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Posted Date: 20 June 2023

doi: 10.20944/preprints202306.1433.v1

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Dynamics of Non-autonomous Stochastic Semi-linear Degenerate Parabolic Equations with Nonlinear Noise†

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Abstract: At present paper, we aim to study the long-time behavior of a stochastic semi-linear degenerate parabolic equation on bounded or unbounded domain and driven by a nonlinear noise and defined. Since the theory of pathwise random dynamical systems can not be applied directly to the equation with nonlinear noise, first, we establish the existence of weak pullback mean random attractors for the equation by applying the theory of mean-square random dynamical systems; then, we prove the existence of (pathwise) pullback random attractors for the Wong-Zakai approximate system of the equation. In addition, we establish the upper semicontinuity of pullback random attractors for the Wong-Zakai approximate system of the equation under consideration driven by a linear multiplicative noise.

MSC (2010): Primary 37L55, 35B40; Secondary 37B55, 35B41.

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1 Introduction

We consider the following stochastic semi-linear degenerate parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + h(t,x,u)\dfrac{\mathrm{d}W}{\mathrm{d}t}, & t > \tau, \\ u(\tau,x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(t,x)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subseteq \mathbb{R}^N (N \geq 2)$ is an arbitrary (bounded or unbounded) domain, λ is positive constants, W is a two-sided Hilbert space valued cylindrical Wiener process or a two-side real-valued Wiener process, the drift term f and diffusion term h are nonlinear functions with respect to u, the given function $g(t,x) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathcal{O}))$. In addition, the variable nonnegative coefficient

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† The work was supported by NSFC(Grant 12271141).

$\sigma(x)$ is allowed to have at most a finite number of (essential) zeros at some points, which is understood the degeneracy of (1.1). As in [3, 8], we assume that the nonnegative function $\sigma(x) : \mathcal{O} \to \mathbb{R}^+ \cup \{0\}$ satisfies the following hypotheses:

$(\mathcal{H}_\alpha)$ $\sigma \in L^1_{\text{loc}}(\mathcal{O})$ and for some $\alpha \in (0, 1)$, $\liminf_{x \to z} |x - z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\mathcal{O}}$, when the domain $\mathcal{O}$ is bounded;

$(\mathcal{H}_{\alpha,\beta})$ σ satisfies condition $(\mathcal{H}_\alpha)$ and $\liminf_{|x| \to \infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$, when the domain $\mathcal{O}$ is unbounded.

The conditions $(\mathcal{H}_\alpha)$ and $(\mathcal{H}_{\alpha,\beta})$ indicates that the diffusion coefficient $\sigma(x)$ is extremely irregular.

One of the most important things in studying evolution partial differential equations is to investigate the long-time behavior of solutions of the equations. In this process, attractors are the ideal objects. At present, abundant results, both in abstract context and concrete models, have been established for the deterministic infinite-dimensional dynamical systems, see, e.g. monographs [2, 14, 25] and papers [3, 4, 11]. However, when one considers the random influences on the systems under investigation, which are always presented as stochastic partial differential equations, and tries to establish the existence of attractors for them, the theory on deterministic infinite-dimensional system can not be applied directly. On the one hand, the stochastic dynamical systems are non-autonomous, and one can not obtain uniform (w.r.t stochastic time symbol) absorbing set as the deterministic case as in e.g. [14]; on the other hand, owing to the influences of stochastic driving system, one can not obtain the fixed invariant set for stochastic dynamical system in general.

In order to overcome these drawbacks, Flandoli etc. in [9, 10] introduce the theory of pathwise random dynamical systems and (pathwise) random attractors for the autonomous stochastic equations, in which the random attractor is a family of compact sets depending on random parameters and has some invariant property under the action of the random dynamical system. Recent theory in [12, 27] are related to non-autonomous pathwise random dynamical systems and pullback random attractors for non-autonomous stochastic equations, where the pullback random attractor is a family of compact sets depending on both random parameters and deterministic time symbols. Up to now, there are many results on the existence and uniqueness of random attractor, one can refer to [16, 20, 36, 37] for the autonomous stochastic equations and [22, 28, 30, 37] for the non-autonomous stochastic equations. In addition, for the result about random attractors for equation (1.1) with linear noise, see, e.g. [5, 13, 16, 36, 37].

However, when one investigates the dynamics of stochastic evolution equations driven by nonlinear noise, the existence of random attractors can not be established directly, since the serious challenge is that the existence of random dynamical system is unknown in general for these kinds of systems. As far as it is known, up to now there are two ways to over come this difficulty in some sense. One method is to investigate the dynamic behavior of the Wong-Zaki approximate system corresponding to original equation. For example, Lu and Wang in [21] get the existence of pullback random attractor for the Wong-Zakai approximate system of a stochastic reaction-diffusion equation with the nonlinear noise in some bounded spatial domain, and later, Wang et al. in [34] extend the result of [21] to unbounded domains by using the method of tail estimates. The another method is established by Kloeden et al. in [18] and Wang in [31], that is, they extend the concept of pathwise random attractor to mean

 doi:10.20944/preprints202306.1433.v1

context and establish the corresponding existence theory of mean random attractor for random dynamical system. There are some relevant works, see e.g. [32, 33].

The first purpose of this article is to establish the existence of weak pullback mean random attractors for Eq. (1.1) by using the theory of [31]. Toward this end, we first need to get the existence and uniqueness of solution for Eq. (1.1). Unlike reference [31], the existence of solution for Eq. (1.1) can not be obtained directly by using the abstract result (Theorem 4.2.4) in [24] since the drift term $f(x,u)$ is allowed to be polynomial growth of arbitrary order with respect to u in this article. We aim to prove the existence and uniqueness of the solution for Eq. (1.1) by using the approach of [32], in which the author prove existence of solutions for a stochastic reaction-diffusion equations involving drift term $f(x,t,u)$ with polynomial growth of any order and nonlinear diffusion term $\sigma(t,u)$, and the embedding $H^k(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 \le p \le \frac{2N}{N-2k}$ $(N \ge 2k)$ plays an essential role in this proof. Hence, we show the embedding result of the corresponding Sobolev space with weight $\sigma(x)$ in Section 2. In Section 3, we show the solution generate a mean random dynamical system and establish the existence of weak pullback random attractors for Eq. (1.1). We shall remark that since the mean random dynamical system is defined on the Banach space $L^p(\Omega, X)$ consisting of all Bochner integrable functions and corresponding probability space $(\Omega, \mathcal{F}, P)$ lacks some topological structure, we only get the weakly compact property and weakly attracting property of mean random attractors for (1.1) in $L^2(\Omega, X)$.

The second goal is to investigate dynamic behavior of the Wong-Zakai approximate system for Eq. (1.1). We prove the existence of pullback random attractor for the Wong-Zakai approximate system for equation (1.1) with nonlinear diffusion term $h(t,x,u)$, which is allowed to be polynomial growth, and we also show that the pullback random attractor of Wong-Zaki approximation for Eq. (1.1) converges to the attractor of Eq. (1.1) as the size of approximation tends to zero, when $h(t,x,u)$ is equal to u. This work will be done in section 4. We remark that when we prove the pullback asymptotic compactness, we use method of weighted sobolev spaces to overcome the non-compactness of usual Sobolev embeddings in the case of unbounded domain, which is different from that of [21].

In what follows of this article, the constant C represents some positive constant and may change from line to line.

2 Preliminaries

2.1 Functional setting

In this subsection, we introduce some function spaces and present some embedding results, which will be used in our proof.

Throughout this article, we let $(X, \|\cdot\|_X)$ be a separable Banach space and $L^p(\Omega, \mathcal{F}; X)$ $(1 < p < \infty)$ be the Banach space consisting of all strongly measurable and Bochner integrable functions Ψ from Ω to X such that

$$\|\Psi\|_{L^p(\Omega,\mathcal{F};X)} = (\int_\Omega \|\Psi\|_X^p \mathrm{d}p)^{\frac{1}{p}} < +\infty. \tag{2.1}$$

Denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ the complete filtered probability space satisfying the usual condition, i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains

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all P-null sets. We use $L^p(\Omega,\mathcal{F}_t;X)$ to represent the subspace of $L^p(\Omega,\mathcal{F};X)$, which consists of all functions belonging to $L^p(\Omega,\mathcal{F};X)$ and being strongly $\mathcal{F}_t$-measurable. For simplicity of notation, we denote by $\|\cdot\|$ the norm in $L^2(\mathcal{O})$ and $L^2(\Omega,\mathcal{F}_t;L^2(\mathcal{O}))$.

To investigate Eq. (1.1), we introduce the weighted Sobolev space $D_0^{1,2}(\mathcal{O},\sigma)$ defined by the completion of $\mathcal{C}_0^\infty(\mathcal{O})$ with norm $\|\cdot\|_{D_0^{1,2}(\mathcal{O},\sigma)}$,

$$\|u\|_{D_0^{1,2}(\mathcal{O},\sigma)} := \Big(\int_{\mathcal{O}} \sigma(x)|\nabla u|^2 \mathrm{d}x\Big)^{\frac{1}{2}}. \tag{2.2}$$

And one can easily check that $D_0^{1,2}(\mathcal{O},\sigma)$ is a Hilbert space with the inner product $(\cdot,\cdot)_\sigma$

$$(u,v)_\sigma := \int_{\mathcal{O}} \sigma(x)\nabla u \nabla v \mathrm{d}x. \tag{2.3}$$

If condition $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_{\alpha,\beta})$ on unbounded domain) holds, the operator $A = -\mathrm{div}(\sigma(x)\nabla u)$ is positive and self-adjoint with domain defined by

$$D(A) := \{u \in D_0^{1,2}(\mathcal{O},\sigma) : \ Au \in L^2(\mathcal{O})\}.$$

Furthermore, one can easily observe that if σ satisfies $(\mathcal{H}_\alpha)$ and $(\mathcal{H}_{\alpha,\beta})$, then there exists a finite set $A = \{a_1, a_2, \cdots, a_k\} \subseteq \bar{\mathcal{O}}$ and δ, $r > 0$ such that the balls $B_i = B_r(a_i)$, $i = 1, 2, \cdots, k$, are disjoint and

$$\sigma(x) \geq \delta|x - a_i|^\alpha \text{ for } x \in B_i \cap \Omega,\ i = 1, 2, \cdots, k, \tag{2.4}$$

$$\sigma(x) \geq \delta \text{ for } x \in \Omega \backslash \cup_i B_i, \tag{2.5}$$

and moreover, if Ω is unbounded, then there exists $R > 0$ such that

$$\sigma(x) \geq \delta|x|^\beta \text{ for } x \in \Omega,\ |x| > R. \tag{2.6}$$

The following spaces will also be needed:

- $D^p(A) := \{u \in D_0^{1,2}(\mathcal{O},\sigma) : Au \in L^p(\mathcal{O})\}$;
- $D_0^{-1}(\mathcal{O},\sigma)$:= the dual space of $D_0^{1,2}(\mathcal{O},\sigma)$;
- $H_0^m(\mathcal{O},\sigma)$:= the closure of $\mathcal{C}_0^\infty(\mathcal{O})$ with norm $\|\cdot\|_{H^m(\mathcal{O},\sigma)}$, defined by

$$\|u\|^2_{H^m(\mathcal{O},\sigma)} := \sum_{1\leq|\kappa|\leq m} \int_{\mathcal{O}} \sigma(x)|D^\kappa u|^2 \mathrm{d}x + \int_{\mathcal{O}} |u|^2 \mathrm{d}x,\ m \in \mathbb{N}^+,$$

  where $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_N)$ is a multi-index of order $|\kappa| = \kappa_1 + \kappa_2 + \cdots + \kappa_N$.

