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Coherent States of the p-Adic Heisenberg Group and Entropic Uncertainty Relations

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Abstract: Properties of coherent states for the Heisenberg group over a field of p-adic numbers are investigated. It turns out that coherent states form an orthonormal basis. The family of all such bases is parametrized by a set of self-dual lattices in the phase space. The Wehrl entropy S_W is considered and its properties are investigated. In particular, it is proved that $S_W \geq 0$ and vanishes only on coherent states. For a pair of different bases of coherent states, an entropic uncertainty relation is obtained. It is shown that the lower bound for the sum of the corresponding Wehrl entropies is given by the distance between the lattices. The proof uses the fact that the bases of coherent states corresponding to a pair of lattices are mutually unbiased.

Keywords: coherent states; Wehrl entropy; entropic uncertainty relations

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1. Introduction

1.1. Coherent states

Let G be a locally compact group and $U(g)$, $g \in G$ be its irreducible unitary representation in a separable complex Hilbert space $\mathcal{H}$.

Fix a unit vector $|\psi\rangle \in \mathcal{H}$ and consider the subgroup $H \subset G$ with the property:

$$U(h)|\psi\rangle = e^{i\omega(h)}|\psi\rangle,\ h \in H.$$

Let $X = G/H$ and $g(x)$ be an arbitrary representative from a coset $x \in X$. The following set:

$$\{|x\rangle = U(g(x))|\psi\rangle,\ x \in X\}$$

is the system of (generalized) coherent states [1]. It is easy to see that $|x\rangle\langle x|$ does not depend on the choice of a representative $g(x)$ from coset x.

If we apply the procedure to the Heisenberg group and vacuum vector as $|\psi\rangle$ then we obtain the well-known coherent states in quantum theory.

1.2. The Husimi function and the Wehrl entropy

For the system of coherent states constructed above, it is possible to determine the Husimi function and the Wehrl entropy. Namely, let ρ be the density operator in the representation space $\mathcal{H}$. The Husimi function $Q_\rho \colon X \to \mathbb{C}$ is defined by the formula:

$$Q_\rho(x) = \langle x|\rho|x\rangle.$$

The group G is locally compact, the left-invariant Haar measure on this group is denoted by μ. By virtue of invariance, the measure naturally defines a measure on X, which we also denote by μ. The Wehrl entropy $S_W(\rho)$ is given by the formula:

$$S_W(\rho) = -\int_X d\mu(x) Q_\rho(x) \log Q_\rho(x).$$

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There is a natural question of the lower bound for the Wehrl entropy:

$$C = \inf_{\rho} S_W(\rho).$$

The constant $C = C(G)$ is a certain characteristic of the group G and its representation U. For example, for the Heisenberg group over a field of real numbers and its standard representation we have $C(Heis_{\mathbb{R}}) = 1$. This conjecture was formulated by Wehrl [2] and subsequently proved by Lieb [3]. Moreover, Lieb proved that the lower bound is achieved on coherent states. Later Carlen [4] proved that the minimum occures only on coherent states.

As an example, we give another interesting result for the group SU(2) and its representation with the highest weight 2J [5]. The following equality is valid:

$$C(SU(2)) = \frac{2J}{2J+1}.$$

Thus, the lower bound of the Wehrl entropy does depend in a nontrivial way on the group and its representation.

In this paper we consider the Heisenberg group over the field $\mathbb{Q}_p$ of p-adic numbers and its standard representation. In particular, the following statement is proved:

$$C(Heis_{\mathbb{Q}_p}) = 0,$$

and the lower bound is achieved on coherent states. The work is motivated by applications in p-adic quantum theory, see [6,7] and references therein.

1.3. Entropic uncertainty relations

Let two orthonormal bases $A = \{|a\rangle\langle a|\}$ and $B = \{|b\rangle\langle b|\}$ be given in the Hilbert space $\mathcal{H}$. Each of these bases defines a quantum-classical channel $\rho \to \{\rho_a = \langle a|\rho|a\rangle\}$, $\rho \to \{\rho_b = \langle b|\rho|b\rangle\}$.

