On Parameterized Theta Function Identities

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ABSTRACT. In history, theta function identities have been studied extensively and deeply. In this paper, applying the residue theorem of an elliptic function, we obtain a parameterized theta function identity. Then, by selecting the appropriate parameter, we derive some famous Jacobi identities, as well as other relatively simple theta function identities.

Key words: Elliptic functions; Theta function identities; Jacobi theta function identities

1. INTRODUCTION

Theta function identities are classical and important objects of study. There are many relations between theta function identities, combinatorics, number theory and modular forms.

Around 1995, Farkas and Kopeliovich [1,2] proved some Ramanujan's identities and modular equations by applying the residue theorem of elliptic functions. In 2001, using similar methods, Liu [4] studied the theta function satisfying the following functional equations $f(z+\pi)=(-1)^n f(z)$ and $f(z+\pi\tau)=\left(-q^{-1}e^{-2iz}\right)^n f(z)$,

where $n=3,4,5$. He arrived at many theta identities, some of which are classical and others are new. For more theta functions, their remarkable history and modern developments, see [5–8, 10–12].

First, we will give a brief view of fundamental facts about classical theta functions.

Let $q=e^{\pi i\tau}$, and $Im(\tau)>0$. The Jacobi theta functions are defined by

$$\theta_1(z\mid\tau)=-iq^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)}e^{(2n+1)zi}=2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin(2n+1)z,$$

$$\theta_2(z\mid\tau)=q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}q^{n(n+1)}e^{(2n+1)zi}=2q^{\frac{1}{4}}\sum_{n=0}^{\infty}q^{n(n+1)}\cos(2n+1)z,$$

$$\theta_3(z\mid\tau)=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2nzi}=1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nz,$$

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$$\theta_4(z \mid \tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nzi} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nz.$$

We will use $\theta_j'(z|q)$ to denote the partial derivative with respect to the variable z.

Lemma 1.1 ([4]). *We have*

$$\frac{\theta_1'(z \mid \tau)}{\theta_1(z \mid \tau)} = \cot z + 4\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin 2nz. \tag{1.1}$$

The $q-$shifted factorial [3] is defined by

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

For convenience, we use $(a)_n$ to denote $(a;q)_n$.

When $z = \frac{\pi}{3}$, we have

$$\theta_1\left(\frac{\pi}{3} \mid \tau\right) = \sqrt{\frac{3(q;q)_\infty^3 \theta_1'(0 \mid 3\tau)}{(q^3;q^3)_\infty \theta_1'(0 \mid \tau)}}, \tag{1.2}$$

$$\theta_2\left(\frac{\pi}{3} \mid \tau\right) = \sqrt{\frac{(q;q)_\infty^3 \theta_2(0 \mid 3\tau)}{(q^3;q^3)_\infty \theta_2(0 \mid \tau)}}, \tag{1.3}$$

$$\theta_3\left(\frac{\pi}{3} \mid \tau\right) = \sqrt{\frac{(q;q)_\infty^3 \theta_3(0 \mid 3\tau)}{(q^3;q^3)_\infty \theta_3(0 \mid \tau)}}, \tag{1.4}$$

$$\theta_4\left(\frac{\pi}{3} \mid \tau\right) = \sqrt{\frac{(q;q)_\infty^3 \theta_4(0 \mid 3\tau)}{(q^3;q^3)_\infty \theta_4(0 \mid \tau)}}. \tag{1.5}$$

With respect to the (quasi) periods π and $\pi\tau$, we have

$$\theta_1(z+\pi \mid \tau) = -\theta_1(z \mid \tau); \quad \theta_1(z+\pi\tau \mid \tau) = -q^{-1}e^{-2iz}\theta_1(z \mid \tau), \tag{1.6}$$

$$\theta_2(z+\pi \mid \tau) = -\theta_2(z \mid \tau); \quad \theta_2(z+\pi\tau \mid \tau) = q^{-1}e^{-2iz}\theta_2(z \mid \tau), \tag{1.7}$$

$$\theta_3(z+\pi \mid \tau) = \theta_3(z \mid \tau); \quad \theta_3(z+\pi\tau \mid \tau) = q^{-1}e^{-2iz}\theta_3(z \mid \tau), \tag{1.8}$$

$$\theta_4(z+\pi \mid \tau) = \theta_4(z \mid \tau); \quad \theta_4(z+\pi\tau \mid \tau) = -q^{-1}e^{-2iz}\theta_4(z \mid \tau). \tag{1.9}$$