Lemma 2.1 ([19]) *There exists a constant c_1 such that the following inequality holds true for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,*

$$\Big(\int_{\mathbb{R}^N} |u|^{2^*_\alpha} \mathrm{d}x\Big)^{\frac{1}{2^*_\alpha}} \leq c_1\Big(\sum_{|\kappa|=m} \int_{\mathbb{R}^N} |x|^\alpha |D^\kappa u|^2 \mathrm{d}x\Big)^{\frac{1}{2}},$$

where $2^*_\alpha = \frac{2N}{N+2\alpha-2m}$ with $N - 2m \geq 0$.

Lemma 2.2 *Let $\sigma(x)$ satisfy assumption $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_{\alpha,\beta})$ on unbounded domain). Then there exists a constant c_2 such that*

$$\Big(\int_{\mathcal{O}} |u|^{2^*_\alpha} \mathrm{d}x\Big)^{\frac{1}{2^*_\alpha}} \leq c_2\Big(\sum_{|\kappa|=m} \int_{\mathcal{O}} \sigma(x)|D^\kappa u|^2 \mathrm{d}x\Big)^{\frac{1}{2}}, \text{ for every } u \in \mathcal{C}_0^\infty(\mathcal{O}).$$

Proof. By using Lemma 2.1, the Rellich-Kondrachov Theorem, and the General Sobolev inequality, we can get the conclusion of Lemma 2.2 in the similar way as in the proof of Proposition 2.5 in [8]. We omit the process here. □

The following embedding results play an important role in our proof in Section 3 and Section 4.

Lemma 2.3 ([8]) *Let $\sigma(x)$ satisfies assumption $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_{\alpha,\beta})$ on unbounded domain). Then it holds the compact embedding $D_0^{1,2}(\mathcal{O}, \sigma) \hookrightarrow\hookrightarrow L^2(\mathcal{O})$.*

Lemma 2.4 *Let $\sigma(x)$ satisfies assumption $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_{\alpha,\beta})$ on unbounded domain). Then it holds the continuous embedding*

$$H_0^m(\mathcal{O}, \sigma) \hookrightarrow L^p(\mathcal{O}), \text{ for } 2 \leq p \leq 2^*_\alpha.$$

Proof. Note that $2^*_\alpha > 2$ for $\alpha \in (0, 1)$. Then we can get by the interpolation theorem and Lemma 2.2 that

$$\|u\|_{L^p(\mathcal{O})} \leq C\|u\|^\theta \|u\|^{1-\theta}_{L^{2^*_\alpha}(\mathcal{O})} \leq C\|u\|_{H^m(\mathcal{O},\sigma)}, \text{ for any } u \in H_0^m(\mathcal{O}, \sigma),$$

where $\theta = \frac{2(2^*_\alpha - p)}{p(2^*_\alpha - 2)}$. The proof is completed. □

2.2 Theory of Random Attractors

In this subsection, we introduce some definitions and known results about weak pullback mean random attractors and pullback random attractors.

Definition 2.1 A family of mappings $\Phi = \{\Phi(t, \tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ is called mean random dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if the following conditions hold for all $\tau \in \mathbb{R}$ and $t, s \in \mathbb{R}^+$:

(i) $\Phi(t, \tau)$ maps $L^p(\Omega, \mathcal{F}_\tau, X)$ to $L^p(\Omega, \mathcal{F}_{t+\tau}, X)$;

(ii) $\Phi(0, \tau)$ is the identity operator on $L^p(\Omega, \mathcal{F}_\tau, X)$;

(iii) $\Phi(t + s, \tau) = \Phi(t, \tau + s) \circ \Phi(s, \tau)$.

Let $\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^p(\Omega, \mathcal{F}_\tau; X) : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets and $\mathcal{D}_0$ be a collection of such families satisfying some conditions. The collection $\mathcal{D}_0$ is said to be inclusion-closed if $\mathcal{D} = \{\mathcal{D}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$, then every family $O = \{O(\tau) : O(\tau) \subseteq \mathcal{D}(\tau), \tau \in \mathbb{R}\} \in \mathcal{D}_0$.

Definition 2.2 A family of sets $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is called a $\mathcal{D}_0$-pullback absorbing set for Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}_0$, there exists $T = T(\tau, \mathcal{D}) > 0$ such that

$$\Phi(t, \tau - t, \mathcal{D}(\tau - t)) \subseteq K(\tau), \quad \forall t \geq T.$$

Moreover, if $K(\tau)$ is a weakly compact nonempty subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for each $\tau \in \mathbb{R}$, then $K = \{K(\tau) : \tau \in \mathbb{R}\}$ is said to be a weakly compact $\mathcal{D}_0$-pullback absorbing set for Φ.

Definition 2.3 A family of sets $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is said to be a $\mathcal{D}_0$-pullback weakly attracting set of Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if for each $\tau \in \mathbb{R}, \mathcal{D} \in \mathcal{D}_0$ and every weak neighborhood $\mathcal{N}^w(K(\tau))$ of $K(\tau)$ in $L^p(\Omega, \mathcal{F}_\tau; X)$, there exists some $T = T(\tau, \mathcal{D}, \mathcal{N}^w(K(\tau))) > 0$ such that

$$\Phi(t, \tau - t, \mathcal{D}(\tau - t)) \subseteq \mathcal{N}^w(K(\tau)), \quad \forall t \geq T.$$

Definition 2.4 We say a family $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is a weak $\mathcal{D}_0$-pullback mean random attractor for Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ if it satisfies the following properties:

- Weak compactness: for any $\tau \in \mathbb{R}$, $\mathcal{A}(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$.
- Pullback weak attraction: for any $\tau \in \mathbb{R}$, $\mathcal{A}(\tau)$ is a $\mathcal{D}_0$-pullback weakly attracting set of Φ.
- Minimality: for any $\tau \in \mathbb{R}$, the family $\mathcal{A}$ is the minimal element of $\mathcal{D}_0$ in the sense that if $B = \{B(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ is another weakly compact $\mathcal{D}_0$-pullback weakly attracting set of Φ, then $\mathcal{A}(\tau) \subseteq B(\tau)$.

The following result about the existence and uniqueness of weak $\mathcal{D}_0$-pullback mean random attractors for Φ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$ comes from [31].

Lemma 2.5 *Suppose that $\mathcal{D}_0$ is an inclusion-closed collection of some families of nonempty bounded subsets of $L^p(\Omega, \mathcal{F}; X)$ and Φ is a weak mean random dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$. If Φ possesses a weakly compact $\mathcal{D}_0$-pullback absorbing set $K \in \mathcal{D}_0$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$, then Φ possesses a unique weak $\mathcal{D}_0$-pullback mean random attractor $\mathcal{A} \in \mathcal{D}_0$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$, which is given by*

$$\mathcal{A}(\tau) = \Omega^w(K, \tau) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \Phi(t, \tau - t, K(\tau - t))}^{w}, \ \forall \tau \in \mathbb{R},$$

where the closure is taken with respect to the weak topology of $L^p(\Omega, \mathcal{F}_\tau; X)$.

Denote by $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ a family of nonempty bounded subsets of X and $\mathcal{D}_1$ a collection of such families satisfying some conditions. Let $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system. We now introduce the pathwise random dynamical systerm as in [6, 9, 27].

Definition 2.5 A mapping $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is said to be a continuous pathwise random dynamical system (or a continuous cocycle) on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if the following conditions hold for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

(i) $\Psi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Psi(0, \tau, \omega, \cdot)$ is the identity operator on X;

(iii) $\Psi(t + s, \tau, \omega, \cdot) = \Psi(t, \tau + s, \theta_s\omega, \cdot) \circ \Psi(s, \tau, , \omega, \cdot)$;

(iv) $\Psi(t, \tau, \omega, \cdot) : X \mapsto X$ is continuous.

Definition 2.6 A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ is said to be a $\mathcal{D}_1$-pullback absorbing set for a cocycle Ψ if for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $\mathcal{D} \in \mathcal{D}_1$, there exists some $T = T(\tau, \mathcal{D}, \omega) > 0$ such that

$$\Psi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all} \quad t \geq T.$$

Moreover, If for every $\tau \in \mathbb{R}$ and $\omega \in \Omega, K(\tau, \omega)$ is a closed nonempty subset of X and is measurable in ω with respect to $\mathcal{F}$, then K is said to be a closed measurable $\mathcal{D}_1$-pullback absorbing set for Ψ.

Definition 2.7 We say that cocycle Ψ is $\mathcal{D}_1$-pullback asymptotically compact in X if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \quad \text{has a convergent subsequence in } X,$$

as $t_n \to +\infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$.

Definition 2.8 A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ is said to be a $\mathcal{D}_1$-pullback random attractor for Ψ if the following properties hold for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$

(i) Measurability and Compactness: $\mathcal{A}$ is measurable in ω with respect to $\mathcal{F}$ and $\mathcal{A}(\tau, \omega)$ is compact in X;

(ii) Invariance: $\mathcal{A}$ is invariant in the sense that $\Psi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega), \ \forall t \geq 0$;

(iii) Pullback attracting: $\mathcal{A}$ attracts $\mathcal{D}_1$ in the sense that for any $\mathcal{D} \in \mathcal{D}_1$,

$$\lim_{t \to +\infty} \text{dist}_X(\Psi(t, \tau - t, \theta_{-t}\omega, \mathcal{D}(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where dist_X is the Hausdorff semi-distance in X.

3 Mean Random Attractors for Stochastic Semilinear Degenerate Parabolic Equation

Let U be a separable Hilbert space and $L_2(U, L^2(\mathcal{O}))$ be the Hilbert space consisting of all Hilbert-Schmidt operators from U to $L^2(\mathcal{O})$ with norm $\|\cdot\|_{L_2(U,L^2(\mathcal{O}))}$. We consider the following non-autonomous stochastic semi-linear degenerate parabolic equation defined on any bounded or unbounded domain $\mathcal{O} \subseteq \mathbb{R}^N$:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + h(t,u)\frac{\mathrm{d}W}{\mathrm{d}t}, & t > \tau, \\ u(\tau,x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(t,x)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{3.1}$$

where W is a two-sided U-valued cylindrical Wiener process defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$, while $\sigma(x)$, λ and $g(t,x)$ are the same as described in Section 1. In this section, the stochastic term in Eq. (3.1) is understood in the sense of Itô integration. Since the Itô integral is martingale, it is convenient for us to take expectation and get the existence of weak pullback mean random attractor.

