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Posted Date: 14 June 2023

doi: 10.20944/preprints202306.1002.v1

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New Subfamily of Bi-Starlike and Bi-Convex Functions Defined by the q-Janowski Function

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Abstract: In this study, we introduce a new class of bi-univalent functions using q-Janowski Function. We derive the Fekete-Szegö functional problems for functions in this new subclass, as well as estimates for the Taylor-Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$. Furthermore, a collection of fresh outcomes are presented by customizing the parameters employed in our initial discoveries.

Keywords: analytic functions; univalent and bi-univalent functions; subordination; q-janowski function; fekete-szegö problem; quantum calculus

MSC: 30C45; 30C50

1. Introduction

The discipline of quantum calculus, alternatively referred to as $q-$calculus, expands the conventional calculus framework to encompass the fundamental tenets of quantum mechanics. The field of mathematics commonly referred to as $q-$calculus is distinguished by the incorporation of a new parameter, denoted by q, that enables the generalization of traditional calculus concepts and methodologies. This area of study has garnered recognition for its widespread application in diverse mathematical domains, particularly in the realm of Geometric function theory.

The incorporation of the parameter q is a fundamental aspect of derivatives, integrals, and functions in the field of q-calculus. The q-derivative is a mathematical operator that utilizes q-analogs of traditional derivatives within a difference quotient. The q-integral is a mathematical construct that can be comprehended as the q-analog of the Riemann integral. q-calculus is a mathematical field that comprises a set of q-special functions that have significant applications in various areas of mathematics and physics. The set of functions under consideration encompasses q-binomial coefficients and q-factorials. Broadly speaking, q-calculus is a powerful tool for examining and solving problems pertaining to discrete and quantum systems.

The utilization of fractional calculus operators has been widely employed in the explanation and solution of issues in the field of applied sciences, as well as in Geometric Function, as documented in the source [1]. The fractional q-calculus is an extended form of the traditional fractional calculus and has been utilized in a variety of fields, including optimal control problems, q-difference and q-integral equations, and ordinary fractional calculus. To obtain additional insights on the topic at hand, it is advisable to refer to a published source [2] and current literature, which may include references such as [3–5].

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2. Preliminaries

Consider the family $\mathcal{A}$ consisting of functions Φ of the form

$$\Phi(\Im) = \Im + \sum_{\Bbbk=2}^{\infty} \alpha_{\Bbbk} \Im^{\Bbbk}, \tag{1}$$

where $\Im$ belongs to the complex unit disk $\mho = \{\Im \in \mathbb{C} : |\Im| < 1\}$, and Φ is analytic in $\mho$. Additionally, Φ must satisfy the normalization condition $\Phi'(0) - 1 = 0 = \Phi(0)$. Furthermore, let P represent the subset of analytic functions in $\mho$ that fulfill the condition $Re\{\varphi(\Im)\}) > 0$ and can be expressed in the form

$$\varphi(\Im) = 1 + \sum_{\Bbbk=1}^{\infty} \varphi_{\Bbbk} \Im^{\Bbbk}, \qquad (\Im \in \mho), \tag{2}$$

where $|\varphi_{\Bbbk}| < 2$ (by Caratheodory's Lemma refer to [6]).

The implementation of differential subordination of analytical functions has the potential to offer considerable benefits to the domain of geometric function theory. Miller and Mocanu [14] proposed the original differential subordination problem, which has subsequently been examined in greater detail in [15]. The book authored by Miller and Mocanu [16] presents a comprehensive overview of the advancements made in the field, accompanied by their respective dates of publication. For a real number λ and β, $-1 \le \beta < \lambda \le 1$, the classes $P[\lambda, \beta]$, $\mathcal{S}^*[\lambda, \beta]$ and $\mathcal{C}[\lambda, \beta]$ where defined by

$$P[\lambda, \beta] := \left\{ \varphi \in P : \text{ if and only if } \varphi(\Im) \prec \frac{1 + \lambda \Im}{1 + \beta \Im}, \ (\Im \in \mho) \right\},$$
$$\mathcal{S}^*[\lambda, \beta] := \left\{ \Phi \in \mathcal{A} : \text{ if } \frac{\Im \Phi'(\Im)}{\Phi(\Im)} \prec \frac{1 + \lambda \Im}{1 + \beta \Im}, \ (\Im \in \mho) \right\},$$
$$\mathcal{C}[\lambda, \beta] := \left\{ \Phi \in \mathcal{A} : \text{ if } 1 + \frac{\Im \Phi''(\Im)}{\Phi'(\Im)} \prec \frac{1 + \lambda \Im}{1 + \beta \Im}, \ (\Im \in \mho) \right\}.$$

Janowski [7] conducted an investigation and analysis of the class $P[\lambda, \beta]$, which is a subset of P. Additionally, other classes such as $\mathcal{S}^*[\lambda, \beta]$ and $\mathcal{C}[\lambda, \beta]$ were also studied and analyzed in previous works such as [9–11], among others.