We also have the following relations:

$$\theta_1\left(z+\frac{\pi}{2} \mid \tau\right) = \theta_2(z \mid \tau); \quad \theta_1\left(z+\frac{\pi\tau}{2} \mid \tau\right) = iA\theta_4(z \mid \tau), \tag{1.10}$$

$$\theta_2\left(z+\frac{\pi}{2} \mid \tau\right) = -\theta_1(z \mid \tau); \quad \theta_2\left(z+\frac{\pi\tau}{2} \mid \tau\right) = A\theta_3(z \mid \tau), \tag{1.11}$$

$$\theta_3\left(z+\frac{\pi}{2} \mid \tau\right) = \theta_4(z \mid \tau); \quad \theta_3\left(z+\frac{\pi\tau}{2} \mid \tau\right) = A\theta_2(z \mid \tau), \tag{1.12}$$

$$\theta_4\left(z+\frac{\pi}{2} \mid \tau\right) = \theta_3(z \mid \tau); \quad \theta_4\left(z+\frac{\pi\tau}{2} \mid \tau\right) = iA\theta_1(z \mid \tau) \tag{1.13}$$

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In the following, we will provide the definition for the residue and the process of computing for the residue of $f(z)$. Let

$$f(z) := \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 + \cdots .$$

Throughout this paper, we will denote the residue of $f(z)$ by $Resf(z)$. Set $F(z) := z^n f(z)$. It is well-known that

$$\operatorname{Res}(f;0) = \frac{1}{(n-1)!} F^{(n-1)}(0).$$

When $2 \le n \le 6$, we have the following theorem.

Theorem 1.2. *Let $F(z)$ be the above definition , and set $\phi(z) = \frac{F'(z)}{F(z)}$, then*

$$\begin{aligned}
n = 1 &: \operatorname{Res}(f;0) = F(0),\\
n = 2 &: \operatorname{Res}(f;0) = F(0)\phi(0),\\
n = 3 &: \operatorname{Res}(f;0) = \frac{1}{2}F(0)\left(\phi^2(0) + \phi'(0)\right)\\
n = 4 &: \operatorname{Res}(f;0) = \frac{1}{6}F(0)\left(\phi^3(0) + 3\phi(0)\phi'(0) + \phi''(0)\right),\\
n = 5 &: \operatorname{Res}(f;0) = \frac{1}{24}F(0)\left(\phi^4(0) + 6\phi^2(0)\phi''(0) + 4\phi(0)\phi''(0)\right.\\
&\qquad \left. +3\phi'(0)^2 + \phi'''(0)\right),\\
n = 6 &: \operatorname{Res}(f;0) = \frac{1}{120}F(0)\left(\phi^5(0) + 10\phi^3(0)\phi'(0) + 10\phi^2(0)\phi''(0)\right.\\
&\qquad + 15\phi(0)\phi'(0)^2 + 5\phi(0)\phi'''(0) + 10\phi'(0)\phi''(0)\\
&\qquad \left. +\phi^{(4)}(n)\right).
\end{aligned}$$

Theorem 1.3. *The sums of all the residues of an elliptic function in the period parallelogram is zero.*

In this paper, we study the case $n = 6$. We get an identity with five paremeters. As applications, we obtain the Jacobi identity and some new identities.

2. MAIN RESULTS

In this section, we apply logarithmic differerentiation to compute the residue of elliptic functions at high order poles. Then we obtain an theta identity with five parameters.