Let $\mathcal{O}$ be a bounded domain (or an unbounded domain) and let the nonnegative function $\sigma(x)$ satisfy $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_{\alpha,\beta})$). We assume that $f : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ is a smooth nonlinear function such that $f(x,0) = 0$ and for all $x \in \mathcal{O}$ and $u \in \mathbb{R}$,

$$\frac{\partial f}{\partial u}(x,u) \geq -\phi_1(x), \tag{3.2}$$

$$f(x,u)u \geq a_1|u|^p - \phi_2(x), \tag{3.3}$$

$$|f(x,u)| \leq a_2|u|^{p-1} + \phi_3(x), \tag{3.4}$$

where a_1, a_2, a_3, $p > 2$ are positive constants, and $\phi_1(x) \in L^\infty(\mathcal{O})$ with $\phi_1(x) \geq 0$, $\phi_2(x) \in L^1(\mathcal{O})$, $\phi_3(x) \in L^{p_1}(\mathcal{O})$ with $\frac{1}{p} + \frac{1}{p_1} = 1$. We also assume $f(x,u)$ is locally Lipschitz continuous in u, i.e., for each bounded interval $I \subseteq \mathbb{R}$, there is $a_I > 0$ such that

$$|f(x,u_1) - f(x,u_2)| \leq a_I|u_1 - u_2|, \ \forall x \in \mathcal{O}, \ u_1, \ u_2 \in I. \tag{3.5}$$

Assume $h : \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \mapsto L_2(U, L^2(\mathcal{O}))$ satisfies the following conditions:

($\mathbf{A_1}$) For any $t \in \mathbb{R}$, $\omega \in \Omega$ and $u \in L^2(\mathcal{O})$, there are positive constants $a_3 < \frac{1}{2}\lambda$ and L such that

$$\|h(t,\omega,u)\|^2_{L_2(U,L^2(\mathcal{O}))} \leq a_3\|u\|^2 + L. \tag{3.6}$$

($\mathbf{A_2}$) For each $r > 0$, there is a positive constant a_r depending on r such that for every $t \in \mathbb{R}$, $\omega \in \Omega$, and u, $v \in L^2(\mathcal{O})$ with $\|u\| \leq r$ and $\|v\| \leq r$,

$$\|h(t,\omega,u) - h(t,\omega,v)\|^2_{L_2(U,L^2(\mathcal{O}))} \leq a_r\|u - v\|^2. \tag{3.7}$$

Moreover, we suppose that for each given $u \in L^2(\mathcal{O})$, $\sigma(\cdot,\cdot,u) : \mathbb{R} \times \Omega \mapsto L_2(U, L^2(\mathcal{O}))$ is progressively measurable.

We now show the solution of Eq. (3.1) can define a mean random dynamical system. The definition of solution for Eq. (3.1) is given as follows in this case.

Definition 3.1 Let $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$ and $T > \tau$. A $L^2(\mathcal{O})$-valued $\mathcal{F}_t$-adapted stochastic process u is called a solution of (3.1) on $[\tau, T]$ with initial data u_τ if

$$u \in L^2(\Omega, C([\tau,T]; L^2(\mathcal{O}))) \cap L^2(\Omega \times [\tau,T]; D_0^{1,2}(\mathcal{O},\sigma)) \cap L^p(\Omega \times [\tau,T]; L^p(\mathcal{O}))$$

and P-a.s. satisfies

$$(u(t),\zeta) + \int_\tau^t (\sigma(x)\nabla u, \nabla\zeta)\mathrm{d}s + \lambda \int_\tau^t (u,\zeta)\mathrm{d}s + \int_\tau^t \int_{\mathcal{O}} f(u)\zeta \mathrm{d}x\mathrm{d}s = \int_\tau^t (g(s),\zeta)\mathrm{d}s$$
$$+ \int_\tau^t (h(s,u)\mathrm{d}W(s),\zeta),\ \forall t \in [\tau, T],\ \zeta \in D_0^{1,2}(\mathcal{O},\sigma) \cap L^p(\mathcal{O}).$$

Using Lemma 2.3, Lemma 2.4, we can get the following result in a similar way that have been used in [32].

Lemma 3.1 *Let $T > \tau$ and $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$. If conditions (3.2)-(3.7) hold, then there exists a unique solution to Eq. (3.1) in the sense of Definition 3.1. Besides,*

$$E(\sup_{t\in[\tau,T]} \|u(t)\|^2) < \infty. \tag{3.8}$$

Note that $u \in L^2(\Omega, C([\tau,T]; L^2(\mathcal{O})))$ for all $T > \tau$, which implies that $u \in C([\tau,\infty); L^2(\Omega, L^2(\mathcal{O})))$. Thus we can define the mean random dynamical system Φ for Eq. (3.1) on $L^2(\Omega, \mathcal{F}; L^2(\mathcal{O}))$ by

$$\Phi(t,\tau,u_\tau) = u(t+\tau,\tau,u_\tau), \quad t > 0, \tau \in \mathbb{R},$$

where $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$ and u is the solution of system (3.1) with initial data u_τ.

Let $\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets. A family $\mathcal{D}$ is said to be tempered if for any $\nu > 0$, there is

$$\lim_{\tau\to-\infty} \mathrm{e}^{\nu\tau} \sup_{u\in D(\tau)} \|u\|^2 = 0. \tag{3.9}$$

We denote by $\mathcal{D}_0$ the collection of all tempered families of nonempty bounded subsets of $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$, that is,

$$\mathcal{D}_0 = \{\mathcal{D} = \{\mathcal{D}(\tau) \subseteq L^p(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : \mathcal{D}(\tau) \neq \emptyset,\ \text{bounded},\ \tau \in R\} : \mathcal{D} \text{ satisfies (3.9)}\}.$$

From now on, we assume:

$$\int_{-\infty}^{\tau} \mathrm{e}^{\lambda s} \|g(s,\cdot)\|^2 \mathrm{d}s < +\infty,\ \ \forall \tau \in \mathbb{R}. \tag{3.10}$$

To get the existence of tempered random attractors, we further assume:

$$\lim_{\tau\to-\infty} \mathrm{e}^{\nu\tau} \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \|g(s+\tau,\cdot)\|^2 \mathrm{d}s = 0,\ \ \forall \nu > 0. \tag{3.11}$$

To investigate the existence of weak $\mathcal{D}_0$-pullback mean random attractors for Eq. (3.1), we need the uniform estimate of solutions, and by the following result, we can construct a weakly compact $\mathcal{D}_0$-pullback absorbing set for Φ.

Lemma 3.2 *Suppose (3.2)-(3.7) and (3.10) hold. Then for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}_0$, there exists some $T = T(\tau, \mathcal{D}) > 0$ such that for all $t \geq T$ and $u_{\tau-t} \in \mathcal{D}(\tau - t)$, the solution u to Eq. (3.1) satisfies*

$$E(\|u(\tau,\tau-t,u_{\tau-t})\|^2) \leq M + M \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \|g(s+\tau)\|^2 \mathrm{d}s, \tag{3.12}$$

where M is a positive constant independent of τ and $\mathcal{D}$.

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Proof. By the Itô formula, we obtain from (3.1) that for each $r \geq \tau - t$,

$$\begin{aligned}
&\|u(r,\tau-t,u_{\tau-t})\|^2 + 2\int_{\tau-t}^{r} \|u(s,\tau-t,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} \mathrm{d}s + 2\lambda \int_{\tau-t}^{r} \|u(s,\tau-t,u_{\tau-t})\|^2 \mathrm{d}s \\
&+ 2\int_{\tau-t}^{r}\int_{\mathcal{O}} f(x,u(s,\tau-t,u_{\tau-t}))u(s,\tau-t,u_{\tau-t})\mathrm{d}x\mathrm{d}s \\
=&\|u_{\tau-t}\|^2 + 2\int_{\tau-t}^{r} (g(s),u(s,\tau-t,u_{\tau-t}))\mathrm{d}s + \int_{\tau-t}^{r} \|h(s,u(s,\tau-t,u_{\tau-t}))\|^2_{L_2(U,L^2(\mathcal{O}))}\mathrm{d}s \\
&+ 2\int_{\tau-t}^{r} (u(s,\tau-t,u_{\tau-t}),h(s,u(s,\tau-t,u_{\tau-t}))\mathrm{d}W(s)), \qquad (3.13)
\end{aligned}$$

Taking the expectation on both sides of (3.13), we get, for almost all $r \geq \tau - t$, that

$$\begin{aligned}
&E(\|u(r,\tau-t,u_{\tau-t})\|^2) + 2\int_{\tau-t}^{r} E(\|u(s,\tau-t,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)})\mathrm{d}s \\
&+ 2\lambda\int_{\tau-t}^{r} E(\|u(s,\tau-t,u_{\tau-t})\|^2)\mathrm{d}s + 2\int_{\tau-t}^{r} E\Big(\int_{\mathcal{O}} f(x,u(s,\tau-t,u_{\tau-t}))u(s,\tau-t,u_{\tau-t})\mathrm{d}x\Big)\mathrm{d}s \\
=&E(\|u_{\tau-t}\|^2) + 2\int_{\tau-t}^{r} E(g(s),u(s,\tau-t,u_{\tau-t}))\mathrm{d}s \\
&+ \int_{\tau-t}^{r} E(\|h(s,u(s,\tau-t,u_{\tau-t}))\|^2_{L_2(U,L^2(\mathcal{O}))})\mathrm{d}s. \qquad (3.14)
\end{aligned}$$

Thus, for almost all $r \geq \tau - t$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}E(\|u(r,\tau-t,u_{\tau-t})\|^2) + 2E(\|u(r,\tau-t,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}) \\
&+2\lambda E(\|u(r,\tau-t,u_{\tau-t})\|^2) + 2E\Big(\int_{\mathcal{O}} f(x,u(r,\tau-t,u_{\tau-t}))u(r,\tau-t,u_{\tau-t})\mathrm{d}x\Big) \\
=&\ 2E(g(r),u(r,\tau-t,u_{\tau-t})) \\
&+E(\|h(r,u(r,\tau-t,u_{\tau-t}))\|^2_{L_2(U,L^2(\mathcal{O}))}). \qquad (3.15)
\end{aligned}$$