For any function Φ in the subfamily $\mathcal{S}$, there exists an inverse function denoted as Φ^{-1} and defined by

$$\Im = \Phi^{-1}(\Phi(\Im)), \quad \varsigma = \Phi(\Phi^{-1}(\varsigma)) \qquad \left(r_0(\Phi) \ge \frac{1}{4}; \ |\varsigma| < r_0(\Phi); \Im \in \mho \right)$$

where

$$\hbar(\varsigma) = \Phi^{-1}(\varsigma) = \varsigma \left(1 - \varsigma^3(\alpha_4 + 5\alpha_2^3 - 5\alpha_3\alpha_2) + \varsigma^2(-\alpha_3 + 2\alpha_2^2) - \alpha_2\varsigma + \cdots \right). \tag{3}$$

In a specified domain $\mho$, a function is considered to be bi-univalent if it satisfies the condition that both $\Phi(\Im)$ and its inverse function $\Phi^{-1}(\varsigma)$ are univalent or injective within $\mho$.

The definition of the subclass Σ in the set $\mathcal{S}$ involves specifying the category of bi-univalent functions in $\mho$, as expressed by equation (1). Examples of the class Σ functions include

$$\Phi_1(\Im) = \frac{\Im}{1 - \Im}, \quad \Phi_2(\Im) = \log\left(\frac{1}{1 - \Im}\right) \quad \text{and} \quad \Phi_3(\Im) = \frac{1}{2}\log\left(\frac{1 + \Im}{1 - \Im}\right).$$

The inverse functions that correspond to the aforementioned functions:

$$\hbar_1(\varsigma) = \frac{\varsigma}{1 + \varsigma}, \quad \hbar_2(\varsigma) = \frac{e^{2\varsigma} - 1}{e^{2\varsigma} + 1} \quad \text{and} \quad \hbar_3(\varsigma) = \frac{e^{\varsigma} - 1}{e^{\varsigma}}.$$

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This article presents an overview of q-calculus, initially introduced by Jackson and subsequently explored by numerous mathematicians [17–23]. It focuses on introducing key concepts and definitions within the realm of q-calculus. Additionally, it highlights the significance of the q-difference operator, widely employed in scientific disciplines such as geometric function theory. Emphasizing the assumption that q lies within the interval $(0,1)$, the study extensively relies on fundamental definitions and properties of q-calculus, as extensively documented by Gasper and Rahman in their work [12].

Definition 1. *Let $0 < q < 1$. The $q-$bracket $[\kappa]_q$ is formally defined as such*

$$[\kappa]_{\langle q\rangle} = \begin{cases} \frac{1-q^{\kappa}}{1-q} & , \quad \text{if} \quad 0<q<1,\ \kappa \in \mathbb{C}\backslash\{0\} \\ q^{\Bbbk-1}+q^{\Bbbk-2}+\cdots+q+1 = \sum\limits_{j=0}^{\Bbbk-1} q^j & , \quad \text{if} \quad 0<q<1,\ \kappa=\Bbbk\in\mathbb{N} \\ 1 & , \quad \text{if} \quad q\to 0^+,\ \kappa\in\mathbb{C}\backslash\{0\} \\ \kappa & , \quad \text{if} \quad q\to 1^-,\ \kappa\in\mathbb{C}\backslash\{0\}. \end{cases}.$$

Definition 2. *The $q-$derivative, also known as the $q-$difference operator, of a function Φ is defined by*

$$\eth\langle\Phi(\Im);q\rangle = \begin{cases} \frac{\Phi(\Im)-\Phi(q\Im)}{\Im-q\Im}, & \text{if} \quad 0<q<1,\ \Im\neq 0 \\ \Phi'(0), & \text{if} \quad q\to 1^-,\ \Im=0 \\ \Phi'(\Im), & \text{if} \quad q\to 1^-,\ \Im\neq 0 \end{cases},$$

and $\eth^{(\kappa)}\langle\Phi(\Im);q\rangle = \eth\left(\eth^{(\kappa-1)}\langle\Phi(\Im);q\rangle\right)$ *for* $\kappa\geq 1$.

It is evident that as the limit $q\to 1^-$ is approached, the q-deformed generator $\eth^{(\kappa)}\langle\Phi(\Im);q\rangle$ tends towards the n^{th} ordinary derivative of the function $\Phi(\Im)$.