Theorem 2.1. *Suppose $f(z)$ is an entire function satisfying the functional equations*

$$f(z+\pi) = (-1)^6 f(z) \quad \textit{and} \quad f(z+\pi\tau) = \left(-q^{-1}e^{-2iz}\right)^6 f(z), \tag{2.1}$$

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then, we have

$$\frac{f''(0)}{2\theta_1'(0\mid\tau)^2}-\frac{f(0)\left(1-24\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\cdot n\right)}{\theta_1'(0\mid\tau)^2}+\frac{q^{\frac{3}{2}}f\left(\frac{\pi\tau}{2}\right)}{\theta_4^2(0\mid\tau)}+\frac{q^{\frac{3}{2}}f\left(\frac{\pi+\pi\tau}{2}\right)}{\theta_3^2(0\mid\tau)}=\frac{f\left(\frac{\pi}{2}\right)}{\theta_2^2(0\mid\tau)}. \tag{2.2}$$

Proof. We consider the following function

$$g(z)=\frac{f(z)}{\theta_1^2(z\mid\tau)\theta_1(2z\mid\tau)},$$

From (1.6) and (2.1), we find that $g(z)$ satisfies

$$g(z+\pi)=g(z)\quad\text{and}\quad g(z+\pi\tau)=g(z).$$

Thus, $g(z)$ is an elliptic function with the periods π and $\pi\tau$. The poles of $g(z)$ are 0, $\frac{\pi}{2},\frac{\pi\tau}{2},\frac{\pi+\pi\tau}{2}$, all of which are simple poles.

Based on Theorem 1.3, we have

$$\operatorname{Res}(g;0)+\operatorname{Res}\left(g;\frac{\pi}{2}\right)+\operatorname{Res}\left(g;\frac{\pi\tau}{2}\right)+\operatorname{Res}\left(g;\frac{\pi+\pi\tau}{2}\right)=0. \tag{2.3}$$

Now let us computer the above residues, respectively.

Suppose $G(z)=z^3g(z)$ and $\phi(z)=\frac{G'(z)}{G(z)}$. Notice that

$$\lim_{z\to0}\theta_1(z\mid\tau)=\lim_{z\to0}z\theta_1'(z\mid\tau),$$
$$\lim_{z\to0}\theta_1(2z\mid\tau)=\lim_{z\to0}2z\theta_1'(2z\mid\tau).$$

By using L'Hoptital'rule, we derive

$$\begin{aligned}G(0)&=\lim_{z\to0}\frac{z^3f(z)}{\theta_1^2(z\mid\tau)\theta_1(2z\mid\tau)}\\&=\lim_{z\to0}\frac{z^3f(z)}{2z^3\theta_1'(z\mid\tau)^2\theta_1'(2z\mid\tau)}\\&=\frac{f(0)}{2\theta_1'(0\mid\tau)^3}.\end{aligned} \tag{2.4}$$

From the expression of $G(z)$, we have

$$\ln G(z)=\ln z^3g(z)=3\ln z+\ln g(z)$$

and

$$\frac{G'(z)}{G(z)}=\frac{3}{z}+\frac{g'(z)}{g(z)}.$$

On the other hand, we have

$$\begin{aligned}\ln g(z)&=\ln\frac{f(z)}{\theta_1^2(z\mid\tau)\theta_1(2z\mid\tau)}\\&=\ln f(z)-2\ln\theta_1(z\mid\tau)-\ln\theta_1(2z\mid\tau),\end{aligned}$$

and

$$\frac{g'(z)}{g(z)} = \frac{f'(z)}{f(z)} - 2\frac{\theta_1'(z \mid \tau)}{\theta_1(z \mid \tau)} - 2\frac{\theta_1'(2z \mid \tau)}{\theta_1(2z \mid \tau)}.$$

Thus,

$$\frac{G'(z)}{G(z)} = \frac{3}{z} + \frac{f'(z)}{f(z)} - 2\frac{\theta_1'(z \mid \tau)}{\theta_1(z \mid \tau)} - 2\frac{\theta_1'(2z \mid \tau)}{\theta_1(2z \mid \tau)}.$$

In view of (1.1), we arrive at

$$\begin{aligned}\phi(z) =&\frac{G'(z)}{G(z)} = \frac{3}{z} + \frac{f'(z)}{f(z)} - 2\left(\frac{1}{z} - \frac{z}{3} - O\left(z^3\right) + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nz\right)\\ &- 2\left(\frac{1}{2z} - \frac{2z}{3} - O\left(z^3\right) + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 4nz\right)\\ =&2z + \frac{f'(z)}{f(z)} - 8\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}(\sin 2nz + \sin 4nz) + O\left(z^3\right).\end{aligned} \tag{2.5}$$