Now, we estimate each item on the right-hand side of (3.15). By (3.3) we have that

$$\begin{aligned}
&\int_{\mathcal{O}} f(u(r,\tau-t,u_{\tau-t}))u(r,\tau-t,u_{\tau-t})\mathrm{d}x \\
\geq& a_1\int_{\mathcal{O}} |u(r,\tau-t,u_{\tau-t})|^p\mathrm{d}x - \|\phi_2\|_{L^1(\mathcal{O})}, \qquad (3.16)
\end{aligned}$$

which implies

$$\begin{aligned}
&2E\Big(\int_{\mathcal{O}} f(u(r,\tau-t,u_{\tau-t}))u(r,\tau-t,u_{\tau-t})\mathrm{d}x\Big) \\
\geq& 2a_1E(\|u(r,\tau-t,u_{\tau-t})\|^p_{L^p(\mathcal{O})}) - 2\|\phi_2\|_{L^1(\mathcal{O})}. \qquad (3.17)
\end{aligned}$$

Note that

$$(g(r),u(r,\tau-t,u_{\tau-t}))$$

$$\leq \frac{\lambda}{4}\|u(r,\tau-t,u_{\tau-t})\|^2+\frac{1}{\lambda}\|g(r)\|^2, \tag{3.18}$$

which implies that

$$\begin{aligned}&2E(g(r),u(r,\tau-t,u_{\tau-t}))\\ \leq&\frac{\lambda}{2}E(\|u(r,\tau-t,u_{\tau-t})\|^2)+\frac{2}{\lambda}\|g(r)\|^2.\end{aligned} \tag{3.19}$$

We deduce from (3.15)-(3.19) and (3.6) that, for almost all $r\geq\tau-t$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}E(\|u(r,\tau-t,u_{\tau-t})\|^2)+\lambda E(\|u(r,\tau-t,u_{\tau-t})\|^2)\\ \leq&\frac{2}{\lambda}\|g(r)\|^2+2\|\phi_2\|_{L^1(\mathcal{O})}+L.\end{aligned} \tag{3.20}$$

Applying Gronwall's inequality to (3.20), we get

$$\begin{aligned}&E(\|u(r,\tau-t,u_{\tau-t})\|^2)\\ \leq&\mathrm{e}^{\lambda(\tau-t-r)}E(\|u_{\tau-t})\|^2)+\frac{1}{\lambda}(2\|\phi_2\|_{L^1(\mathcal{O})}+L)+\mathrm{e}^{-\lambda r}\int_{\tau-t}^{r}\mathrm{e}^{\lambda s}\frac{2}{\lambda}\|g(s)\|^2\mathrm{d}s.\end{aligned} \tag{3.21}$$

Then we get that

$$\begin{aligned}&E(\|u(\tau,\tau-t,u_{\tau-t})\|^2)\\ \leq&\mathrm{e}^{-\lambda t}E(\|u_{\tau-t}\|^2)+\frac{1}{\lambda}(2\|\phi_2\|_{L^1(\mathcal{O})}+L)+\mathrm{e}^{-\lambda\tau}\frac{2}{\lambda}\int_{-\infty}^{\tau}\mathrm{e}^{\lambda s}\|g(s)\|^2\mathrm{d}s.\end{aligned} \tag{3.22}$$

Since $u_{\tau-t}\in\mathcal{D}(\tau-t)$ and $\mathcal{D}=\{\mathcal{D}(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}_0$, we get

$$\mathrm{e}^{-\lambda\tau}\mathrm{e}^{\lambda(\tau-t)}E(\|u_{\tau-t}\|^2)\to 0 \quad as \quad t\to+\infty.$$

Therefore, there exists $T=T(\tau,\mathcal{D})>0$ such that for all $t\geq T$,

$$\mathrm{e}^{-\lambda t}E(\|u_{\tau-t}\|^2)\leq 1. \tag{3.23}$$

By (3.22) and (3.23), we get, for all $t\geq T$, there exists some positive constant M independent of τ and $\mathcal{D}$ such that

$$E(\|u(\tau,\tau-t,u_{\tau-t})\|^2)\leq M+M\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\|g(s+\tau)\|^2\mathrm{d}s.$$

This completes the proof. □

Corollary 3.1 *Let* (3.2)-(3.7) *and* (3.10)-(3.11) *hold. Then the mean random dynamical system* Φ *for Eq.* (3.1) *possesses a weakly compact* $\mathcal{D}_0$*-pullback absorbing set* $K_0=\{K_0(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}_0$, *which is given by,*

$$K_0(\tau)=\{u\in L^2(\Omega,\mathcal{F}_\tau;L^2(\mathcal{O})):E(\|u\|^2)\leq\mathcal{R}_0(\tau)\}, \tag{3.24}$$

where

$$\mathcal{R}_0(\tau):=M+M\int_{-\infty}^{0}\mathrm{e}^{\lambda s}\|g(s+\tau)\|^2\mathrm{d}s \tag{3.25}$$

with M *being the same constant as in Lemma 3.2.*

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Proof. We know that for each $\tau \in \mathbb{R}, K_0(\tau)$ in (3.24) is a bounded and closed convex subset of $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$, and therefore it is weakly compact in $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$. Lemma 3.2 indicates that for every $\tau \in \mathbb{R}$ and $\mathcal{D} \in \mathcal{D}_0$, there exists $T = T(\tau, D) > 0$ such that

$$\Phi(t, \tau - t, D(\tau - t)) \subseteq K_0(\tau), \forall t \geq T. \tag{3.26}$$

In addition, from (3.11) and (3.25), we get for any $\nu > 0$

$$\lim_{\tau \to -\infty} \mathrm{e}^{\nu\tau} \sup_{u \in K_0(\tau)} \|u\| = 0,$$

that is $K_0 \in \mathcal{D}_0$. Hence, K_0 is a weakly compact $\mathcal{D}_0$-pullback absorbing set for Φ. □

Theorem 3.1 *Suppose* (3.2)-(3.7) *and* (3.10)-(3.11) *hold. Then the mean random dynamical system* Φ *for problem* (3.1) *possesses a unique weak* $\mathcal{D}_0$*-pullback mean random attractor* $\bar{\mathcal{A}}_0 = \{\bar{\mathcal{A}}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ *in* $L^2(\Omega, \mathcal{F}; L^2(\mathcal{O}))$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathrm{P})$.

Proof. From Lemma 2.5 and Corollary 3.1, we can easily get the existence and uniqueness of weak $\mathcal{D}_0$-pullback mean random attractor $\bar{\mathcal{A}}_0 \in \mathcal{D}_0$ of Φ for Eq. (3.1). □

4 Wong-ZaKai Approximations of Stochastic Semi-linear Degenerate Parabolic Equation

In this section, we consider the following stochastic semi-linear degenerate parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(t,x) + h(t,x,u) \circ \dfrac{\mathrm{d}W}{\mathrm{d}t}, & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{4.1}$$

Here $W = \omega(t)$ is a two-sided real-valued Wiener process on a probability space and the other terms are the same as described in section 1. The symbol " $\circ$ " indicates that the stochastic term in Eq. (4.1) is understood in the sense of Stratonovich's integration.

We remark that, in this section, we consider the stochastic term of Eq. (4.1) in the sense of Stratonovich's integration because the Stratonovich's interpretation is more appropriate than Itô's when we consider the pathwise dynamical behavior (fixed any $\omega \in \Omega$) of the Wong-Zakai approximate system corresponding to the equation (see [35] for details).

4.1 Random dynamical systems for Wong-Zakai Approximations

In this subsection, we first define a continuous cocycle Ψ for Wong-Zakai approximate system of Eq. (4.1), and then prove that there exists a unique pullback random attractor for the cocycle Ψ.

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Let $\mathcal{O}$ be a bounded domain (or an unbounded domain) and let the nonnegative function $\sigma(x)$ satisfy $(\mathcal{H}_\alpha)$ (or $(\mathcal{H}_{\alpha,\beta})$). In what follows, We assume that $f : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$ is a smooth nonlinear function such that for all $x \in \mathcal{O}$ and $u \in \mathbb{R}$,

$$f(x,u)u \geq \alpha_1|u|^p - \beta_1(x), \tag{4.2}$$

$$|f(x,u)| \leq \alpha_2|u|^{p-1} + \beta_2(x), \tag{4.3}$$

$$\frac{\partial f(x,u)}{\partial u} \geq \alpha_3|u|^{p-2} - \beta_3(x), \tag{4.4}$$

where $p > 2$, α_1, α_2, α_3 are positive numbers, $\beta_1(x) \in L^1(\mathcal{O})$, $\beta_2(x) \in L^{p_1}(\mathcal{O})$ with $\frac{1}{p_1} + \frac{1}{p} = 1$, $\beta_3(x) \in L^\infty(\mathcal{O})$. Let h be a continuous function and for all $t, u \in \mathbb{R}$, $x \in \mathcal{O}$, satisfy

$$|h(t,x,u)| \leq \psi_1(t,x)|u|^{q-1} + \psi_2(t,x), \tag{4.5}$$

$$|\frac{\partial}{\partial u}h(t,x,u)| \leq \psi_3(t,x)|u|^{q-2} + \psi_4(t,x), \tag{4.6}$$

where $2 \leq q < p$, $\psi_1 \in L_{\rm loc}^{\frac{p}{p-q}}(\mathbb{R}; L_{\rm loc}^{\frac{p}{p-q}}(\mathcal{O}))$ and $\psi_2 \in L_{\rm loc}^{p_1}(\mathbb{R}; L^{p_1}(\mathcal{O}))$, and $\psi_3, \psi_4 \in L^\infty(\mathbb{R}; L^\infty(\mathcal{O}))$.

In the sequel, let $(\Omega, \mathcal{F}, \mathrm{P})$ be the classical Wiener probability space, where

$$\Omega = C_0(\mathbb{R},\mathbb{R}) := \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\} \tag{4.7}$$

with the open compact topology. The Brownian motion has the form $W(t,\omega) = \omega(t)$. Consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, P)$ defined by

$$\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t). \tag{4.8}$$

Then from [1], we get that $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system and there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\widetilde{\Omega} \subseteq \Omega$ of full P measure such that for each $\omega \in \widetilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0 \qquad \text{as} \qquad \mathrm{t} \to \pm\infty. \tag{4.9}$$

For brevity, we identify the space $\widetilde{\Omega}$ with Ω. For any given $\delta \neq 0$, define the random variable $\mathcal{G}_\delta$ by

$$\mathcal{G}_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \qquad \forall\, \omega \in \Omega. \tag{4.10}$$

We get from (4.8) and (4.10) that

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{\omega(t+\delta) - \omega(t)}{\delta} \quad \text{and} \quad \int_0^t \mathcal{G}_\delta(\theta_s\omega)\mathrm{d}s = \int_t^{t+\delta} \frac{\omega(s)}{\delta}\mathrm{d}s + \int_\delta^0 \frac{\omega(s)}{\delta}\mathrm{d}s. \tag{4.11}$$

By the continuity of ω and (4.11), the following result have been proved in [21].