Definition 3. *[8] By letting $-1\leq\beta<\lambda\leq 1$. A function $f\in\mathcal{A}$ is in the class $\mathcal{S}^*(\lambda,\beta;q)$ if it satisfies the following subordination:*

$$\frac{\Im\eth\langle\Phi(\Im);q\rangle}{\Phi(\Im)} \prec \frac{1+\lambda_q\Im}{1+\beta_q\Im}, \qquad (q\in(0,1),\Im\in\mho) \tag{4}$$

where the values of λ_q and β_q are given as

$$\lambda_q = \frac{(\lambda+1)+(\lambda-1)q}{2} \quad \text{and} \quad \beta_q = \frac{(\beta+1)+(\beta-1)q}{2}.$$

The function $\Phi\in\mathcal{A}$ possesses a geometric meaning in the set $\mathcal{S}^*(\lambda,\beta;q)$ if and only if the quotient $\frac{\Im\eth\langle\Phi(\Im);q\rangle}{\Phi(\Im)}$ spans across all values within a circular region centered at $\frac{1-\lambda_q\beta_q}{1-\beta_q^2}$ and with a radius of $\frac{\lambda_q-\beta_q}{1-\beta_q^2}$.
Motivated by Ismail et al. [8], we introduce the class $\mathcal{C}(\lambda,\beta;q)$ by

Definition 4. *By letting $-1\leq B<\lambda\leq 1$. A function $f\in\mathcal{A}$ is in the class $\mathcal{C}(\lambda,\beta;q)$ if it satisfies the following subordination:*

$$1+\frac{\Im\eth^{(2)}\langle\Phi(\Im);q\rangle}{\eth\langle\Phi(\Im);q\rangle} \prec \frac{1+\lambda_q\Im}{1+\beta_q\Im}, \qquad (q\in(0,1),\Im\in\mho) \tag{5}$$

where the values of λ_q and β_q are given as

$$\lambda_q = \frac{(\lambda+1)+(\lambda-1)q}{2} \quad and \quad \beta_q = \frac{(\beta+1)+(\beta-1)q}{2}.$$

The primary objective of this study is to initiate an investigation into the characteristics of bi-univalent functions that are associated with q-univalent-preserving property. In order to attain this objective, the subsequent definitions are taken into account.

3. Definitions and Examples

In this section, we present some fresh subclasses that belong to the realm of bi-univalent functions. These subclasses are namely bi-starlike and bi-convex are defined using the subordination principal to the q-Janowski Function.

Definition 5. *By imposing $-1 \leq \beta < \lambda \leq 1$ and $q \in (0,1)$. A function f belonging to the family Σ, as defined in equation (1), is considered to be a member of the bi-starlike class, denoted by $\mathcal{S}^*_\Sigma(\lambda, \beta; q)$ if it satisfies certain subordination conditions. These conditions can be expressed as follows:*

$$\frac{\Im\,\partial\langle\Phi(\Im);q\rangle}{\Phi(\Im)} \prec \frac{1+\lambda_q\Im}{1+\beta_q\Im}, \qquad (\Im \in \mho) \tag{6}$$

and

$$\frac{\varsigma\,\partial\langle\hbar(\varsigma);q\rangle}{\hbar(\varsigma)} \prec \frac{1+\lambda_q\varsigma}{1+\beta_q\varsigma}, \qquad (\varsigma \in \mho) \qquad , \tag{7}$$

where the function $\hbar(\varsigma) = \Phi^{-1}(\varsigma)$ is defined by the equation (3), and the values of λ_q and β_q are given as

$$\lambda_q = \frac{(\lambda+1)+(\lambda-1)q}{2} \quad and \quad \beta_q = \frac{(\beta+1)+(\beta-1)q}{2}.$$

Definition 6. *By imposing $-1 \leq \beta < \lambda \leq 1$ and $q \in (0,1)$. A function f belonging to the family Σ, as defined in equation (1), is considered to be a member of the bi-convex class, denoted by $\mathcal{C}_\Sigma(\lambda, \beta; q)$ if it satisfies certain subordination conditions. These conditions can be expressed as follows:*

$$1+\frac{\Im\,\partial^{(2)}\langle\Phi(\Im);q\rangle}{\partial\langle\Phi(\Im);q\rangle} \prec \frac{1+\lambda_q\Im}{1+\beta_q\Im}, \qquad (\Im \in \mho) \tag{8}$$

and

$$1+\frac{\varsigma\,\partial^{(2)}\langle\hbar(\varsigma);q\rangle}{\partial\langle\hbar(\varsigma);q\rangle} \prec \frac{1+\lambda_q\varsigma}{1+\beta_q\varsigma}, \qquad (\varsigma \in \mho) \qquad , \tag{9}$$

where the function $\hbar(\varsigma) = \Phi^{-1}(\varsigma)$ is defined by the equation (3), and the values of λ_q and β_q are given as

$$\lambda_q = \frac{(\lambda+1)+(\lambda-1)q}{2} \quad and \quad \beta_q = \frac{(\beta+1)+(\beta-1)q}{2}.$$

Example 1. *Let $-1 \leq \beta < \lambda \leq 1$. A function f belonging to the family Σ, as defined in equation (1), is considered to be a member of the bi-starlike class $\lim\limits_{q\to 1^-} \mathcal{S}^*_\Sigma(\lambda, \beta; q) = \mathcal{S}^*_\Sigma(\lambda, \beta)$ if it satisfies the following subordination conditions:*

$$\frac{\Im\Phi'(\Im)}{\Phi(\Im)} \prec \frac{1+\lambda\Im}{1+\beta\Im}, \qquad (\Im \in \mho) \tag{10}$$

and

$$\frac{\varsigma\hbar'(\varsigma)}{\hbar(\varsigma)} \prec \frac{1+\lambda\varsigma}{1+\beta\varsigma}, \qquad (\varsigma \in \mho) \qquad , \tag{11}$$

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where the function $\hbar(\varsigma) = \Phi^{-1}(\varsigma)$ is defined by the equation (3).