Furthermore, we have

$$\phi'(z) = \left(\frac{G'(z)}{G(z)}\right)' = 2 + \left(\frac{f'(z)}{f(z)}\right)' - 8\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}(2n\cos 2nz + \mid 4n\cos 4nz) + O\left(z^2\right). \tag{2.6}$$

Based on Theorem 1.2, (2.4), (2.5), and (2.6), we have

$$\begin{aligned}\operatorname{Res}(g;0) &= \frac{1}{2}G(0)\left(\phi^2(0) + \phi'(0)\right)\\ &= \frac{1}{2}\cdot\frac{f(0)}{2\theta_1'(0 \mid \tau)^3}\cdot\left(\left(\frac{f'(0)}{f(0)}\right)^2 + \left(\frac{f'(0)}{f(0)}\right)' + 2 - 48\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\cdot n\right)\\ &= \frac{f''(0)}{4\theta_1'(0 \mid \tau)^3} - \frac{f(0)\left(1 - 24\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\cdot n\right)}{2\theta_1'(0 \mid \tau)^3}.\end{aligned} \tag{2.7}$$

In terms of the definition of residue, we have the following results.

$$\begin{aligned}\operatorname{Res}\left(g;\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}}\frac{\left(z-\frac{\pi}{2}\right)f(z)}{\theta_1^2(z \mid \tau)\theta_1(2z \mid \tau)}\\ &= \lim_{z\to 0}\frac{zf\left(z+\frac{\pi}{2}\right)}{\theta_1^2\left(z+\frac{\pi}{2} \mid \tau\right)\theta_1(2z+\pi \mid \tau)}\\ &= -\frac{f\left(\frac{\pi}{2}\right)}{2\theta_1'(0 \mid \tau)\theta_2^2(0 \mid \tau)},\end{aligned} \tag{2.8}$$

$$\operatorname{Res}\left(g;\frac{\pi\tau}{2}\right) = \frac{q^{\frac{3}{2}}f\left(\frac{\pi\tau}{2}\right)}{2\theta_1'(0 \mid \tau)\theta_4^2(0 \mid \tau)}, \tag{2.9}$$

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and

$$\operatorname{Res}\left(g;\frac{\pi+\pi\tau}{2}\right)=\frac{q^{\frac{3}{2}}f\left(\frac{\pi+\pi\tau}{2}\right)}{2\theta_1'(0\mid\tau)\theta_3^2(0\mid\tau)}. \tag{2.10}$$

Substituting the (2.7) -(2.10) into (2.3), we obtain (2.2). We complete the proof. □

Theorem 2.2. *For any complex numbers* a, b, c, d, e, *let* $f(z)=\theta_1(z+a\mid\tau)\theta_1(z+b\mid\tau)\theta_1(z+c\mid\tau)\theta_1(z+d\mid\tau)\theta_1(z+e\mid\tau)\theta_1(z-a-b-c-d-e\mid\tau)$, *we have*

$$\begin{aligned}&\frac{f''(0)}{2\theta_1'(0\mid\tau)^2}\\&+\frac{\theta_1(a\mid\tau)\theta_1(b\mid\tau)\theta_1(c\mid\tau)\theta_1(d\mid\tau)\theta_1(e\mid\tau)\theta_1(a+b+c+d+e\mid\tau)\left(1-24\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\cdot n\right)}{\theta_1'(0\mid\tau)^2}\\&+\frac{\theta_3(a\mid\tau)\theta_3(b\mid\tau)\theta_3(c\mid\tau)\theta_3(d\mid\tau)\theta_3(e\mid\tau)\theta_3(a+b+c+d+e\mid\tau)}{\theta_3^2(0\mid\tau)}\\&=\frac{\theta_4(a\mid\tau)\theta_4(b\mid\tau)\theta_4(c\mid\tau)\theta_4(d\mid\tau)\theta_4(e\mid\tau)\theta_4(a+b+c+d+e\mid\tau)}{\theta_4^2(0\mid\tau)}\\&+\frac{\theta_2(a\mid\tau)\theta_2(b\mid\tau)\theta_2(c\mid\tau)\theta_2(d\mid\tau)\theta_2(e\mid\tau)\theta_2(a+b+c+d+e\mid\tau)}{\theta_2^2(0\mid\tau)}.\end{aligned} \tag{2.11}$$

