, we have found that every $(3+1)D$ observable system must contain inevitably natural and universal but hidden string as well as extra dimensions within the quantum spacetime of this new scenario of Quantum Mechanics. This string has been found fundamentally eleven-dimensional (rather than lower dimensional) by nature in this quantum spacetime, that is why eleven is naturally the maxinium spacetime dimension in which one can formulate a consistent Supergravity theory.

Unification of gravity with Standard Model was tried previously by numerous authors, but no one yet thought to unify non-abelian gravitational, Dark Energy, Dark Matter, Electroweak and Quantum Chromodynamic gauge fields all together under a single Yang-Mills Lagrangian. In this work, we have not only unified them all together under a single formalism, but also successfully presented here the solutions of two open problems, viz.

1. *Why Dark Matter contents* 26.8% *of the critical density in the Universe against* 4.9% *of the critical density of baryonic matters?*
2. *Why is the energy density of matter nearly equal to the Dark Energy density today?*

We have developed here the Gravitational Electroweak Dark (GED) interaction, where gravity and Dark Energy are unified with Electroweak interaction. In non-abelian Dark Energy gauge symmetry, Casimir energy is considered to associate with an abelian gauge group to complete the Dark Energy scenario properly. Likewise, in the second scheme of unification, gravity and Dark Energy are unified with Quantum Chromodynamics and we have developed the Gravitational Chromodynamic Dark (GCD) interaction, where we get another set of Dark particles, which are quite different from the particles for non-abelian Dark gauge group of GED. Finally, combining Dark Matter with all other non-abelian gauge symmetries, we have developed an Yang-Mills Lagrangian to explain non-abelian Dark Matter gauge symmetry.

Combining GED, GCD and Dark Matter gauge symmetry, we have a Universal Model for all kind of baryonic and non-baryonic particle interactions as,

$$SU(3)_{GED} \otimes SU(4)_{GCD} \otimes SU(5)_{DM} \subset SU(7)_{UM}\,,$$

which makes it clear that Dark Energy field is homogeneous whether the matter is baryonic or Dark, or their mixture, and the effective universal relativistic cosmological constant Λ_{eff} at the surrounding is always positive – that is why the Universe is expanding and accelerating, even at the present epoch.

2. General Quantum Gravity

Let us construct a quantum version of Einstein field equation out of the classical General Relativity [1]. Let $\mathcal{H}$ be a Hilbert space. Let (M^n, g) be a manifold, where M^n is an n-dimensional differentiable manifold and g is a metric, which is either as a positive-definite section of the bundle of symmetric (covariant) 2-tensors $T^*M \otimes_S T^*M$ or as positive-definite bilinear maps, $\mathrm{g}\left((i\hbar)^{-1}x\right) : T_{\left((i\hbar)^{-1}x\right)}M \times T_{\left((i\hbar)^{-1}x\right)}M \to \mathcal{H}$ for all $(i\hbar)^{-1}x \in M$. Here, $T^*M \otimes_S T^*M$ is the subspace of $T^*M \otimes T^*M$ generated by elements of the form $X \otimes Y + Y \otimes X$. Let $\{(i\hbar)^{-1}x^i\}_{i=1}^n$ be local coordinates in a neighborhood U of some point of M. In U the vector fields $\left\{\left(i\hbar\,\vec{\nabla}_i\right)^{-1}\right\}_{i=1}^n$ form a local basis for TM and the 1-forms $\left\{i\hbar\,\vec{\nabla}_i\right\}_{i=1}^n$ form a dual basis for T^*M, that is, $i\hbar\,\vec{\nabla}_j\left(i\hbar\,\vec{\nabla}_i\right)^{-1} = \delta^i_j$. The metric may then be written in local coordinates as $\mathrm{g} = \mathrm{g}_{ij}\left(i\hbar\,\vec{\nabla}_i\right) \otimes \left(i\hbar\,\vec{\nabla}_j\right)$. Let ∇^{g} denote the Levi-Civita connection of the metric g. The Christoffel symbols are the components of the Levi-Civita

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connection and are defined in U by $\nabla_{(i\hbar\vec{\nabla}_i)}\left(i\hbar\vec{\nabla}_j\right) \doteqdot \Gamma^k_{ij}\left(i\hbar\vec{\nabla}_k\right)$, and for $\left[\left(i\hbar\vec{\nabla}_i\right),\left(i\hbar\vec{\nabla}_j\right)\right] = 0$, we see that they are given by,

$$\Gamma^k_{ij}\,\psi(\vec{r},t) \quad = \quad \frac{1}{2}\,\mathrm{g}^{k\ell}\left[\left(i\hbar\vec{\nabla}_i\right)\mathrm{g}_{j\ell} + \left(i\hbar\vec{\nabla}_j\right)\mathrm{g}_{i\ell} - \left(i\hbar\vec{\nabla}_\ell\right)\mathrm{g}_{ij}\right]\psi(\vec{r},t)\,. \tag{1}$$

Let the curvature $(3,1)$-tensor Rm is defined by, $\mathrm{Rm}(X,Y)Z \doteqdot \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. Thus, the curvature tensor, $R^\ell_{ijk} = \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \Gamma^p_{jk}\Gamma^\ell_{ip} - \Gamma^p_{ik}\Gamma^\ell_{jp}$, is purely Quantum Mechanical due to (1). Let the tensor Rc is the trace of Rm curvature tensor: $\mathrm{Rc}(Y,Z) \doteqdot \mathrm{trace}\,(X \mapsto \mathrm{Rm}(X,Y)Z)$, defined by $R_{ij} \doteqdot \mathrm{Rc}\left(\left(i\hbar\vec{\nabla}_i\right),\left(i\hbar\vec{\nabla}_j\right)\right)$, and the scalar curvature R is the trace of Rc tensor: $\mathrm{R} \doteqdot \sum_{a=1}^{n}\mathrm{Rc}(e_a,e_a)$ where $e_a \in T_{((i\hbar)^{-1}x)}M^n$ is a unit vector spanning $L \subset T_{((i\hbar)^{-1}x)}M^n$. Then, the Einstein-like purely quantum tensor $\mathrm{Rc} - \frac{1}{2}\,\mathrm{g}\,\mathrm{R}$ directly acts on a quantum space. Thus, Einstein-like field equation $\left[\mathrm{Rc} - \frac{1}{2}\,\mathrm{g}\,\mathrm{R}\right]\psi(\vec{r},t) = (i\hbar)^2\,8\pi\,G\,\mathrm{T}\,\psi(\vec{r},t)$, is now "purely" Quantum Mechanical for (1). But the Ricci tensor $R_{ij} \doteqdot \mathrm{Rc}\left(\left(i\hbar\vec{\nabla}_i\right),\left(i\hbar\vec{\nabla}_j\right)\right) \doteqdot (i\hbar)^2\,\mathrm{Rc}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$, thus, Einstein field equation (in quantum spacetime) should become as,

$$\begin{aligned} \left[\mathrm{Rc} - \frac{1}{2}\,\mathrm{g}\,\mathrm{R}\right]\psi(\vec{r},t) &= (i\hbar)^2\,8\pi\,G\,\mathrm{T}\,\psi(\vec{r},t)\,, \\ \therefore\quad (i\hbar)^2\left[\mathrm{Rc} - \frac{1}{2}\,\mathrm{g}\,\mathrm{R}\right]\psi(\vec{r},t) &= (i\hbar)^2\,8\pi\,G\,\mathrm{T}\,\psi(\vec{r},t)\,, \\ \Rightarrow\quad \left[\mathrm{Rc} - \frac{1}{2}\,\mathrm{g}\,\mathrm{R}\right] &= 8\pi\,G\,\mathrm{T}\,, \end{aligned} \tag{2}$$

where $\left[\mathrm{Rc} - \frac{1}{2}\,\mathrm{g}\,\mathrm{R}\right]$ is Einsteinian and non-renormalizable. In the first line of (2), all mass dimensions vanish due to $\hbar^2\,G$ and if divergences are to be present, they could now be disposed of by the technique of renormalization (though, this will not play a role in our present discussion). Hence, (2) should be used as a renormalizable Einstein field equation (in quantum spacetime) for general purposes. But (2) is not sufficient to predict the properties of particles (baryonic and non-baryonic) to a great extent, such as, it never helps us to represent any kind of particle interactions, etc. So, we need to develop General Quantum Gravity (GQG), which is a formalization of quantized gravity that emerges from General Relativity through Quantum Mechanics. In the basic formalisms of GQG, we are going to develop two different aspects of GQG, such as:

1. GQG in Semi-Quantum Minkowski Spacetime, and
2. GQG in Quantum Non-Minkowski Spacetime.

In the first case, GQG gives us an Einstein field equation in a Semi-Quantum Minkowski Spacetime with Semi-Quantum Lorentzian metric tensor, whereas, the second one yields a purely Quantum Mechanical Einstein field equation in a Quantum Non-Minkowski Spacetime with Non-Lorentzian metric tensor. But in both cases, we always get the Schrödinger equation as a byproduct, though, unlike its classical form, this Schrödinger equation is now in a $(3+1)D$ quantum spacetime. Additionally, this second case formalism helps us to explain Superstring/M-theory from a completely different geometric perspective.

Basics of these two different aspects of GQG are discussed as follows:

2.1. GQG in Semi-Quantum Minkowski Spacetime

Let the line element of Minkowski spacetime,

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \equiv \frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu, \tag{3}$$

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where $P^{\mu} = m\,\mathrm{v}^{\mu}$ is the 'Four-momentum', hence $P^{\mu}\,\mathrm{v}^{\nu} \equiv p^{0}\,\mathrm{v}^{0} + p^{i}\,\mathrm{v}^{j}$ for $i, j = 1, 2, 3$, and $\mu, \nu = 0, 1, 2, 3$, whereas v^{μ} is the 'four-velocity'. Be careful that it is $P^{\mu}\,\mathrm{v}^{\nu} \neq p^{0}\,\mathrm{v}^{0} - p^{i}\,\mathrm{v}^{j}$, because $g_{\mu\nu}$ takes the '$-$' sign. Also note that $m \neq m_0$ in (3) for the rest mass m_0. Thus, (3) gives us an energy-momentum invariant line element as,

$$m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = mc^2 - m\sum\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = E - \sum p^i\,\mathrm{v}^j = p^0\,\mathrm{v}^0 - \sum p^i\,\mathrm{v}^j = g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}, \tag{4}$$

when $T = \frac{1}{2}\,m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2$ is the kinetic energy [2], that is, we have a new line element as,

$$\mathrm{d}\,\mathcal{S}_{\mathrm{v}}^2 = m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^0\,\mathrm{v}^0 - \sum p^i\,\mathrm{v}^j = g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}, \tag{5}$$

then, the rearrangement of (4) gives the following equation by using (3) as,

$$E = p^i\,\mathrm{v}^j + m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^i\,\mathrm{v}^j + P^{\mu}\,\mathrm{v}^{\nu}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\mu}}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\nu}} = p^i\,\mathrm{v}^j + P^{\mu}\,\mathrm{v}^{\nu}\,g_{\mu\nu}\,. \tag{6}$$

Let us consider the representation of a wave field $\psi(\vec{r}, t)$ by superposition of a free particle (de Broglie wave) for (6) as follows,

$$\begin{aligned}\psi(\vec{r}, t) &= \frac{1}{(2\pi\,\hbar)^2}\,\exp\left[\frac{i}{\hbar}\left\{\left(\vec{p}\cdot\vec{r} + g_{\mu\nu}\,\vec{P}\cdot\vec{R}\right) - Et\right\}\right] \\ &\equiv \frac{1}{(2\pi\,\hbar)^2}\,\exp\left[\frac{i}{\hbar}\left\{\left(\vec{p}\cdot\vec{r} + m\,t\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right) - Et\right\}\right],\end{aligned} \tag{7}$$

where $\vec{P} \to \vec{P}^{\,\mu}$ and $\vec{R} \to \vec{r}^{\,\mu}$, as $\vec{r}^{\,\mu} = (\mathrm{v}^{\mu}t)$, in addition, note it that we have taken here E as total energy for (6). Thus, from (7), we can get the (total) energy operator $\hat{E} \to i\,\hbar\,\partial_t$ (it is analogous with, but not exactly the same as, the classical Quantum Mechanics, as it is now the total energy and related to $(3+1)D$ instead of $3D$ due to the presence of $g_{\mu\nu}$ in (7) in either ways), the three momentum operator $\hat{\boldsymbol{p}} \to -\,i\,\hbar\,\vec{\nabla}_i$, the 'Four-momentum' operator,

$$\begin{aligned} &\hat{\boldsymbol{\mathcal{P}}}^{\mu} \to i\,\hbar\,\vec{\nabla}_{\mu} = \left\{i\,\hbar\,\frac{\partial}{\partial(ct)},\,-\,i\,\hbar\,\frac{\partial}{\partial x^i}\right\} = \left(\frac{1}{c}\,\hat{E},\,\hat{\boldsymbol{p}}\right), \\ \text{thus,}\quad &\hat{\boldsymbol{\mathcal{P}}}_{\mu} \to -\,i\,\hbar\,\vec{\nabla}_{\mu} = \left\{-\,i\,\hbar\,\frac{\partial}{\partial(i\,ct)},\,-\,i\,\hbar\,\frac{\partial}{\partial x^i}\right\} = \left(\frac{i}{c}\,\hat{E},\,\hat{\boldsymbol{p}}\right),\end{aligned} \tag{8}$$

and the mass operator,

$$\hat{m} \to -\,i\,\hbar\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\frac{\partial}{\partial t} \equiv -\,i\,\hbar\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)\frac{\partial}{\partial s}, \tag{9}$$

where, $\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)$ is evidently relativistic, but $\frac{\partial}{\partial s}$ is no doubt Quantum Mechanical, so as,

$$\begin{aligned} s &= \int_a^u \left(g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\right)^{1/2}\mathrm{d}u\,, \\ \text{and}\quad \frac{\partial}{\partial s} &= \left\{\frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^i\partial x^j}\right\}^{1/2} = \left\{g^{\mu\nu}\,\partial_{\mu}\,\partial_{\nu}\right\}^{1/2} = \Box^{1/2}.\end{aligned} \tag{10}$$

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For constant velocity, we can develop an uncertainty principle describing the intrinsic indeterminacy with which m and s can be determined as,

$$\Delta m \, \Delta s \geq \frac{\hbar}{2} .$$

The mass-energy relation, i.e., $E = m\,c^2$, of (4) yields the Quantum Mechanical definition for the mass operator $\hat{m}$ of (9) for (8) and (10) as,

$$\begin{aligned} i\,\hbar \frac{\partial}{\partial t}\,\psi(\vec{r},t) &= -\,i\,\hbar\,c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) \frac{\partial}{\partial s}\,\psi(\vec{r},t) = -\,c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) \hat{\boldsymbol{\mathcal{P}}}^{\mu}\,\psi(\vec{r},t) \\ &= c\,\gamma\,\hat{\boldsymbol{\mathcal{P}}}_{\mu}\,\psi(\vec{r},t)\,, \end{aligned} \tag{11}$$

since $c\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) = 1/\sqrt{1-\frac{v^2}{c^2}} = \gamma$. Thus, (11) tells us that the total energy of a system directly relates to its geometry, more precisely, to $\partial/\partial s$. Additionally, (11) yields the following for (10) and $c = 1$,

$$\begin{aligned} \mathrm{d}s\,\psi(\vec{r},t) &= -\,i\,\hbar\left(\frac{\mathrm{d}t}{\hat{E}}\right)\frac{\partial}{\partial s}\,\psi(\vec{r},t)\,, \\ \text{hence,}\quad g_{\mu\nu}\,(\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu})^{1/2}\,\psi(\vec{r},t) &= -\,i\,\hbar\left(\frac{\mathrm{d}t}{\hat{E}}\right)(\partial_{\mu}\,\partial_{\nu})^{1/2}\,\psi(\vec{r},t)\,, \\ \therefore\quad g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\psi(\vec{r},t) &= -\,i\,\hbar\left(\frac{\mathrm{d}t}{\hat{E}}\right)\frac{\partial}{\partial x^{\mu}}\,\psi(\vec{r},t)\,,\quad \text{as } \mu=\nu \\ &= \left(\frac{\mathrm{d}t}{\hat{E}}\right)\hat{\boldsymbol{\mathcal{P}}}_{\mu}\,\psi(\vec{r},t)\,, \\ \Rightarrow\quad g_{\mu\nu}\,P^{\mu}\,\psi(\vec{r},t) &= \left(\frac{m}{\hat{E}}\right)\hat{\boldsymbol{\mathcal{P}}}_{\mu}\,\psi(\vec{r},t) \\ &= \hat{\boldsymbol{\mathcal{P}}}_{\mu}\,\psi(\vec{r},t)\,,\quad \text{since, } c=1\,. \end{aligned} \tag{12}$$

For the second term of $\hat{m}$ in (9), we can rewrite (11) by squaring its both sides after considering $c = 1$ as follows,

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \left(i\,\hbar\,\frac{\partial}{\partial t}\right)^2 \psi(\vec{r},t) = -\,\hbar^2\,\frac{\partial^2}{\partial s^2}\,\psi(\vec{r},t)\,, \tag{13}$$

which clarifies more precisely that $(\mathrm{d}s/\mathrm{d}t)^2$ is evidently relativistic, but $\partial^2/\partial s^2$ is definitely Quantum Mechanical. Thus, (13), so as (12), are very peculiar equations where the LHS spacetime is classical relativistic but the RHS spacetime is Quantum Mechanical, and the total energy is directly related to the RHS spacetime. If the LHS spacetime of (13) changes (not more than $(3+1)D$ and not less than $(1+1)D$, unless it is a vacuum state) and the total energy remains fixed (so as time), then the spacetime of RHS should not remain as same as before, but changes inversely against the LHS spacetime. Though, the increment of RHS spacetime should not be observable, i.e., all extra dimensions would have to remain hidden inside the overall observable system, in other words, inside the LHS observable spacetime of (13). We will discuss it below in more details very soon.

For (10), we can rewrite the mass-energy relation (13) as follows,

$$\hbar^2\,\Box\,\psi(\vec{r},t) + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \left(i\,\hbar\,\frac{\partial}{\partial t}\right)^2 \psi(\vec{r},t) = 0\,.$$

Thus, it is somehow a kind of Klein-Gordon-like (but not an exactly similar) equation of the relativistic waves due to the above quantum scenario derived from (7). (By the way, we will see at the very

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end of this Subsection that Klein-Gordon equation is a subset of a Second Order Equation of GQG in Semi-Quantum Minkowski Spacetime). This equation may yield the first order equation as,

$$\left[i\hbar\,\gamma^{\mu}\,\vec{\nabla}_{\mu} - \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)\,\left(i\hbar\,\frac{\partial}{\partial t}\right)\right]\psi(\vec{r},t) = 0\,,$$

where, γ^{μ} are Dirac's gamma matrices.

Remark 1. *Definitely, the above quantum scenario derived from (7) is analogous to, but not exactly similar to, the classical Quantum Mechanics since E is taken as total energy (and m is not rest mass) in (6). So, an expectation of the exactness between classical Quantum Mechanics and the present quantum scenario must lead a confusion and may yield wrong or faulty conclusions in a great extent. Readers are requested to be careful about it.*

The wave field $\psi(\vec{r},t)$ in (7) must satisfy the eigenfunctions for a discrete Lorentz transformation as,

$$\begin{aligned}
\Psi &= \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i = \frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i^{\dagger} = -\Psi^{\dagger}\,, \\
\Psi^{\dagger} &= \frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i = \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i^{\dagger} = -\Psi\,,
\end{aligned} \tag{14}$$

when $\psi^{\dagger}$ is the complex conjugate of ψ. Then, using summation convention and (10), we can write the joint state of both spacetimes as,

$$\begin{aligned}
\left.\frac{\partial^2}{\partial s^2}\Psi\right|_{s_0} &= \left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i\partial x^j}\right\}\left\{\frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\psi_i\right\}\Bigg|_{x_0^{\mu}} \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[\psi_0-\psi_i] - \frac{\partial^2}{\partial x^i\partial x^j}[\psi_0-\psi_i]\right\}\Bigg|_{x_0^{\mu}} \\
&= \left(\frac{\partial^2}{\partial t^2}\Psi - \frac{\partial^2}{\partial x^i\,\partial x^j}\Psi\right)\Bigg|_{x_0^{\mu}} = g^{\mu\nu}(s)\,\partial_{\mu}\,\partial_{\nu}\Psi\Big|_{x_0^{\mu}}\,.
\end{aligned} \tag{15}$$

But, (15) also intends to,

$$\begin{aligned}
\left.\frac{\partial^2}{\partial s^2}\Psi\right|_{s_0} &= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[\psi_0-\psi_i] - \frac{\partial^2}{\partial x^i\partial x^j}[\psi_0-\psi_i]\right\}\Bigg|_{x_0^{\mu}} \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[\psi_0] - \frac{\partial^2}{\partial x^i\,\partial x^j}[-\psi_i]\right\}\Bigg|_{x_0^{\mu}} + \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_i] - \frac{\partial^2}{\partial x^i\,\partial x^j}[\psi_0]\right\}\Bigg|_{x_0^{\mu}} \\
&= \frac{1}{\sqrt{2}}\left\{g^{\mu\nu}(\rho)\,\partial_{\mu}\partial_{\nu}\,[\psi_0-\psi_i]\right\}\Big|_{x_0^{\mu}} + \frac{1}{\sqrt{2}}\left\{\widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_{\mu}\widehat{\partial}_{\nu}\,[\psi_0-\psi_i]\right\}\Big|_{x_0^{\mu}} \\
&= g^{\mu\nu}(\rho)\,\partial_{\mu}\partial_{\nu}\,\Psi\Big|_{x_0^{\mu}} + \widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_{\mu}\widehat{\partial}_{\nu}\,\Psi\Big|_{x_0^{\mu}} = \rho\,\Psi\Big|_{x_0^{\mu}} + \Gamma\,\Psi\Big|_{x_0^{\mu}}\,,
\end{aligned} \tag{16}$$

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for $\partial_\mu^2 = \widehat{\partial}_\mu^2 = \left[\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial x^i \partial x^j}\right]$ and $g_{\mu\nu} = \widehat{g}_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$, where $\widehat{\partial}$ implies ∂_t's dependency on ψ_i and ∂_x's dependency on ψ_0, respectively. Here, ρ and Γ have been chosen arbitrarily. Hence, the complex conjugate of (16) is,

$$\begin{aligned} \frac{\partial^2}{\partial s^2}\Psi^\dagger\bigg|_{s_0} &= \left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i \partial x^j}\right\}\left\{\frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\psi_i\right\}\bigg|_{x_0^\mu} \\ &= g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\,\Psi^\dagger\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,\Psi^\dagger\Big|_{x_0^\mu} = \rho\,\Psi^\dagger\Big|_{x_0^\mu} + \Gamma\,\Psi^\dagger\Big|_{x_0^\mu}. \end{aligned} \tag{17}$$