Example 2. *Let $-1 \leq \beta < \lambda \leq 1$. A function f belonging to the family Σ, as defined in equation (1), is considered to be a member of the bi-convex class, denoted by $\lim\limits_{q\to 1^-} \mathcal{C}_\Sigma(\lambda,\beta;q) = \mathcal{C}_\Sigma(\lambda,\beta)$ if it satisfies the following subordination conditions:*

$$1+\frac{\Im\Phi''(\Im)}{\Phi'(\Im)} \prec \frac{1+\lambda\Im}{1+\beta\Im}, \qquad (\Im \in \mho) \tag{12}$$

and

$$1+\frac{\varsigma\hbar''(\varsigma)}{\hbar'(\varsigma)} \prec \frac{1+\lambda\varsigma}{1+\beta\varsigma}, \qquad (\varsigma \in \mho) \qquad , \tag{13}$$

where the function $\hbar(\varsigma) = \Phi^{-1}(\varsigma)$ is defined by the equation (3).

Fekete and Szegö established a precise limit for the functional $\mu a_2^2 - a_3$ in their 1933 publication [56]. The limit was derived using real values of μ $(0 \leq \mu \leq 1)$ and has been commonly known as the classical Fekete-Szeg"o outcome. Establishing precise boundaries for a given function within a compact family of functions $\Phi \in \mathcal{A}$, and for any complex η, poses a formidable challenge.

4. The bounds of the coefficients within the bi-starlike class

Estimations for the initial coefficients of functions were discovered. Nevertheless, the issue of establishing precise coefficient limits for $|\alpha_n|$, $(n = 3,4,5,\cdots)$, is yet to be resolved, as indicated in several sources ([24–52]). Initially, the estimates for the coefficients of the class $\mathcal{S}^*_\Sigma(\lambda,\beta;q)$, as defined in Definition 5, are provided.

Theorem 1. *If Φ is an element of Σ defined by (1), it can be said that Φ is a member of the class $\mathcal{S}^*_\Sigma(\lambda,\beta;q)$, as per the following statement:*

$$|\alpha_2| \leq \frac{\lambda_q-\beta_q}{q\sqrt{2\left(1+\lambda_q\right)}}, \quad \text{and} \quad |\alpha_3| \leq \frac{\left(\lambda_q-\beta_q\right)^2}{4q^2} + \frac{\lambda_q-\beta_q}{2q[2]_{\langle q\rangle}}.$$

Proof. If Φ belongs to the class $\mathcal{S}^*_\Sigma(\lambda,\beta;q)$, according to Definition 5, under the given conditions, there exist analytic functions Υ and Ω such that $\Upsilon(0) = \Omega(0) = 0$, and $|\Upsilon(\Im)| < 1$ and $|\Omega(\varsigma)| < 1$ hold for all $\Im$ and ς in the unit disk $\mho$. In light of these conditions, the function Φ can be expressed as follows:

$$\frac{\Im\,\eth\langle\Phi(\Im);q\rangle}{\Phi(\Im)} = \frac{1+\lambda_q\Upsilon(\Im)}{1+\beta_q\Upsilon(\Im)} \tag{14}$$

and

$$\frac{\varsigma\,\eth\langle\hbar(\varsigma);q\rangle}{\hbar(\varsigma)} = \frac{1+\lambda_q\Omega(\varsigma)}{1+\beta_q\Omega(\varsigma)} \tag{15}$$

For $\Upsilon(\Im) = 1+\sum\limits_{\Bbbk=1}^{\infty} c_\Bbbk\Im^\Bbbk = 1+c_1\Im+c_2\Im^2+c_3\Im^3+\cdots$, we have

$$\frac{1+\lambda_q\Upsilon(\Im)}{1+\beta_q\Upsilon(\Im)} = 1+(\lambda_q-\beta_q)c_1\Im+\left((\lambda_q-\beta_q)c_2-\beta_q(\lambda_q-\beta_q)c_1^2\right)\Im^2+\cdots,$$

Next we calculate the values of c_1 and c_2. Taking

$$\frac{1+\Upsilon(\Im)}{1-\Upsilon(\Im)} = p(\Im) = 1+\ell_1\Im+\ell_2\Im^2+\cdots, \tag{16}$$

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then, we get

$$1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + \cdots = 1 + \ell_1 \Im + \ell_2 \Im^2 + \cdots, \tag{17}$$

Comparing Equations (16) and (17), we have

$$c_1 = \frac{\ell_1}{2} \quad \text{and} \quad c_2 = \frac{1}{4}(2\ell_2 - \ell_1^2), \tag{18}$$

By utilizing equations (14) and (15) along (18), we can derive the following expression.