Denote by $H(\rho_{a,b})$ the Shannon entropy for probability distributions $\rho_{a,b}$ respectively. Then the following formula is valid:

$$H(\rho_a) + H(\rho_b) \geq K,$$

where the constant K depends only on the bases A and B. The optimal estimate is achieved for a pair of mutually unbiased bases.

Relations of this type are called entropy uncertainty relations. For the first time, the entropy uncertainty relation is obtained by Hirschman [8]. This result was further developed in articles [9–11] and many others. For more detailed information, I refer to the review [13].

In this paper, entropy uncertainty relations are considered in the context of p-adic quantum mechanics. In this case, the coherent states form an orthonormal basis. The entropy uncertainty relation for a pair of different bases of coherent states is obtained. Moreover, the obtained estimate is optimal, since the bases of coherent states are mutually unbiased.

1.4. p-adic numbers

We fix a prime number p. Any rational number $x \in \mathbb{Q}$ is uniquely representable as

$$x = p^k \frac{m}{n},\ k, m, n \in \mathbb{Z},\ p \nmid m,\ p \nmid n.$$

Let's define the norm $|\cdot|_p$ on $\mathbb{Q}$ by the formula $|x|_p = p^{-k}$, Completion of the field of rational numbers with this norm is the field $\mathbb{Q}_p$ of p-adic numbers.The p-adic norm of a rational integer $n \in \mathbb{Z}$ is always less than or equal to one, $|n|_p \leq 1$, the completion of rational integers $\mathbb{Z}$ with the p-adic norm is denoted by $\mathbb{Z}_p$. $\mathbb{Z}_p = x \in \mathbb{Q}_p \colon |x|_p \leq 1$, that is, it is a disk of a unit radius.

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For the p-adic norm, the strong triangle inequality holds:

$$|x+y|_p \le \max\{|x|_p, |y|_p\}.$$

The non-Archimedean norm defines totally disconnected topology on $\mathbb{Q}_p$ (the disks are open and closed simultaneously). Two disks either do not intersect, or one lies in the other.

Locally constant functions are continuous, for example:

$$h_{\mathbb{Z}_p}(x) = \begin{cases} 1, x \in \mathbb{Z}_p \\ 0, x \notin \mathbb{Z}_p \end{cases}$$

is a continuous function.

$\mathbb{Q}_p$ is Borel isomorphic to the real line $\mathbb{R}$. The shift-invariant measure dx by $\mathbb{Q}_p$ is normalized in such a way that $\int_{\mathbb{Z}_p} dx = 1$.

For any nonzero p-adic number, the canonical representation holds:

$$\mathbb{Q}_p \ni x = \sum_{k=-n}^{+\infty} x_k p^k,\ n \in \mathbb{Z}_+,\ x_k \in \{0, 1, \dots, p-1\}.$$

We define integer $[x]_p$ and fractional $\{x\}_p$ parts of $x \in \mathbb{Q}_p^*$ by the following expressions:

$$\underbrace{p^{-n}x_{-n} + p^{-n+1}x_{-n+1} + \dots + p^{-1}x_{-1}}_{\{x\}_p} + \underbrace{x_0 + px_1 + \dots + p^k x_k + \cdots}_{[x]_p}$$

The following function, which takes values in a unit circle $\mathbb{T}$ in $\mathbb{C}$, is the additive character of the field of p-adic numbers.

$$\chi_p(x) = \exp\left(2\pi i \{x\}_p\right),\ \chi_p(x+y) = \chi_p(x)\chi_p(y).$$

p-Adic integers $\mathbb{Z}_p$ form a group with respect to addition (a consequence of the non-Archimedean norm) and it is profinite (procyclic) group. This is the inverse limit of finite cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$, $n \in \mathbb{N}$.