But, if we take, $\mathrm{d}s^{-2}\varphi = (\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j)^{-1}\varphi$, where φ is an operator, we can say that, $(\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j)^{-1} \neq \mathrm{d}t^{-2} - \sum(\mathrm{d}x^i\,\mathrm{d}x^j)^{-1}$, so, we may assume without any objection that, $(\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j)^{-1} \equiv \frac{\partial^2}{\partial t^2} - \sum \frac{\partial^2}{\partial x^i \partial x^j} - \Pi$, for some value of Π. Thus,

$$\begin{aligned} \mathrm{d}s^{-2}\varphi &= (\mathrm{d}t^2 - \mathrm{d}x^i\,\mathrm{d}x^j)^{-1}\varphi = \left\{\frac{\partial^2\varphi}{\partial t^2} - \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right)\right\} - \Pi\varphi \\ &= \frac{\partial^2}{\partial s^2}\varphi - \Pi\varphi = g^{\mu\nu}\,\partial_\mu\,\partial_\nu\varphi - \Pi\varphi\,, \end{aligned} \tag{18}$$

thus, from (16), if $g^{\mu\nu}\,\partial_\mu\partial_\nu \equiv g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu$, we can write as,

$$\frac{\partial^2}{\partial s^2}\Psi\bigg|_{s_0} - \Gamma\,\Psi\Big|_{x_0^\mu} \quad\Longleftrightarrow\quad \mathrm{d}s^{-2}\varphi + \Pi\varphi\,,$$

this yields, $\frac{\partial^2}{\partial s^2}\Psi\,\big|_{s_0} \equiv \mathrm{d}s^{-2}\varphi$ if $\Gamma \to -\Pi$, which gives the equivalency of (16) and (18). No more combinations are possible from (16) apart from ρ, Γ and (15) itself. The arrangement of ρ and Γ implies that $g_{\mu\nu}(s) = \frac{1}{2}\big(g_{\mu\nu}(\rho) + \widehat{g}_{\mu\nu}(\Gamma)\big)$. Thus, (16) should be rewritten by using (15), (16) and additionally replacing $g_{\mu\nu}(s)$ with $g_{\mu\nu}(s) = \frac{1}{2}\big(g_{\mu\nu}(\rho) + \widehat{g}_{\mu\nu}(\Gamma)\big)$ as follows,

$$\frac{1}{2}\big(g^{\mu\nu}(\rho) + \widehat{g}^{\mu\nu}(\Gamma)\big)\,\partial_\mu\,\partial_\nu\,\Psi\Big|_{x_0^\mu} = g^{\mu\nu}(\rho)\,\partial_\mu\,\partial_\nu\Psi\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\,\widehat{\partial}_\nu\Psi\Big|_{x_0^\mu}. \tag{19}$$

Note that, Π exclusively has to depend upon spacetime. Comparing (16) with (18), let us say that,

$$\frac{\partial^2}{\partial s^2}\Psi\bigg|_{s_0} = \rho\Psi\Big|_{x_0^\mu} - \Pi\Psi\Big|_{x_0^\mu}, \tag{20}$$

where $\Gamma \to -\Pi$, suppose. Since $g_{\mu\nu}(s) \neq \frac{1}{2}\big(g_{\mu\nu}(\rho) - \widehat{g}_{\mu\nu}(\Pi)\big) \equiv 0$ as long as $g_{\mu\nu} = \widehat{g}_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$, then it must be as follows, if $g_{\mu\nu}(s) \neq 0$,

$$\frac{1}{2}\big(g^{\mu\nu}(\rho) + \widehat{g}^{\mu\nu}(\Pi)\big)\,\partial_\mu\,\partial_\nu\,\Psi\Big|_{x_0^\mu} = g^{\mu\nu}(\rho)\,\partial_\mu\,\partial_\nu\Psi\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\,\widehat{\partial}_\nu\,[-\Psi]\Big|_{x_0^\mu}. \tag{21}$$

It is impossible to decompose (19) and (21) since both of their LHS are only depended upon ∂, thus,

1. The spacetimes of ρ and Γ (so as Π) are not easily dissociative even upto a very high energy scale.
2. Since $g_{\mu\nu}(s)$ is independent of $\widehat{\partial}$, the spacetime of Γ (so as Π) must be an internal hidden property of the overall system (in other words, inside the observable spacetime of ρ). The observable spacetime is always ∂-dependent.

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But, the RHS of (21) gives us,

$$\{|\Psi\rangle \in V \otimes V : F|\Psi\rangle = |\Psi\rangle\} = \mathrm{Sym}^2\, V\,,$$
$$\{|\Psi\rangle \in V \otimes V : F|\Psi\rangle = -|\Psi\rangle\} = \mathrm{Anti}^2\, V\,,$$

where, the swap operator $F|\Psi\rangle = \exp\,[i\,\theta]$ for some phase $\exp\,[i\,\theta]$, whereas V is a vector space. Then the corresponding eigenspaces are called the symmetric and antisymmetric subspaces and are denoted by the state spaces $\mathrm{Sym}^2\, V$ and $\mathrm{Anti}^2\, V$, respectively. Note that, we have not intended here that ρ and Π individually are two indistinguishable particles for the state spaces $\mathrm{Sym}^2\, V$ and $\mathrm{Anti}^2\, V$; the above equations are just the generalization forms of their kinds, because ρ and Π do not have distinguished (opposite) spins until otherwise they are dissociated as free particles; so, the observable spin is always the spin of ρ, since Π is an internal hidden property of the overall system and the observable spacetime is always ∂-dependent. Thus, (21) tells us that, if we allow Π to be dissociated as a free particle at very high energy, the internal hidden spacetime of Π then must be transformed into a fermionic particle, whereas, the overall ∂-dependent system remains bosonic, since, the observable spacetime is always ∂-dependent.

Similarly, (17) yields,

$$\begin{aligned} \frac{\partial^2}{\partial s^2}\Psi^\dagger\bigg|_{s_0} &= \rho\,\Psi^\dagger\Big|_{x_0^\mu} - \Pi\,\Psi^\dagger\Big|_{x_0^\mu}\,, \\ \therefore\quad \frac{1}{2}\big(g^{\mu\nu}(\rho) + \widehat{g}^{\mu\nu}(\Pi)\big)\,\partial_\mu\partial_\nu\,[-\Psi]\,\Big|_{x_0^\mu} &= g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\,[-\Psi]\,\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\Psi\Big|_{x_0^\mu}\,, \end{aligned} \tag{22}$$

this tells us that the internal hidden spacetime of Π must be now bosonic, whereas, the overall system is fermionic, since, the observable spacetime is always ∂-dependent. So, whatever (21) and (22) want to tell us is that the ∂-dependent overall system has Supersymmetry and since the spacetimes of ρ and its supersymmetric partner Π are not easily dissociative even upto a very high energy scale; thus, Π must require extremely high energy to dissociate itself from the overall system as a free particle. Instead of being a free supersymmetric partner, Π actually works quite differently inside of the observable spacetime ρ, though, at the same time, Π is still satisfying all the properties of Supersymmetry. We will show you Π's actual purpose very soon in the below. But Supersymmetry needs extra dimensions and we should have to discuss it now.

Before proceeding with anything, we can develop a $\widehat{\partial}$-dependent scenario as follows,

$$\begin{aligned} \frac{\widehat{\partial}^2}{\widehat{\partial} s^2}\Psi\bigg|_{s_0} &= \Pi\Psi\Big|_{x_0^\mu} - \rho\Psi\Big|_{x_0^\mu}\,, \\ \therefore\quad \frac{1}{2}\big(\widehat{g}^{\mu\nu}(\Pi) + g^{\mu\nu}(\rho)\big)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,\Psi\Big|_{x_0^\mu} &= \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\Psi\Big|_{x_0^\mu} + g^{\mu\nu}(\rho)\,\partial_\mu\,\partial_\nu\,[-\Psi]\,\Big|_{x_0^\mu}\,, \end{aligned} \tag{23}$$

and the complex conjugate of Ψ is,

$$\begin{aligned} \frac{\widehat{\partial}^2}{\widehat{\partial} s^2}\Psi^\dagger\bigg|_{s_0} &= \Pi\,\Psi^\dagger\Big|_{x_0^\mu} - \rho\,\Psi^\dagger\Big|_{x_0^\mu}\,, \\ \therefore\quad \frac{1}{2}\big(\widehat{g}^{\mu\nu}(\Pi) + g^{\mu\nu}(\rho)\big)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,[-\Psi]\,\Big|_{x_0^\mu} &= \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,[-\Psi]\,\Big|_{x_0^\mu} + g^{\mu\nu}(\rho)\,\partial_\mu\,\partial_\nu\Psi\Big|_{x_0^\mu}\,. \end{aligned} \tag{24}$$

It is not important which state spaces satisfy such bosonic or fermionic representations of (23) and (24), here, the most important thing is that the overall system as a free observable particle is must not be baryonic because now only the internal hidden spacetime of ρ has 'proper' spacetime arrangement for its ∂-dependency, whereas, the overall (observable) system's spacetime arrangement is quite 'improper'

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as it is $\widehat{\partial}$-dependent. Despite of ρ's ∂-dependency, here, being a supersymmetric partner, if it is allowed to be free at very high energy, it must not be baryonic either. We should not be confused with it. We will discuss about its property in Section 4 below.

The internal hidden spacetime of Π in (21) and (22) also provides us some additional geometry for its $\widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,[\pm\Psi]$ structures. Suppose, for $\widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,[\Psi]$, we have, $\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\,[-\psi_i]-\frac{\partial^2}{\partial x^i\,\partial x^j}\,[\psi_0]\right\}$, where, neither spacetime arrangements are matched with one another in either combinations within the curvey brackets. These 'wrong' arrangements must have a noticeable effect on the acceptable spacetime, i.e., its temporal part must influence over the spatially depended ψ_i, or its spatial part must influence over the temporally depended ψ_0, or vice versa. In other words, the acceptable spacetime should not have to be four-dimensional in this case. Let us check it. Suppose, for $\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\,[-\psi_i]-\frac{\partial^2}{\partial x^i\,\partial x^j}\,[\psi_0]\right\}$, we can consider a dimension function (see [3]; we follow Nagata's work throughout this paragraph and all the proofs for this paragraph are easily obtainable by using his book [3]),

$$\dim\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\;\forall\, i\in\{1,2,3\}\;, \tag{25}$$

and the space $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ satisfies a normal T_1-space. Let $\mathcal{U}$ be a collection in an initially $(3+1)D$ topological spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$, i.e., $\dim\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)=(3+1)$, which is actually hidden inside an observable $(3+1)D$ spacetime, i.e., $\dim\left(\partial^2_{[0,i]}\otimes\Psi_{[0,i]}\right)=(3+1)$ (be careful here about the subscripts, do not confuse observable spacetime with hidden spacetime), and p a point of $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$, then the order of $\mathcal{U}$ at p should be denoted by, $\operatorname{ord}\mathcal{U}=\sup\left\{\operatorname{ord}_p\mathcal{U}\mid p\in\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\right\}$, where, $\operatorname{ord}_p\mathcal{U}$ is the number of members of $\mathcal{U}$ which contain p. If for any finite open covering $\mathcal{U}$ of the topological spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ there exists an open covering $\mathfrak{B}$ such that $\mathfrak{B}<\mathcal{U}$, $\operatorname{ord}\mathfrak{B}\leq n+1$, then $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ has covering dimension $\leq n$, i.e., $\dim\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\leq n$. If $\mathcal{U}$ can be decomposed as $\mathcal{U}=\bigcup_{i=1}^{\infty}\mathcal{U}_i$ for locally finite (star-finite, discrete, etc.) collections $\mathcal{U}_i$, then $\mathcal{U}$ is called a σ-locally finite (σ-star-finite, σ-discrete, etc.) collection. The topological spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ has strong inductive dimension -1, i.e., $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)=-1$, if $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)=\emptyset$. If for any disjoint closed sets F and G of the topological spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ there exists an open set U such that $F\subset U\subset\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)-G$, and $\operatorname{Ind}B(U)\leq n-1$, where $B(U)$ denotes the boundary of U, then $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ has strong inductive dimension $\leq n$, i.e., $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\leq n$. Let V is an open set and $\bar{V}$ is a closed set of $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$. If $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\leq n$, then there exists a σ-locally finite open basis $\mathfrak{B}$ of $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ such that, $\operatorname{Ind}B(V)\leq n-1$ for every $V\in\mathfrak{B}$. If a spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ has a σ-locally finite open basis $\mathfrak{B}$ such that, $B(V)=\emptyset$ for every $V\in\mathfrak{B}$, then $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\leq 0$. Again, $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\leq n$ if and only if there exists a σ-locally finite open basis $\mathfrak{B}$ such that $\operatorname{Ind}B(V)\leq n-1$ for every $V\in\mathfrak{B}$. For every subset $A=\bigcup\{B(V)\mid V\in\mathfrak{B}\}$, for any integer $n\geq 0$, of a spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$, we have, $\operatorname{Ind}A\leq\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$. Hence, if and only if $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)=\bigcup_{i=1}^{n+1}A_i$ for some $n+1$ subsets A_i with $\operatorname{Ind}A_i\leq 0$, $i=1,\ldots,n+l$. For the spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$, we have then $\dim\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)=\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$. Let A be a subset of a spacetime $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ and I the unit segment. If U is an open set of the topological product $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)\times I$ such that $U\supset A\times I$, then there exists an open set V of $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ such that $A\subset V$, and $V\times I\subset U$. Let F be a closed set of $\left(\widehat{\partial}^2_{[0,i]}\otimes\Psi_{[i,0]}\right)$ with $\dim F\leq n$. Let F_α and U_α,

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$\alpha < \tau$, be closed and open sets, respectively such that $F_\alpha \subset U_\alpha$, and $\{U_\alpha \mid \alpha < \tau\}$ is locally finite. Then there exist open sets V_α satisfying $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\dim B_k \leq n-k$, $k = 1, \ldots, n+1$, where, $B_k = \{p \mid p \in F, \text{ord}_p B(\mathfrak{B}) \geq k\}$, and $\mathfrak{B} = \{V_\alpha \mid \alpha < \tau\}$. Let F, F_α and U_α satisfy the same condition as above, then there exist open sets V_α, W_α, $\alpha < \tau$ satisfying, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset W_\alpha \subset U_\alpha$, and $\text{ord}_p \{W_\alpha - \bar{V}_\alpha \mid \alpha < \tau\} \leq n$ for every $p \in F$. Let G_k, $k = 0, \ldots, n$, be closed sets with $\dim G_k \leq n-k$ of the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Let $\{F_\alpha \mid \alpha < \tau\}$ be a closed collection and $\{U_\alpha \mid \alpha < \tau\}$ a locally finite open collection such that $F_\alpha \subset U_\alpha$. Then there exists an open collection, $\mathfrak{B} = \{V_\alpha \mid \alpha < \tau\}$, such that, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\text{ord}_p B(\mathfrak{B}) \leq n-k$ for every $p \in G_k$. A mapping f of the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ into a spacetime S is a closed (open) mapping if the image of every closed (open) set of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ is closed (open) in S. Then the continuous mappings which lower dimensions of the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ should be defined as follows,

Theorem 1. *Let f be a closed continuous mapping of the $(3+1)D$ spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ onto the spacetime S such that $\dim f^{-1}(q) \leq k$ for every $q \in S$. Then,*

$$dim \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq dim\, S + dim\, K\,, \tag{26}$$

where $\dim K \leq k$ for the space K, when $0 < \dim S \leq 2$, since i should not be zero in (25).

Proof. Using Theorem III.6 of [3], we can easily prove this theorem. □

Since, the temporal axis is unaltered in Lorentz transformation, as we have already seen it in (14), we can express the maximal continuous mapping of the $(3+1)D$ spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ onto the spacetime S of (26) as,

$$\begin{aligned} X^\mu(\tau,\sigma) \quad &\leq \quad \left\{\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \mapsto S \;\middle|\; S \leq \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)\right\}, \\ \text{thus,} \quad \mathfrak{X}^\mu \quad &\leq \quad S \cup K \text{ inside the } (3+1)D \text{ observable spacetime } \left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right), \end{aligned}$$

since i should not be zero in (25), if the considered state is not vacuum; then the spacetime S definitely intends the basic structure of a 2-dimensional worldsheet $X^\mu(\tau,\sigma)$ with the joint states, $\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial\tau^2}[-\psi_\sigma] - \frac{\partial^2}{\partial\sigma^2}[\psi_\tau]\right\}$ for the spacetime $\mathfrak{X}^\mu$, where S is a $(1+1)D$ spacetime, but for the space K, we will like to discuss it below in more details in the Theorem 2. Obviously, a string can sweep out the 2-dimensional worldsheet $X^\mu(\tau,\sigma)$ for the spacetime $\mathfrak{X}^\mu$.

If the internal hidden spacetime of Π is considered as the LHS spacetime of (13) and let it to be changed from $(3+1)D$ to $(1+1)D$ when its total energy remains fixed (so as its time), then the spacetime $\mathfrak{X}^\mu$ of RHS of (13) changes inversely against the spacetime of Π. Since the spacetime of Π is hidden inside the overall system of (21), i.e., in other words, inside the observable spacetime of ρ, then the increment of RHS spacetime $\mathfrak{X}^\mu$ of (13) should not be observable by any means, i.e., the extra dimensions of $\mathfrak{X}^\mu$ remain hidden forever inside the observable spacetime of ρ. As these internal hidden extra dimensions inside the observable spacetime of ρ are considered as the representation of the spacetime S and the space K, thus, we can conclude,

1. Strings (i.e., the spacetime S for the hidden spacetime $\mathfrak{X}^\mu$) are natural and universal but forever hidden inside every $(3+1)D$ observable system, i.e., the spacetime $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$, in Quantum Mechanics.
2. Every $(3+1)D$ observable system in Quantum Mechanics must contain forever hidden extra dimensions (i.e., the space K for the hidden spacetime $\mathfrak{X}^\mu$) independent of any external observer whether she/he considers any string in this system or not (for more details, see (27) below and its following text therein).

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But the space K should raise more extra hidden dimensions by a closed continuous mapping beyond $\dim K \leq k$ by adopting the following,

Theorem 2. *Let f be a closed continuous mapping of a space R onto a space K such that for each point q of K, $B(f^{-1}(q))$ contains at most $m+1$ points $(m \geq 0)$; then $\dim K \leq \dim R + \dim M$, when $\dim R \leq \left[dim\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - dim\, S\right]$ and $\dim M \leq m$, where $dim\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq (3+1)$.*

Proof. Using Theorem III.7 of [3], we can easily prove this theorem. □

Then, we can say for the overall spacetime $\mathfrak{X}^{\mu}_{\text{OVERALL}}$ that,

$$\mathfrak{X}^{\mu}_{\text{OVERALL}} \leq \left\{ \left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right) \cup S \cup K \,\middle|\, S \leq \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right) \text{ and } K \leq \left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - S\right] \cup M \;\forall\; \dim M \leq m \;\exists\; m \geq 0 \right\}, \tag{27}$$

for which,

$$\mathfrak{X}^{\mu} \leq S \cup K \leq \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \cup M \quad \forall \quad \dim M \leq m \;\exists\; m \geq 0 \text{ inside the } (3+1)D \text{ observable spacetime } \left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right). \tag{28}$$

Note here that stringy spacetime S vanishes in the overall spacetime $\mathfrak{X}^{\mu}_{\text{OVERALL}}$ of (27) for the space K leaving behind the forever hidden extra dimensions m in $\mathfrak{X}^{\mu}_{\text{OVERALL}}$. Thus, in other words, strings are experimentally unobservable forever, whereas, their actions should be mandatory in the purpose of string interactions. Also notice that Supersymmetry (now having extra dimensions m for $\mathfrak{X}^{\mu}$ due to (28)) remains unchanged in $\mathfrak{X}^{\mu}_{\text{OVERALL}}$ of (27). Thus, with these extra dimensions, the above scenario is now perfect for Supersymmetry and String Theory without any further objections and/or adjustments.

Along with Theorem 1, what (28) actually wants to say us is,

$$\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq \mathfrak{X}^{\mu} \leq \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \cup M,$$

when $\mathfrak{X}^{\mu} \leq S \cup K$, which yields,

$$\left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - S\right] \leq K \leq \left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - S\right] \cup M. \tag{29}$$

Since $S \leq \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)$ in (27), let the LHS of (29) gives,

$$\left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)\right] \leq \left\{ \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \cup \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \right\}. \tag{30}$$

The most disturbing thing here is that the temporal axis is a part of S spacetime but not the part of K space, but both $\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ spaces are influenced by the (mutual) temporal axis, despite the fact that neither of them have contained any temporal axis within themselves. On the other hand, it is evidence that only an influence should not be sufficient to emerge a temporal axis within M (or K) space. Moreover, Theorem 2 yields no temporal axis for M (or K) space either. But the influenced of the temporal axis should not ease to be avoided in (30).

From Theorem 2, if we think that the dimension of M space depends only on $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = \left\{\left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right), \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right), \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right\}$, then we should be mistaken, M is not independent from either elements of the set $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Thinking otherwise, let $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ are related to new quantities Q_i and T_i, differently, which are the curvilinear coordinates of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$.