$$\frac{\Im \eth\langle \Phi(\Im); q\rangle}{\Phi(\Im)} = 1 + \left(\frac{\lambda_q - \beta_q}{2}\right) c_1 \Im + \left(\frac{\lambda_q - \beta_q}{2}\right)\left(c_2 - (1+\beta_q)\frac{c_1^2}{2}\right) \Im^2 + \cdots, \tag{19}$$

and

$$\frac{\varsigma \eth\langle \hbar(\varsigma); q\rangle}{\hbar(\varsigma)} = 1 + \left(\frac{\lambda_q - \beta_q}{2}\right) d_1 \varsigma + \left(\frac{\lambda_q - \beta_q}{2}\right)\left(d_2 - (1+\beta_q)\frac{d_1^2}{2}\right) \varsigma^2 + \cdots + \cdots. \tag{20}$$

There is a commonly accepted understanding that if.

$$|\Upsilon(\Im)| = \left|c_1 \Im + c_2 \Im^2 + c_3 \Im^3 + \cdots\right| < 1, \quad (\Im \in \mho)$$

and

$$|\Omega(\varsigma)| = \left|d_1 \varsigma + d_2 \varsigma^2 + d_3 \varsigma^3 + \cdots\right| < 1, \quad (\varsigma \in \mho),$$

then for all $j \in \mathbb{N}$, we have

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1. \tag{21}$$

In view of (1), (3), from (19) and (20), we obtain

$$\begin{aligned} &1 + q\alpha_2 \Im + q\left([2]_{\langle q\rangle} \alpha_3 - \alpha_2^2\right) \Im^2 + \cdots \\ &\quad = 1 + \left(\frac{\lambda_q - \beta_q}{2}\right) c_1 \Im + \left(\frac{\lambda_q - \beta_q}{2}\right)\left(c_2 - (1+\beta_q)\frac{c_1^2}{2}\right) \Im^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} &1 - q\alpha_2 \varsigma + q\left(([2]_{\langle q\rangle} + q)\alpha_2^2 - [2]_{\langle q\rangle} \alpha_3\right) \varsigma^2 + \cdots \\ &= 1 + \left(\frac{\lambda_q - \beta_q}{2}\right) d_1 \varsigma + \left(\frac{\lambda_q - \beta_q}{2}\right)\left(d_2 - (1+\beta_q)\frac{d_1^2}{2}\right) \varsigma^2 + \cdots + \cdots. \end{aligned}$$

By comparing the pertinent coefficients in (19) and (20), we arrive at the following.

$$q\,\alpha_2 = \frac{(\lambda_q - \beta_q)\, c_1}{2}, \tag{22}$$

$$-q\,\alpha_2 = \frac{(\lambda_q - \beta_q)\, d_1}{2}, \tag{23}$$

$$q\left([2]_{\langle q\rangle} \alpha_3 - \alpha_2^2\right) = \left(\frac{\lambda_q - \beta_q}{2}\right)\left(c_2 - (1+\beta_q)\frac{c_1^2}{2}\right) \tag{24}$$

and

$$q\left(([2]_{\langle q\rangle} + q)\alpha_2^2 - [2]_{\langle q\rangle} \alpha_3\right) = \left(\frac{\lambda_q - \beta_q}{2}\right)\left(d_2 - (1+\beta_q)\frac{d_1^2}{2}\right). \tag{25}$$

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It follows from (22) and (24) that

$$c_1 = -d_1 \Longleftrightarrow c_1^2 = d_1^2, \tag{26}$$

and

$$\alpha_2^2 = \frac{(\lambda_q - \beta_q)^2}{8q^2}(c_1^2 + d_1^2) \Longleftrightarrow c_1^2 + d_1^2 = \frac{8q^2}{(\lambda_q - \beta_q)^2}\alpha_2^2. \tag{27}$$

Adding (24) and (25), we get

$$2q^2\,\alpha_2^2 = \left(\frac{\lambda_q - \beta_q}{2}\right)\left((c_2 + d_2) - (1 + \beta_q)\frac{c_1^2 + d_1^2}{2}\right). \tag{28}$$

Substituting the value of $(c_1^2 + d_1^2)$ from (27), we obtain

$$\alpha_2^2 = \frac{(\lambda_q - \beta_q)^2\,(c_2 + d_2)}{4q^2\,(1 + \lambda_q)}. \tag{29}$$

Applying for the coefficients c_2 and d_2 along the equation (21), we obtain

$$|\alpha_2| \le \frac{\lambda_q - \beta_q}{q\sqrt{2\,(1 + \lambda_q)}}.$$

The subtraction of equation (25) from equation (24) yields the following result:

$$2q[2]_{\langle q \rangle}(\alpha_3 - \alpha_2^2) = \frac{\lambda_q - \beta_q}{2}\left((c_2 - d_2) - (1 + \beta_q)\frac{c_1^2 - d_1^2}{2}\right). \tag{30}$$

Then, in view of (26) and (27), Eq. (30) becomes

$$\alpha_3 = \frac{(\lambda_q - \beta_q)^2}{8q^2}(c_1^2 + d_1^2) + \frac{\lambda_q - \beta_q}{4q[2]_{\langle q \rangle}}(c_2 - d_2)$$

Thus applying (21), we conclude that

$$|\alpha_3| \le \frac{(\lambda_q - \beta_q)^2}{4q^2} + \frac{\lambda_q - \beta_q}{2q[2]_{\langle q \rangle}}.$$

This completes the proof of Theorem. □

The Fekete-Szeg"o inequality for functions belonging to the class $\mathcal{S}_\Sigma^*(\lambda, \beta; q)$ is examined in view of Zaprawa's [57] finding.