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Lemma 4.1 *Let $\tau \in \mathbb{R}$, $T > 0$, and $\omega \in \Omega$. Then for each $\epsilon > 0$, there is a constant $\delta' = \delta'(\epsilon, \tau, \omega, T) > 0$ such that for every $0 < |\delta| < \delta'$ and $t \in [\tau, \tau + T]$,*

$$|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t)| < \epsilon. \tag{4.12}$$

Let's consider the Wong-Zakai approximate system of Eq. (4.1):

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\mathrm{div}(\sigma(x)\nabla u)) + \lambda u + f(x,u) = g(t,x) + h(t,x,u)\mathcal{G}_\delta(\theta_t\omega), & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{4.13}$$

Notice that system (4.13) can be viewed as deterministic equation parameterized by $\omega \in \Omega$. Let assumptions (4.2)-(4.6) hold, and then by the Galerkin method similar to [3], we can prove that for any $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\mathcal{O})$, Eq. (4.13) possesses a unique solution

$$u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty); L^2(\mathcal{O})) \cap L^2_{\mathrm{loc}}((0,\infty); D_0^{1,2}(\mathcal{O}, \sigma)) \cap L^p_{\mathrm{loc}}((0,\infty); L^p(\mathcal{O})). \tag{4.14}$$

In addition, the solution $u(\cdot, \tau, \omega, u_\tau)$ is continuous in $u_\tau \in L^2(\mathcal{O})$ and is $\big(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O}))\big)$-measurable in $\omega \in \Omega$. Hence, we can define a continuous cocycle $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ by

$$\Psi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), \quad \forall\ \tau \in \mathbb{R},\ t > 0,\ \omega \in \Omega,\ u_\tau \in L^2(\mathcal{O}). \tag{4.15}$$

Let $\mathcal{D}_1 = \{\mathcal{D}_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $L^2(\mathcal{O})$. A family $\mathcal{D}_1$ is said to be tempered if for any $\nu > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is

$$\lim_{t \to -\infty} \mathrm{e}^{\nu t} \sup_{u \in D(\tau + t, \theta_t\omega)} \|u\| = 0.$$

We denote by $\mathcal{D}_1$ the class of all tempered families of nonempty bounded subsets of $L^2(\mathcal{O})$.

Now, we devote to prove existence of $\mathcal{D}_1$-pullback random attractors for the cocycle Ψ corresponding to Eq. (4.13) in $L^2(\mathcal{O})$.

Lemma 4.2 *Suppose (4.2)-(4.6) and (3.10)-(3.11) hold. Then the continuous cocycle Ψ of problem (4.13) possesses a closed measurable $\mathcal{D}_1$-pullback absorbing set $K_1 = \{K_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, which is given by*

$$K_1(\tau, \omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \le R(\tau, \omega)\}, \tag{4.16}$$

where

$$R(\tau, \omega) = M_1 + M_1 \int_{-\infty}^0 \mathrm{e}^{\lambda s}(\|g(s + \tau)\|^2 + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})\mathrm{d}s \tag{4.17}$$

with M_1 is a positive constant independent of τ, ω and $\mathcal{D}_1$.

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Proof. We first prove that, for any given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K_1(\tau,\omega)$ given by (4.16) is a pullback absorbing set for the cocycle Ψ. Taking inner product of Eq. (4.13) with u in $L^2(\mathcal{O})$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}+\lambda\|u\|^2+\int_{\mathcal{O}}f(x,u)u\mathrm{d}x\\ =&(g,u)+\mathcal{G}_\delta(\theta_t\omega)\int_{\mathcal{O}}h(t,x,u)u\mathrm{d}x.\end{aligned}\tag{4.18}$$

By (4.2) we obtain that

$$\int_{\mathcal{O}}f(x,u)udx\geq\alpha_1\int_{\mathcal{O}}|u|^pdx-\|\beta_1\|_{L^1(\mathcal{O})}.\tag{4.19}$$

By (4.5) and Young's inequality, we get

$$\begin{aligned}&\mathcal{G}_\delta(\theta_t\omega)\int_{\mathcal{O}}h(t,x,u)u\mathrm{d}x\\ \leq&|\mathcal{G}_\delta(\theta_t\omega)|\int_{\mathcal{O}}(|\psi_1(t,x)||u|^q+|\psi_2(t,x)||u|)\mathrm{d}x\\ \leq&\frac{\alpha_1}{2}\int_{\mathcal{O}}|u|^p\mathrm{d}x+C\int_{\mathcal{O}}|\psi_1(t,x)\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}dx+C\int_{\mathcal{O}}|\psi_2(t,x)\mathcal{G}_\delta(\theta_t\omega)|^{p_1}\mathrm{d}x\\ \leq&\frac{\alpha_1}{2}\int_{\mathcal{O}}|u|^p\mathrm{d}x+C|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}\|\psi_1(t)\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}}+C|\mathcal{G}_\delta(\theta_t\omega)|^{p_1}\|\psi_2(t)\|^{p_1}_{L^{p_1}}\\ \leq&\frac{\alpha_1}{2}\int_{\mathcal{O}}|u|^p\mathrm{d}x+C|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}+C|\mathcal{G}_\delta(\theta_t\omega)|^{p_1}.\end{aligned}\tag{4.20}$$

From Cauchy's inequality, we have

$$(g(t,x),u)\leq\frac{\lambda}{2}\|u\|^2+\frac{1}{2\lambda}\|g\|^2.\tag{4.21}$$

Therefore, it follows easily from (4.18)-(4.21) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\|u\|^2+2\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}+\lambda\|u\|^2+\alpha_1\|u\|^p_{L^p(\mathcal{O})}\\ \leq&2\|\beta_1\|_{L^1(\mathcal{O})}+\frac{1}{\lambda}\|g\|^2+C|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}}+C|\mathcal{G}_\delta(\theta_s\omega)|^{p_1}.\end{aligned}\tag{4.22}$$

Multiplying (4.22) by $\mathrm{e}^{\lambda s}$, replacing ω by $\theta_{-\tau}\omega$ and then integrating with respect to s over $(\tau-t,\tau)$ with $t\geq 0$, we get that

$$\begin{aligned}&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+2\int_{\tau-t}^{\tau}\mathrm{e}^{\lambda(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}\mathrm{d}s\\ &+\alpha_1\int_{\tau-t}^{\tau}\mathrm{e}^{\lambda(s-\tau)}\|u\|^p_{L^p}\mathrm{d}s\\ \leq&\mathrm{e}^{-\lambda t}\|u_{\tau-t}\|^2+\frac{2\|\beta_1\|_{L^1(\mathcal{O})}}{\lambda}+\frac{1}{\lambda}\int_{\tau-t}^{\tau}\mathrm{e}^{\lambda(s-\tau)}\|g(s)\|^2\mathrm{d}s\\ &+C\int_{\tau-t}^{\tau}\mathrm{e}^{\lambda(s-\tau)}(|\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^{\frac{p}{p-q}}+|\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^{p_1})\mathrm{d}s\end{aligned}$$

$$\leq e^{-\lambda t}\|u_{\tau-t}\|^2 + \frac{2\|\beta_1\|_{L^1(\mathcal{O})}}{\lambda} + \frac{1}{\lambda}\int_{-\infty}^{0} e^{\lambda s}\|g(s+\tau)\|^2 ds$$
$$+ C\int_{-\infty}^{0} e^{\lambda s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds. \tag{4.23}$$

The last two integrals in (4.23) are well defined due to (3.10), (4.9), (4.11) and the continuity of ω. For every $u_{\tau-t} \in D_1(\tau-t, \theta_{-t}\omega)$ and $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, we have

$$\limsup_{t\to+\infty} e^{-\lambda t}\|u_{\tau-t}\|^2 = 0. \tag{4.24}$$

Hence, there exists some $T_1 = T_1(\sigma,\tau,\omega,D_1) > 0$ such that for all $t \geq T_1$,

$$\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 + 2\int_{\tau-t}^{\tau} e^{\lambda(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}ds$$
$$+ \alpha_1\int_{\tau-t}^{\tau} e^{\lambda(s-\tau)}\|u\|^p_{L^p}ds$$
$$\leq 1 + \frac{2\|\beta_1\|_{L^1(\mathcal{O})}}{\lambda} + \frac{1}{\lambda}\int_{-\infty}^{0} e^{\lambda s}\|g(s+\tau)\|^2 ds + C\int_{-\infty}^{0} e^{\lambda s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds$$
$$\leq M_1 + M_1\int_{-\infty}^{0} e^{\lambda s}(\|g(s+\tau)\|^2 + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds, \tag{4.25}$$

where M_1 is a positive constant independent of τ, ω and D_1. Then by (4.25) we get that, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and every $D_1 \in \mathcal{D}_1$, $K_1(\tau,\omega)$ given by (4.16) satisfies

$$\Psi(t,\tau-t,\theta_{-t}\omega, D_1(\tau-t,\theta_{-t}\omega)) \subseteq K_1(\tau,\omega).$$

We next prove that $K_1 \in \mathcal{D}_1$. Let ν be an arbitrary positive constant. Then for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we can get from (4.17) that

$$e^{\nu t}\|K_1(\tau+t,\theta_t\omega)\|^2 \leq e^{\nu t}R(\tau+t,\theta_t\omega)$$
$$= M_1 e^{\nu t} + M_1 e^{\nu t}\int_{-\infty}^{0} e^{\lambda s}\Big(\|g(s+t+\tau)\|^2 + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}\Big)ds. \tag{4.26}$$

First, we can get from (3.11) that

$$\lim_{t\to-\infty} e^{\nu t}\int_{-\infty}^{0} e^{\lambda s}\|g(s+\tau+t)\|^2 ds$$
$$= \lim_{t\to-\infty} e^{-\lambda\tau}e^{\nu t}\int_{-\infty}^{\tau} e^{\lambda s}\|g(s+t)\|^2 ds = 0. \tag{4.27}$$

Let $\tilde{\nu} = \min\{\lambda,\nu\}$, and then we can get from (4.9) and (4.11) that for any $t \leq 0$,

$$e^{\nu t}\int_{-\infty}^{0} e^{\lambda s}(|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1})ds$$
$$\leq \int_{-\infty}^{t} e^{\tilde{\nu} s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})ds. \tag{4.28}$$

Note that

$$\int_{-\infty}^{0} e^{\tilde{\nu}s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})\mathrm{d}s < +\infty,$$

which implies that

$$\int_{-\infty}^{t} e^{\tilde{\nu}s}(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1})\mathrm{d}s \to 0 \quad \text{as} \quad t \to -\infty. \tag{4.29}$$

It follows from (4.26)-(4.29) that K_1 is tempered, i.e. $K_1 \in \mathcal{D}_1$. Moreover, since for each $\tau \in \mathbb{R}$, $R(\tau, \cdot) : \Omega \mapsto \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, then $K_1(\tau, \cdot)$ is also measurable. Hence, $K_1 \in \mathcal{D}_1$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set for Ψ. The proof is completed. □

Lemma 4.3 *Let* (4.2)-(4.6) *hold. Then for each* $\tau \in \mathbb{R}$, $t > \tau$, $\omega \in \Omega$ *and for each bounded sequence* $\{u_{0,n}\}_{n=1}^{\infty} \subseteq L^2(\mathcal{O})$, *the sequence* $\{u(t, \tau, \omega, u_{0,n})\}_{n=1}^{\infty}$ *possesses a convergent subsequence in* $L^2(\mathcal{O})$.