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Let the corresponding members $\{T_1, T_2, T_3\} \subset T$ are determining γ, then if each pair of members from the either sides of these curvilinear coordinates joining the pairs of points Q_i and T_i $(i = 1, 2, 3)$ meet in points $\mathfrak{m}_i \in M$ separately, then the three points of intersection U_i of the pairs of coordinates q_i and t_i $(i = 1, 2, 3)$ lie on a line. Let each of the pairs of coordinates Q_i, T_i $(i = 1, 2, 3)$ consists of two distinct coordinates and in which $q_i \neq t_i \ \forall \ i$. Let the coordinate vectors of $\mathfrak{m}_i$ be denoted by $\mathfrak{z}_i$, that of Q_i by $\mathfrak{r}_i$ $(i = 1, 2, 3)$ and that of T_i by $\mathfrak{y}_i$ $(i = 1, 2, 3)$. Then $\mathfrak{z}_i$ can be represented by a linear combination of the $\mathfrak{r}_i$ and $\mathfrak{y}_i$ for each $i = 1, 2, 3$, say,

$$\mathfrak{z}_i = \mathfrak{r}_1 + \mathfrak{y}_1 = \mathfrak{r}_2 + \mathfrak{y}_2 = \mathfrak{r}_3 + \mathfrak{y}_3 \,.$$

Hence,

$$\begin{aligned} \mathfrak{r}_1 - \mathfrak{r}_2 &= \mathfrak{y}_2 - \mathfrak{y}_1 \,, \\ \mathfrak{r}_2 - \mathfrak{r}_3 &= \mathfrak{y}_3 - \mathfrak{y}_2 \,, \\ \mathfrak{r}_3 - \mathfrak{r}_1 &= \mathfrak{y}_1 - \mathfrak{y}_3 \,. \end{aligned}$$

Let us choose two set of coordinates,

$$\begin{aligned} a_i &= (a_1, a_2, a_3) = \frac{1}{\sqrt{2}} \left\{ \frac{\partial^2}{\partial t^2} [-\varphi_i] - \frac{\partial^2}{\partial x^i \, \partial x^j} [\varphi_0] \right\} = \left(\widehat{\partial}^2_{[0,i]} \otimes \boldsymbol{\varphi}_{[i,0]} \right) \in \Lambda \,, \\ b_i &= (b_1, b_2, b_3) = \frac{1}{\sqrt{2}} \left\{ \frac{\partial^2}{\partial t^2} [-\varrho_i] - \frac{\partial^2}{\partial x^i \, \partial x^j} [\varrho_0] \right\} = \left(\widehat{\partial}^2_{[0,i]} \otimes \boldsymbol{\varrho}_{[i,0]} \right) \in \Lambda \,, \end{aligned} \tag{31}$$

for $i, j = 1, 2, 3$, such that $\{a_i\} \in \mathfrak{r}_i$, $\{b_i\} \in \mathfrak{y}_i$ and $\{a, b\}$ is a basis of Λ, whereas $\Lambda \cap Q_i = \{O\}$, where Q_i is the interior of $\boldsymbol{Q}$ and O is the origin, i.e., Λ is admissible for $\boldsymbol{Q}$. Let the quadratic form,

$$\begin{aligned} & \mathcal{Q}\left(\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right), \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right) \right) \\ &= \sum_{1 \le i,j \le 3} \frac{1}{2} \left[(a_i - a_j) \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right) + (b_i - b_j) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right) \right] \\ &= \frac{1}{2} \left[A \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right)^2 + 2B \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right) + C \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right)^2 \right] , \end{aligned}$$

say, is reduced. The last fact means that $2|B| \le A \le C$, so that $3A^2 \le 4\left(A^2 - B^2\right) \le 4\left(AC - B^2\right)$. Since Λ is admissible for $\boldsymbol{Q}$, the coordinates $a + mc$ (m an integer) do not belong to int $\boldsymbol{Q}$. Thus,

$$\{|(m + a_1)(m + a_2)(m + a_3)| \ge 1 \ \forall \text{ integers } m\} \,,$$

this implies that,

$$\begin{aligned} A &= (a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 \\ &= 2\left(a_1^2 + a_2^2 + a_3^2\right) - 2\left(a_1 \, a_2 + a_2 \, a_3 + a_3 \, a_1\right) \,. \end{aligned}$$

Note it here that $a_i \, a_j \neq 0$ if $i = j$ and $a_i \, a_j = 0$ if $i \neq j$ do not hold due to (31). So as,

$$\begin{aligned} C &= (b_1 - b_2)^2 + (b_2 - b_3)^2 + (b_3 - b_1)^2 \\ &= 2\left(b_1^2 + b_2^2 + b_3^2\right) - 2\left(b_1 \, b_2 + b_2 \, b_3 + b_3 \, b_1\right) \,. \end{aligned}$$

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and we can easily find that $B = 0$. Here,

$$\begin{aligned}
\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)^2 &= \left[\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] - \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right]\right\}\right]^2 \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] - \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right]\right\} \times \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] - \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right]\right\} \\
&= \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] \times \frac{\partial^2}{\partial t^2}\left[-\psi_2\right] - \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right] \times \frac{\partial^2}{\partial t^2}\left[-\psi_2\right] - \right. \\
&\qquad \left. - \frac{\partial^2}{\partial t^2}\left[-\psi_2\right] \times \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right] + \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right] \times \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right]\right\} \\
&= \frac{1}{2}\left\{\left(\frac{\partial^2}{\partial t^2}\left[-\psi_2\right]\right)^2 + \left(\frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right]\right)^2\right\} - \frac{\partial^2}{\partial t^2}\left[\psi_0\right]\frac{\partial^2}{\partial x^2\,\partial x^2}\left[-\psi_2\right] \\
&= \frac{1}{2}\left(\widehat{\partial}^4_0\,\Psi^2_2 + \widehat{\partial}^4_2\,\Psi^2_0\right) - \partial^2_0\,\Psi_0\,\partial^2_2\,\Psi_2\,.
\end{aligned} \tag{32}$$

Similarly,

$$\begin{aligned}
\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)^2 &= \left[\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_3\right] - \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right]\right\}\right]^2 \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_3\right] - \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right]\right\} \times \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_3\right] - \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right]\right\} \\
&= \frac{1}{2}\left(\widehat{\partial}^4_0\,\Psi^2_3 + \widehat{\partial}^4_3\,\Psi^2_0\right) - \partial^2_0\,\Psi_0\,\partial^2_3\,\Psi_3\,.
\end{aligned} \tag{33}$$

In the same way,

$$\begin{aligned}
&\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] - \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right]\right\} \times \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_3\right] - \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right]\right\} \\
&= \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] \times \frac{\partial^2}{\partial t^2}\left[-\psi_3\right] - \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right] \times \frac{\partial^2}{\partial t^2}\left[-\psi_3\right] - \right. \\
&\qquad \left. - \frac{\partial^2}{\partial t^2}\left[-\psi_2\right] \times \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right] + \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right] \times \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right]\right\} \\
&= \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_2\right] \times \frac{\partial^2}{\partial t^2}\left[-\psi_3\right] + \frac{\partial^2}{\partial x^2\,\partial x^2}\left[\psi_0\right] \times \frac{\partial^2}{\partial x^3\,\partial x^3}\left[\psi_0\right]\right\} - \\
&\qquad - \frac{1}{2}\frac{\partial^2}{\partial t^2}\left[\psi_0\right]\left\{\frac{\partial^2}{\partial x^2\,\partial x^2}\left[-\psi_3\right] + \frac{\partial^2}{\partial x^3\,\partial x^3}\left[-\psi_2\right]\right\} \\
&= \frac{1}{2}\left(\widehat{\partial}^4_0\,\Psi^2_{\{2,3\}} + \widehat{\partial}^4_{\{2,3\}}\,\Psi^2_0\right) - \frac{1}{2}\,\partial^2_0\,\Psi_0\left(\partial^2_{\{2,3\}} \otimes \Psi_{\{3,2\}}\right)\,.
\end{aligned} \tag{34}$$

In the last line we have used subscripts $\{\,,\,\}$, which are quite different from the subscripts $[\,,\,]$ we have been using yet and their purposes are quite obvious here. Since, the temporal axis is a part of S spacetime but not the part of K space, so both $\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ spaces, as well as a_i and b_i spaces of (31), are influenced by the (mutual) temporal axis though, neither of them have contained any temporal axis within themselves, then we can say that all axes of $a_i\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_i\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$) in K space are interrelated with the (mutual) temporal axis coming from string spacetime S, since the temporal axis is a part of $S = \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)$ but not the part of K space, thus, $a_i\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_i\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$) in K space have

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individual existences as independent axes $x^{(1+i)}$ and $x^{(1+(i+\ell))}$ (for $\ell = \max i$) influenced by the (mutual) temporal axis x^0. Let us assume that $a_i \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_i \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$) in K space have maximal weight as 1 of each dimension as an independent axis for $x^{(1+i)}$ and $x^{(1+(i+\ell))}$, which yields,

$$\begin{aligned} \dim \left[(a_1 + a_2 + a_3) \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\right] &\leq 3\,, \\ \dim \left[(b_1 + b_2 + b_3) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right] &\leq 3\,. \end{aligned} \tag{35}$$

Hence, they have the "proper" dimensions. Comparing the last line of (34) with (32) and (33), we can determine that if (32) and (33) give us some "proper" dimensions, then (34) definitely gives us an "improper" dimension, as both $\left(\widehat{\partial}^4_0 \, \Psi^2_{\{2,3\}} + \widehat{\partial}^4_{\{2,3\}} \, \Psi^2_0\right)$ and $\left(\partial^2_{\{2,3\}} \otimes \Psi_{\{3,2\}}\right)$ are depended on x^2 and x^3 axes, simultaneously. Since a and b are satisfying (31), then $a_i a_j$ and $b_i b_j$ (for $i, j = 1, 2, 3$, $i \neq j$) must give us "improper" dimensions, too. If we consider these "improper" dimensions $(a_i\, a_j)^{1/2} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $(b_i\, b_j)^{1/2} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$, $i \neq j$) in K space have individual existences as independent axes $x_a^{(1+(i+\ell)+\frac{1}{2}j)}$ and $x_b^{(1+(i+\ell)+\frac{1}{2}j)}$ (since they are depended on x^2 and x^3 axes, simultaneously) influenced by the (mutual) temporal axis x^0, then, on the contrary of (35), let us assume that they have maximal weight as 0.5 of each dimension for $x_a^{(1+(i+\ell)+\frac{1}{2}j)}$ and $x_b^{(1+(i+\ell)+\frac{1}{2}j)}$, so as they can give $x^{(1+(i+\ell)+j)} = x_a^{(1+(i+\ell)+\frac{1}{2}j)} + x_b^{(1+(i+\ell)+\frac{1}{2}j)}$, thus, we can say that,

$$\begin{aligned} \dim \left[\left\{(a_1\, a_2)^{1/2} + (a_2\, a_3)^{1/2} + (a_3\, a_1)^{1/2}\right\} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\right] &\leq 1.5\,, \\ \dim \left[\left\{(b_1\, b_2)^{1/2} + (b_2\, b_3)^{1/2} + (b_3\, b_1)^{1/2}\right\} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right] &\leq 1.5\,. \end{aligned}$$

Hence, altogether they have,

$$\begin{aligned} \dim \Bigg[&\left\{(a_1\, a_2)^{1/2} + (a_2\, a_3)^{1/2} + (a_3\, a_1)^{1/2}\right\} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \\ &\qquad \bigcup (a_1 + a_2 + a_3) \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \Bigg] \quad \leq \quad 4.5\,, \\ \dim \Bigg[&\left\{(b_1\, b_2)^{1/2} + (b_2\, b_3)^{1/2} + (b_3\, b_1)^{1/2}\right\} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \\ &\qquad \bigcup (b_1 + b_2 + b_3) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \Bigg] \quad \leq \quad 4.5\,. \end{aligned}$$

Since $B = 0$, the K space yields,

$$\begin{aligned} K \quad = \quad \Bigg[&(a_1 + a_2 + a_3) \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \\ &\bigcup \left\{(a_1\, a_2)^{1/2} + (a_2\, a_3)^{1/2} + (a_3\, a_1)^{1/2}\right\} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \\ &\bigcup (b_1 + b_2 + b_3) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \\ &\bigcup \left\{(b_1\, b_2)^{1/2} + (b_2\, b_3)^{1/2} + (b_3\, b_1)^{1/2}\right\} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \Bigg]\,, \end{aligned}$$

i.e.,

$$\dim K \leq (4.5 + 4.5) = 9\,.$$

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Thus, $\mathfrak{X}^{\mu} \leq S \cup K$ has the spacetime axes as (using summation convention),

$$\begin{aligned}
&\Big[\left(\widehat{\partial}^{2}_{[0,1]} \otimes \Psi_{[1,0]}\right), a_i \left(\widehat{\partial}^{2}_{[0,2]} \otimes \Psi_{[2,0]}\right), b_i \left(\widehat{\partial}^{2}_{[0,3]} \otimes \Psi_{[3,0]}\right), \\
&\qquad \left\{ \left(a_i a_j\right)^{1/2} \left(\widehat{\partial}^{2}_{[0,2]} \otimes \Psi_{[2,0]}\right) + \left(b_i b_j\right)^{1/2} \left(\widehat{\partial}^{2}_{[0,3]} \otimes \Psi_{[3,0]}\right) \right\} \Big] \\
\mapsto \quad & \left(x^{0}, x^{1}, x^{(1+i)}, x^{(1+(i+\ell))}, x^{(1+(i+\ell)+j)} \right) \in \mathfrak{X}^{\mu} ,
\end{aligned} \tag{36}$$

where $\left(\widehat{\partial}^{2}_{[0,1]} \otimes \Psi_{[1,0]}\right) \mapsto (x^{0}, x^{1})$, whereas, other maps are obvious, for $i, j = 1,2,3$, $i \neq j$ and $\ell = \max i$. So, (36) achieves,

$$\dim \mathfrak{X}^{\mu} \leq \dim (S \cup K) \leq (2+9) = 11 ,$$

i.e., string has eleven-dimensions by nature, that is why eleven is the maxinium spacetime dimension in which one can formulate a consistent supersymmetric theory.

Now, returning to our main purpose and using the first two terms of (7), we can generate the following wave equation for (6) as,

$$i\hbar\, \mathrm{v}^{0}\, \vec{\nabla}_{0}\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^{j}\, \vec{\nabla}_{i}\, \psi(\vec{r},t) - i\hbar\, g_{\mu\nu}\, \mathrm{v}^{\nu}\, \vec{\nabla}_{\mu}\, \psi(\vec{r},t) = 0, \tag{37}$$

where $\vec{\nabla}_{0} = \left(\partial / \partial\, (\mathrm{v}^{0}\, t)\right)$ for $x^{0} = (\mathrm{v}^{0}\, t)$, while the 'Four-momentum' operator is $\hat{\boldsymbol{\mathcal{P}}}^{\mu} \to i\hbar\, \vec{\nabla}_{\mu}$, and the three momentum operator is $\hat{\boldsymbol{p}} \to -i\hbar\, \vec{\nabla}_{i}$.

Remark 2. *The signature of the metric $g_{\mu\nu}$, i.e., $(+,-,-,-)$, has been absorbed and retained unaltered by the last term of (37), as long as it satisfies (5) and (6). Thus, readers are requested to be careful not to presume space and time separately in (37), what we usually accept in the conventional Quantum Mechanics.*

Again rearranging (37) by using (8), we may get,

$$i\hbar\, \mathrm{v}^{0} \left(1 - g_{00}\right) \vec{\nabla}_{0}\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^{j} \left(1 + g_{ij}\right) \vec{\nabla}_{i}\, \psi(\vec{r},t) = 0,$$

or, simply discarding $(1 - g_{00}) = (1 + g_{ij}) = 0$, we can have the First Variance of the First Order Equation of GQG in Semi-Quantum Minkowski Spacetime as follows for $x^{0} = (\mathrm{v}^{0}\, t)$,

$$\begin{aligned}
& i\hbar\, \mathrm{v}^{0}\, \vec{\nabla}_{0}\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^{j}\, \vec{\nabla}_{i}\, \psi(\vec{r},t) = 0, \\
\therefore \quad & i\hbar\, \frac{\partial}{\partial t}\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^{j}\, \vec{\nabla}_{i}\, \psi(\vec{r},t) = 0 .
\end{aligned} \tag{38}$$

Evidently, (38) may take the form $\hat{E}\, \psi(\vec{r},t) = -i\hbar\, \mathrm{v}^{j}\, \vec{\nabla}_{i}\, \psi(\vec{r},t)$ for the energy operator $\hat{E} \to i\hbar\, \partial_{t}$. Setting the Hamiltonian operator as $\hat{H}\, \psi(\vec{r},t) = -i\hbar\, \mathrm{v}^{j}\, \vec{\nabla}_{i}\, \psi(\vec{r},t) \equiv \hat{\boldsymbol{p}}\, \mathrm{v}^{j}\, \psi(\vec{r},t)$, where the three momentum operator $\hat{\boldsymbol{p}} \to -i\hbar\, \vec{\nabla}_{i}$, we can therefore have, $\hat{E}\, \psi(\vec{r},t) = \hat{H}\, \psi(\vec{r},t)$. Interested readers can easily check it that (38) is nothing but the gravitational form of the classical Schrödinger equation, where E is total energy, and now the equation has been rewritten with v^{j} along with the signature of the metric $(+,-,-,-)$.

It is also possible to develop a Second Variance of the First Order Equation of GQG in Semi-Quantum Minkowski Spacetime from (37) as follows,

$$i\hbar\, \mathrm{v}^{\nu}\, \vec{\Delta}_{\mu}\, \psi(\vec{r},t) - i\hbar\, g_{\mu\nu}\, \mathrm{v}^{\nu}\, \vec{\nabla}_{\mu}\, \psi(\vec{r},t) = 0,$$

where $i\hbar\, \vec{\Delta}_{\mu} \to \left[\hat{\boldsymbol{p}}_{0}, -\hat{\boldsymbol{p}}\right]^{\mathrm{T}} \to \left[i\hbar\, \vec{\nabla}_{0}, i\hbar\, \vec{\nabla}_{i}\right]^{\mathrm{T}}$.

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Now, let us multiply both sides of (37) by $(\mathrm{d}t^2/m)$, so as,

$$\begin{aligned} i\hbar\,\frac{\mathrm{d}t^2}{m}\,\mathrm{v}^0\,\vec{\nabla}_0\,\psi(\vec{r},t)+i\hbar\,\frac{\mathrm{d}t^2}{m}\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t) &= i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \\ &= \frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,\psi(\vec{r},t)\,, \end{aligned} \tag{39}$$

which has the form of a general inhomogeneous Lorentz transformation (or Poincaré transformation).

Note it that (39) is exactly equivalent to $\mathrm{d}s^2 \equiv (\mathrm{d}t^2/m)\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu$ of (3), i.e., $\mathrm{d}s^2 \equiv (\mathrm{d}t^2/m)\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu \mapsto \mathrm{d}s^2\,\psi(\vec{r},t) \equiv (\mathrm{d}t^2/m)\,g_{\mu\nu}\,\mathrm{v}^\nu\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,\psi(\vec{r},t)$, for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}^\mu \to i\hbar\,\vec{\nabla}_\mu$. In other words, for (39), we can say that the quantum line element is,

$$\begin{aligned} \mathrm{d}s^2\,\psi(\vec{r},t) &\equiv i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) = \frac{\mathrm{v}^\mu}{\mathrm{v}^\mu}\left(i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t) \\ &= \frac{1}{\mathrm{v}^\mu}\left(i\hbar\,\frac{\mathrm{d}x^\mu}{m}\,g_{\mu\nu}\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t) \\ &= \frac{1}{(\mathrm{v}^\mu\,m)}\left(i\hbar\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t)\,, \end{aligned}$$

hence, by considering $\mathcal{E} = (\mathrm{v}^\mu\,m)^{-1}$, we have,

$$\mathrm{d}s^2\,\psi(\vec{r},t) = i\hbar\,\mathcal{E}\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)\,. \tag{40}$$

Proposition 1. *From above discussion, we can deduce that:*

1. *In (12), relativistic spacetime is showing a relation with the quantum spacetime.*
2. *The wave field $\psi(\vec{r},t)$ itself in (7) is relativistic due to $g_{\mu\nu}$ in its term.*
3. *Equally, (37) also assures us that Quantum Mechanics and Relativity are must be correlated for the presence of $g_{\mu\nu}$ in the last term of (37).*
4. *Lastly, the exact equivalency of (39) and (3), i.e.,*

$$ds^2 \equiv (dt^2/m)\,g_{\mu\nu}\,v^\nu\,P^\mu \quad\longmapsto\quad ds^2\,\psi(\vec{r},t) \equiv (dt^2/m)\,g_{\mu\nu}\,v^\nu\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,\psi(\vec{r},t)\,,$$

for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}^\mu \to i\hbar\,\vec{\nabla}_\mu$, can say us that Quantum Mechanics and Relativity are correlated in the wave field $\psi(\vec{r},t)$.

So, we have a sufficient reason to replace the relativistic one with the quantum mechanical relation, and vice versa, from (12) as,

$$\begin{aligned} g_{\mu\nu}\,P^\mu &\iff \hat{\boldsymbol{\mathcal{P}}}_\mu = -\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,, \\ \text{i.e.,}\quad g_{\mu\nu}\,m\,v^\mu &\iff -\,i\hbar\,\vec{\nabla}_\mu\,. \end{aligned} \tag{41}$$

We will use Proposition 1 throughout our work. This proposition is quite straightforward than some commonly used textbook procedures, for example, [4].

Remark 3. *For the Proposition 1, the relation between $\mathcal{E} = (v^\mu\,m)^{-1}$ and $\hat{\boldsymbol{\mathcal{P}}}^\mu$ of (40) should become as,*

$$i\hbar\,\mathcal{E}\,\vec{\nabla}_\mu = i\hbar\,\frac{g_{\mu\nu}}{g_{\mu\nu}\,(v^\mu\,m)}\,\vec{\nabla}_\mu = i\hbar\,\frac{g_{\mu\nu}}{\hat{\boldsymbol{\mathcal{P}}}_\mu}\,\vec{\nabla}_\mu = i\hbar\,\frac{1}{g^{\mu\nu}\,\hat{\boldsymbol{\mathcal{P}}}_\mu}\,\vec{\nabla}_\mu = i\hbar\,\frac{1}{\hat{\boldsymbol{\mathcal{P}}}^\mu}\,\vec{\nabla}_\mu = 1\,. \tag{42}$$

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Let us consider $\vec{\nabla}'_\mu = (\delta/\delta x^\mu)$, etc., and let us also consider that for Proposition 1, let $g_{\mu\nu}$ would transform classical-to-quantum as,

$$\begin{aligned} g_{\mu\nu} &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right) = g_{\alpha\beta}\left(\frac{m}{\partial t}\right)\left(\frac{\partial t}{m}\right)\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right) = g_{\alpha\beta}\left(\frac{P^\alpha}{P^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right) \\ &= g_{\alpha\beta}\left(\frac{-g_{\alpha\beta}P^\alpha}{-g_{\mu\nu}P^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right), \\ g^{(\mathrm{Q})}_{\mu\nu}\,\psi(\vec{r},t) &= g^{(\mathrm{Q})}_{\alpha\beta}\left(\frac{\hat{\mathcal{P}}^\alpha}{\hat{\mathcal{P}}^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right) \longrightarrow g^{(\mathrm{Q})}_{\alpha\beta}\left(\frac{i\hbar\vec{\nabla}_\alpha}{i\hbar\vec{\nabla}_\mu}\frac{\partial x^\beta}{\partial x^\nu}\right)\psi(\vec{r},t)\,, \end{aligned} \tag{43}$$

where $g^{(\mathrm{Q})}_{\mu\nu}$ is a 'semi-quantum Lorentzian metric tensor' in a semi-Quantum Minkowski spacetime, i.e.,

$$g^{(\mathrm{Q})}_{\mu\nu} = g^{(\mathrm{Q})}_{\alpha\beta}\left(\frac{i\hbar\vec{\nabla}_\alpha}{i\hbar\vec{\nabla}_\mu}\frac{\partial x^\beta}{\partial x^\nu}\right) \longleftrightarrow g_{\mu\nu} = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right), \tag{44}$$

so as the 'semi-quantum Lorentzian metric tensor' $g^{(\mathrm{Q})}_{\mu\nu}$ and the classical Lorentzian metric tensor $g_{\mu\nu}$ should differ from each other and can establish a relation (44). Without any loss of generality, we may assume that the 'quantum metric tensor' is symmetric: $g^{(\mathrm{Q})}_{\mu\nu} = g^{(\mathrm{Q})}_{\nu\mu}$, and $\det\left(g^{(\mathrm{Q})}_{\mu\nu}\right) \neq 0$. It has an inverse matrix $\left(g^{(\mathrm{Q})}\right)^{-1}$ whose components are themselves the components of matrix $g^{(\mathrm{Q})}$, as their product gives: $\left(g^{(\mathrm{Q})}\right)^{-1} g^{(\mathrm{Q})}$ = identity matrix, i.e., in terms of components, $g^{(\mathrm{Q})}_{\mu\nu}\,g^{\mu\gamma}_{(\mathrm{Q})} = g^{\gamma\mu}_{(\mathrm{Q})}\,g^{(\mathrm{Q})}_{\mu\nu} = \delta^\gamma_\nu$, where, δ^γ_ν is the Kronecker delta.