Theorem 2. *Given that Φ is an element of Σ defined by (1) and belongs to the class $\mathcal{S}_\Sigma^*(\lambda, \beta; q)$, and μ is a real number, we can state the following*

$$\left|\alpha_3 - \epsilon\,\alpha_2^2\right| \le \begin{cases} \frac{\lambda_q - \beta_q}{4q[2]_{\langle q \rangle}}, & |1 - \epsilon| \le \frac{q(1+\lambda_q)}{[2]_{\langle q \rangle}(\lambda_q - \beta_q)}, \\ \\ \frac{(\lambda_q - \beta_q)^2|1-\epsilon|}{2q^2(1+\lambda_q)}, & |1 - \epsilon| \ge \frac{q(1+\lambda_q)}{[2]_{\langle q \rangle}(\lambda_q - \beta_q)}. \end{cases}$$

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Proof. If $\Phi \in \mathcal{S}^*_\Sigma(\lambda, \beta; q)$ is given by (1), from (29) and (30), we have

$$
\begin{aligned}
\alpha_3 - \epsilon\,\alpha_2^2 &= (1-\epsilon)\frac{\left(\lambda_q - \beta_q\right)^2 (c_2 + d_2)}{4q^2\,(1+\lambda_q)} + \frac{\lambda_q - \beta_q}{4q[2]_{\langle q\rangle}}(c_2 - d_2) \\
&= \frac{\lambda_q - \beta_q}{4q}\left[(1-\epsilon)\frac{(\lambda_q - \beta_q)\,(c_2 + d_2)}{q\,(1+\lambda_q)} + \frac{1}{[2]_{\langle q\rangle}}(c_2 - d_2)\right] \\
&= \frac{\lambda_q - \beta_q}{4q}\left[\left(\frac{(\lambda_q - \beta_q)\,(1-\epsilon)}{q\,(1+\lambda_q)} + \frac{1}{[2]_{\langle q\rangle}}\right)c_2 + \left(\frac{(\lambda_q - \beta_q)\,(1-\epsilon)}{q\,(1+\lambda_q)} - \frac{1}{[2]_{\langle q\rangle}}\right)d_2\right] \\
&= \frac{\lambda_q - \beta_q}{4q}\left[\left(H(\epsilon) + \frac{1}{[2]_{\langle q\rangle}}\right)c_2 + \left(H(\epsilon) - \frac{1}{[2]_{\langle q\rangle}}\right)d_2\right],
\end{aligned}
$$

where

$$
H(\epsilon) = \frac{(\lambda_q - \beta_q)\,(1-\epsilon)}{q\,(1+\lambda_q)}.
$$

Then, we conclude that

$$
\left|\alpha_3 - \epsilon\,\alpha_2^2\right| \le \begin{cases} \frac{\lambda_q - \beta_q}{4q[2]_{\langle q\rangle}}, & |H(\epsilon)| \le \frac{1}{1+q}, \\ \frac{\lambda_q - \beta_q}{2q}\left|H(\epsilon)\right|, & |H(\epsilon)| \ge \frac{1}{1+q}. \end{cases}
$$

Which completes the proof of Theorem 2. □

5. The bounds of the coefficients within the bi-convex class

In next theorem, the estimates for the coefficients of the class $\mathcal{C}_\Sigma(\lambda, \beta; q)$, as defined in Definition 6, are provided.

Theorem 3. *If Φ is an element of Σ defined by (1), it can be said that Φ is a member of the class $\mathcal{C}_\Sigma(\lambda, \beta; q)$, as per the following statement:*

$$
|\alpha_2| \le \frac{\lambda_q - \beta_q}{\sqrt{(1+q)\left[q^2\left(\lambda_q - \beta_q\right) + (1+q)(1+\beta_q)\right]}},
$$

and

$$
|\alpha_3| \le \frac{\left(\lambda_q - \beta_q\right)^2}{4(1+q)^2} + \frac{\lambda_q - \beta_q}{2(1+q)[3]_{\langle q\rangle}}.
$$