Proof. Taking $T > t$, and integrating (4.22) over $[\tau, T]$, we can get that

$$\{u(\cdot, \tau, \omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^p\big((\tau, T); L^p(\mathcal{O})\big) \cap L^2\big((\tau, T); D_0^{1,2}(\mathcal{O}, \sigma)\big). \tag{4.30}$$

We can also infer from (4.3), (4.5) and (4.30) that, for $s \in [\tau, T]$,

$$\begin{aligned} &\{f(\cdot, u(\cdot, \tau, \omega, u_{0,n}))\}_{n=1}^{\infty} \text{ and } \{h(\cdot, \cdot, u(\cdot, \tau, \omega, u_{0,n}))\mathcal{G}_\delta(\theta_s\omega)\}_{n=1}^{\infty} \\ &\qquad \text{are bounded in } L^{p_1}((\tau, T); L^{p_1}(\mathcal{O})). \end{aligned} \tag{4.31}$$

Then it follows from (4.30), (4.31), and Eq. (4.13), that

$$\{\frac{\partial}{\partial t}u(\cdot, \tau, \omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^2\big((\tau, T); D_0^{-1,2}(\mathcal{O}, \sigma)\big) + L^{p_1}\big((\tau, T); L^{p_1}(\mathcal{O})\big). \tag{4.32}$$

By Lemma 2.3, we note that the embedding $D_0^{1,2}(\mathcal{O}, \sigma) \hookrightarrow L^2(\mathcal{O})$ is compact (in both cases of bounded and unbounded domain). Then we can get from (4.30), (4.32) and Aubin-Lions compactness lemma that there exist some $w \in L^2((\tau, T); L^2(\mathcal{O}))$ and a subsequence of $\{u(s, \tau, \omega, u_{0,n})\}_{n=1}^{\infty}$ such that

$$u(\cdot, \tau, \omega, u_{0,n_k}) \to w \text{ in } L^2((\tau, T); L^2(\mathcal{O})). \tag{4.33}$$

By choosing a further subsequence (relabelled the same), we infer from (4.33) that

$$u(s, \tau, \omega, u_{0,n_k}) \to w(s) \text{ in } L^2(\mathcal{O}), \text{ a.e. } s \in [\tau, T]. \tag{4.34}$$

Finally, since $t \in (\tau, T)$, we can by the continuity of solutions on initial data in $L^2(\mathcal{O})$ and (4.34) get

$$u(t, \tau, \omega, u_{0,n_k}) = u(t, s, \omega, u(s, \tau, \omega, u_{0,n_k})) \to u(t, s, \omega, w(s)),$$

i.e., $u(t, \tau, \omega, u_{0,n})$ possesses a convergent subsequence in $L^2(\mathcal{O})$. We complete the proof. □

Lemma 4.4 *Suppose* (4.2)-(4.6) *and* (3.10) *hold. Then the continuous cocycle* Ψ *for Eq.* (4.13) *is* $\mathcal{D}_1$*-pullback asymptotically compact in* $L^2(\mathcal{O})$.

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Proof. For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, $t_n \to +\infty$ as $n \to \infty$ and $u_{0,n} \in D_1(\tau - t_n, \theta_{-t_n}\omega)$, we shall prove the sequence $\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ has a convergent subsequence in $L^2(\mathcal{O})$. Note that $t_n \to +\infty$ as $n \to \infty$ and $u_{0,n} \in D_1(\tau - t_n, \theta_{-t_n}\omega)$. We can get from Lemma 4.2 that there exist $N_1 = N_1(\tau, \omega, D_1) > 0$ such that for all $n \geq N_1$ that

$$\|u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\| \leq C(\tau, \omega), \tag{4.35}$$

which implies that

$$\{u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^2(\mathcal{O}). \tag{4.36}$$

It follows from (4.36) and Lemma 4.3 that the sequence

$$\{u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty} \text{ is precompact in } L^2(\mathcal{O}),$$

which along with $\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) = u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})$, it implies the result. □

Theorem 4.1 *Suppose* (4.2)-(4.6) *and* (3.10)-(3.11) *hold. Then the continuous cocycle* Ψ *associated with system* (4.13) *possesses a unique* $\mathcal{D}_1$*-pullback random attractor* $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ *in* $L^2(\mathcal{O})$.

Proof. From Lemma 4.2, Lemma 4.4 as well as [34, Proposition 2.1], the existence of unique $\mathcal{D}_1$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ follows.

□

4.2 Stochastic Semi-linear Degenerate Parabolic Equation Driven by linear Multiplicative Noise

In this subsection, we discuss the following stochastic semi-linear degenerate parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x, u) = g(t, x) + u \circ \dfrac{dw}{dt}, & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x, t)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{4.37}$$

and consider the following Wong-Zakai approximate system for Eq. (4.37):

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} - \operatorname{div}(\sigma(x)\nabla u_\delta) + \lambda u_\delta + f(x, u_\delta) = g(t, x) + u_\delta \mathcal{G}_\delta(\theta_t\omega), & t > \tau, \\ u_\delta(\tau, x) = u_{\delta,\tau}(x), & \tau \in \mathbb{R}, \\ u_\delta(x, t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{4.38}$$

We will investigate the relations between the solutions of Eq. (4.37) and Eq. (4.38). To this end, we need to transform the stochastic equation (4.37) into a pathwise deterministic one. Let

$$v(t, \tau, \omega) = \mathrm{e}^{-\omega(t)} u(t, \tau, \omega), \tag{4.39}$$

with

$$v_\tau = \mathrm{e}^{-\omega(\tau)} u_\tau.$$

Then by (4.37) and (4.39), we get

$$\begin{cases} \dfrac{\partial v}{\partial t} + Av + \lambda v + \mathrm{e}^{-\omega(t)} f(x,u) = \mathrm{e}^{-\omega(t)} g(t,x), & t > \tau, \\ u(\tau, x) = u_\tau(x), & \tau \in \mathbb{R}, \\ u(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau, \end{cases} \tag{4.40}$$

where $Av = -\mathrm{div}(\sigma(x)\nabla v)$. We also introduce a similar transform for Eq. (4.38) as we did for Eq. (4.37). Let

$$v_\delta(t,\tau,\omega,v_{\delta,\tau}) = \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} u_\delta(t,\tau,\omega,u_{\delta,\tau}) \tag{4.41}$$

with

$$v_{\delta,\tau} = \mathrm{e}^{-\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} u_{\delta,\tau}.$$

Then we have

$$\begin{cases} \dfrac{\partial v_\delta}{\partial t} + Av_\delta + \lambda v_\delta + \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} f(x,u_\delta) = \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} g(t,x), & t > \tau, \\ v(\tau,x) = v_{\delta,\tau}(x), & \tau \in \mathbb{R}, \\ v(x,t)|_{\partial\mathcal{O}} = 0, & t > \tau. \end{cases} \tag{4.42}$$

For any $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathcal{O})$, let (4.2)-(4.4) hold. Then by the classic Galerkin method, we can get the existence and uniqueness of solution $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty), L^2(\mathcal{O}))$ for system (4.40). In addition $v(\cdot,\tau,\omega,v_\tau)$ is continuous in $v_\tau \in L^2(\mathcal{O})$ and is $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O})))$-measurable in $\omega \in \Omega$. Thus, we can define a continuous cocycle $\widetilde{\Psi}_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ for system (4.37) by

$$\widetilde{\Psi}_0(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = \mathrm{e}^{\omega(t)-\omega(-\tau)} v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau). \tag{4.43}$$

Similarly, we can also define a continuous cocycle $\widetilde{\Psi}_\delta(t,\tau,\omega,u_{\delta,\tau})$ for system (4.38).

Lemma 4.5 *Assume* (4.2)-(4.4) *and* (3.10)-(3.11) *hold. Then the continuous cocycle* $\widetilde{\Psi}_0$ *for system* (4.37) *possesses a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $\widetilde{B}_0 = \{\widetilde{B}_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$*, which is given by*

$$\widetilde{B}_0(\tau,\omega) = \{u \in L^2(\mathcal{O}) : \|u\|^2 \le \widetilde{R}_0(\tau,\omega)\}, \tag{4.44}$$

where

$$\widetilde{R}_0(\tau,\omega) = 4\int_{-\infty}^0 \mathrm{e}^{\frac{3}{2}\lambda s - 2\omega(s)} \left(\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})}\right)\mathrm{d}s. \tag{4.45}$$

Proof. Taking inner product of Eq. (4.40) with $v(t,\tau,\omega) = \mathrm{e}^{-\omega(t)} u(t,\tau,\omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \|v\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|v\|^2 + \mathrm{e}^{-\omega(t)} \int_{\mathcal{O}} f(x,u) v \mathrm{d}x = \mathrm{e}^{-\omega(t)}(g,v). \tag{4.46}$$

It follows from (4.2) and (4.39) that

$$\mathrm{e}^{-\omega(t)}\int_{\mathcal{O}} f(x,u)v\mathrm{d}x \geqslant \alpha_1\mathrm{e}^{-2\omega(t)}\|u\|^p_{L^p(\mathcal{O})} - \|\beta_1\|_{L^1(\mathcal{O})}\mathrm{e}^{-2\omega(t)}. \tag{4.47}$$

By Cauchy's inequality, we obtain

$$\mathrm{e}^{-\omega(t)}(g,v) \leqslant \frac{\lambda}{4}\|v\|^2 + \frac{1}{\lambda}\mathrm{e}^{-2\omega(t)}\|g\|^2. \tag{4.48}$$

Then combining (4.46)-(4.48), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\|v\|^2 + 2\|v\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \frac{3}{2}\lambda\|v\|^2 + 2\alpha_1\mathrm{e}^{-2\omega(s)}\|u\|^p_{L^p(\mathcal{O})}\\ &\leqslant 2\|\beta_1\|_{L^1(\mathcal{O})}\mathrm{e}^{-2\omega(s)} + \frac{2}{\lambda}\mathrm{e}^{-2\omega(s)}\|g\|^2.\end{aligned} \tag{4.49}$$

Multiplying $\mathrm{e}^{\frac{3}{2}\lambda s}$ on both sides of (4.49) and then integrating with respect to s over $[\tau-t,\tau]$ with $t>0$, we get

$$\begin{aligned}&\|v(\tau,\tau-t,\omega,v_{\tau-t}\|^2 + 2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\|v\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}\mathrm{d}s + 2\alpha_1\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)-2\omega(s)}\|u\|^p_{L^p(\mathcal{O})}\mathrm{d}s\\ &\leq 2\|\beta_1\|_{L^1(\mathcal{O})}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)-2\omega(s)}\mathrm{d}s + \frac{2}{\lambda}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)-2\omega(s)}\|g(s)\|^2\mathrm{d}s + \|v_{\tau-t}\|^2\mathrm{e}^{-\frac{3}{2}\lambda t}.\end{aligned} \tag{4.50}$$