Hence, (40) should be rewritten as,

$$\begin{aligned} \mathrm{d}s^2\,\psi(\vec{r},t) &= i\hbar\,\mathcal{E}\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \\ &= i\hbar\,\mathcal{E}\,g^{(\mathrm{Q})}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \to \mathcal{E}\,g^{(\mathrm{Q})}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\hat{\mathcal{P}}^\mu\,\psi(\vec{r},t)\,, \end{aligned}$$

then, let us vary the length of a curve [5–8] as,

$$\begin{aligned} \delta\,L|\gamma|\,\psi(\vec{r},t) &\equiv \int \delta\left\{i\hbar\,\mathcal{E}\,g^{(\mathrm{Q})}_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu\,\vec{\nabla}_\mu\right\}^{1/2}\mathrm{d}\tau\,\psi(\vec{r},t) \\ &= \frac{1}{2}\int\Big\{i\hbar\,\mathcal{E}\,\partial_\epsilon\,g^{(\mathrm{Q})}_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu\,\vec{\nabla}_\mu - \\ &\qquad - 2\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(i\hbar\,\mathcal{E}\,g^{(\mathrm{Q})}_{\epsilon\nu}\,\dot{x}^\nu\,\vec{\nabla}_\epsilon\right)\Big\}\,\delta x^\epsilon\,\mathrm{d}\tau\,\psi(\vec{r},t)\,. \end{aligned}$$

This gives,

$$\begin{aligned} &\left\{i\hbar\,\mathcal{E}\,\partial_\epsilon\,g^{(\mathrm{Q})}_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu\,\vec{\nabla}_\mu - i\hbar\,\mathcal{E}\,\partial_\mu\,g^{(\mathrm{Q})}_{\epsilon\nu}\,\dot{x}^\mu\,\dot{x}^\nu\,\vec{\nabla}_\nu - i\hbar\,\mathcal{E}\,\partial_\nu\,g^{(\mathrm{Q})}_{\epsilon\mu}\,\dot{x}^\mu\,\dot{x}^\nu\,\vec{\nabla}_\epsilon\right\}\psi(\vec{r},t) \\ &\qquad = 2\,i\hbar\,\mathcal{E}\,g^{(\mathrm{Q})}_{\epsilon\delta}\,\ddot{x}^\delta\,\vec{\nabla}_\epsilon\,\psi(\vec{r},t)\,, \end{aligned}$$

then the Christoffel symbol $\Gamma^\delta_{\mu\nu}$ should be defined by,

$$\Gamma^\delta_{\mu\nu}\,\psi(\vec{r},t) = \frac{1}{2}\,i\hbar\,\mathcal{E}\,g^{\delta\epsilon}_{(\mathrm{Q})}\left(\partial_\mu\,g^{(\mathrm{Q})}_{\epsilon\nu}\,\vec{\nabla}_\nu + \partial_\nu\,g^{(\mathrm{Q})}_{\epsilon\mu}\,\vec{\nabla}_\epsilon - \partial_\epsilon\,g^{(\mathrm{Q})}_{\mu\nu}\,\vec{\nabla}_\mu\right)\psi(\vec{r},t)\,,$$

such that the Christoffel symbols are symmetric in the lower indices: $\Gamma^\delta_{\mu\nu} = \Gamma^\delta_{\nu\mu}$.

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In this way, if we proceed further, we can develop a non renormalizable Einstein-like field equation, which is useless for our goal. To avoid this problem, let us redevelop the whole scenario from the quantum mechanical perspective as follows. Let the Christoffel symbol $\Gamma^{\delta}_{\mu\nu}$ should be redefined by,

$$\Gamma^{\delta}_{\mu\nu}\,\psi(\vec{r},t) = -\frac{1}{2}\hbar^2\,\mathcal{E}\,g^{\delta\epsilon}_{(\mathrm{Q})}\left(\vec{\nabla}_{\mu}\,g^{(\mathrm{Q})}_{\epsilon\nu}\,\vec{\nabla}_{\nu} + \vec{\nabla}_{\nu}\,g^{(\mathrm{Q})}_{\epsilon\mu}\,\vec{\nabla}_{\epsilon} - \vec{\nabla}_{\epsilon}\,g^{(\mathrm{Q})}_{\mu\nu}\,\vec{\nabla}_{\mu}\right)\psi(\vec{r},t)\,,$$

such that the Christoffel symbols are symmetric in the lower indices: $\Gamma^{\delta}_{\mu\nu} = \Gamma^{\delta}_{\nu\mu}$. After a little exercise, we can yield the curvature tensor,

$$\mathcal{R}^{\sigma}_{\ \nu\gamma\delta}\,\psi(\vec{r},t) = \left(\frac{\partial\Gamma^{\sigma}_{\nu\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\sigma}_{\nu\gamma}}{\partial x^{\delta}} + \Gamma^{\sigma}_{\gamma\epsilon}\,\Gamma^{\epsilon}_{\nu\delta} - \Gamma^{\sigma}_{\delta\epsilon}\,\Gamma^{\epsilon}_{\nu\gamma}\right)\psi(\vec{r},t)\,,$$

thus we find,

$$\begin{aligned}\mathcal{R}_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t) &= -\frac{1}{2}\,i\hbar^3\,\mathcal{E}\left(\vec{\nabla}_{\nu}\,\vec{\nabla}_{\gamma}\,g^{(\mathrm{Q})}_{\lambda\delta}\,\vec{\nabla}_{\lambda} + \vec{\nabla}_{\lambda}\,\vec{\nabla}_{\delta}\,g^{(\mathrm{Q})}_{\nu\gamma}\,\vec{\nabla}_{\nu} - \right.\\ &\left. - \vec{\nabla}_{\lambda}\,\vec{\nabla}_{\gamma}\,g^{(\mathrm{Q})}_{\delta\nu}\,\vec{\nabla}_{\delta} - \vec{\nabla}_{\nu}\,\vec{\nabla}_{\delta}\,g^{(\mathrm{Q})}_{\lambda\gamma}\,\vec{\nabla}_{\gamma}\right)\psi(\vec{r},t)\,, \end{aligned} \tag{45}$$

which satisfies the properties like symmetry, antisymmetry and cyclicity as usual, but it is not equivalent to Riemannian tensor $R_{\lambda\nu\gamma\delta}$, i.e., $R_{\lambda\nu\gamma\delta} \neq \mathcal{R}_{\lambda\nu\gamma\delta}$ for $i\hbar\,\vec{\nabla}_X$ in (45). Though, the Riemann tensor (in quantum spacetime) should be derivable from (45) as,

$$\begin{aligned}\left(\frac{1}{i\hbar}\right)^2\mathcal{R}_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t) &= R_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t)\\ &= \frac{1}{2}\,i\hbar\,\mathcal{E}\left(\frac{\partial^2}{\partial x^{\nu}\,\partial x^{\gamma}}\,g^{(\mathrm{Q})}_{\lambda\delta}\,\vec{\nabla}_{\lambda} + \frac{\partial^2}{\partial x^{\lambda}\,\partial x^{\delta}}\,g^{(\mathrm{Q})}_{\nu\gamma}\,\vec{\nabla}_{\nu} - \right.\\ &\left. - \frac{\partial^2}{\partial x^{\lambda}\,\partial x^{\gamma}}\,g^{(\mathrm{Q})}_{\delta\nu}\,\vec{\nabla}_{\delta} - \frac{\partial^2}{\partial x^{\nu}\,\partial x^{\delta}}\,g^{(\mathrm{Q})}_{\lambda\gamma}\,\vec{\nabla}_{\gamma}\right)\psi(\vec{r},t)\,. \end{aligned} \tag{46}$$

Now, let us take the classical-to-quantum energy momentum tensor, for example, as follows for (42),

$$\begin{aligned} T_{\mu\nu} &= -\frac{1}{\kappa^2}\,(\partial^{\lambda}\partial_{\lambda}\,g_{\mu\nu} - \partial^{\lambda}\partial_{\nu}\,g_{\mu\lambda} - \partial^{\lambda}\partial_{\mu}\,g_{\nu\lambda} + \partial_{\mu}\partial_{\nu}\,g^{\sigma}_{\sigma} - \\ &\quad - \eta_{\mu\nu}\,\partial_{\lambda}\partial^{\lambda}\,g^{\sigma}_{\sigma} + \eta_{\mu\nu}\,\partial^{\lambda}\partial^{\sigma}\,g_{\lambda\sigma})\,,\\ \mathcal{T}_{(1)\,\mu\nu}\,\psi(\vec{r},t) &\to -\frac{1}{\kappa^2}\,i\hbar\,\mathcal{E}\left(\vec{\nabla}^{\lambda}\vec{\nabla}_{\lambda}\,g^{(\mathrm{Q})}_{\mu\nu}\,\vec{\nabla}_{\mu} - \vec{\nabla}^{\lambda}\vec{\nabla}_{\nu}\,g^{(\mathrm{Q})}_{\mu\lambda}\,\vec{\nabla}_{\lambda} - \vec{\nabla}^{\lambda}\vec{\nabla}_{\mu}\,g^{(\mathrm{Q})}_{\nu\lambda}\,\vec{\nabla}_{\nu} + \right.\\ &\quad + \vec{\nabla}_{\mu}\vec{\nabla}_{\nu}\,g^{(\mathrm{Q})\,\sigma}_{\sigma}\,\vec{\nabla}_{\mu} - \eta_{\mu\nu}\,\vec{\nabla}_{\lambda}\vec{\nabla}^{\lambda}\,g^{(\mathrm{Q})\,\sigma}_{\sigma}\,\vec{\nabla}_{\lambda} + \\ &\quad \left. + \eta_{\mu\nu}\,\vec{\nabla}^{\lambda}\vec{\nabla}^{\sigma}\,g^{(\mathrm{Q})}_{\lambda\sigma}\,\vec{\nabla}_{\sigma}\right)\psi(\vec{r},t)\\ &= i\hbar\,\mathcal{E}\,T_{\mu\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)\,, \end{aligned}$$

or the electrodynamic,

$$\begin{aligned} T_{\zeta\eta} &= -F_{\zeta\mu}\,F^{\mu}_{\eta} + \frac{1}{4}\,g_{\zeta\eta}\,F^{\mu\nu}\,F_{\mu\nu} = -g_{\zeta\eta}\,F_{\zeta\mu}\,F^{\zeta\mu} + \frac{1}{4}\,g_{\zeta\eta}\,F^{\mu\nu}\,F_{\mu\nu}\,,\\ \mathcal{T}_{(1)\,\zeta\eta}\,\psi(\vec{r},t) &\to -i\hbar\,\mathcal{E}\left(g^{(\mathrm{Q})}_{\zeta\eta}\,F_{\zeta\mu}\,F^{\zeta\mu} + \frac{1}{4}\,g^{(\mathrm{Q})}_{\zeta\eta}\,F^{\mu\nu}\,F_{\mu\nu}\right)\vec{\nabla}_{\zeta}\,\psi(\vec{r},t)\\ &= i\hbar\,\mathcal{E}\,T_{\zeta\eta}\,\vec{\nabla}_{\zeta}\,\psi(\vec{r},t)\,, \end{aligned}$$

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etc., and so on, hence, we have the quantum energy momentum tensors $\mathcal{T}_{(1)\,\mu\nu}$, $\mathcal{T}_{(1)\,\zeta\eta}$, etc., those are what the graviton field couples to. Let G is the gravitational coupling and now let us develop an unusual classical-to-quantum G (namely $\mathcal{G}_{(1)}$) in Planck scale using [9] by accepting $\ell_p \mapsto \ell_P^\mu$ as,

$$\begin{aligned} G &= \frac{(\mathrm{d}\ell_P)^2}{m_P}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2} = \frac{m_P}{m_P^2}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,g_{\zeta\eta}\,\mathrm{d}\ell_P^\zeta\,\mathrm{d}\ell_P^\eta\,, \\ \mathcal{G}_{(1)}\,\psi(\vec{r},t) &\rightarrow i\,\hbar\,\mathcal{E}\,\frac{F_P}{m_P^2}\,g^{(\mathrm{Q})}_{\zeta\eta}\,\mathrm{d}\ell_P^\zeta\,\mathrm{d}\ell_P^\eta\,\vec{\nabla}_\zeta\,\psi(\vec{r},t) = i\,\hbar\,\mathcal{E}\,G\,\vec{\nabla}_\zeta\,\psi(\vec{r},t)\,, \end{aligned} \tag{47}$$

where $F_P = m_P\left\{\mathrm{d}^2\ell_P^\zeta/(\mathrm{d}t_P)^2\right\}$. Without much ado, we can easily obtain the Einstein field equation of the GQG in Semi-Quantum Minkowski Spacetime, which can give us the classical Einstein field equation while using (46) and (42) as follows,

$$\begin{aligned} \left[\mathcal{R}_{\mu\nu} - \frac{1}{2}\,i\,\hbar\,\mathcal{E}\,g^{(\mathrm{Q})}_{\mu\nu}\,\mathcal{R}\,\vec{\nabla}_\mu\right]\psi(\vec{r},t) &= (i\,\hbar)^2\,8\pi\,\mathcal{G}_{(1)}\,\mathcal{T}_{(1)\,\mu\nu}\,\psi(\vec{r},t) \\ &= -\,(i\,\hbar)^2\,8\pi\,\hbar^2\,\mathcal{E}^2\,G\,T_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)\,, \\ \therefore\quad (i\,\hbar)^2\left[R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R\right]\psi(\vec{r},t) &= -\,(i\,\hbar)^2\,8\pi\,\hbar^2\,\mathcal{E}^2\,G\,T_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t) \\ &= (i\,\hbar)^2\,8\pi\,G\,T_{\mu\nu}\,\psi(\vec{r},t)\,, \\ \Rightarrow\quad \left[R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R\right] &= 8\pi\,G\,T_{\mu\nu}\,. \end{aligned} \tag{48}$$

Both LHS and RHS of (48) independently have their mass dimensions. For the RHS factor $\hbar^4\,\mathcal{E}^2\,G$ of (48), both the gravitational coupling G (which has the dimension of inverse second order of mass) and $\mathcal{E} = (\mathrm{v}^\mu\,m)^{-1}$ have lost their mass dimensions due to $\hbar^4$. Consequently, if divergences are to be present, they could now be disposed of by the technique of renormalization. So, the last line of (48) is not renormalizable by nature, whereas, the second equation shows us that classical Einstein field equation with $(i\,\hbar)^2$ is renormalizable in quantum spacetime. (We will develop another renormalizable scenario by using a purely quantum form of gravity, which will be discussed in Section 2.2 below.)

Remark 4. *It is necessary to remember that, we should not introduce the cosmological constant Λ in (48), because we can get Dark Energy from GQG in Quantum Non-Minkowski Spacetime quite naturally (see the last equation of (73) in Section 2.2 below for more details). Otherwise, introduction of the cosmological constant Λ in (48) should intend to double entry of Dark Energy in the same gravitational field of GQG, which should obviously be faulty. Though, in (75) below, we will develop an Einstein-like field equation containing Λ, which is slightly different from the classical Einstein field equation.*

As (48) yields the non-renormalizable relation,

$$8\pi\hbar^2\,\mathcal{E}^2\,g^{(\mathrm{Q})}_{\mu\nu}\,G\,T_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}^\mu\,\psi(\vec{r},t) + \left[R_{\mu\nu} - \frac{1}{2}\,g^{(\mathrm{Q})}_{\mu\nu}\,R\right]\psi(\vec{r},t) = 0\,, \tag{49}$$

then, by considering d'Alembertian operator $\Box = \vec{\nabla}_\mu\,\vec{\nabla}^\mu$, as well as $\mathcal{U} = (8\,\pi\,G\,\mathrm{T}_{\mu\nu})$, we can get the Second Order Equation of GQG in Semi-Quantum Minkowski Spacetime from (49) as,

$$\hbar^2\,\mathcal{E}^2\,\mathcal{U}\,g^{(\mathrm{Q})}_{\mu\nu}\,\Box\,\psi(\vec{r},t) + \left[R_{\mu\nu} - \frac{1}{2}\,g^{(\mathrm{Q})}_{\mu\nu}\,R\right]\psi(\vec{r},t) = 0\,. \tag{50}$$

The wavefunction $\psi(\vec{r},t)$ in (50) is emphatically defining a bosonic field. Thus, as $\mathcal{U}^{-1}\Big[R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}^{(\mathrm{Q})}\, R\Big] = 1$, we can immediately develop a fermionic field (or the Third Variance of the First Order Equation of Semi-Quantum Minkowski GQG) out of (50) for (43) and (12) as,

$$\begin{aligned} & i\hbar\,\gamma^{\mu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t) - g_{\mu\nu}^{(\mathrm{Q})}\,\frac{1}{\mathcal{E}}\left(g^{\mu\nu}_{(\mathrm{Q})}\right)^{3/2}\psi(\vec{r},t) = 0\,,\\ \therefore\quad & i\hbar\,\gamma^{\mu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t) - \hat{\mathcal{P}}_{\mu}\left(g^{\mu\nu}_{(\mathrm{Q})}\right)^{3/2}\psi(\vec{r},t) = 0\,,\\ \Rightarrow\quad & i\hbar\,\gamma^{\mu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t) - \hat{\mathcal{P}}^{\mu}\left(g^{\mu\nu}_{(\mathrm{Q})}\right)^{1/2}\psi(\vec{r},t) = 0\,, \end{aligned} \tag{51}$$

where, γ^{μ} are Dirac's gamma matrices.

The classical Dirac's equation should be derivable from (51), but here, $\mathcal{E} = \left(\mathrm{v}^{\mu}\, m_{\mathrm{F}}\right)^{-1}$ is not intended to have a factor of rest (fermionic) mass, since $m_{\mathrm{F}} \neq m_{\mathrm{F}0}$ in (3). Thus, we can say that Dirac's equation is a subset of the Third Variance of the First Order Equation of GQG in Semi-Quantum Minkowski Spacetime, i.e., (51). Similarly, we can also say that the Klein-Gordon equation is a subset of the Second Order Equation of GQG in Semi-Quantum Minkowski Spacetime, i.e., (50). (An analogous formalism is equally applicable for the following Section 2.2.)

2.2. GQG in Quantum Non-Minkowski Spacetime

Let the line element of Minkowski spacetime,

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} \equiv \left(\frac{\mathrm{d}t}{m}\right)^2 g_{\mu\nu}\,P^{\mu}\,P^{\nu}\,,\\ \therefore\quad m^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= m^2c^2 - m^2\sum\frac{\mathrm{d}x^i}{\mathrm{d}t}\,\frac{\mathrm{d}x^j}{\mathrm{d}t} = m\,E - \sum p^i\,p^j\\ &= p^0\,p^0 - \sum p^i\,p^j = g_{\mu\nu}\,P^{\mu}\,P^{\nu}\,,\\ \text{differently,}\quad m^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= m^2\left(1 - \frac{\mathrm{v}^2}{c^2}\right)c^2 = m_0^2\,c^2\,, \end{aligned} \tag{52}$$

for the rest mass m_0, when,

$$\mathrm{d}\,\mathcal{S}_P^2 = m^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^0\,p^0 - \sum p^i\,p^j = g_{\mu\nu}\,P^{\mu}\,P^{\nu}\,, \tag{53}$$

then, rearrangement of (52) gives,

$$m\,E = p^i\,p^j + m^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^i\,p^j + P^{\mu}\,P^{\nu}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\mu}}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\nu}} = p^i\,p^j + P^{\mu}\,P^{\nu}\,g_{\mu\nu}\,. \tag{54}$$

Then, considering the representation of a wave field $\psi(\vec{r},t)$ by superposition of a free particle (de Broglie wave) for (54) as follows,

$$\begin{aligned} \psi(\vec{r},t) &= \frac{1}{(2\pi\hbar)^2}\exp\left[\frac{i}{\hbar\,m}\left\{m\left(\vec{p}\cdot\vec{r} + g_{\mu\nu}\,\vec{P}\cdot\vec{R}\right) - mEt\right\}\right]\\ &\equiv \frac{1}{(2\pi\hbar)^2}\exp\left[\frac{i}{\hbar\,m}\left\{m\left(\vec{p}\cdot\vec{r} + m\,t\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right) - mEt\right\}\right]\,, \end{aligned} \tag{55}$$

we can generate the following wave equation using (54) combining with (52) as,

$$-\,\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t) + \hbar^2\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t) + \hbar^2\,g_{\mu\nu}\,\vec{\nabla}_{\mu}\,\vec{\nabla}_{\nu}\,\psi(\vec{r},t) = 0\,, \tag{56}$$

which may give,

$$-\hbar^2\left(1-g_{00}\right)\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\left(1+g_{ij}\right)\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0\,,$$

or, simply discarding $(1-g_{00})=(1+g_{ij})=0$, we can get the First Variance of the Second Order Equation of GQG in Quantum Non-Minkowski Spacetime as,

$$\begin{aligned} &-\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0\,,\\ \therefore\quad &-\hbar^2\,g_{00}\,\vec{\nabla}_0\,\vec{\nabla}^0\,\psi(\vec{r},t)+\hbar^2\,g_{ij}\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t)=0\,. \end{aligned} \tag{57}$$

Here, $\psi(\vec{r},t)$ is definitely a bosonic field. But the uppermost equation of (57) may give us the gravitational form of the classical Schrödinger equation by using (11) for $c=1$ as follows,

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+\frac{\hbar^2}{m}\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0\,. \tag{58}$$

Putting differently,

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+\frac{\hbar^2}{m}\,g_{ij}\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t)=0\,.$$

Hence, we get the gravitational form of the classical Schrödinger equation for the total energy E, and now it is in a $(3+1)D$ quantum spacetime.

Let us now prescribe classical-to-quantum $g_{\mu\nu}$ as follows by using Proposition 1 as,

$$\begin{aligned} g_{\mu\nu} &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\,\frac{\partial x^\beta}{\partial x^\nu}\right)=g_{\alpha\beta}\left(\frac{m}{\partial t}\right)^2\left(\frac{\partial t}{m}\right)^2\left(\frac{\partial x^\alpha}{\partial x^\mu}\,\frac{\partial x^\beta}{\partial x^\nu}\right)\\ &= g_{\alpha\beta}\left(\frac{P^\alpha}{P^\mu}\,\frac{P^\beta}{P^\nu}\right)=g_{\alpha\beta}\left(\frac{-\,g_{\alpha\beta}\,P^\alpha}{-\,g_{\mu\nu}\,P^\mu}\,\frac{-\,g_{\alpha\beta}\,P^\beta}{-\,g_{\mu\nu}\,P^\nu}\right),\\ \mathrm{g}_{\mu\nu}\,\psi(\vec{r},t) &\rightarrow \mathrm{g}_{\alpha\beta}\left(\frac{\hat{\mathcal{P}}^\alpha}{\hat{\mathcal{P}}^\mu}\,\frac{\hat{\mathcal{P}}^\beta}{\hat{\mathcal{P}}^\nu}\right)\psi(\vec{r},t)\rightarrow \mathrm{g}_{\alpha\beta}\left(\frac{i\,\hbar\,\vec{\nabla}_\alpha}{i\,\hbar\,\vec{\nabla}_\mu}\,\frac{i\,\hbar\,\vec{\nabla}_\beta}{i\,\hbar\,\vec{\nabla}_\nu}\right)\psi(\vec{r},t)\,. \end{aligned} \tag{59}$$

To avoid any confusion between the classical Lorentzian metric tensor $g_{\mu\nu}$ and the quantum Lorentzian tensor of (59), let us assume that,

$$\mathrm{g}_{\mu\nu}=\mathrm{g}_{\alpha\beta}\left(\frac{i\,\hbar\,\vec{\nabla}_\alpha}{i\,\hbar\,\vec{\nabla}_\mu}\,\frac{i\,\hbar\,\vec{\nabla}_\beta}{i\,\hbar\,\vec{\nabla}_\nu}\right)\longleftrightarrow g_{\mu\nu}=g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\,\frac{\partial x^\beta}{\partial x^\nu}\right). \tag{60}$$

This approach is quantizing gravity. The 'quantum metric tensor' $\mathrm{g}_{\mu\nu}$ is symmetric, i.e., $\mathrm{g}_{\mu\nu}=\mathrm{g}_{\nu\mu}$, and $\det\left(\mathrm{g}_{\mu\nu}\right)\neq 0$. Components of its inverse matrix g^{-1} are themselves the components of matrix g, namely, $\mathrm{g}_{\mu\nu}\,\mathrm{g}^{\mu\gamma}=\mathrm{g}^{\gamma\mu}\,\mathrm{g}_{\mu\nu}=\delta^\gamma_\nu$, where, δ^γ_ν is the Kronecker delta.