$$\mathbb{Z}/p\mathbb{Z} \longleftarrow \cdots \longleftarrow \mathbb{Z}/p^n\mathbb{Z} \longleftarrow \mathbb{Z}/p^{n+1}\mathbb{Z} \longleftarrow \cdots$$

Consider the group $\hat{\mathbb{Z}}_p$ of characters $\mathbb{Z}_p$. This group has the form

$$\hat{\mathbb{Z}}_p = \mathbb{Q}_p \,/\, \mathbb{Z}_p = \mathbb{Z}(p^\infty) = \{\exp(2\pi i m/p^n),\ m, n \in \mathbb{N}\}\,.$$

This is the Prüfer group. It is a direct limit of finite cyclic groups (i.e. quasicyclic) of order p^n.

$$\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \cdots \to \mathbb{Z}/p^n\mathbb{Z} \to \cdots\,.$$

2. *p*-Adic coherent states

2.1. Canonical commutation relations

Let $V = \mathbb{Q}_p^2$ be a two-dimensional vector space over $\mathbb{Q}_p$ and Δ be a non-degenerate symplectic form on this space. On the set $V \times \mathbb{Q}_p$, we define a group operation according to the rule:

$$(u, s)(v, t) = (u + v, s + t + \Delta(u, v)),\ u, v \in V,\ s, t \in \mathbb{Q}_p\,.$$

The set $V \times \mathbb{Q}_p$ equipped with such an operation is the p-adic Heisenberg group $Heis(\mathbb{Q}_p)$. The set of elements $\{(0, s),\ s \in \mathbb{Q}_p\}$ forms a commutative subgroup Z (center) of the Heisenberg group.

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Now let's apply Perelomov's construction to the groups $Heis(\mathbb{Q}_p)$ and Z as the group G and its subgroup H respectively.

More familiar (completely equivalent) is the language of representations of canonical commutation relations (or Weyl systems).

Let $\mathcal{H}$ be a separable complex Hilbert space. A map W from V to a set of unitary operators on $\mathcal{H}$ satisfying the condition

$$W(u)W(v) = \chi_p(\Delta(u,v))W(v)W(u),\ u,v \in V$$

is called a representation of canonical commutation relations (CCR). We will also require continuity in a strong operator topology and irreducibility. When these conditions are met, such a representation is unique up to unitary equivalence.

Let's choose an arbitrary unit vector $|\psi\rangle \in \mathcal{H}$. The set of vectors in $\mathcal{H}$ of the form

$$\{|z\rangle = W(x)|\psi\rangle,\ z \in V\}$$

is called a system of (generalized) coherent states.

The next element of the construction is the choice of the vector $|\psi\rangle$. The Heisenberg group over the field of real numbers is a Lie group. The standard approach is the transition to the corresponding Lie algebra. The vacuum vector is defined as the eigenvector of the annihilation operator. In the p-adic case, there is no structure of a smooth manifold on the Heisenberg group and, accordingly, there is no corresponding Lie algebra. We will use a different approach to the construction of the vacuum vector.

2.2. Vacuum vector

p-Adic integers $\mathbb{Z}_p$ form a ring. Let L be a two-dimensional $\mathbb{Z}_p$-submodule of the space V. Such submodules will be called lattices.

On the set of lattices, we introduce the operations $\vee$ and $\wedge$:

$$L_1 \vee L_2 = L_1 + L_2 = \{z_1 + z_2,\ z_1 \in L_1,\ z_2 \in L_2\},$$

$$L_1 \wedge L_2 = L_1 \cap L_2.$$

We also define the involution $*$:

$$L^* = \{z \in V : \Delta(z,u) \in \mathbb{Z}_p\ \forall u \in L\}.$$

It's easy to see that $(L_1 \wedge L_2)^* = L_1 \vee L_2$. The lattice L invariant with respect to the involution is called self-dual, $L = L^*$.