Replacing ω in (4.50) by $\theta_{-\tau}\omega$ and using

$$u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) = \mathrm{e}^{(\omega(-\tau+s)-\omega(-\tau))}v(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}), \tag{4.51}$$

we get that

$$\begin{aligned}&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\mathrm{e}^{2\omega(-\tau)} + 2\alpha_1\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)-2(\omega(-\tau+s)-\omega(-\tau))}\|u\|^p_{L^p(\mathcal{O})}\mathrm{d}s\\ &+2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\mathrm{e}^{-2\omega(s-\tau)+2\omega(-\tau)}\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}\mathrm{d}s\\ &\leq 2\|\beta_1\|_{L^1(\mathcal{O})}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)-2(\omega(-\tau+s)-\omega(-\tau))}\mathrm{d}s + \mathrm{e}^{2\omega(-\tau)-2\omega(-t)}\|u_{\tau-t}\|^2\mathrm{e}^{-\frac{3}{2}\lambda t}\\ &+\frac{2}{\lambda}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)-2(\omega(-\tau+s)-\omega(-\tau))}\|g(s)\|^2\mathrm{d}s.\end{aligned} \tag{4.52}$$

Then from (4.52), we get

$$\begin{aligned}&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 + 2\alpha_1\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s-2\omega(s)}\|u\|^p_{L^p(\mathcal{O})}\mathrm{d}s\\ &+2\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s}\mathrm{e}^{-2\omega(s)}\|u\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}\mathrm{d}s\\ &\leq 2\|\beta_1\|_{L^1(\mathcal{O})}\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s-2\omega(s)}\mathrm{d}s + \frac{2}{\lambda}\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s-2\omega(s)}\|g(s+\tau)\|^2\mathrm{d}s\end{aligned}$$

$$+\,\mathrm{e}^{-2\omega(-t)}\mathrm{e}^{-\frac{3}{2}\lambda t}\|u_{\tau-t}\|^2. \tag{4.53}$$

By (3.10) and (4.9), we have

$$2\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s-2\omega(s)}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s<\infty. \tag{4.54}$$

Note that if $u_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ and $D_1\in\mathcal{D}_1$, then by (4.9) we have

$$\limsup_{t\to+\infty}\mathrm{e}^{-2\omega(-t)}\mathrm{e}^{-\frac{3}{2}\lambda t}\|u_{\tau-t}\|^2=0. \tag{4.55}$$

Then there exists some $T_4=T_4(\tau,\omega,D_1)>0$ such that for all $t\geq T_4$,

$$\mathrm{e}^{-2\omega(-t)}\mathrm{e}^{-\frac{3}{2}\lambda t}\|u_{\tau-t}\|^2\leq 2\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s-2\omega(s)}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s, \tag{4.56}$$

which along with (4.43) implies that

$$\widetilde{\Psi}_0(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))\subseteq\widetilde{B}_0(\tau,\omega),\ \forall t\geq T_4,$$

where $\widetilde{B}_0(\tau,\omega)$ is given by (4.44). In addition, by (3.11), (4.9) and the continuity of $\omega(t)$, we can easily get that $\widetilde{B}_0$ is tempered, that is, $\widetilde{B}_0\in\mathcal{D}_1$. Hence, $\widetilde{B}_0\in\mathcal{D}_1$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set for $\widetilde{\Psi}_0$. The proof is completed. □

Theorem 4.2 *Suppose* (4.2)-(4.4) *and* (3.10)-(3.11) *hold. Then the continuous cocycle* $\widetilde{\Psi}_0$ *for system* (4.37) *is* $\mathcal{D}_1$*-pullback asymptotically compact and possesses a unique* $\mathcal{D}_1$*-pullback random attractor* $\widetilde{\mathcal{A}}_0=\{\widetilde{\mathcal{A}}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$ *in* $L^2(\mathcal{O})$.

Proof. The proof of $\mathcal{D}_1$-pullback asymptotical compactness of cocycle Ψ_0 in $L^2(\mathcal{O})$ is similar to that of Lemma 4.4. And then by [34, Proposition 2.1] and Lemma 4.5, we can easily get the cocycle Ψ_0 possesses a unique $\mathcal{D}_1$-pullback random attractor $\mathcal{A}_0$. □

Lemma 4.6 *Suppose* (4.2)-(4.4) *and* (3.10)-(3.11) *hold. Then the continuous cocycle* $\widetilde{\Psi}_\delta$ *for Eq.* (4.38) *possesses a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $\widetilde{B}_\delta=\{\widetilde{B}_\delta(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$,

$$\widetilde{B}_\delta(\tau,\omega)=\{u_\delta\in L^2(\mathcal{O}):\|u_\delta\|^2\leq\widetilde{R}_\delta(\tau,\omega)\}, \tag{4.57}$$

where

$$\widetilde{R}_\delta(\tau,\omega)=4\int_{-\infty}^{0}\mathrm{e}^{\frac{3}{2}\lambda s}\mathrm{e}^{2\int_s^0\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}(\frac{1}{\lambda}\|g(s+\tau)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s. \tag{4.58}$$

In addition, we have for every $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$

$$\lim_{\delta\to0}\widetilde{R}_\delta(\tau,\omega)=\widetilde{R}_0(\tau,\omega), \tag{4.59}$$

where $\widetilde{R}_0(\tau,\omega)$ *is given by* (4.45).

Proof. By (4.42) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\delta\|^2+\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}+\lambda\|v_\delta\|^2+\int_{\mathcal{O}}\mathrm{e}^{-\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}f(x,u_\delta)v_\delta\mathrm{d}x$$
$$=\mathrm{e}^{-\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}(g,v_\delta). \quad (4.60)$$

By (4.2) and (4.41), we get

$$-\int_{\mathcal{O}}\mathrm{e}^{-\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}f(x,u_\delta)v_\delta\mathrm{d}x$$
$$\leq-\alpha_1\mathrm{e}^{-2\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\|u_\delta\|^p_{L^p(\mathcal{O})}+\|\beta_1\|_{L^1(\mathcal{O})}\mathrm{e}^{-2\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}. \quad (4.61)$$

By Cauchy's inequality, we obtain

$$\mathrm{e}^{-\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}(g,v_\delta)\leq\frac{\lambda}{4}\|v_\delta\|^2+\frac{1}{\lambda}\mathrm{e}^{-2\int_0^t\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\|g\|^2. \quad (4.62)$$

Then it follows from (4.60)-(4.62) that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|v_\delta\|^2+2\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}+\frac{3}{2}\lambda\|v_\delta\|^2+2\alpha_1\mathrm{e}^{-2\int_0^s\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\|u_\delta\|^p_{L^p(\mathcal{O})}$$
$$\leq 2\mathrm{e}^{-2\int_0^s\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}(\frac{1}{\lambda}\|g\|^2+\|\beta_1\|_{L^1(\mathcal{O})}). \quad (4.63)$$

For all $\tau\in\mathbb{R},t\in\mathbb{R}^+$ and $\omega\in\Omega$, multiplying $\mathrm{e}^{\frac{3}{2}\lambda s}$ and then integrating with respect to s from $\tau-t$ to τ, we have

$$\|v_\delta(\tau,\tau-t,\omega,v_{\delta,\tau-t})\|^2+2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}\mathrm{d}s$$
$$+2\alpha_1\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\mathrm{e}^{-2\int_0^s\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\|u_\delta\|^p_{L^p(\mathcal{O})}\mathrm{d}s$$
$$\leq\mathrm{e}^{-\frac{3}{2}\lambda t}\|v_{\delta,\tau-t}\|^2+2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\mathrm{e}^{-2\int_0^s\mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}(\frac{1}{\lambda}\|g(s)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s. \quad (4.64)$$

Replacing ω in (4.64) by $\theta_{-\tau}\omega$, we get

$$\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2+2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\|v_\delta\|^2_{D_0^{1,2}(\mathcal{O},\sigma)}\mathrm{d}s$$
$$+2\alpha_1\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\mathrm{e}^{-2\int_0^s\mathcal{G}_\delta(\theta_{r-\tau}\omega)\mathrm{d}r}\|u_\delta\|^p_{L^p(\mathcal{O})}\mathrm{d}s$$
$$\leq\mathrm{e}^{-\frac{3}{2}\lambda t}\|v_{\delta,\tau-t}\|^2+2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\mathrm{e}^{-2\int_0^s\mathcal{G}_\delta(\theta_{r-\tau}\omega)\mathrm{d}r}(\frac{1}{\lambda}\|g(s)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s. \quad (4.65)$$

By (4.41) and (4.65) we get

$$\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2$$
$$\leq\mathrm{e}^{-\frac{3}{2}\lambda t}\mathrm{e}^{2\int_{\tau-t}^{\tau}\mathcal{G}_\delta(\theta_{r-\tau}\omega)\mathrm{d}r}\|u_{\delta,\tau-t}\|^2$$
$$+2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{3}{2}\lambda(s-\tau)}\mathrm{e}^{2\int_s^{\tau}\mathcal{G}_\delta(\theta_{r-\tau}\omega)\mathrm{d}r}(\frac{1}{\lambda}\|g(s)\|^2+\|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s$$

$$\begin{aligned}
&\leq e^{-\frac{3}{2}\lambda t} e^{2\int_{-t}^{0} \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} \|u_{\delta,\tau-t}\|^2 \\
&\quad + 2\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} (\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s. \qquad (4.66)
\end{aligned}$$

By (3.10), (4.9), (4.11) and the continuity of $\omega(t)$ we get

$$2\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} (\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s < \infty. \qquad (4.67)$$

Note that if $u_{\delta,\tau-t} \in D_1(\tau-t, \theta_{-t}\omega)$ and $D_1 \in \mathcal{D}_1$, then by (4.9), (4.11) and the continuity of $\omega(t)$, we get

$$\limsup_{t\to+\infty} e^{-\frac{3}{2}\lambda t} e^{2\int_{-t}^{0} \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} \|u_{\delta,\tau-t}\|^2 = 0, \qquad (4.68)$$

which implies that there exists $T_5 = T_5(\tau, \omega, D_1, \delta) > 0$ such that for all $t \geq T_5$,

$$\begin{aligned}
&e^{-\frac{3}{2}\lambda t} e^{2\int_{-t}^{0} \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} \|u_{\delta,\tau-t}\|^2 \\
\leq& 2\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} (\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s. \qquad (4.69)
\end{aligned}$$

By (4.66)-(4.69), we get

$$\|u_\delta(\tau, \tau-t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2 \leq 4\int_{-\infty}^{0} e^{\frac{3}{2}\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)dr} (\frac{1}{\lambda}\|g(s+\tau)\|^2 + \|\beta_1\|_{L^1(\mathcal{O})})\mathrm{d}s. \qquad (4.70)$$

In other words, we get for all $t \geq T_5$,

$$u_\delta(\tau, \tau-t, \theta_{-\tau}\omega, D(\tau-t, \theta_{-t}\omega)) \subseteq \widetilde{B}_\delta(\tau, \omega), \qquad (4.71)$$

where $\widetilde{B}_\delta(\tau, \omega)$ is given by (4.57). In addition, $\widetilde{B}_\delta$ is tempered due to (3.11), (4.9), (4.11). Therefore, $\widetilde{B}_\delta$ is a closed measurable $\mathcal{D}_1$-pullback absorbing set of Ψ_δ. The proof of (4.59) is similar to that of [21, Lemma 3.7] and the details are omitted here. □

Theorem 4.3 *Suppose* (4.2)-(4.4) *and* (3.10)-(3.11) *hold. Then the cotinuous cocycle* $\widetilde{\Psi}_\delta$ *for Eq.* (4.38) *is* $\mathcal{D}_1$*-pullback asymptotically compact and possesses a unique* $\mathcal{D}_1$*-pullback random attractor* $\widetilde{\mathcal{A}}_\delta = \{\widetilde{\mathcal{A}}_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ *in* $L^2(\mathcal{O})$.