Now, applying the representation of wave field $\psi(\vec{r},t)$ either of (55), or (7), into (54), we can get,

$$\begin{aligned} \hat{E}\,\psi(\vec{r},t) &= \frac{1}{m}\left(\hat{p}^i\,\hat{p}^j+\mathrm{g}_{\mu\nu}\,\hat{\mathcal{P}}^\mu\hat{\mathcal{P}}^\nu\right)\psi(\vec{r},t)\,,\\ \text{i.e.,}\quad i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t) &= -\,\frac{\hbar^2}{m}\left(\vec{\nabla}_i\,\vec{\nabla}_j+\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\right)\psi(\vec{r},t)\,. \end{aligned} \tag{61}$$

Note that, (61) should be used as an alternate of (58).

After using the first (or, second) term of mass operator $\hat{m}$ from (9), the (61) yields,

$$\hbar^2 \frac{\partial^2}{\partial s^2} \psi(\vec{r},t) = -\hbar^2 \left(\vec{\nabla}_i \vec{\nabla}_j + g_{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \right) \psi(\vec{r},t) , \tag{62}$$
$$\therefore \quad \hbar^2 \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right) \psi(\vec{r},t) = 0 .$$

Interested readers may compare (62) with (13) and can easily retain that, in (13), $\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = c^2 - (\mathrm{v}^i)^2 \sim c^2 \to 1$ as $c \gg \mathrm{v}^i$ and $c = 1$.

Now, the Second Variance of the Second Order Equation of GQG in Quantum Non-Minkowski Spacetime from (56) should be,

$$-\hbar^2 \vec{\Delta}_\mu \vec{\Delta}_\nu \psi(\vec{r},t) + \hbar^2 g_{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi(\vec{r},t) = 0 ,$$

where $i\hbar \vec{\Delta}_\mu \to [\hat{p}_0, -\hat{p}]^{\mathrm{T}} \to [i\hbar \vec{\nabla}_0, i\hbar \vec{\nabla}_i]^{\mathrm{T}}$.

Let us try to develop an Einstein field equation, which is "purely" Quantum Mechanical (i.e., it has neither a Minkowski spacetime and not its metric is Lorentzian) in comparison to (48). We can start with (61), which immediately tells us that (53) is possible to be written as,

$$\mathrm{d}\,\mathcal{S}^2_{\mathcal{P}}\, \psi(\vec{r},t) = g_{\mu\nu}\, \hat{\boldsymbol{\mathcal{P}}}^{\mu} \hat{\boldsymbol{\mathcal{P}}}^{\nu} \psi(\vec{r},t) = \left(\hat{m}\,\hat{E} - \hat{\boldsymbol{p}}^{i}\, \hat{\boldsymbol{p}}^{j} \right) \psi(\vec{r},t) , \tag{63}$$

then using (10) into it, we have,

$$\begin{aligned} \mathrm{d}\,\mathcal{S}^2_{\mathcal{P}}\, \psi(\vec{r},t) &= g_{\mu\nu}\, \hat{\boldsymbol{\mathcal{P}}}^{\mu} \hat{\boldsymbol{\mathcal{P}}}^{\nu} \psi(\vec{r},t) = -\hbar^2 \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i \partial x^j} \right) \psi(\vec{r},t) \\ &= -\hbar^2 \frac{\partial^2}{\partial s^2} \psi(\vec{r},t) = -\hbar^2 g^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi(\vec{r},t) . \end{aligned} \tag{64}$$

This line element has neither a Minkowski spacetime and not its metric is Lorentzian, since $g_{\mu\nu}$ is satisfying (59). Note here that, the LHS of (64), i.e., $\mathrm{d}\,\mathcal{S}^2_{\mathcal{P}}$, is relativistic, whereas, the RHS components of (64), i.e., $-\hbar^2 (\partial^2/\partial s^2)$, etc., are Quantum Mechanical. We can compare this line element with (13), but this time, the relativity-to-quantum relation (or vice versa) in (64) has been explained more uniquely and explicitly than (13). Though, despite this advantage, we should like to achieve our goal by not accepting (64) but by accepting the way similar to what we have already discussed in the previous subsection. Interested readers can check the workability of (64) by themselves.

Multiplying both sides of (56) by $(\mathrm{d}t/m)^2$ and comparing it with (52), we have the quantum line element for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}^{\mu} \to i\hbar \vec{\nabla}_\mu$ as follows,

$$\mathrm{d}s^2 \psi(\vec{r},t) = -\hbar^2 \left(\frac{\mathrm{d}t}{m} \right)^2 g_{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi(\vec{r},t) \to \left(\frac{\mathrm{d}t}{m} \right)^2 g_{\mu\nu}\, \hat{\boldsymbol{\mathcal{P}}}^{\mu} \hat{\boldsymbol{\mathcal{P}}}^{\nu} \psi(\vec{r},t) . \tag{65}$$

Similar to the Section 2.1, after a little exercise, we can develop,

$$\begin{aligned} \mathcal{R}_{\lambda\nu\gamma\delta}\, \psi(\vec{r},t) &= -\frac{1}{2} \hbar^2 \left(\vec{\nabla}_\nu \vec{\nabla}_\gamma g_{\lambda\delta} + \vec{\nabla}_\lambda \vec{\nabla}_\delta g_{\nu\gamma} - \vec{\nabla}_\lambda \vec{\nabla}_\gamma g_{\delta\nu} - \vec{\nabla}_\nu \vec{\nabla}_\delta g_{\lambda\gamma} \right) \psi(\vec{r},t) \\ \therefore \quad \left(\frac{1}{i\hbar} \right)^2 \mathcal{R}_{\lambda\nu\gamma\delta}\, \psi(\vec{r},t) &= R_{\lambda\nu\gamma\delta}\, \psi(\vec{r},t) \\ &= \frac{1}{2} \left(\partial_\nu \partial_\gamma g_{\lambda\delta} + \partial_\lambda \partial_\delta g_{\nu\gamma} - \partial_\lambda \partial_\gamma g_{\delta\nu} - \partial_\nu \partial_\delta g_{\lambda\gamma} \right) \psi(\vec{r},t) . \end{aligned}$$

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Let us develop another unusual classical-to-quantum gravitational coupling G (namely $\mathcal{G}_{(2)}$) in Planck scale using Proposition 1 and by accepting $\ell_p \mapsto \ell_P^{\mu}$ as,

$$\begin{aligned} G &= \frac{(\mathrm{d}\ell_P)^2}{m_P}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2} = \frac{m_P}{m_P^2}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,g_{\mu\nu}\,\mathrm{d}\ell_P^{\mu}\,\mathrm{d}\ell_P^{\nu} = \frac{F_P}{m_P^2}\,\mathrm{d}\ell_P^{\mu}\,\mathrm{d}\ell_{P\,\mu}\,, \\ \mathcal{G}_{(2)}\,\psi(\vec{r},t) &\to -\hbar^2\,F_P\left(\frac{\mathrm{d}t_P}{m_P^2}\right)^2 \vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\,\psi(\vec{r},t) = -\hbar^2\,\frac{\ell_P^{\mu}}{m_P^3}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\,\psi(\vec{r},t) \\ &= G\,\psi(\vec{r},t)\,, \end{aligned} \tag{66}$$

for $F_P = m_P\left\{\mathrm{d}^2\ell_P^{\mu}/(\mathrm{d}t_P)^2\right\}$. Now, let us take the classical-to-quantum energy momentum tensor as, for example,

$$\begin{aligned} T_{\mu\nu} &= -\frac{1}{\kappa^2}\left(\partial^{\lambda}\partial_{\lambda}\,g_{\mu\nu} - \partial^{\lambda}\partial_{\nu}\,g_{\mu\lambda} - \partial^{\lambda}\partial_{\mu}\,g_{\nu\lambda} + \partial_{\mu}\partial_{\nu}\,g^{\sigma}_{\sigma} -\right. \\ &\qquad \left. -\,\eta_{\mu\nu}\,\partial_{\lambda}\partial^{\lambda}\,g^{\sigma}_{\sigma} + \eta_{\mu\nu}\,\partial^{\lambda}\partial^{\sigma}\,g_{\lambda\sigma}\right), \\ \mathcal{T}_{(2)\,\mu\nu}\,\psi(\vec{r},t) &\to -\frac{1}{\kappa^2}\left(\vec{\nabla}^{\lambda}\vec{\nabla}_{\lambda}\,\mathrm{g}_{\mu\nu} - \vec{\nabla}^{\lambda}\vec{\nabla}_{\nu}\,\mathrm{g}_{\mu\lambda} - \vec{\nabla}^{\lambda}\vec{\nabla}_{\mu}\,\mathrm{g}_{\nu\lambda} + \vec{\nabla}_{\mu}\vec{\nabla}_{\nu}\,\mathrm{g}^{\sigma}_{\sigma} -\right. \\ &\qquad \left. -\,\eta_{\mu\nu}\,\vec{\nabla}_{\lambda}\vec{\nabla}^{\lambda}\,\mathrm{g}^{\sigma}_{\sigma} + \eta_{\mu\nu}\,\vec{\nabla}^{\lambda}\vec{\nabla}^{\sigma}\,\mathrm{g}_{\lambda\sigma}\right)\psi(\vec{r},t) \\ &= T_{\mu\nu}\,\psi(\vec{r},t)\,, \end{aligned}$$

etc. Then, we can obtain the Einstein field equation of the GQG in Quantum Non-Minkowski Spacetime, which can give us the classical Einstein field equation as,

$$\begin{aligned} \left[\mathcal{R}_{\mu\nu} - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\,\mathcal{R}\right]\psi(\vec{r},t) &= (i\hbar)^2\,8\pi\,\mathcal{G}_{(2)}\,\mathcal{T}_{(2)\,\mu\nu}\,\psi(\vec{r},t)\,, \\ \therefore\quad (i\hbar)^2\left[R_{\mu\nu} - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\,R\right]\psi(\vec{r},t) &= (i\hbar)^2\,8\pi\,G\,T_{\mu\nu}\,\psi(\vec{r},t)\,, \\ \Rightarrow\quad \left[R_{\mu\nu} - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\,R\right] &= 8\pi\,G\,T_{\mu\nu}\,. \end{aligned} \tag{67}$$

The first line of (67) is renormalizable, but its last line is not renormalizable by nature. Though, (67) yields the gravitational coupling-free field equation as,

$$\begin{aligned} \left[\mathcal{R}_{\mu\nu} - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\,\mathcal{R}\right]\psi(\vec{r},t) &= -(i\hbar)^2\,8\pi\,\hbar^2\,\frac{\ell_P^{\mu}}{m_P^3}\,T_{\mu\nu}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\,\psi(\vec{r},t) \\ &= 8\pi\,\hbar^4\,\frac{\ell_P^{\mu}}{m_P^3}\,T_{\mu\nu}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\,\psi(\vec{r},t)\,. \end{aligned} \tag{68}$$

This equation does not have any mass dimensions, so, it should be renormalizable by nature.

Interested readers can easily check that (48) and (67) are exactly the same expression but comprised with different components, that is, (48) has a mixed expression of classical and quantum geometries due to $g^{(\mathrm{Q})}_{\mu\nu}$, whereas, (67) has a purely quantum geometric expression due to $\mathrm{g}_{\mu\nu}$. In other words, we can say that (48) is in a Semi-Quantum Minkowski Spacetime with Semi-Quantum Lorentzian metric tensor, whereas, (67) is in a Quantum Non-Minkowski Spacetime with Non-Lorentzian metric tensor.

If we transform our spacetime into Planck scale, i.e., $x^{\mu} \to \ell_P^{\mu}$ and $t \to t_P$, and consider $m_P = \sum m = N\,m$, where m is the mass of a certain particle and N is a very large constant number since Planck mass is a very big number, thus, m_P is not considered here as the mass of a particular particle but the amount of N number of certain particle with mass m, then, by considering $\left(\ell_P^{\mu}\right)^{-1} = m_P = N\,m$ and d'Alembertian operator $\Box_P = \vec{\nabla}_{P\,\mu}\, \vec{\nabla}^{\,\mu}_P$, we can get the Third Variance of the Second Order Equation of GQG in Quantum Non-Minkowski Spacetime from (68) as follows,

$$\frac{8\pi\,\hbar^2}{(N\,m)^4}\, T_{\mu\nu}\, \Box_P\, \psi(\vec{r},t) + \left[R_{\mu\nu} - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\, R\right] \psi(\vec{r},t) = 0\,. \tag{69}$$

Thus, the First Order Equation of GQG in Quantum Non-Minkowski Spacetime should be as follows for the fermionic mass m_{F},

$$\begin{aligned} i\,\hbar\,\gamma^{\mu}\, \vec{\nabla}_{P\,\mu}\, \psi(\vec{r},t) - G^{1/2}\,(N_{\mathrm{F}}\, m_{\mathrm{F}})^2 \left(\frac{1}{8\pi\, G\, T_{\mu\nu}} \left[R_{\mu\nu} - \frac{1}{2}\,\mathrm{g}_{\mu\nu}\, R\right]\right)^{1/2} \psi(\vec{r},t) = 0\,, \\ \therefore \quad i\,\hbar\,\gamma^{\mu}\, \vec{\nabla}_{P\,\mu}\, \psi(\vec{r},t) - (N_{\mathrm{F}}\, m_{\mathrm{F}}) \left(\frac{1}{8\pi}\right)^{1/2} \psi(\vec{r},t) = 0\,, \end{aligned} \tag{70}$$

for $(8\pi\, G\, T_{\mu\nu})^{-1} \left[R_{\mu\nu} - \frac{1}{2}\, g^{(\mathrm{Q})}_{\mu\nu}\, R\right] = 1$ and $8\pi\, G = m_P^{-2} = (N_{\mathrm{F}}\, m_{\mathrm{F}})^{-2}$, where, γ^{μ} are Dirac's gamma matrices. Definitely, (70) gives us Dirac-like equation.

By the way, we can also rewrite (70) as follows for the fermionic mass $m_{\mathrm{F}} \mapsto \mathfrak{m}_{(\varrho)\,\mathrm{F}}$,

$$(8\pi)^{1/2}\, i\,\hbar\,\gamma^{\mu}\, \vec{\nabla}_{P\,\mu}\, \psi(\vec{r},t) \quad - \quad \left(N_{(\varrho)\,\mathrm{F}}\, \mathfrak{m}_{(\varrho)\,\mathrm{F}}\right) \psi(\vec{r},t) = 0\,. \tag{71}$$

Whatever matter satisfies (71), it must result almost $(8\pi)^{1/2}$ times high population density than the critical density of matter which satisfies (70), i.e., the fermions of (71) have almost five-times higher critical density than the fermions of (70), which is quite unusual. So, the fermion with mass $\mathfrak{m}_{(\varrho)\,\mathrm{F}}$ in (71) is not as same as the fermion with mass m_{F} in (70), i.e., they must be completely different particles. The only possible candidate having such characteristics as (71) is Dark Matter, which accounts for 26.8% of the critical density in the Universe against 4.9% of the critical density of baryonic matters, in other words, the critical density of Dark Matter is almost $(8\pi)^{1/2}$ times higher than the critical density of baryonic matters – it exactly matches with (71).

Again, returning to (65) and using $m^2\,(\mathrm{d}s^2/\mathrm{d}t^2)$ of (52) so as $m^2\,(\mathrm{d}s^2/\mathrm{d}t^2) = m^2\,(1 - v^2) \equiv m_0^2$ for the rest mass m_0, we can get,

$$\begin{aligned} m_0^2\, \psi(\vec{r},t) &= -\hbar^2\, \mathrm{g}_{\mu\nu}\, \vec{\nabla}_{\mu}\, \vec{\nabla}_{\nu}\, \psi(\vec{r},t)\,, \\ \therefore \quad & \hbar^2\, \mathrm{g}_{\mu\nu}\, \vec{\nabla}_{\mu}\, \vec{\nabla}_{\nu}\, \psi(\vec{r},t) + m_0^2\, \psi(\vec{r},t) = 0\,. \end{aligned}$$

Then, considering $\vec{\nabla}_{\nu} = \mathrm{g}_{\mu\nu}\, \vec{\nabla}^{\mu}$ and d'Alembertian operator $\Box = \vec{\nabla}_{\mu}\, \vec{\nabla}^{\mu}$, we have,

$$\hbar^2\, \mathrm{g}^2_{\mu\nu}\, \Box\, \psi(\vec{r},t) + m_0^2\, \psi(\vec{r},t) = 0\,.$$

Thus, we can immediately develop a fermionic field equation as,

$$i\,\hbar\,\gamma^{\mu}\, \mathrm{g}_{\mu\nu}\, \vec{\nabla}_{\mu}\, \psi(\vec{r},t) - m_{\mathrm{F}\,0}\, \psi(\vec{r},t) = 0\,, \tag{72}$$

for fermionic rest mass $m_{\mathrm{F}\,0}$.

Now, let us replace m^2 of (52) with the Planck mass $m_P^2 = \sum \mathfrak{m}^2_{(\Lambda)} = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2$ for a certain particle with mass $\mathfrak{m}_{(\Lambda)}$, where $N_{(\Lambda)}$ is a very large constant number and $\left\{N_{(\Lambda)},\, \mathfrak{m}_{(\Lambda)}\right\} \neq \left\{N_{(\varrho)},\, \mathfrak{m}_{(\varrho)}\right\}$ as $\mathfrak{m}_{(\Lambda)} \ll \mathfrak{m}_{(\varrho)}$; then $m_P^2 = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2$ satisfies as: $m_P^2\,\Lambda = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 \Lambda = \frac{1}{2}\,\langle T\rangle$, where the cosmological constant $\Lambda = 8\pi\, G\, \rho_{\Lambda}$, so as we have $\left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 (\mathrm{d}s^2/\mathrm{d}t^2) = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 (1 - v^2) \equiv \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\right)^2$ for the Planck rest mass $m_{P\,0} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}$ and $c = 1$, thus, for,

$$\left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 \Lambda\,(1-v^2) = \frac{1}{2}\,\langle T\rangle\,(1-v^2) \quad \Longleftrightarrow \quad \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\right)^2 \Lambda = \frac{1}{2}\,\langle T_0\rangle\,,$$

where, $\langle T\rangle\,(1-v^2) \equiv \langle T_0\rangle$, and following the argument cited above,

$$\begin{aligned} & \hbar^2\, g_{\mu\nu}\, \vec{\nabla}_{P\,\mu}\, \vec{\nabla}_{P\,\nu}\, \psi(\vec{r},t) + \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\right)^2 \psi(\vec{r},t) = 0\,, \\ \therefore \quad & 2\,\hbar^2\,\Lambda\, g^2_{\mu\nu}\,\Box_P\,\psi(\vec{r},t) + \langle T_0\rangle\,\psi(\vec{r},t) = 0\,, \\ \text{and,} \quad & \hbar^2\,\Lambda\, g^2_{\mu\nu}\,\Box_P\,\psi(\vec{r},t) + \rho_{\Lambda\,0}\,\psi(\vec{r},t) = 0\,, \end{aligned} \tag{73}$$

by replacing $\frac{1}{2}\,\langle T\rangle$ with ρ_Λ for the cosmological constant $\Lambda = 8\pi\,G\,\rho_\Lambda = m_P^{-2}\,\rho_\Lambda = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^{-2}\rho_\Lambda$. So, by using $8\pi\,G = m_P^{-2} = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^{-2}$ and by switching $(1-v^2)^{1/2}$ right to left in the term: $N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\,\langle T_0\rangle^{1/2} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\,\langle T\rangle^{1/2}\,(1-v^2)^{1/2} \equiv N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\,\langle T\rangle^{1/2}$, we can develop a fermionic field equation as follows,

$$\begin{aligned} & i\,\hbar\,\sqrt{2\Lambda}\,\gamma^\mu\, g_{\mu\nu}\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r},t) - \langle T_0\rangle^{1/2}\,\psi(\vec{r},t) = 0\,, \\ \therefore \quad & i\,\hbar\,\left(2\,\rho_\Lambda\right)^{1/2}\gamma^\mu\, g_{\mu\nu}\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r},t) - N_{(\Lambda)\,\mathrm{F}}\,\mathfrak{m}_{(\Lambda)\,\mathrm{F}\,0}\,\langle T\rangle^{1/2}\,\psi(\vec{r},t) = 0\,, \end{aligned} \tag{74}$$

for fermionic rest mass $\mathfrak{m}_{(\Lambda)\,\mathrm{F}\,0}$ as $N_{(\Lambda)\,\mathrm{F}} \neq N_{(\Lambda)}$. The interesting thing in (74) is that Dark Energy has a direct relationship with gravity. In other words, Dark Energy would be obtainable from the breaking of particle symmetry where gravity counts, or vice versa (see, Section 3.1 below, for example).

Note that the last equation of (73) is definitely applicable simultaneously whether the matter is baryonic or non-baryonic.

Since (67) and (48) are exactly the same, by combining the last equation of (73) with (69), we can get a non-renormalizable field equation as.

$$\hbar^2\left(\frac{8\pi}{(N\,m)^4}\,T_{\mu\nu} \quad + \quad \Lambda\, g^2_{\mu\nu}\right)\Box_P\,\psi(\vec{r},t) + \left[R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\,R + \rho_{\Lambda\,0}\right]\psi(\vec{r},t) = 0\,, \tag{75}$$

which should be renormalizable by reintroducing $(i\hbar)^2$ to the both terms of (75). This field equation, which is actually a Klein-Gordon-type equation, is slightly different form the classical Einstein field equation.

Again, either from the first equation of (73), or by placing $\frac{1}{2}\,\langle T\rangle = \rho_\Lambda$ in (74), we can find,

$$i\,\hbar\,\gamma^\mu\, g_{\mu\nu}\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r},t) - N_{(\Lambda)\,\mathrm{F}}\,\mathfrak{m}_{(\Lambda)\,\mathrm{F}\,0}\,\psi(\vec{r},t) = 0\,, \tag{76}$$

which is the Planck scale counterpart of (71) and (72), in other words, (71), (72) and (76) are counterbalancing each other's actions upon the Universe.

Since, the cosmological constant $\Lambda = 8\pi\,G\,\rho_\Lambda = m_P^{-2}\,\rho_\Lambda = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^{-2}\rho_\Lambda$, then again replacing m^2 of (52) with $m_P^2 = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 = (\rho_\Lambda/\Lambda)$ gives us, $(\rho_\Lambda/\Lambda)\,(\mathrm{d}s^2/\mathrm{d}t^2) = (\rho_\Lambda/\Lambda)\,(1 - v^2) \equiv (\rho_{\Lambda\,0}/\Lambda)$, for $c = 1$. But, we can say, $E_\Lambda^2 = (\rho_{\Lambda\,0}/\Lambda)\,c^4$, i.e., $E_\Lambda = (\rho_{\Lambda\,0}/\Lambda)^{1/2}\,c^2$, as the *rightful and lawful* 'Dark Energy' for relativistic ρ_Λ.