Proof. Based on (1.6)-(1.9) and the following

$$\begin{aligned}\theta_1(-z\mid\tau)&=-\theta_1(z\mid\tau),\\\theta_2(-z\mid\tau)&=\theta_2(z\mid\tau),\\\theta_3(-z\mid\tau)&=\theta_3(z\mid\tau)\\\theta_4(-z\mid\tau)&=\theta_4(z\mid\tau),\\\theta_1'(-z\mid\tau)&=-\theta_1'(z\mid\tau),\\\theta_1''(-z\mid\tau)&=\theta_1''(z\mid\tau),\end{aligned}$$

we can derive

$$\begin{aligned}f(z+\pi)&=(-1)^6f(z)=f(z)\\f(z+\pi\tau)&=\left(-q^{-1}e^{-2iz}\right)^6f(z)\end{aligned}$$

so, $f(z)$ satisfies (2.1).

On the other hand, we have

$$\begin{aligned}&f\left(\frac{\pi}{2}\right)=\theta_2(a\mid\tau)\theta_2(b\mid\tau)\theta_2(c\mid\tau)\theta_2(d\mid\tau)\theta_2(e\mid\tau)\theta_2(a+b+c+d+e\mid\tau),\\&f\left(\frac{\pi\tau}{2}\right)=-q^{-\frac{3}{2}}\theta_4(a\mid\tau)\theta_4(b\mid\tau)\theta_4(c\mid\tau)\theta_4(d\mid\tau)\theta_4(e\mid\tau)\theta_4(a+b+c+d+e\mid\tau),\\&f\left(\frac{\pi+\pi\tau}{2}\right)=q^{-\frac{3}{2}}\theta_3(a\mid\tau)\theta_3(b\mid\tau)\theta_3(c\mid\tau)\theta_3(d\mid\tau)\theta_3(e\mid\tau)\theta_3(a+b+c+d+e\mid\tau).\end{aligned}$$

Substituting the above three identities into (2.2), we derive (2.11). We compete the proof. □

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Let $a = b = c = d = e = 0$ in (2.11), we obtain the famous Jacobi identity [13].

$$\theta_2^4(0|\tau) + \theta_4^4(0|\tau) = \theta_3^4(0|\tau).$$

Let $a = 0, b = c = d = \frac{\pi}{3}, e = 0$ in (2.11), we have the following theta identity.

$$\theta_2^3\left(\frac{\pi}{3} \mid \tau\right)\theta_2(0 \mid \tau) + \theta_3^3\left(\frac{\pi}{3} \mid \tau\right)\theta_3(0 \mid \tau) = \theta_4^3\left(\frac{\pi}{3} \mid \tau\right)\theta_4(0 \mid \tau).$$

Based on (1.2)-(1.5), we have

$$\theta_2(0 \mid 3\tau)\sqrt{\frac{\theta_2(0 \mid 3\tau)}{\theta_2(0 \mid \tau)}} + \theta_3(0 \mid 3\tau)\sqrt{\frac{\theta_3(0 \mid 3\tau)}{\theta_3(0 \mid \tau)}} = \theta_4(0 \mid 3\tau)\sqrt{\frac{\theta_4(0 \mid 3\tau)}{\theta_4(0 \mid \tau)}}.$$

Let $\tau \to -\frac{1}{3\tau}$, we get

$$\theta_3(0 \mid \tau)\sqrt{\frac{\theta_3(0 \mid \tau)}{\theta_3(0 \mid 3\tau)}} + \theta_4(0 \mid \tau)\sqrt{\frac{\theta_4(0 \mid \tau)}{\theta_4(0 \mid 3\tau)}} = \theta_2(0 \mid \tau)\sqrt{\frac{\theta_2(0 \mid \tau)}{\theta_2(0 \mid 3\tau)}}.$$

Notice that the above two identities are relative with modular equations which are found by Ramanujan [9].

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