Proof. If Φ belongs to the class $\mathcal{C}_\Sigma(\lambda, \beta; q)$, according to Definition 6, under the given conditions, the function Φ can be expressed as follows:

$$
1 + \frac{\Im\,\eth^{(2)}\langle \Phi(\Im); q\rangle}{\eth\langle \Phi(\Im); q\rangle} = \frac{1 + \lambda_q \Upsilon(\Im)}{1 + \beta_q \Upsilon(\Im)} \tag{31}
$$

and

$$
1 + \frac{\Im\,\eth^{(2)}\langle \hbar(\varsigma); q\rangle}{\eth\langle \hbar(\varsigma); q\rangle} = \frac{1 + \lambda_q \Omega(\varsigma)}{1 + \beta_q \Omega(\varsigma)} \tag{32}
$$

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By utilizing equations (31) and (32) along (16-18), we can derive the following expression.

$$1+\frac{\Im\eth^{(2)}\langle\Phi(\Im);q\rangle}{\eth\langle\Phi(\Im);q\rangle}=1+\left(\frac{\lambda_q-\beta_q}{2}\right)c_1\Im+\left(\frac{\lambda_q-\beta_q}{2}\right)\left(c_2-(1+\beta_q)\frac{c_1^2}{2}\right)\Im^2+\cdots, \tag{33}$$

and

$$1+\frac{\Im\eth^{(2)}\langle\hbar(\varsigma);q\rangle}{\eth\langle\hbar(\varsigma);q\rangle}=1+\left(\frac{\lambda_q-\beta_q}{2}\right)d_1\varsigma+\left(\frac{\lambda_q-\beta_q}{2}\right)\left(d_2-(1+\beta_q)\frac{d_1^2}{2}\right)\varsigma^2+\cdots. \tag{34}$$

In view of (1), (3), from (33) and (34), we obtain

$$\begin{aligned}&1+(1+q)\alpha_2\Im+(1+q)\left([3]_{\langle q\rangle}\alpha_3-(1+q)\alpha_2^2\right)\Im^2+\cdots\\&\quad=1+\left(\frac{\lambda_q-\beta_q}{2}\right)c_1\Im+\left(\frac{\lambda_q-\beta_q}{2}\right)\left(c_2-(1+\beta_q)\frac{c_1^2}{2}\right)\Im^2+\cdots\end{aligned}$$

and

$$\begin{aligned}&1-(1+q)\alpha_2\varsigma-[2]_{\langle q\rangle}\left([3]_{\langle q\rangle}\alpha_3-\left([3]_{\langle q\rangle}+q^2\right)\alpha_2^2\right)\varsigma^2+\cdots\\&\quad=1+\left(\frac{\lambda_q-\beta_q}{2}\right)d_1\varsigma+\left(\frac{\lambda_q-\beta_q}{2}\right)\left(d_2-(1+\beta_q)\frac{d_1^2}{2}\right)\varsigma^2+\cdots+\cdots.\end{aligned}$$

By comparing the pertinent coefficients in (19) and (20), we arrive at the following.

$$(1+q)\,\alpha_2=\frac{(\lambda_q-\beta_q)\,c_1}{2}, \tag{35}$$

$$-(1+q)\,\alpha_2=\frac{(\lambda_q-\beta_q)\,d_1}{2}, \tag{36}$$

$$(1+q)\left([3]_{\langle q\rangle}\alpha_3-(1+q)\alpha_2^2\right)=\left(\frac{\lambda_q-\beta_q}{2}\right)\left(c_2-(1+\beta_q)\frac{c_1^2}{2}\right) \tag{37}$$

and

$$(1+q)\left(-[3]_{\langle q\rangle}\alpha_3+\left([3]_{\langle q\rangle}+q^2\right)\alpha_2^2\right)=\left(\frac{\lambda_q-\beta_q}{2}\right)\left(d_2-(1+\beta_q)\frac{d_1^2}{2}\right). \tag{38}$$

It follows from (35) and (37) that

$$c_1=-d_1\Longleftrightarrow c_1^2=d_1^2, \tag{39}$$

and

$$\alpha_2^2=\frac{(\lambda_q-\beta_q)^2}{8(1+q)^2}(c_1^2+d_1^2)\Longleftrightarrow c_1^2+d_1^2=\frac{8(1+q)^2}{(\lambda_q-\beta_q)^2}\alpha_2^2. \tag{40}$$

Adding (37) and (38), we get

$$2(1+q)q^2\,\alpha_2^2=\left(\frac{\lambda_q-\beta_q}{2}\right)\left((c_2+d_2)-(1+\beta_q)\frac{c_1^2+d_1^2}{2}\right). \tag{41}$$

Substituting the value of $(c_1^2+d_1^2)$ from (40), we obtain

$$\alpha_2^2=\frac{(\lambda_q-\beta_q)^2\,(c_2+d_2)}{4(1+q)\left[q^2\,(\lambda_q-\beta_q)+(1+q)(1+\beta_q)\right]}. \tag{42}$$

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Applying for the coefficients c_2 and d_2 along the equation (21), we obtain

$$|\alpha_2| \leq \frac{\lambda_q - \beta_q}{\sqrt{(1+q)\left[q^2\left(\lambda_q - \beta_q\right) + (1+q)(1+\beta_q)\right]}}.$$