The grate difference between (71) and (76) is that the nature of the former one is non-baryonic, whereas, the later one is independent of matter's constructive property, i.e., its effects can be observable simultaneously both in the cases of baryonic and non-baryonic matters. Another difference is that (71) is effective at $m_P = N_{(\varrho)}\mathfrak{m}_{(\varrho)}$ scale, whereas, (76) is effective at $m_{P\,0} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}$ scale, i.e., Dark Energy had originated at much earlier cosmological epochs than Dark Matter. Similarly, Dark Matter

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had originated at much earlier cosmological epochs than baryonic matters of (72) at m_0 scale. Thus, we have a quite fair chronology of the formation of cosmological matters in the Universe.

2.2.1. Superstring/M-theory

Instead of considering $\mathfrak{X}^\mu$ of (36), let us develop $\mathfrak{X}^\mu$ with purely quantum spacetime axes as follows,

$$\begin{aligned} &\Big[\left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right), a_i \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right), b_i \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right), \\ &\qquad \left\{ (a_i a_j)^{1/2} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) + (b_i b_j)^{1/2} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \right\} \Big] \qquad (77) \\ &\mapsto \left((i\hbar)^{-1}\mathfrak{x}^0, (i\hbar)^{-1}\mathfrak{x}^1, (i\hbar)^{-1}\mathfrak{x}^{(1+i)}, (i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell))}, (i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell)+j)} \right) \in \mathfrak{X}^\mu , \end{aligned}$$

for $i, j = 1, 2, 3$, $i \neq j$ and $\ell = \max i$, where,

$$(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell)+j)} = \left((i\hbar)^{-1}\mathfrak{x}_a^{(1+(i+\ell)+\frac{1}{2}j)} + (i\hbar)^{-1}\mathfrak{x}_b^{(1+(i+\ell)+\frac{1}{2}j)} \right) .$$

For the selection of the axis from (77), we use fully democratic way, e.g., if $\left\{ (a_3 a_1)^{1/2} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) + (b_3 b_1)^{1/2} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \right\}$ gives the 11^{th} dimension, then $a_3 \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_3 \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ should give us the 9^{th} and 10^{th} dimensions, respectively, and so on.

Exclusively for this sub-subsection, we will switch to the Minkowski signature $(-, +, \ldots, +)$ to express a point particle and a string propagating in a D-dimensional curved spacetime.

The bosonic part of the action of the $N = 1$ supergravity theory in 11 dimensions should be [10],

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}(i\hbar)^{-1}\mathfrak{x} \sqrt{-\mathrm{g}} \left(\mathsf{R} - \frac{1}{48} F_4^2 \right) - \frac{1}{12\kappa^2} \int F_4 \wedge F_4 \wedge A_3 , \qquad (78)$$

where $F_4 = \mathrm{d}\, A_3$. Definitely, this $D = 11$ supergravity is now in $\mathfrak{X}^\mu$ spacetime of (77) with purely quantum gravity g, which is satisfying (59) but now with the signature $(-, +, \ldots, +)$. The overall scenario is condensed inside the observable $(3 + 1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$ for $i = 1, 2, 3$. In other words, the eleven-dimensional Supergravity is necessarily a natural phenomenon within the quantum spacetime of GQG.

Similarly, the bosonic part of the action of type IIA theory ($N = 2$, $d = 10$) should be (in the string frame),

$$\begin{aligned} S_{\text{IIA}} \;=\; & \frac{1}{2\kappa^2} \int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x} \sqrt{-\mathrm{g}} \Bigg(\exp\left(-2\phi\right) \left[\mathsf{R} + 4(\partial\phi)^2 - \frac{1}{12} H_3^2 \right] - \\ & - \frac{1}{4} F_2^2 - \frac{1}{48} F_4^2 \Bigg) - \frac{1}{4\kappa^2} \int F_4 \wedge F_4 \wedge B_2 , \end{aligned} \qquad (79)$$

where $F_4 = \mathrm{d}\, C_3 + H_3 \wedge C_1$, $F_2 = \mathrm{d}\, C_1$, $H_3 = \mathrm{d}\, B_2$ and ϕ is the dilaton. And the bosonic part of the action of type IIB theory ($N = 2$, $d = 10$) should be (in the string frame),

$$\begin{aligned} S_{\text{IIB}} \;=\; & \frac{1}{2\kappa^2} \int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x} \sqrt{-\mathrm{g}} \Bigg(\exp\left(-2\phi\right) \left[\mathsf{R} + 4(\partial\phi)^2 - \frac{1}{12} H_3^2 \right] - \frac{1}{2} (\partial\phi)^2 - \\ & - \frac{1}{12} \left[F_3 + aH_3\right]^2 - \frac{1}{480} F_5^2 \Bigg) + \frac{1}{4\kappa^2} \int \left(C_4 + \frac{1}{2} B_2 \wedge C_2 \right) \wedge F_3 \wedge H_3 , \end{aligned}$$

where $F_5 = \mathrm{d}C_4 + H_3 \wedge C_2$, $F_3 = \mathrm{d}C_2$, $H_3 = \mathrm{d}B_2$, while a is the RR axion and ϕ is the dilaton. Whereas, the bosonic part of the type I action ($N = 1$, $d = 10$) should be,

$$S_{\mathrm{I}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x}\,\sqrt{-\mathrm{g}}\left(\exp\left(-2\phi\right)\left[\mathrm{R} + 4(\partial\phi)^2\right] - \frac{1}{12}\tilde{F}_3^2 - \frac{1}{4}\exp\left(-\phi\right)\mathrm{Tr}F^2\right),$$

where $\tilde{F}_3$ is the modified field strength for the two form, ϕ is the dilaton and $F_2 = \mathrm{d}A + A \wedge A$, where A is the gauge potential in the adjoint representation of SO(32). As the two heterotic supergravity theories are obtained as the low energy limit of heterotic string theory with gauge group SO(32) and $E_8 \times E_8$, respectively, the bosonic part of the actions should be,

$$S_{\mathrm{Heterotic}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x}\,\sqrt{-\mathrm{g}}\,\exp\left(-2\phi\right)\left(\mathrm{R} + 4(\partial\phi)^2 - \frac{1}{12}\tilde{H}_3^2 - \frac{1}{4}\mathrm{Tr}F^2\right),$$

where $\tilde{H}_3$ is the modified field strength for the two form, ϕ is the dilaton and $F_2 = \mathrm{d}A + A \wedge A$, where A is the gauge potential in the adjoint representation of SO(32) or $E_8 \times E_8$, respectively. In the similar way, a solitonic supergravity solution for p-branes in 11 dimensions which interpolates between a vacuum with SO(1,2)×SO(8) symmetry yields,

$$\begin{aligned} \mathrm{d}s^2 &= H^{-2/3}\,\eta_{\mu\nu}\,(i\hbar)^{-2}\mathrm{d}\,\mathfrak{x}^{\mu}\,\mathrm{d}\,\mathfrak{x}^{\nu} + H^{1/3}\,\delta_{mn}\,(i\hbar)^{-2}\mathrm{d}\,\mathfrak{x}^{m}\,\mathrm{d}\,\mathfrak{x}^{n}\,, \\ A_3 &= H^{-1}\,(i\hbar)^{-3}\mathrm{d}\,\mathfrak{x}^{0}\wedge\mathrm{d}\,\mathfrak{x}^{1}\wedge\mathrm{d}\,\mathfrak{x}^{2}\,, \\ H &= 1 + \frac{R^6}{r^6}\,, \end{aligned}$$

where $\mu, \nu = 0, 1, 2$, $m, n = 3, \ldots, 10$, and H is a harmonic function on the transverse space, whereas r is the radius for the eight-dimensional space transverse to the membrane. Hence, the complete scenario of the Superstring/M-theory is condensed inside the observable $(3+1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$ for $i = 1, 2, 3$. We know that conventionally M-theory on $\mathbb{R}^{11}$ does not contain any strings, however, as we have replaced Minkowski spacetime with $\mathfrak{X}^{\mu}$ spacetime for (77), hence, we have found that the new scenario of M-theory on $\mathfrak{X}^{\mu}$ is now definitely contain strings, which are condensed inside the observable $(3+1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$ for $i = 1, 2, 3$.

Let Planck mass $m_P = \sum \mathfrak{m} = N\mathfrak{m}$ for a certain particle with mass $\mathfrak{m}$, where N is a very large constant number. Since M-theory is the strong coupling limit of the type IIA string theory, it must be an inherently non-perturbative theory, with no arbitrary coupling constant, but only a length scale ℓ_p, then the relation between this length scale and the IIA length scale and coupling can be obtained by comparing the 11 and 10-dimensional gravitational constants κ_{11} and κ_{10}. But we know from (66) $\mathcal{G}_{(2)} \to -\hbar^2\,\frac{\ell_P^{\mu}}{(N\mathfrak{m})^3}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}$ in the observable $(3+1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$ for $i = 1, 2, 3$. Then, κ_{11} and κ_{10} are related as, $\kappa_{11}^2 = 2\pi\,R\,\kappa_{10}^2$, where, $2\kappa_{10}^2 = (2\pi)^7\,g^2\,\alpha'^4$, whereas, by accepting $\ell_p \mapsto \ell_P^{\mu}$,

$$\begin{aligned} 2\kappa_{11}^2 &= (2\pi)^8\left(-\hbar^{-2}\,\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\right]^{-1}\right)^9, \\ \therefore\quad 2\,\hbar^{18}\,\kappa_{11}^2\,\Box_P^9 &+ (2\pi)^8\,\mathcal{G}_{(2)}^9\,(N\mathfrak{m})^{27} = 0\,, \end{aligned} \tag{80}$$

which yields the first order equation as,

$$i\hbar^9\,\gamma^{\mu}\,\vec{\nabla}_{P\,\mu}^{9/2} - \frac{(2\pi)^4\,\mathcal{G}_{(2)}^{9/2}}{2^{1/2}\,\kappa_{11}}\,(N_{\mathrm{F}}\,\mathfrak{m}_{\mathrm{F}})^{27/2} = 0\,,$$

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for fermionic mass $\mathfrak{m}_{\,\mathrm{F}}$ as $N_{\mathrm{F}} \neq N$, when $\Box_{\,P} = \vec{\nabla}_{P\,\mu}\,\vec{\nabla}_{P}^{\,\mu}$ and $\gamma^{\,\mu}$ are Dirac's gamma matrices. This shows that the eleventh direction is more dynamic than what we used to think about it. Apart from this, (80) establishes the dynamic relationship between string and the observable $(3+1)D$ spacetime's gravitational constant $\mathcal{G}_{\,(2)}$. We can again obtain the following relation using (80),

$$\left(-\,\hbar^{\,-2}\,\mathcal{G}_{\,(2)}\,\left(N\mathfrak{m}\right)^3 \left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_{P}^{\,\mu}\right]^{-1}\right)^2 = g^{\,2/3}\,\alpha^{\,\prime}\,,$$

that,

$$\begin{aligned} \alpha^{\,\prime} &= \frac{1}{R}\left(-\,\hbar^{\,-2}\,\mathcal{G}_{\,(2)}\,\left(N\mathfrak{m}\right)^3 \left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_{P}^{\,\mu}\right]^{-1}\right)^3 , \\ g &= R^{\,3/2}\left(-\,\hbar^{\,-2}\,\mathcal{G}_{\,(2)}\,\left(N\mathfrak{m}\right)^3 \left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_{P}^{\,\mu}\right]^{-1}\right)^{-3/2} . \end{aligned} \tag{81}$$

Hence, in presence of a bosonic field (80), when the 11-dimensional radius is much smaller than the 11-dimensional length scale we effectively have a 10-dimensional theory, which is type IIA string theory. But this idea should be applicable for any dimensional radious $R_{n\mathrm{th}}$ and any dimensional length scale since we have taken $\ell_p \mapsto \ell_P^{\,\mu}$; thus, we can rewrite (81) as follows,

$$\begin{aligned} \alpha^{\,\prime} &= \frac{\Theta^{\,3}_{n\mathrm{th}}}{R_{n\mathrm{th}}}\left(-\,\hbar^{\,-2}\,\mathcal{G}_{\,(2)}\,\left(N\mathfrak{m}\right)^3 \left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_{P}^{\,\mu}\right]^{-1}\right)^3 , \\ g &= \left(\Theta^{-1}_{n\mathrm{th}}\,R_{n\mathrm{th}}\right)^{3/2}\left(-\,\hbar^{\,-2}\,\mathcal{G}_{\,(2)}\,\left(N\mathfrak{m}\right)^3 \left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_{P}^{\,\mu}\right]^{-1}\right)^{-3/2} , \end{aligned}$$

here $\Theta_{n\mathrm{th}}$ is a coefficient for the various energy lavels of $\ell_P^{\,\mu}$, for example, $\Theta_{11} = 1$. This suggests that the low energy limit of the strong coupling limit of IIA string theory (which is M-theory) must be the 11-dimensional supergravity for any dimensional radius $R_{n\mathrm{th}}$ and any dimensional length scale in presence of a bosonic field (80). Note it that the above all equations can yield their first order fermionic equations derivable from them, if we like so.

Let no fields depend on the $10^{\,\mathrm{th}}$ space direction $(i\hbar)^{-1}\mathrm{d}\,\mathfrak{r}^{\,10}$, then splitting the metric and having the three form as,

$$\begin{aligned} \mathrm{d}s^2_{11} &= \exp\left(-2\phi/3\right)\mathrm{d}s^2_{\mathrm{IIA}} + \exp\left(4\phi/3\right)(i\hbar)^{-1}\left(\mathrm{d}\,\mathfrak{r}^{\,10} + C_\mu\,\mathrm{d}\,\mathfrak{r}^{\,\mu}\right)^2 , \\ A_{\mu\nu,\,10} &= B_{\mu\nu}\,, \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho}\,, \end{aligned}$$

and inserting these relations into the 11-dimensional supergravity action (78), a straightforward calculation gives the type IIA supergravity action (79), in the string frame. If no fields depend on the n^{th} space direction $(i\hbar)^{-1}\mathrm{d}\,\mathfrak{r}^{\,n}$, then we have,

$$\begin{aligned} \mathrm{d}s^2_{11} &= \exp\left(-2\phi/3\right)\mathrm{d}s^2_{\mathrm{IIA}} + \Theta_{n\mathrm{th}}\,\exp\left(4\phi/3\right)(i\hbar)^{-1}\left(\mathrm{d}\,\mathfrak{r}^{\,n^{\mathrm{th}}} + C_\mu\,\mathrm{d}\,\mathfrak{r}^{\,\mu}\right)^2 , \\ A_{\mu\nu,\,n^{\mathrm{th}}} &= B_{\mu\nu}\,, \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho}\,. \end{aligned}$$

Hence, type IIA supergravity can be obtainable from a dimensional reduction of 11-dimensional supergravity (M-theory), where the reduced dimension should be depended on any n^{th} space direction $(i\hbar)^{-1}\mathrm{d}\,\mathfrak{r}^{\,n^{\mathrm{th}}}$. So, obtaining type IIA string theory from M-theory by dimensional reduction is now not only restricted for the $10^{\,\mathrm{th}}$ space direction but universally for any n^{th} space direction.

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The relations between branes in type IIA string theory and M-theory may descrbe in the following new format of tensions establishing the relation with the observable $(3+1)D$ spacetime's gravitational constant $\mathcal{G}_{(2)}$ as,

M-brane:

$$
\begin{aligned}
\mathrm{MW}: &\quad \frac{1}{R_{n\text{th}}}, \\
\mathrm{M2}: &\quad \frac{1}{(2\pi)^2}\left[\Theta_{n\text{th}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)\right]^{-3}, \\
\mathrm{M5}: &\quad \frac{1}{(2\pi)^5}\left[\Theta_{n\text{th}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)\right]^{-6}, \\
\mathrm{KK6}: &\quad \frac{(2\pi R_{n\text{th}})^2}{(2\pi)^8}\left[\Theta_{n\text{th}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)\right]^{-9}.
\end{aligned}
$$

Type IIA-brane:

$$
\begin{aligned}
\mathrm{D0}: &\quad \frac{1}{R_{n\text{th}}} = \left[\left(\Theta_{n\text{th}}^{-1}R_{n\text{th}}\right)^{3/2}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3/2}\right]^{-1}\times \\
&\qquad\qquad \times\left[\frac{\Theta_{n\text{th}}^3}{R_{n\text{th}}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3\right]^{-1/2}, \\
\mathrm{F1}: &\quad \frac{2\pi R_{n\text{th}}}{(2\pi)^2}\Theta_{n\text{th}}^{-3}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3} \\
&\qquad = \left[(2\pi)\frac{\Theta_{n\text{th}}^3}{R_{n\text{th}}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3\right]^{-1}, \\
\mathrm{D2}: &\quad \frac{1}{(2\pi)^2}\left[\Theta_{n\text{th}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)\right]^{-3} \\
&\qquad = \frac{1}{(2\pi)^2}\left[\left(\Theta_{n\text{th}}^{-1}R_{n\text{th}}\right)^{3/2}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3/2}\right]^{-1}\times \\
&\qquad\qquad \times\left[\frac{\Theta_{n\text{th}}^3}{R_{n\text{th}}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3\right]^{-3/2}, \\
\mathrm{D4}: &\quad \frac{2\pi R_{n\text{th}}}{(2\pi)^5}\left[\Theta_{n\text{th}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)\right]^{-6} \\
&\qquad = \frac{1}{2\pi}\left[\left(\Theta_{n\text{th}}^{-1}R_{n\text{th}}\right)^{3/2}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3/2}\right]^{-1}\times \\
&\qquad\qquad \times\left[\frac{\Theta_{n\text{th}}^3}{R_{n\text{th}}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3\right]^{-5/2}, \\
\mathrm{NS5}: &\quad \frac{1}{(2\pi)^5}\left[\Theta_{n\text{th}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)\right]^{-6} \\
&\qquad = \frac{1}{(2\pi)^5}\left[\left(\Theta_{n\text{th}}^{-1}R_{n\text{th}}\right)^{3/2}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3/2}\right]^{-2}\times \\
&\qquad\qquad \times\left[\frac{\Theta_{n\text{th}}^3}{R_{n\text{th}}}\left(-\hbar^{-2}\mathcal{G}_{(2)}\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3\right]^{-3},
\end{aligned}
$$

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$$
\begin{aligned}
\mathrm{D6}: \quad \frac{(2\pi\, R_{n^{\mathrm{th}}})^{2}}{(2\pi)^{8}} & \left[\Theta_{n^{\mathrm{th}}}\left(-\hbar^{-2}\,\mathcal{G}_{(2)}\,(N\mathfrak{m})^{3}\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}^{\mu}_{P}\right]^{-1}\right)\right]^{-9} \\
&= \frac{1}{(2\pi)^{5}}\left[\left(\Theta^{-1}_{n^{\mathrm{th}}}\,R_{n^{\mathrm{th}}}\right)^{3/2}\left(-\hbar^{-2}\,\mathcal{G}_{(2)}\,(N\mathfrak{m})^{3}\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}^{\mu}_{P}\right]^{-1}\right)^{-3/2}\right]^{-1}\times \\
&\quad\times\left[\frac{\Theta^{3}_{n^{\mathrm{th}}}}{R_{n^{\mathrm{th}}}}\left(-\hbar^{-2}\,\mathcal{G}_{(2)}\,(N\mathfrak{m})^{3}\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}^{\mu}_{P}\right]^{-1}\right)^{3}\right]^{-7/2}.
\end{aligned}
$$

Note that above all tension relations of type IIA-branes are highly dynamic in nature.

Since, type IIA and IIB string theories are T-dual when compactified on a small and large circle, respectively, then it must be possible to relate M-theory and type IIB string theory by using the same consideration as above. Thus, we obtain the following relations between M-theory and type IIB string theory parameters as,

$$
\begin{aligned}
g_{\mathrm{B}} &= \frac{R_{n^{\mathrm{th}}}}{R_{(n-1)^{\mathrm{th}}}}\,, \\
R_{\mathrm{B}} &= \frac{1}{R_{(n-1)^{\mathrm{th}}}\,R_{n^{\mathrm{th}}}}\left[\Theta_{n^{\mathrm{th}}}\left(-\hbar^{-2}\,\mathcal{G}_{(2)}\,(N\mathfrak{m})^{3}\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}^{\mu}_{P}\right]^{-1}\right)\right]^{3}.
\end{aligned}
$$

This last relation can also yield its first order equation which is quite interesting in nature.

We have skipped a very interesting thing in our above discussion that would be occurred if we construct flux compactifications of M-theory to three-dimensional $\mathfrak{X}$ spacetime preserving $N = 2$ supersymmetry. Here, a scalar function depending on the coordinates of the internal dimensions $\Delta(y)$ (called the warp factor) is included as the explicit form for the metric ansatz as,

$$
\mathrm{d}s^{2} \;=\; \Delta(y)^{-1}\eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\mu} + \Delta(y)^{1/2}g_{mn}(y)\mathrm{d}y^{m}\mathrm{d}y^{u}\,,
$$

where,

$$
x^{\mu} = \left((i\hbar)^{-1}\mathfrak{x}^{0},\,(i\hbar)^{-1}\mathfrak{x}^{1},\,(i\hbar)^{-1}\mathfrak{x}^{(1+i')}\right), \qquad \text{for } i' = 1\,,
$$

are the coordinates of the three-dimensional spacetime M_3 and,

$$
y^{m} = \left((i\hbar)^{-1}\mathfrak{x}^{(1+i'')},\,(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell))},\,(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell)+j)}\right), \quad \text{for } \begin{cases} i'' = 2,3\,, \\ i = 1,2,3\,, \end{cases}
$$

are the coordinates of the internal eight-manifold M. This opens a new scope for the understanding of string geometry.

Relating M-theory on a line interval and $E_8 \times E_8$ heterotic string theory is quite obvious now, so, we have omitted it here.

3. Dark Energy Gauge Symmetries

In the above subsections, we have gotten a group of Dirac-like equations for fermions, such as,

1. In the case of baryonic matter, it yields for (51),

$$
i\,\hbar\,\gamma^{\mu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t) - \hat{\boldsymbol{\mathcal{P}}}^{\mu}\left(g^{\mu\nu}_{(\mathrm{Q})}\right)^{1/2}\psi(\vec{r},t) = 0\,,
$$

   or, in Planck scale, for (70),

$$
i\,\hbar\,\gamma^{\mu}\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r},t) - (N_{\mathrm{F}}\,m_{\mathrm{F}})\left(\frac{1}{8\pi}\right)^{1/2}\psi(\vec{r},t) = 0\,,
$$

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or, simply, for (72),

$$i\hbar\,\gamma^{\mu}\,g_{\mu\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t) - m_{\mathrm{F}\,0}\,\psi(\vec{r},t) = 0\,.$$

2. In the case of Dark Matter, it yields for (71),

$$(8\pi)^{1/2}\,i\hbar\,\gamma^{\mu}\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r},t) \quad - \quad \left(N_{(\varrho)\,\mathrm{F}}\,\mathfrak{m}_{(\varrho)\,\mathrm{F}}\right)\psi(\vec{r},t) = 0\,.$$

3. In the case of Dark Energy, it yields for (76),

$$i\hbar\,\gamma^{\mu}\,g_{\mu\nu}\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r},t) - N_{(\Lambda)\,\mathrm{F}}\,\mathfrak{m}_{(\Lambda)\,\mathrm{F}\,0}\,\psi(\vec{r},t) = 0\,.$$

If we try to develop an Yang-Mills Lagrangian for gravity, we can choose either (70) or (72). Though, gravity is feeblest in Electroweak or Quantum Chromodynamic interactions, but gravity is always related to their interacting particles, thus, we should like to go with (72) for Electroweak or Quantum Chromodynamic Yang-Mills Lagrangians, as well as for gravitational Yang-Mills Lagrangian, too. Since the last equation of (73) is applicable simultaneously whether the matter is baryonic or non-baryonic, then we should like to choose (76) to include Dark Energy in the Yang-Mills Lagrangian along with gravitational and Electroweak or Quantum Chromodynamic interactions.