Proof. The proof is similar to that of Theorem 4.2. □

Now, we show the solution of Eq. (4.38) converges to the solution of Eq. (4.37) as $\delta \to 0$. Toward this end, we further assume the following assumption hold: there exists some $\alpha_4 > 0$ such that for all $x \in \mathcal{O}$, $u \in \mathbb{R}$,

$$|\frac{\partial f}{\partial u}(x, u)| \leq \alpha_4(1 + |u|^{p-2}). \qquad (4.72)$$

Lemma 4.7 *Suppose* (4.2)-(4.4) *and* (3.10)-(3.11) *hold. Let* u *and* u_δ *be the solutions of Eq.* (4.37) *and Eq.* (4.38)*, respectively, with initial data* u_τ *and* $u_{\delta,\tau}$*. If* $u_{\delta,\tau} \to u_\tau$ *in* $L^2(\mathcal{O})$ *as* $\delta \to 0$*, then for every* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$ *and* $T > 0$*, there exists some* $\tilde{\delta}_0 = \tilde{\delta}_0(\tau, \omega, T) > 0$ *such that for any* $0 < |\delta| < \tilde{\delta}_0$ *and* $t \in [\tau, \tau+T]$*,* $u_\delta(t, \tau, \omega, u_{\delta,\tau}) \to u(t, \tau, \omega, u_\tau)$ *in* $L^2(\mathcal{O})$.

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Proof. Let $\xi = v_\delta - v$ and then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \|\xi\|^2_{D_0^{1,2}(\mathcal{O},\sigma)} + \lambda\|\xi\|^2 \\ = \int_{\mathcal{O}} (\mathrm{e}^{-\omega(t)} f(x,u) - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} f(x,u_\delta))\xi dx + (\mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - \mathrm{e}^{-\omega(t)})(g(t),\xi). \quad (4.73)$$

By using (4.3)-(4.4) and (4.72), we have

$$\begin{aligned}
&(\mathrm{e}^{-\omega(t)} f(x,u) - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} f(x,u_\delta))\xi \\
=&\big(\mathrm{e}^{-\omega(t)} f(x,\mathrm{e}^{\omega(t)}v) - \mathrm{e}^{-\omega(t)} f(x, v_\delta \mathrm{e}^{\omega(t)})\big)\xi + \big(\mathrm{e}^{-\omega(t)} f(x, v_\delta \mathrm{e}^{\omega(t)}) - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} f(x, v_\delta \mathrm{e}^{\omega(t)})\big)\xi \\
&+ \big(\mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} f(x, v_\delta \mathrm{e}^{\omega(t)}) - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} f(x, v_\delta \mathrm{e}^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r})\big)\xi \\
=&\frac{\partial f}{\partial s}(\mathrm{e}^{-\omega(t)}(v\mathrm{e}^{\omega(t)} - v_\delta \mathrm{e}^{\omega(t)}))\xi + f(x, v_\delta \mathrm{e}^{\omega(t)})(\mathrm{e}^{-\omega(t)} - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r})\xi \\
&+ \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} v_\delta(\mathrm{e}^{\omega(t)} - \mathrm{e}^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r})\frac{\partial f}{\partial s}\xi \\
=&-\frac{\partial f}{\partial s}\xi^2 + f(x, v_\delta \mathrm{e}^{\omega(t)})\xi(\mathrm{e}^{-\omega(t)} - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}) + v_\delta(\mathrm{e}^{\omega(t)-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} - 1)\frac{\partial f}{\partial s}\xi \\
\leq& \mid \beta_3 \mid |\xi|^2 + \big(\alpha_2 \mathrm{e}^{(p-1)\omega(t)}|v_\delta|^{p-1}|\xi| + |\beta_2||\xi|\big)\big|\mathrm{e}^{-\omega(t)} - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\big| \\
&+ \alpha_4\Big|1 - \mathrm{e}^{\omega(t)-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\Big|\Big(|v_\delta|^{p-1}\Big|\mathrm{e}^{\omega(t)} + \mathrm{e}^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}\Big|^{p-2}|\xi| + |v_\delta||\xi|\Big).
\end{aligned} \quad (4.74)$$

From Lemma 4.1, we find that for any $\epsilon > 0$, there exists some $\tilde{\delta}_1 = \tilde{\delta}_1(\epsilon, \tau, \omega, T) > 0$ such that

$$|1 - \mathrm{e}^{\omega(t)-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}| < \epsilon,\ |\mathrm{e}^{-\omega(t)} - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r}| < \epsilon,\ \forall 0 < |\delta| < \tilde{\delta}_1,\ t \in [\tau, \tau + T]. \quad (4.75)$$

It follows from (4.74) and (4.75) that

$$\int_{\mathcal{O}} (\mathrm{e}^{-\omega(t)} f(x,u) - \mathrm{e}^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} f(x,u_\delta))\xi \mathrm{d}x \leq C\|\xi\|^2 + C\epsilon(\|v_\delta\|^p_{L^p(\mathcal{O})} + \|v\|^p_{L^p(\mathcal{O})} + 1). \quad (4.76)$$

By Cauchy's inequality, we have

$$(\mathrm{e}^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} - \mathrm{e}^{-\omega(t)})(g(t),\xi) \leq \big|\mathrm{e}^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\mathrm{d}r} - \mathrm{e}^{-\omega(t)}\big|(\frac{1}{2}\|g\|^2 + \frac{1}{2}\|\xi\|^2). \quad (4.77)$$

Combing (4.73)-(4.77), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 \leq C\|\xi\|^2 + C\epsilon(\|v_\delta\|^p_{L^p(\mathcal{O})} + \|v\|^p_{L^p(\mathcal{O})} + \|g\|^2 + 1). \quad (4.78)$$

Applying Gronwall's inequality to (4.78), we get for all $0 < |\delta| < \tilde{\delta}_1$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
\|\xi(t)\|^2 \leq& \mathrm{e}^{C(t-\tau)}\|\xi(\tau)\|^2 + C\epsilon \mathrm{e}^{C(t-\tau)} \int_\tau^t \Big(1 + \|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^p_{L^p(\mathcal{O})} \\
&+ \|v(s,\tau,\omega,v_\tau)\|^p_{L^p(\mathcal{O})} + \|g(s)\|^2\Big)\mathrm{d}s.
\end{aligned} \quad (4.79)$$

By (4.39), (4.41), (4.50), (4.64) and (4.79), we get that there exists some $\tilde{\delta}_2 \in (0, \tilde{\delta}_1)$ and $\tilde{c_1} = \tilde{c}_1(\tau, T, \omega) > 0$ such that for all $0 < |\delta| < \tilde{\delta}_2$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}&\|v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau)\|^2 \\ \le& e^{\tilde{c}_1(t-\tau)}\|v_{\delta,\tau} - v_\tau\|^2 + \tilde{c}_1\epsilon e^{\tilde{c}_1(t-\tau)}\Big(1 + \|v_\tau\|^2 + \|v_{\delta,\tau}\|^2 + \int_\tau^t \|g(s)\|^2 ds\Big).\end{aligned} \tag{4.80}$$

Using (4.39) and (4.41) again, we get

$$\begin{aligned}&\|u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau)\| \\ \le& \|v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau)\|\Big|e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr}\Big| + \Big|e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\Big|\|v(t, \tau, \omega, v_\tau)\|.\end{aligned} \tag{4.81}$$

Note that $u_{\delta,\tau} = v_{\delta,\tau}e^{\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)dr}$ and $u_\tau = v_\tau e^{\omega(\tau)}$. Then by the continuity of $\omega(t)$, (4.12), (4.50), and (4.80)-(4.81), we can obtain the desired convergence. □

Lemma 4.8 *Suppose* (4.2)-(4.4) *and* (3.10)-(3.11) *hold. For any given* $\tau \in \mathbb{R}$, $T > 0$ *and* $\omega \in \Omega$, *if* $\delta_n \to 0$ *and* $u_n \in \widetilde{\mathcal{A}}_{\delta_n}(\tau, \omega)$, *then the sequence* $\{u_n\}_{n=1}^\infty$ *has a convergent subsequence in* $L^2(\mathcal{O})$.

Proof. By using Lemma 2.3 and the similar method as that of Lemma 3.10 in [21], we can get the result. □

Theorem 4.4 *Suppose* (4.2)-(4.3) *and* (3.10)-(3.11) *hold. Then for any given* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *the following relationship holds:*

$$\lim_{\delta \to 0} d_{L^2(\mathcal{O})}(\widetilde{\mathcal{A}}_\delta(\tau, \omega), \widetilde{\mathcal{A}}_0(\tau, \omega)) = 0.$$

Proof. By Lemma 4.5 and Lemma 4.6, we obtain that, for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta \to 0} \|\widetilde{B}_\delta(\tau, \omega)\|^2 = \|\widetilde{B}_0(\tau, \omega)\|^2 \le \widetilde{B}_0(\tau, \omega),$$

where $\widetilde{B}_0(\tau, \omega)$ is given by (4.45) and $\widetilde{B}_0 = \{\widetilde{B}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$. Let $\delta \to 0$ and $u_\delta \to u_\tau$, and then from Lemma 4.7, we get, for every $\tau \in \mathbb{R}$, $t \in \mathbb{R}^+$, and $\omega \in \Omega$, that $\Psi_\delta(\tau, t, \omega, u_{\delta,\tau}) \to \Psi_0(\tau, t, \omega, u_\tau)$ in $L^2(\mathcal{O})$. Then, by Lemma 4.8 and Theorem 3.1 in [28] we can get the result. □

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