We normalize the measure on V in such a way that the volume of a self-dual lattice is equal to one. Symplectic group $Sp(V) = SL_2(\mathbb{Q}_p)$ acts transitively on the set of self-dual lattices.

By $\mathcal{L}$ we denote the set of self-dual lattices. On the set $\mathcal{L}$, we define metric d by the formula

$$d(L_1, L_2) = \frac{1}{2}\log \#\,(L_1 \vee L_2 / L_1 \wedge L_2)$$

log everywhere further denotes the logarithm to the base p, # is the number of elements of the set.

Example 1. *Let $\{e,f\}$ be a symplectic basis in V, $\Delta(e,f) = 1$.Then the lattices*

$$L_1 = \mathbb{Z}_p\, e \oplus \mathbb{Z}_p\, f,\ \ L_2 = p^n\, \mathbb{Z}_p\, e \oplus p^{-n}\, \mathbb{Z}_p\, f$$

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are self-dual. If $n \geq 0$, then

$$L_1 \wedge L_2 = p^n \mathbb{Z}_p e \oplus \mathbb{Z}_p f, \;\; L_1 \vee L_2 = \mathbb{Z}_p e \oplus p^{-n} \mathbb{Z}_p f$$
$$d(L_1, L_2) = \frac{1}{2} \log \# (L_1 \vee L_2 / L_1 \wedge L_2) = \frac{1}{2} \log p^{2n} = n$$

Note that for any pair of self-dual lattices, such a basis exists.

The set of self-dual lattices can be represented as a graph. The distance d takes values in the set of non-negative integers. The vertices of the graph are elements of the set $\mathcal{L}$, and the edges are pairs of self-dual lattices $\{L_2, L_2\}$: $d(L_1, L_2) = 1$.

The graph of self-dual lattices is constructed according to the following rule. Let K_{p+1} denote a complete graph with $p+1$ vertices.The countable family of copies of the graph K_{p+1} is glued together in such a way that each vertex of each graph in this family belongs to exactly $p+1$ graphs K_{p+1}.

By replacement of each complete graph K_{p+1} by a star graph S_{p+1} we get a Bruhat-Tits tree.

We proceed with the construction of the vacuum vector. Let us choose a self-dual lattice $L \in \mathcal{L}$ and consider the operator

$$P_L = \int_L dz W(z).$$

Lemma 1. *The P_L operator is a one-dimensional projection.*

$$P_L^2 = \int_L dz W(z) \int_L dz' W(z') =$$
$$= \int_l dz \int_L dz' W(z+z') = \int_L dz W(z) = P_L.$$

The one-dimensionality of the projection P_L directly follows from the irreducibility of the representation W.

Our desired vacuum state will be this projection. We fix the notation $P_L = |0_L\rangle\langle 0_L|$.

Definition 1. *The family of vectors $\{|z_L\rangle = W(z)|0_L\rangle,\, z \in V\}$ in $\mathcal{H}$ is said to be the system of (L-)coherent states.*

We denote by h_L the indicator function of the lattice L,

$$h_L(z) = \begin{cases} 1, z \in L \\ 0, z \notin L \end{cases}$$

Theorem 1. *Coherent states satisfy the following relation:*

$$|\langle z_L | z'_L \rangle| = h_L(z - z').$$

In other words, the coherent states $|z_L\rangle\langle z_L|$ and $|z'_L\rangle\langle z'_L|$ coincide if $z - z' \in L$ and are orthogonal otherwise.