Here, we only touch upon the bare minima of Electroweak or Quantum Chromodynamic interactions in presence of gravity and Dark Energy. We have left a number of features for the interested readers to check them out with their own interests.

3.1. Gravitational Electroweak Dark Energy Interactions

Let $\hbar = c = 1$. For baryonic matter, let us include gravity and Dark Energy in the Yang-Mills Lagrangian of the Electroweak symmetry $SU(2)_L \otimes U(1)_Y$. Let us consider a gauge group $U(1)_G$ as the symmetry associated with gravity group and it is unbroken, since it does not directly interact with the Higgs.

Let us consider Casimir energy is associated with the right-handed fermions, so by choosing the isospin quantum numbers of different Standard Model fermions and by considering the $U(1)_{\mathscr{C}}$ is the symmetry associated with the Casimir hypercharge, $Y_{\mathscr{C}} = -1$, we can consider the Casimir hypercharge field χ. So, for the overall invariance $SU(2)_L \otimes U(1)_Y \otimes U(1)_G \otimes U(1)_{\mathscr{C}}$, we can assume the unbroken Gravitational Electroweak Dark Energy (GED) gauge group must be mathematically at least $SU(3)_{GED}$. If so, then, Casimir energy must be unified with the Dark Energy interactions minimally as a Yang-Mills field with an $SU(2)_D \otimes U(1)_{\mathscr{C}}$ gauge group, where $SU(2)_D$ is gauged Dark Energy isospin. So, the Gravitational Electroweak Dark Energy interactions (GED) can trigger the symmetry breaking,

$$\begin{aligned} SU(3)_{GED} \quad &\rightarrow \quad SU(2)_L \otimes U(1)_Y \otimes U(1)_G \otimes SU(2)_D \otimes U(1)_{\mathscr{C}} \\ &\rightarrow \quad U(1)_{em} \otimes U(1)_G \otimes SU(2)_D \otimes U(1)_{\mathscr{C}}\,, \end{aligned}$$

which describes the formal operations that can be applied to the Electroweak, gravitational and Dark Energy gauge fields without changing the dynamics of the system. Let $SU(2)_D \otimes U(1)_{\mathscr{C}}$ fields are the Dark Energy isospin fields Y_1, Y_2, and Y_3, and the Dark Energy hypercharge field χ. We need to remember here that fermionic isospin states in reactions/decays governed by the Dark Energy interaction are conserved, i.e., the transition from $|00\rangle$ to $|10\rangle$ is not allowed in Dark Energy interaction, which is quite familiar with Electromagnetic sector of Electroweak interaction, whereas, no conservation of isospin is occurred in GED. We should remember that bosons do not give Dark Energy by nature. On the other hand, all bosons gravitate (e.g., opposite moving photons gravitate).

Let the Yang-Mills Lagrangian of the Gravitational Electroweak Dark Energy (GED) interaction is,

$$\begin{aligned}\mathscr{L}_{GED} &= \mathrm{g}_{\mu\nu}\left[\bar{\ell}\, i\, \gamma^{\mu}\, D_{\mu}\, \ell + \bar{e}_R\, i\, \gamma^{\mu}\, D_{\mu}\, e_R + \bar{f}\, i\, \gamma^{\mu}\, \mathcal{D}_{\mu\,(G)}\, f + \right. \\ &\left. + \bar{\ell}\, i\, \gamma^{\mu}\, \mathscr{D}^{L}_{\mu\,(D)}\, \ell + \bar{e}_R\, i\, \gamma^{\mu}\, \mathscr{D}^{R}_{\mu\,(D)}\, e_R\right] - \\ &- \frac{1}{4}\left[\mathbf{W}_{\mu\nu}\, \mathbf{W}^{\mu\nu} + B_{\mu\nu}\, B^{\mu\nu} + \mathcal{G}_{\mu\nu}\, \mathcal{G}^{\mu\nu} + \mathbf{Y}_{\mu\nu}\, \mathbf{Y}^{\mu\nu} + \chi_{\mu\nu}\, \chi^{\mu\nu}\right],\end{aligned} \tag{82}$$

where, f indicates all kind of fermions. By using (72) for Electroweak interaction and gravity, whereas (76) for Dark Energy interaction, here, covariant derivatives and field tensors are given by,

$$\begin{aligned}D_{\mu}\, \ell &= \left\{\vec{\nabla}_{\mu} + i\, g_{W}\, \mathbf{W}_{\mu} \cdot \frac{\boldsymbol{\tau}}{2} + i\, g_{B}\left(\frac{Y_B}{2}\right) B_{\mu}\right\} \ell \\ &= \left\{\vec{\nabla}_{\mu} + i\, g_{W}\, \mathbf{W}_{\mu} \cdot \frac{\boldsymbol{\tau}}{2} + i\, g_{B}\left(-\frac{1}{2}\right) B_{\mu}\right\} \ell, \\ D_{\mu}\, e_R &= \left\{\vec{\nabla}_{\mu} + i\, g_{B}\left(-\frac{1}{2}\right) B_{\mu}\right\} e_R, \\ \mathcal{D}_{\mu\,(G)}\, f &= \left\{\vec{\nabla}_{\mu} + i\, g_{G}\left(\frac{Y_G}{2}\right) \mathsf{G}_{\mu}\right\} f = \left\{\vec{\nabla}_{\mu} + i\, g_{G}\left(+\frac{2}{2}\right) \mathsf{G}_{\mu}\right\} f, \\ \mathscr{D}^{L}_{\mu\,(D)}\, \ell &= \left\{\vec{\nabla}_{P\,\mu} + i\, g_{D}\, \mathbf{Y}_{\mu} \cdot \frac{\boldsymbol{\kappa}}{2} + i\, g_{\mathscr{C}}\left(\frac{Y_{\mathscr{C}}}{2}\right) \chi_{\mu}\right\} \ell \\ &= \left\{\vec{\nabla}_{P\,\mu} + i\, g_{D}\, \mathbf{Y}_{\mu} \cdot \frac{\boldsymbol{\kappa}}{2} + i\, g_{\mathscr{C}}\left(-\frac{1}{2}\right) \chi_{\mu}\right\} \ell, \\ \mathscr{D}^{R}_{\mu\,(D)}\, e_R &= \left\{\vec{\nabla}_{P\,\mu} + i\, g_{\mathscr{C}}\left(-\frac{1}{2}\right) \chi_{\mu}\right\} e_R, \\ W^{i}_{\mu\nu} &= \left(\vec{\nabla}_{\mu}\, W^{i}_{\nu} - \vec{\nabla}_{\nu}\, W^{i}_{\mu}\right) + g_{W}\, \epsilon^{jik}\, W^{j}_{\mu}\, W^{k}_{\nu}, \\ B_{\mu\nu} &= \left(\vec{\nabla}_{\mu}\, B_{\nu} - \vec{\nabla}_{\nu}\, B_{\mu}\right), \\ \mathcal{G}_{\mu\nu} &= \left(\vec{\nabla}_{\mu}\, \mathsf{G}_{\nu} - \vec{\nabla}_{\nu}\, \mathsf{G}_{\mu}\right), \\ \mathrm{Y}^{i}_{\mu\nu} &= \left(\vec{\nabla}_{P\,\mu}\, \mathrm{Y}^{i}_{\nu} - \vec{\nabla}_{P\,\nu}\, \mathrm{Y}^{i}_{\mu}\right) + g_{D}\, \xi^{jik}\, \mathrm{Y}^{j}_{\mu}\, \mathrm{Y}^{k}_{\nu}, \\ \chi_{\mu\nu} &= \left(\vec{\nabla}_{P\,\mu}\, \chi_{\nu} - \vec{\nabla}_{P\,\nu}\, \chi_{\mu}\right),\end{aligned} \tag{83}$$

here, the components $Y_{\mathscr{C}}$ and $g_{\mathscr{C}}$ are belonged to Casimir fields, where $Y_B = -1$ and $Y_{\mathscr{C}} = -1$ are the diagonal matrices with the hypercharges for electromagnetic and Casimir fields, respectively, while $Y_G = 2$ is an operator for gravity field, in their diagonal entries, and $i = 1, 2, 3$. In (82), fermions and vector fields all are massless.

Now, to introduce spontaneous symmetry-breaking and to generate masses for the gauge bosons by the choice of Higgs vacuum, let us consider a complex scalar field with a quartic interaction, where the Lagrangian has the form,

$$\mathscr{L} = \vec{\nabla}_{\mu}\, \phi^{\dagger}\, \vec{\nabla}^{\mu}\, \phi + \vec{\nabla}_{P\,\mu}\, \phi^{\dagger}\, \vec{\nabla}^{\mu}_{P}\, \phi - \vartheta^{2}\, \phi^{\dagger}\phi - \lambda\, (\phi^{\dagger}\phi)^{2} - \vartheta^{2}_{P}\, \phi^{\dagger}\phi - \lambda_{P}\, (\phi^{\dagger}\phi)^{2},$$

from which we find the equation of motion as,

$$\{\Box + \vartheta^{2}\}\, \phi + \{\Box_{P} + \vartheta^{2}_{P}\}\, \phi + 2\, \lambda\, \phi\, (\phi\, \phi^{\dagger}) + 2\, \lambda_{P}\, \phi\, (\phi\, \phi^{\dagger}) = 0.$$

So, we have the non-vanishing one,

$$\begin{aligned}\Xi &= \{(-\vartheta^{2})/(2\,\lambda)\}^{1/2} = \{v^{2}/2\}^{1/2}, \\ \Xi_{P} &= \{(-\vartheta^{2}_{P})/(2\,\lambda_{P})\}^{1/2} = \{v^{2}_{P}/2\}^{1/2}.\end{aligned}$$

Let us introduce a set of matrices in such a way,

$$\begin{bmatrix} M \\ N \\ O \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} v \\ w \\ x \end{bmatrix} = v \begin{bmatrix} a \\ d \\ g \end{bmatrix} + w \begin{bmatrix} b \\ e \\ h \end{bmatrix} + x \begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} av + bw + cx \\ dv + ew + fx \\ gv + hw + ix \end{bmatrix},$$

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} c & f & i \\ g & h & i \end{bmatrix} \begin{bmatrix} y \\ z \\ x \end{bmatrix} = y \begin{bmatrix} c \\ g \end{bmatrix} + z \begin{bmatrix} f \\ h \end{bmatrix} + x \begin{bmatrix} i \\ i \end{bmatrix} = \begin{bmatrix} cy + fz + ix \\ gy + hz + ix \end{bmatrix}. \tag{84}$$

Let the Lagrangian with gauge-boson masses is,

$$\mathscr{L}_m = \frac{v^2 g_W^2}{4} W_\mu^+ W_\nu^- + \frac{v_P^2 g_D^2}{4} \Upsilon_\mu^a \Upsilon_\nu^b + \\ + \begin{pmatrix} W_\mu^3 & B_\mu & \mathsf{G}_\mu & \Upsilon_\mu^3 & \chi_\mu \end{pmatrix} \begin{bmatrix} \mathcal{M} \\ \mathcal{M}_P \end{bmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \\ \mathsf{G}_\mu \\ \Upsilon_\mu^3 \\ \chi_\mu \end{pmatrix},$$

where we have written the masses for the charged $W_\mu^\pm$ and $\Upsilon_\mu^{a,b}$ fields defined as $W_\mu^\pm = \left(W_\mu^1 \mp i\, W_\mu^2\right)/\sqrt{2}$ and $\Upsilon_\mu^{a,b} = \left(\Upsilon_\mu^1 \mp i\, \Upsilon_\mu^2\right)/\sqrt{2}$, respectively, and where,

$$\mathcal{M} = \frac{v^2}{8} \begin{pmatrix} g_W^2 & -g_W g_B & g_W g_D \\ -g_W g_B & g_B^2 & -g_B g_{\mathscr{C}} \\ g_W g_D & -g_B g_{\mathscr{C}} & 4\, g_G^2 \end{pmatrix},$$

$$\mathcal{M}_P = \frac{v_P^2}{8} \begin{pmatrix} g_W^2 & -g_W g_B & g_W g_D \\ -g_W g_B & g_B^2 & -g_B g_{\mathscr{C}} \\ g_W g_D & -g_B g_{\mathscr{C}} & 4\, g_G^2 \end{pmatrix}, \tag{85}$$

having $\det(\mathcal{M}) = 0$ and $\det(\mathcal{M}_{\mathcal{P}}) = 0$, hence allowing a massless photon and a massless graviton. Now, omitting $W_\mu^\pm$ as they are quite obvious, we choose only $\Upsilon_\mu^{a,b}$, whose masses can be seen from $\mathscr{L}_m$ are given as,

$$M_{P\,\Upsilon^{a,b}} = \frac{v_P}{2} \left(g_W g_D\right)^{1/2}.$$

The neutral gauge bosons mix after symmetry-breaking and the mass eigenstates are the neutral weak boson Z_μ, photon A_μ, Dark Energy boson $\mathfrak{D}_\mu$, the Casimir energy $\mathscr{C}_\mu$ and graviton $\mathcal{G}_\mu$ – first four of which are given in terms of W_μ^3, B_μ, Υ_μ^3 and χ_μ as,

$$\begin{pmatrix} Z_\mu \\ A_\mu \\ \mathcal{G}_\mu \\ \mathfrak{D}_\mu \\ \mathscr{C}_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W & \sin\theta_D \\ \sin\theta_W & \cos\theta_W & -\sin\theta_D \\ \sin\theta_D & \sin\theta_D & 4\cos\theta_{W,D} \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \\ \mathsf{G}_\mu \\ \Upsilon_\mu^3 \\ \chi_\mu \end{pmatrix},$$

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i.e., alike (84),

$$
\begin{aligned}
Z_\mu &= \cos\theta_{W} W_\mu^3 - \sin\theta_{W} B_\mu + \sin\theta_{D} \mathsf{G}_\mu\,, \\
A_\mu &= \sin\theta_{W} W_\mu^3 + \cos\theta_{W} B_\mu - \sin\theta_{D} \mathsf{G}_\mu\,, \\
\mathcal{G}_\mu &= \sin\theta_{D} W_\mu^3 + \sin\theta_{D} B_\mu + 4\,\cos\theta_{W} \mathsf{G}_\mu\,, \\
\mathfrak{D}_\mu &= \sin\theta_{D} \mathrm{Y}_\mu^3 - \sin\theta_{D} \chi_\mu + 4\,\cos\theta_{D} \mathsf{G}_\mu\,, \\
\mathscr{C}_\mu &= \sin\theta_{D} \mathrm{Y}_\mu^3 + \sin\theta_{D} \chi_\mu + 4\,\cos\theta_{D} \mathsf{G}_\mu\,,
\end{aligned}
\tag{86}
$$

with the Electroweak mixing angle θ_{W} and the Dark Energy mixing angle θ_{D}. Let us check their values. The condition that the fields A_μ and $\mathscr{C}_\mu$ should be the eigenvector of (85) with zero eigenvalue is written as,

$$
0 = \begin{pmatrix} g_W^2 & -g_W g_B & g_W g_D \\ -g_W g_B & g_B^2 & -g_B g_{\mathscr{C}} \\ g_W g_D & -g_B g_{\mathscr{C}} & 4\,g_G^2 \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \\ 1 \end{pmatrix} = \begin{pmatrix} g_W^2 \sin\theta - g_W g_B \cos\theta + g_W g_D \\ -g_W g_B \sin\theta + g_B^2 \cos\theta - g_B g_{\mathscr{C}} \\ g_W g_D \sin\theta - g_B g_{\mathscr{C}} \cos\theta + 4\,g_G^2 \end{pmatrix},
$$

and the vanishing of its RHS requires,

$$
\tan\theta = \frac{g_B}{g_W} - \frac{2\,g_G}{g_W \cos\theta}\,,
\tag{87}
$$

where, $\tan\theta_{W} = \frac{g_B}{g_W}$, then, $\theta_{W} = \tan^{-1}\left(\frac{g_B}{g_W}\right)$. Let us consider $\tan\theta$ of (87) as, $\tan\theta \equiv \tan\theta_{D}$, then, $\theta_{D} = \tan^{-1}\left(\frac{g_B}{g_W} - \frac{2\,g_G}{g_W \cos\theta_{W}}\right)$ and the Electroweak masses are found to be $M_W = (v/2)\,g_W$, and $M_Z = (v/2)\,\left(g_W^2 + g_B^2\right)^{1/2}$, while $M_\gamma = 0$, similarly, the Dark Energy masses are found to be $M_{\mathrm{Y}} = (v_{P}/2)\,\left(g_W g_D\right)^{1/2}$ and $M_{\mathfrak{D}} = (v_{P}/2)\,\left(g_W^2 + g_B^2 + 4\,g_G^2\right)^{1/2}$, while $M_{\mathscr{C}} = 0$. This gives

$$
\frac{M_{\mathrm{Y}}}{M_{\mathfrak{D}}} = \frac{\left(g_W g_D\right)^{1/2}}{\left(g_W^2 + g_B^2 + 4\,g_G^2\right)^{1/2}} = \cos\theta_{D}\,.
$$

Now, if we write down the charged-current and neutral-current interactions in GED theory, we can see the condition for which the field A_μ couples to the electron via the electromagnetic current is $g_W \sin\theta_{W} = g_{B} \cos\theta_{W} = e$ in Electroweak symmetry breaking, whereas, the condition for which the field $\mathscr{C}_\mu$ couples to the fermion via the Casimir current is $\left(g_W g_D\right)^{1/2} \sin\theta_{D} = g_{\mathscr{C}} \cos\theta_{D} = \chi$ in Dark Energy symmetry breaking.

By the way, the last two terms of (86) make it clear that Dark energy field is non-decaying. Whereas, (86) also gives us that graviton has a direct relationship with Dark Energy mixing angle θ_{D}, thus, Dark Energy is not separable from baryonic matter fields (and non-baryonic matter fields, too, if we consider Section 4, that will be discussed below). Earlier, in (74), we have already seen a similar result, where Dark Energy has a direct relationship with gravity.

3.2. Gravitational Chromodynamic Dark Energy Interactions

In the gauge theory of Gravitational Chromodynamic Dark Energy interactions (GCD), if we regard the non-abelian gauge groups of Dark Energy symmetry is only $SU(2)_D \otimes U(1)_{\mathscr{C}}$ familiar to the earlier subsection, then we have to consider minimally $SU(4)$ for GCD. But, this consideration is not mathematically sufficient for the $SU(4)$ picture. So, we must have to consider the Dark Energy gauge groups minimally as $SU(2)_{D'} \otimes SU(2)_{D''} \otimes U(1)_{\mathscr{C}}$ to meet the fulfillments of the gauge symmetry $SU(4)$ for GCD, where $U(1)_{\mathscr{C}}$ is Casimir gauge group, but the non-abelian gauge groups $SU(2)_{D'} \otimes SU(2)_{D''}$ are now quite different than the previously discussed Dark Energy fields χ_μ of non-abelian

 doi:10.20944/preprints202306.1472.v4

gauge group $SU(2)_D$ of GED. The gauge groups $SU(2)_{D'} \otimes SU(2)_{D''}$ actually give two different kind of non-abelian gauge groups for two different families of quarks, namely, for light quarks u, d and s, we have Dark Energy gauge group $SU(2)_{D'}$, whereas, for heavy quarks c, t and b, we have Dark Energy gauge group $SU(2)_{D''}$. With abelian gauge group $U(1)_{\mathscr{C}}$, the non-abelian gauge groups $SU(2)_{D'}$ and $SU(2)_{D''}$ give the overall Dark Energy symmetry as $SU(2)_{D'} \otimes SU(2)_{D''} \otimes U(1)_{\mathscr{C}}$. Then, we can assume the gauge symmetry for GCD is,

$$SU(4)_{GCD} \to SU(3)_C \otimes U(1)_G \otimes SU(2)_{D'} \otimes SU(2)_{D''} \otimes U(1)_{\mathscr{C}},$$

where the $SU(3)_C$ symmetry of the colour degree of freedom is now with gravitational (i.e., $U(1)_G$) and Dark Energy (i.e., $SU(2)_{D'} \otimes SU(2)_{D''} \otimes U(1)_{\mathscr{C}}$) gauge symmetries.

Using (72) for Chromodynamic and gravitational interactions, whereas (76) for Dark Energy interaction, let the Gravitational Chromodynamic Dark Energy (GCD) Lagrangian, which describes quarks and gluons in interactions in the presence of gravitational and the Dark Energy fields for baryonic matter, is as follows,

$$\begin{aligned}
\mathscr{L}_{GCD} &= \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu\, \mathrm{g}_{\mu\nu}\, (D_\mu)_{jk} - m^{(q)}_{\mathrm{F}\,0}\, \delta_{jk} \right\} \psi_k^{(q)} + \\
&+ \sum_q \bar{\psi}_j^{(q)}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathcal{D}_{\mu\,(G)} \right)_{jk} \psi_k^{(q)} + \\
&+ \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu\, \mathrm{g}_{\mu\nu}\, (\mathscr{D}_{\mu\,(D)})_{jk} - \mathrm{N}_{(\Lambda)\,\mathrm{F}}\, \mathfrak{m}^{(q)}_{(\Lambda)\,\mathrm{F}\,0}\, \delta_{jk} \right\} \psi_k^{(q)} + \\
&+ \sum_{q_R} \bar{\psi}_j^{(q_R)}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu}\, (\mathscr{D}^R_{\mu\,(D)})_{jk}\, \psi_k^{(q_R)} - \\
&- \frac{1}{4} \sum_a \mathrm{G}^a_{\mu\nu} \left(\mathrm{G}^a \right)^{\mu\nu} - \frac{1}{4} \mathcal{G}_{\mu\nu}\, \mathcal{G}^{\mu\nu} - \\
&- \frac{1}{4} \sum_q \Upsilon^q_{\mu\nu} \left(\Upsilon^q \right)^{\mu\nu} - \frac{1}{4} \chi_{\mu\nu}\, \chi^{\mu\nu},
\end{aligned} \tag{88}$$

where, $q = u, d, s, \ldots$ is the flavor label, $j = 1, 2, 3$ is the quark color index, $a = 1, 2, \ldots, 8$ is the gluon color index, where, q_R indicates right-handed quarks and

$$\begin{aligned}
D_\mu &= \vec{\nabla}_\mu + i\, g_s\, \boldsymbol{A}_\mu \cdot \frac{\boldsymbol{\lambda}}{2}, \\
\mathcal{D}_{\mu\,(G)} &= \vec{\nabla}_\mu + i\, g_G \left(\frac{Y_G}{2} \right) \mathsf{G}_\mu = \vec{\nabla}_\mu + i\, g_G \left(+ \frac{2}{2} \right) \mathsf{G}_\mu, \\
\mathscr{D}_{\mu\,(D)} &= \vec{\nabla}_{P\,\mu} + i\, g_D\, \mathbf{Y}_\mu \cdot \frac{\boldsymbol{\kappa}_D}{2} + i\, g_{\mathscr{C}} \left(\frac{Y_{\mathscr{C}}}{2} \right) \chi_\mu \\
&= \vec{\nabla}_{P\,\mu} + i\, g_D\, \mathbf{Y}_\mu \cdot \frac{\boldsymbol{\kappa}_D}{2} + i\, g_{\mathscr{C}} \left(-\frac{1}{2} \right) \chi_\mu, \\
\mathscr{D}^R_{\mu\,(D)} &= \vec{\nabla}_{P\,\mu} + i\, g_{\mathscr{C}} \left(-\frac{1}{2} \right) \chi_\mu, \\
\mathrm{G}^a_{\mu\nu} &= \left(\vec{\nabla}_\mu A^a_\nu - \vec{\nabla}_\nu A^a_\mu \right) + g_s\, f^{abc}\, A^b_\mu\, A^c_\nu, \\
\mathcal{G}_{\mu\nu} &= \left(\vec{\nabla}_\mu\, \mathsf{G}_\nu - \vec{\nabla}_\nu\, \mathsf{G}_\mu \right), \\
\Upsilon^q_{\mu\nu} &= \left(\vec{\nabla}_{P\,\mu}\, \Upsilon^q_\nu - \vec{\nabla}_{P\,\nu}\, \Upsilon^q_\mu \right) + g_D\, \xi^{qrs}\, \Upsilon^r_\mu\, \Upsilon^s_\nu, \\
\chi_{\mu\nu} &= \left(\vec{\nabla}_{P\,\mu}\, \chi_\nu - \vec{\nabla}_{P\,\nu}\, \chi_\mu \right),
\end{aligned}$$

where $\boldsymbol{\kappa}_D \neq \boldsymbol{\kappa}$, A^a_μ is the field of gluon and g_s is the coupling constant, $\boldsymbol{\lambda}$ the Gell-Mann matrices in the space of colour whereas f^{abc} is the $SU(3)_C$ structure constant, here, Υ^q_μ is the Dark Energy fields and $q = 1, 2, \ldots, 6$ for six quark flavours. In (88), Dark Energy fields are massless and they require Higgs mechanism to gain masses. We left it again for the interested readers and avoid further repetition here.

4. Dark Matter Gauge Symmetry

If a matter satisfies (23) and (24) so as it yields the internal hidden spacetime of ρ which has 'proper' spacetime for its ∂-dependency, whereas, the overall (observable) system's spacetime is seemingly quite 'improper' as it is $\widehat{\partial}$-dependent, then this kind of matter must be non-baryonic, let us consider such matter as Dark Matter.

Let us consider a renormalizable Lagrangian, $\mathcal{L} = \mathcal{L}_{(\text{GED+GCD})} + \mathcal{L}_{\varrho_i} + \mathcal{L}_{\text{mix}}$, where $\mathcal{L}_{(\text{GED+GCD})}$ is for baryonic GED and GCD interactions, $\mathcal{L}_{\varrho_i}$ is the Lagrangian for Dark Matter particles ϱ_i, and $\mathcal{L}_{\text{mix}}$ contains possible non-gravitational interactions coupling between baryonic matters and ϱ_i particles. Let these ϱ_i particles are stable and behave as almost collisionless matter of Dark light quarks and/or antiquarks because of (23) and (24). This additional set of ϱ_i particles must be associated with a Dark Matter gauge group, $\text{SU}(3)'_C$, denoted with a prime ($'$). Candidates for $\text{SU}(3)'_C$ gauge group are almost non-luminous and non-absorbing matter, otherwise if they interact/decay with any of the baryonic particles, these non-baryonic particles would ensue the Universe very unstable with unbelievably high acceleration rate. Thus,

1. The non-baryonic Dark Matter sector, which corresponds to the gauge group $\text{SU}(3)'_C$, is *not* an exact copy of the Standard Model baryonic sector $\text{SU}(3)_C$. Though, in both cases, the symmetry interchanges the common type of gauge boson(s) and can be a full invariance of the two (almost) phenomenologically equivalent theories, although they are originated from fundamentally different physics maybe at very different cosmological epochs.
2. The particles for $\text{SU}(3)'_C$ gauge group would have only felt the gravitational attraction of other baryonic/non-baryonic objects.

So, the Dark sector particles are solely graviton, neutral Dark weak boson, darkgluons and darkquarks only, viz. $(\mathcal{G}, Z'^0, g', u', d', \ldots)$, but not leptonic $(e', \gamma', W'^{\pm}, \ldots)$, even no Casimir $\mathscr{C}'$ particle, and neither any charged Higgs fields. Colored and/or electrically charged particles are prevented from mixing with their Dark analogues by color and electric charge conservation laws, though, the physical gluons may couple to antidarkquarks with extremely weak gluonic strength (to be discussed shortly, vide (90) and its following discussion).

Let the above Lagrangian $\mathcal{L}$ respects an exact parity symmetry, which we also refer to as mirror symmetry [11]:

$$\begin{aligned} &(\vec{x}, t) \to (-\vec{x}, t)\,, && g^\mu \leftrightarrow g'_\mu, && \Phi_H \leftrightarrow \Phi'_H, \\ &q_{\Bbbk L} \leftrightarrow \gamma_0\, q'_{\Bbbk R}, && u_{\Bbbk R} \leftrightarrow \gamma_0\, u'_{\Bbbk L}, && d_{\Bbbk R} \leftrightarrow \gamma_0\, d'_{\Bbbk L}, \end{aligned}$$

where g_μ are the SU(3)$_C$ gluons, Φ_H is the Standard Model neutral Higgs doublet with its Dark partner Φ'_H, the fermion fields are quarks $q_{\Bbbk L} \equiv (u_\Bbbk, d_\Bbbk)_L$, $u_{\Bbbk R}$, and $d_{\Bbbk R}$ whereas darkquarks $q'_{\Bbbk R} \equiv (u'_\Bbbk, d'_\Bbbk)_R$, $u'_{\Bbbk L}$, and $d'_{\Bbbk L}$ represent the (anti-) darkquark families, while $\Bbbk = 1, 2, 3$ is the generation index, and γ_0 is a Dirac gamma matrix.

Suppose, $\mathcal{L}_{\varrho_i}$ is the combination of Dark Matter GED and Dark Matter GCD interactions. Thus, applying (71) into (82)-like and (88)-like equations for Dark Matter GED and Dark Matter GCD interactions, we get,

$$
\begin{aligned}
\mathscr{L}_{\varrho_i} \;=\; & \sum_{q'_L} \bar{\psi}_j^{(q'_L)} \, (8\pi)^{1/2} \, i\,\gamma^{\mu} \left(D'_{\mu}\right)^{W}_{jk} \psi_k^{(q'_L)} + \sum_{q'_R} \bar{\psi}_j^{(q'_R)} \, (8\pi)^{1/2} \, i\,\gamma^{\mu} \left(D'_{\mu}\right)^{W}_{jk} \psi_k^{(q'_R)} + \\
& + \sum_{q'} \bar{\psi}_j^{(q')} \left\{ (8\pi)^{1/2} \, i\,\gamma^{\mu} \left(D'_{\mu}\right)_{jk} - \left(N_{(\varrho)\,\mathrm{F}} \, \mathfrak{m}^{(q')}_{(\varrho)\,\mathrm{F}}\right)_{jk} \delta_{jk} \right\} \psi_k^{(q')} + \\
& + \sum_{q'} \bar{\psi}_j^{(q')} \, i\,\gamma^{\mu} \, \mathrm{g}_{\mu\nu} \left(\mathcal{D}_{\mu\,(G)}\right)_{jk} \psi_k^{(q')} + \\
& + \sum_{q'_L} \bar{\psi}_j^{(q'_L)} \, i\,\gamma^{\mu} \, \mathrm{g}_{\mu\nu} \left(\mathscr{D}^{L}_{\mu\,(D)}\right)_{jk} \psi_k^{(q'_L)} + \sum_{q'_R} \bar{\psi}_j^{(q'_R)} \, i\,\gamma^{\mu} \, \mathrm{g}_{\mu\nu} \left(\mathscr{D}^{R}_{\mu\,(D)}\right)_{jk} \psi_k^{(q'_R)} + \\
& + \sum_{q'} \bar{\psi}_j^{(q')} \left\{ i\,\gamma^{\mu} \, \mathrm{g}_{\mu\nu} \left(\mathscr{D}_{\mu\,(D)}\right)_{jk} - \left(N_{(\Lambda)\,\mathrm{F}} \, \mathfrak{m}^{(q')}_{(\Lambda)\,\mathrm{F}\,0}\right)_{jk} \delta_{jk} \right\} \psi_k^{(q')} - \\
& - \frac{1}{4} W'_{\mu\nu} \left(W'\right)^{\mu\nu} - \frac{1}{4} \sum_{a} G'^{\,a}_{\mu\nu} \left(G'^{\,a}\right)^{\mu\nu} - \frac{1}{4} \mathcal{G}_{\mu\nu} \, \mathcal{G}^{\mu\nu} - \\
& - \frac{1}{4} \sum_{i} \mathrm{Y}'^{\,i}_{\mu\nu} \left(\mathrm{Y}'^{\,i}\right)^{\mu\nu} - \frac{1}{4} \sum_{q'} \mathrm{Y}^{q'}_{\mu\nu} \left(\mathrm{Y}^{q'}\right)^{\mu\nu} ,
\end{aligned}
\tag{89}
$$

where Dark Matter Weak and Dark Matter Chromodynamic covariant derivatives are,

$$
\begin{aligned}
\left(D'_{\mu}\right)^{W} &= \vec{\nabla}_{P\,\mu} + i\, g'_{W} \, \boldsymbol{W}'_{\mu} \cdot \frac{\boldsymbol{\tau}'}{2} , \\
D'_{\mu} &= \vec{\nabla}_{P\,\mu} + i\, g'_{s} \, \boldsymbol{\mathcal{A}}'_{\mu} \cdot \frac{\boldsymbol{\lambda}'}{2} ,
\end{aligned}
$$

when the other covariant derivatives are analogous to Section 3, but they do not contain either photon or Casimir energy field. In (89), both Dark Matter Weak fields and Dark Energy fields are massless and they require (neutral) Higgs mechanism to gain masses, but we omit it to avoid further repetition. Here we also presume that Dark Energy gauge groups $SU(2)_{D'} \otimes SU(2)_{D''}$ are analogous to the GED and GCD Dark Energy gauge groups. So, due to (89), the Dark Matter gauge symmetry is,

$$
SU(5)_{DM} \longrightarrow SU(3)_{C'} \otimes SU(2^{\star})_{L} \otimes U(1)_{G} \otimes SU(2)_{D} \otimes SU(2)_{D'} \otimes SU(2)_{D''} ,
$$

here $SU(2^{\star})_L$ gives only Dark Z'^{0} boson but no other Dark (charged) weak vector bosons. As a policy of desperation, we can consider that one of the Dark Matter particles is definitely *axion*.

Considering that both of the ordinary (i.e., baryonic) and Dark particles are too close to interact in spacetime no matter what their cosmological epochs are, then for the physical effects of gluon-darkgluon kinetic mixing, let us take account of $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)'_C$ quantum chromodynamics for the quark $\psi^{(q)}$ and gluon field $\mathcal{A}_{\mu}$, antidarkquark $\psi^{(q')}$ and antigluon field $\mathcal{A}'_{\mu}$ as follows by

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using (88) and (89), when gravitational and Dark Energy interactions are quite obvious and unaltered, so as omitted here, then we have the Lagrangian as,

$$\begin{aligned}
\mathcal{L}_{\text{mix}} &= \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu\, \mathrm{g}_{\mu\nu}\, (D_\mu)_{jk} - m_{\mathrm{F}\,0}^{(q)}\, \delta_{jk} \right\} \psi_k^{(q)} + \\
&\quad + \sum_{q'} \bar{\psi}_j^{(q')} \left\{ (8\pi)^{1/2}\, i\,\gamma^\mu \left(D'_\mu \right)_{jk} - \left(N_{(\varrho)\,\mathrm{F}}\, \mathfrak{m}_{(\varrho)\,\mathrm{F}}^{(q')} \right)_{jk} \delta_{jk} \right\} \psi_k^{(q')} + \\
&\quad - \frac{1}{4} \sum_a G_{\mu\nu}^a \left(G^a \right)^{\mu\nu} - 2\pi \sum_a G'^a_{\mu\nu} \left(G'^a \right)^{\mu\nu} - \frac{\epsilon}{2\,(8\pi)^{1/2}} \sum_a G_{\mu\nu}^a \left(G'^a \right)^{\mu\nu} - \\
&\quad - \frac{g_s^2\, \Theta_s}{64\,\pi^2}\, f^{\mu\nu\lambda\rho} \sum_a \left(G^a \right)^{\mu\nu} \left(G^a \right)^{\lambda\rho} - \frac{g_s'^2\, \Theta_s}{8\,\pi}\, f^{\mu\nu\lambda\rho} \sum_a \left(G'^a \right)^{\mu\nu} \left(G'^a \right)^{\lambda\rho} - \\
&\quad - \frac{\epsilon\, \left(g_s^2\, g_s'^2 \right)^{1/2} \Theta_s}{32\,\pi^2 (8\pi)^{1/2}}\, f^{\mu\nu\lambda\rho} \sum_a \left(G^a \right)^{\mu\nu} \left(G'^a \right)^{\lambda\rho},
\end{aligned} \tag{90}$$

involving a sufficiently small dimensionless parameter ϵ, and there is no symmetry reason for suppressing this term, with the minimal particle content [12]. Here $\mathcal{A}_\mu^C \equiv A_\mu^C$ and $\mathcal{A}'^C_\mu \equiv \frac{1}{2} A'^C_\mu$. Note here, that the $\mathrm{SU}(3)_C$-$\mathrm{SU}(3)'_C$ kinetic mixing term is gauge invariant. The kinetic mixing can be removed with a non-orthogonal transformation [13],

$$\mathcal{A}_\mu \to \tilde{\mathcal{A}}_\mu \equiv \mathcal{A}_\mu + \epsilon\, \mathcal{A}'_\mu \;, \qquad \mathcal{A}'_\mu \to \tilde{\mathcal{A}}'_\mu \equiv \mathcal{A}'_\mu \sqrt{1 - \epsilon^2} \;.$$

We can transform to a basis where only one of these states couples to gluons. By the orthogonal state we call the *sterile gluon* $\mathcal{A}_2^\mu$,

$$\mathcal{A}_1^\mu = \mathcal{A}^\mu \sqrt{1 - \epsilon^2}, \qquad \mathcal{A}_2^\mu = \mathcal{A}'^\mu + \epsilon\, \mathcal{A}^\mu,$$

the Lagrangian for an ordinary matter environment shows (to leading order in ϵ) that the physical gluon couples to antidarkquarks with gluonic strength $g_s \epsilon$, which is extremely weak for the sufficiently small dimensionless parameter ϵ, while g'_s doesn't couple to ordinary matter at all. Similarly, considering the *sterile darkgluon* $\mathcal{A}'^\mu_1$,

$$\mathcal{A}'^\mu_1 = \mathcal{A}^\mu + \epsilon\, \mathcal{A}'^\mu, \qquad \mathcal{A}'^\mu_2 = \mathcal{A}'^\mu \sqrt{1 - \epsilon^2},$$

the Lagrangian for a dark matter environment becomes (to leading order in ϵ),

$$\begin{aligned}
\mathcal{L}_{\text{mix}} &= \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu\, \mathrm{g}_{\mu\nu}\, (D_\mu)_{jk} - m_{\mathrm{F}\,0}^{(q)}\, \delta_{jk} \right\} \psi_k^{(q)} + \\
&\quad + \sum_{q'} \bar{\psi}_j^{(q')} \left\{ (8\pi)^{1/2}\, i\,\gamma^\mu \left(D'_\mu \right)_{jk} - \left(N_{(\varrho)\,\mathrm{F}}\, \mathfrak{m}_{(\varrho)\,\mathrm{F}}^{(q')} \right)_{jk} \delta_{jk} \right\} \psi_k^{(q')} + \\
&\quad + \sum_q \bar{\psi}_j^{(q)}\, g_s \gamma^\mu\, \lambda_{jk}^C \left(\mathcal{A}'^{C\,\mu}_1 - \epsilon\, \mathcal{A}'^{C\,\mu}_2 \right) \psi_k^{(q)} + \\
&\quad + \sum_{q'} \bar{\psi}_j^{(q')}\, g'_s \gamma^\mu\, \lambda'^C_{jk}\, \mathcal{A}'^{C\,\mu}_2\, \psi_k^{(q')},
\end{aligned}$$

in terms of the ordinary matter physical states,

$$\mathcal{A}'^\mu_2 = \left(\mathcal{A}_2^\mu - \epsilon\, \mathcal{A}_1^\mu \right),$$

(to leading order in ϵ) and suppressing the $G_{\mu\nu}^a\,(G^a)^{\mu\nu}$ and $G'^a_{\mu\nu}\,(G'^a)^{\mu\nu}$ related components in $\mathcal{L}_{\text{mix}}$. Evidently, an antidarkquark would emit the state $\mathcal{A}'^\mu_2$, thus, the flux of g'_s detectable in an ordinary matter detector is reduced by a factor ϵ^2. Since ϵ is sufficiently small, this makes such emission should not be detected in the present state collider experiments.

Remark 5. *So, the above inspection implies that, antiquark-quark → antidarkquark-darkquark or quark-quark (or quark-antiquark) → darkquark-antidarkquark annihilation channels may occur within the nucleons and gives us the effects something like Refs. [14–16], but antidarkquark-darkquark → quark-antiquark or darkquark-antidarkquark → quark-antiquark annihilation channels should never take place on the inside of a nucleon (baryonic or dark), and neither any decay* $q'_1 \to q'_2 + (f\bar{f})$ *through intermediated spin-1 gauge bosons, where* f *and* $\bar{f}$ *stand for light Standard Model particles (assuming the conservation of a dark-parity) unless* $(f\bar{f})$ *are leptons.*

Remark 5 is sufficient to explain the effectively more collisionless bow shock phenomena of the mass component of the merging galaxy clusters like as the 1E 0657-56 cluster [17], commonly known as the Bullet Cluster, and the similar collisionless behavior that has been observed in other merging galaxy clusters, for example, two high-redshift clusters, CL 0152-1357 [18] and MS 1054-0321 [19], and several other merging clusters, viz. A754, A1750, A1914, A2034, A2142 [20], A2744 [21], A2163 [22], MACS J0025.4-1222 [23], and A1758 [20,24], etc.

5. Conclusion and Discussion

In this work, we have quantized the classical theory of General Relativity and contributing a very natural geometric way, we have wrote a fundamental theory of renormalizable quantum gravity coupled to matter fields (baryonic and non-baryonic).

Present physics is unable to provide us a more acceptable scenario of Einstein field equation which is developed in a quantum spacetime. Moreover, it is commonly believed in contemporary physics that gravity is the bending of spacetime, but in GQG, Einstein field equations (48) and (67), and then GED and GCD interactions, all they assure us that, *gravity is the bending of spacetime intermediated by gravitons in its quantum gravity field, whose geometric part bends spacetime, whereas its quantum part interacts with the spacetime by exchanging gravitons*.

Two different aspects of quantum gravity in Section 2.1 and Section 2.2, respectively, are developed in different spacetimes, viz. one Einstein field equation (48) is developed in a Semi-Quantum Minkowski spacetime, while, another Einstein field equation (67) is developed in a purely Quantum Non-Minkowski spacetime. This is a remarkable achievement of GQG.

Classical Einstein field equations have failed to predict the fundamental properties of particles (baryonic and non-baryonic) to a great extent, whereas, it is clear from (48) and (67), that Einstein field equations in quantum spacetime not only provide us a thorough information of all types of baryonic and non-baryonic particles, but it also can successfully predict Dark Matter and Dark Energy within itself. Except this work, there has no evidence of any kind of simple bosonic and fermionic fields (neither even supersymmetric or stringy fields) that provides us that Dark Energy and Dark Matter are appearing from the same quantum gravitational fields, simultaneously. GQG fields give us a picture of Dark Matter in (71), through which we develop Section 4, where Dark Energy appears quite naturally from the Dark Matter Yang-Mills Lagrangian fields.

Combining GED, GCD and Dark Matter gauge symmetry, we have a Universal Model for all kind of baryonic and non-baryonic particle interactions as,

$$SU(3)_{GED} \otimes SU(4)_{GCD} \otimes SU(5)_{DM} \subset SU(7)_{UM},$$

from where, Yang-Mills Lagrangians (82), (88) and (89) make it clear that Dark Energy field is homogeneous, as well as non-decaying, in all kind of matter fields. So, it is also clear that whether the matter is baryonic or Dark, or their mixture as like as (90), the effective universal relativistic cosmological constant Λ_{eff} at the surrounding is always positive – that is why the Universe is expanding and accelerating, even at the present epoch. Dark Energy particles, that the fundamental interactions possessed in abundance due to the Universal Model of particle interactions $SU(3)_{GED} \otimes SU(4)_{GCD} \otimes SU(5)_{DM} \subset SU(7)_{UM}$, give an excellent fit to observations with the present day $\sim$ 68.3% content of the Dark Energy, i.e., it is providing us the solution of the 'coincidence problem': *Why is the energy density of matter nearly equal to the Dark Energy density today?* So, from GQG, we have a suitable solution that why Dark Energy has become dominant after galaxy formation. (Note: See the text immediate after (71) for the relation between baryonic and Dark Matter abundance.)

The overall scenarios of unification of gravity, Dark Energy, Dark Matter with fundamental interactions of particles or fields in GQG have left behind sufficient amount of calculations in Section 3 and Section 4 that give us a prospective opportunity for their future uses and they also ensure us that graviton, Dark Energy and Dark Matter now become possible to be testable (directly or indirectly) at laboratory scales (i.e., in a standard particle collider) without regarding Planck scales. So, we have a growing possibility to find their signatures in the particle colliders, like LHC, in the future by adopting this study.

In our present work, we have not chosen Superstring/M-theory intentionally, but it has come up quite naturally in GQG spacetime geometry, whereas we have intentionally omitted the formalism of Loop Quantum Gravity since its construction looks quite artificial in comparison to the emergence of Superstring/M-theory in GQG, though we can easily develop an effective formalism of Loop Quantum Gravity with the help of (2) or Section 2.2.

We did not know much about M-theory yet. Not even knew the origin of eleven dimensions in the higher dimensional spacetime. This work is the first time answer of the origin of eleven dimensions within a quantum spacetime, and surprisingly, which actually is a quite natural phenomenon, i.e., eleven-dimensional Supergravity is necessarily a natural phenomenon within the quantum spacetime of GQG – better to say that within any kind of quantum spacetime. This finding not only makes an effective impact in the field of studying Superstring/M-theory, but also opens a new possibility for Loop Quantum Gravity to study it in a deeper level.

A new kind of Quantum Mechanics, which has been introduced here, is truly inspiring us for a new beginning of both theoretical and experimental fields related to Quantum Mechanics, no matter whether it is optics, or condense matter, or astrophysics, or even gravity itself.

An extreme possibility to study both Dark Energy and Dark Matter from a completely different angle of view, which has been discussed here, can make a hope for a completely new pathway of future theoretical and experimental studies in the relevant fields of astroparticle and astrophysics.

Conflicts of Interest: The authors declare no conflict of interest.

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