Indeed, let $u = z - z'$. Then

$$|\langle z_L | z'_L \rangle| = |\chi_p(1/2\Delta(z,u))\langle 0_L | W(u) 0_L\rangle| = |\langle 0_L | W(u) 0_L \rangle|.$$

If $u \in L$ the statement of the theorem follows from the definition of a vacuum vector. If $u \notin L$, then by virtue of the self-duality of the lattice L, there exists $v \in L$ that $\chi_p(\Delta(u,v)) \neq 1$. We have

$$\langle 0_L | W(u) 0_L \rangle = \langle 0_L | W(-v) W(u) W(v) 0_L \rangle =$$
$$= \chi_p(\Delta(u,v)) \langle 0_L | W(u) 0_L \rangle,$$

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which is true only if $\langle 0_L|W(u)0_L\rangle = 0$.

Therefore, non-matching (and pairwise orthogonal) coherent states are parametrized by elements of the set $V/L = \left(\mathbb{Q}_p\,/\,\mathbb{Z}_p\right)^2 \cong \mathbb{Z}(p^\infty)\times\mathbb{Z}(p^\infty)$. This makes the following modification of Definition 1 natural.

Definition 2. *The set $\{|\alpha_L\rangle = W(\alpha)|0_L\rangle,\ \alpha \in V/L\}$ is said to be the basis of coherent states for the p-adic Heisenberg group.*

3. Entropy

3.1. Lower bound of the Wehrl entropy

Let ρ be a density matrix, that is, ρ is a positive operator with a unit trace in $\mathcal{H}$.

We will be interested in the following objects:

- the Husimi function $Q^L_\rho(z) = \langle z|\rho|z\rangle$,
- the Werhl entropy $S^L_W(\rho) = -\int_V Q^L_\rho(z)\log Q^L_\rho(z)dz$.

Proposition 1. *The following equality is valid:*

$$Q^L_\rho(z+z') = Q^L_\rho(z),\ z \in V,\ z' \in L.$$

That is the Husimi function is constant on adjacency classes from V/L and is thus defined on the set V/L.

Indeed:

$$Q^L_\rho(z+z') = \langle W(z+z')0_L|\rho|W(z+z')0_L\rangle =$$
$$= \langle \chi_p(\Delta(z,z'))z_L|\rho|\chi_p(\Delta(z,z'))z_L\rangle = Q^L_\rho(z).$$

Corollary 1. *For the Wehrl entropy the following equality is valid:*

$$S^L_W(\rho) = -\sum_{\alpha\in V/L} Q^L_\rho(\alpha)\log Q^L_\rho(\alpha).$$

Proposition 2. *The Husimi function defines the probability distribution on the set V/L, that is*

$$Q^L_\rho(\alpha) \geq 0,\ \sum_{\alpha\in V/L} Q^L_\rho(\alpha) = 1.$$

Since coherent states form an orthonormal basis, we have

$$\sum_{\alpha\in V/L} Q^L_\rho(\alpha) = \operatorname{Tr}\rho = 1.$$

The following map

$$\rho \to \rho_{out} = \Phi_L[\rho] = \sum_{\alpha\in V/L} Q^L_\rho(\alpha)|\alpha_L\rangle\langle\alpha_L|$$

defines the quantum-classical channel Φ_L.

This is a complete ideal quantum measurement associated with an orthonormal basis of p-adic coherent states.

This measurement is heterodyne, that is, the coordinate and momentum of the particle are measured simultaneously with the maximum accuracy allowed by the uncertainty relation.

Denote by $S(\rho) = -\operatorname{Tr}\rho\log\rho$ the von Neumann entropy. Jensen's inequality and concavity of $-x\log x$ gives an inequality $S(\rho) \leq S^L_W(\rho)$. (as for coherent states for the real (i.e. over the field $\mathbb{R}$) Heisenberg group ([2]).

Proposition 3. *The Wehrl entropy has the following property.*

$$S_W^L(\rho) = S\left(\Phi_L[\rho]\right).$$

In other words, the Wehrl entropy of a state ρ is the von Neumann entropy of the measurement result in the basis of coherent states.

This is a consequence of the orthogonality of p-adic coherent states and does not hold in the real case.

In the real case, there is an estimate of $S_W(\rho) \le S(\rho_{out})$ (Berezin-Lieb inequality [12]).

Theorem 2. *The following lower bound is valid for the Wehrl entropy*

$$0 \le S_W^L(\rho),$$

and $S_W^L(\rho) = 0$ if and only if ρ is a L-coherent state.

In the real case, the inequality $1 \le S_W(\rho)$ is valid and equality occurs only on coherent states (the Wehrl-Lieb inequality) [2,3]. This is a non-trivial result.

In the p-adic case, this is a very simple statement, which directly follows from a simple observation that the Husimi function of a coherent state takes only two values – zero and one,

$$Q^L_{|\beta_L\rangle\langle\beta_L|}(\alpha) = \delta_{\alpha\beta}.$$

3.2. Entropic uncertainty relation

The channel Φ_L looks very simple in the representation of characteristic functions. The characteristic function π_ρ of the quantum state ρ is given by the relation

$$\pi_\rho(z) = \mathrm{Tr}\,\rho W(z),\ z \in V$$

and defines the state uniquely.

The following two statements are given without proof. More detailed information can be found in [15].

Proposition 4. *The channel Φ_L multiplies the characteristic function of the state by the indicator function of the lattice L:*

$$\pi_{\Phi_L[\rho]}(z) = \pi_\rho(z) h_L(z),\ z \in V.$$

Now let two self-dual lattices L_1 and L_2 be given. These lattices are corresponded to measurements in the bases of coherent states. From the previous sentence, it's easy to see what sequential measurements look like. Namely:

$$\pi_\rho \xrightarrow{\Phi_{L_1}} \pi_\rho h_{L_1} \xrightarrow{\Phi_{L_2}} \pi_\rho h_{L_1} h_{L_2} = \pi_\rho h_{L_2} h_{L_1} = \pi_\rho h_{L_1 \wedge L_2}$$

The composition of channels is thus naturally denoted by $\Phi_{L_1\wedge L_2}[\rho]$.

Proposition 5. *The channel $\Phi_{L_1\wedge L_2}$ is an incomplete ideal measurement. The orthogonal decomposition of the unit corresponding to this measurement is $\{P_a,\ a \in V/L_1 \vee L_2\}$. All projections P_a have the same dimension equal to $p^{d(L_1,L_2)}$.*

The entropy of the output state for the channel $\Phi_{L_1\wedge L_2}[\rho]$ is denoted by $S_W^{L_1\wedge L_2}$. This entropy can no longer be zero, it is bounded from below by the distance between the corresponding lattices.

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Corollary 2. *The entropy $S_W^{L_1 \wedge L_2}$ satisfies the inequality*

$$d(L_1, L_2) \leq S_W^{L_1 \wedge L_2}(\rho).$$

Proposition 6. *The following inequalities are valid:*

$$S_W^{L_1}(\rho) \leq S_W^{L_1 \wedge L_2}(\rho).$$

In other words, entropy does not decrease with successive measurements:

$$S(\rho) \leq S\left(\Phi_{L_1}[\rho]\right) \leq S\left(\Phi_{L_2} \circ \Phi_{L_1}[\rho]\right).$$

The following chain of relations is valid:

$$S_W^{L_1 \wedge L_2}(\rho) = S\left(\Phi_{L_1 \wedge L_2}[\rho]\right) = S\left(\Phi_{L_2} \circ \Phi_{L_1}[\rho]\right) = \\ = S_W^{L_2}\left(\Phi_{L_1}[\rho]\right) \geq S\left(\Phi_{L_1}[\rho]\right) = S_W^{L_1}(\rho).$$

In the proof, we used Proposition 3 and the inequality between von Neumann entropy and Wehrl entropy. Similarly, we prove that $S_W^{L_1 \wedge L_2}(\rho) \geq S_W^{L_2}(\rho)$.

Equality is achieved on coherent states. Let ρ be an L_2-coherent state. Then $\rho = \Phi_{L_2}[\rho]$. Therefore:

$$S_W^{L_1}(\rho) = S_W^{L_1}\left(\Phi_{L_2}[\rho]\right) = S\left(\Phi_{L_1} \circ \Phi_{L_2}[\rho]\right) = S_W^{L_1 \wedge L_2}(\rho).$$

Theorem 3. *The inequality (entropic uncertainty relation) is valid.*

$$d(L_1, L_2) \leq S_W^{L_1}(\rho) + S_W^{L_2}(\rho).$$

Equality occurs for L_1- or L_2-coherent states.

Applying the results of Lieb and Frank [14], we can get a stronger estimate:

$$d(L_1, L_2) + S(\rho) \leq S_W^{L_1}(\rho) + S_W^{L_2}(\rho).$$

Equality is also occurs for L_1- or L_2-coherent states (these states are pure and the von Neumann entropy is zero on them).

The proof of the above result is based on paper Lieb and Frank [14]. The result of Lieb and Frank is as follows. Let two orthonormal bases be given $\{|a_j\rangle\}$ and $\{|b_k\rangle\}$ and $p_j = \langle a_j|\rho|a_j\rangle$, $q_k = \langle b_k|\rho|b_k\rangle$. Then the estimate is valid

$$-\sum_j p_j \log p_j - \sum_k q_k \log q_k \geq S(\rho) - \log\left(\sup_{j,k} |\langle a_j|b_k\rangle|^2\right).$$

3.3. Mutually unbiased bases

Entropy uncertainty relations are closely related to mutually unbiased bases [13,16].

Mutually unbiased bases in Hilbert space $\mathbb{C}^D$ are two orthonormal bases $\{|e_1\rangle, \ldots, |e_D\rangle\}$ and $\{|f_1\rangle, \ldots, |f_D\rangle\}$ such that the square of the magnitude of the inner product between any basis states $|e_j\rangle$ and $|f_k\rangle$ equals the inverse of the dimension D

$$|\langle e_j|f_k\rangle|^2 = \frac{1}{D}, \quad \forall j, k \in \{1, \ldots, D\}.$$

Let L_1 and L_2 be a pair of self-dual lattices, $d(L_1, L_2) \geq 1$.

It turns out that the corresponding bases of L_1- and L_2-coherent states are mutually unbiased on finite-dimensional subspaces of dimension $p^{d(L_1,L_2)}$.

Theorem 4. *For bases of L_1- and L_2-coherent states $\{|\alpha_{L_1}\rangle,\ \alpha \in V/L_1\}$ and $\{|\beta_{L_2}\rangle,\ \beta \in V/L_2\}$ the following formula is valid*

$$|\langle \alpha_{L_1}|\beta_{L_2}\rangle|^2 = p^{-d(L_1,L_2)} h_{L_1 \vee L_2}(\alpha - \beta).$$

The theorem means the following. Our Hilbert space of representation of CCR $\mathcal{H}$ decomposes into an orthogonal direct sum of finite-dimensional subspaces of the same dimension $p^{d(L_1,L_2)}$:

$$\mathcal{H} = \bigoplus_{a \in V/(L_1 \vee L_2)} \mathcal{H}_a, \ \dim \mathcal{H}_a = p^{d(L_1,L_2)}.$$

In each of these subspaces, the subbasis of L_1- and L_2-coherent states are mutually unbiased.

More information and proof Theorem 4 can be found in [17].

It follows from Theorem 4 that for a pair of bases L_1- and L_2-coherent states, the equality is valid

$$\sup_{\alpha \in V/L_1, \beta \in V/L_2} |\langle \alpha_{L_1}|\beta_{L_2}\rangle|^2 = p^{-d(L_1,L_2)},$$

from which the statement of Theorem 3 immediately follows.

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