The subtraction of equation (38) from equation (37) yields the following result:

$$2(1+q)[3]_{\langle q\rangle}(\alpha_3 - \alpha_2^2) = \frac{\lambda_q - \beta_q}{2}\left((c_2 - d_2) - (1+\beta_q)\frac{c_1^2 - d_1^2}{2}\right). \tag{43}$$

Then, in view of (39) and (40), Eq. (43) becomes

$$\alpha_3 = \frac{\left(\lambda_q - \beta_q\right)^2}{8(1+q)^2}(c_1^2 + d_1^2) + \frac{\lambda_q - \beta_q}{4(1+q)[3]_{\langle q\rangle}}(c_2 - d_2)$$

Thus applying (21), we conclude that

$$|\alpha_3| \leq \frac{\left(\lambda_q - \beta_q\right)^2}{4(1+q)^2} + \frac{\lambda_q - \beta_q}{2(1+q)[3]_{\langle q\rangle}}.$$

This completes the proof of Theorem. □

In the last theorem, we examine the Fekete-Szeg"o inequality for functions belonging to the class $\mathcal{C}_\Sigma(\lambda, \beta; q)$.

Theorem 4. *Given that Φ is an element of Σ defined by (1) and belongs to the class $\mathcal{C}_\Sigma(\lambda, \beta; q)$, and ϵ is a real number, we can state the following*

$$\left|\alpha_3 - \epsilon\,\alpha_2^2\right| \leq \begin{cases} \frac{\lambda_q - \beta_q}{4(1+q)[3]_{\langle q\rangle}}, & |1-\epsilon| \leq \frac{q^2\left(\lambda_q - \beta_q\right) + (1+q)(1+\beta_q)}{[3]_{\langle q\rangle}\left(\lambda_q - \beta_q\right)}, \\[2ex] \frac{\lambda_q - \beta_q}{2(1+q)}\left|\mathcal{L}(\epsilon)\right|, & |1-\epsilon| \geq \frac{q^2\left(\lambda_q - \beta_q\right) + (1+q)(1+\beta_q)}{[3]_{\langle q\rangle}\left(\lambda_q - \beta_q\right)}, \end{cases}$$

where

$$\mathcal{L}(\epsilon) = \frac{\left(\lambda_q - \beta_q\right)(1-\epsilon)}{q^2\left(\lambda_q - \beta_q\right) + (1+q)(1+\beta_q)}.$$

Proof. If $\Phi \in \mathcal{C}_\Sigma(\lambda, \beta; q)$ is given by (1), from (42) and (43),

$$\alpha_2^2 = \frac{\left(\lambda_q - \beta_q\right)^2(c_2 + d_2)}{4(1+q)\left[q^2\left(\lambda_q - \beta_q\right) + (1+q)(1+\beta_q)\right]}$$

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we have

$$\begin{aligned}
\alpha_3-\epsilon\,\alpha_2^2 &= (1-\epsilon)\frac{\left(\lambda_q-\beta_q\right)^2(c_2+d_2)}{4(1+q)\left[q^2\left(\lambda_q-\beta_q\right)+(1+q)(1+\beta_q)\right]}+\frac{\lambda_q-\beta_q}{4(1+q)[3]_{\langle q\rangle}}(c_2-d_2)\\
&=\frac{\lambda_q-\beta_q}{4(1+q)}\left[(1-\epsilon)\frac{\left(\lambda_q-\beta_q\right)(c_2+d_2)}{q^2\left(\lambda_q-\beta_q\right)+(1+q)(1+\beta_q)}+\frac{1}{[3]_{\langle q\rangle}}(c_2-d_2)\right]\\
&=\frac{\lambda_q-\beta_q}{4(1+q)}\left[\left(\frac{\left(\lambda_q-\beta_q\right)(1-\epsilon)}{q^2\left(\lambda_q-\beta_q\right)+(1+q)(1+\beta_q)}+\frac{1}{[3]_{\langle q\rangle}}\right)c_2\right.\\
&\qquad\left.+\left(\frac{\left(\lambda_q-\beta_q\right)(1-\epsilon)}{q^2\left(\lambda_q-\beta_q\right)+(1+q)(1+\beta_q)}-\frac{1}{[3]_{\langle q\rangle}}\right)d_2\right]\\
&=\frac{\lambda_q-\beta_q}{4(1+q)}\left[\left(\mathcal{L}(\epsilon)+\frac{1}{[3]_{\langle q\rangle}}\right)c_2+\left(\mathcal{L}(\epsilon)-\frac{1}{[3]_{\langle q\rangle}}\right)d_2\right],
\end{aligned}$$

where

$$\mathcal{L}(\epsilon)=\frac{\left(\lambda_q-\beta_q\right)(1-\epsilon)}{q^2\left(\lambda_q-\beta_q\right)+(1+q)(1+\beta_q)}.$$

Then, we conclude that

$$\left|\alpha_3-\epsilon\,\alpha_2^2\right|\leq\begin{cases}\frac{\lambda_q-\beta_q}{4(1+q)[3]_{\langle q\rangle}}, & |\mathcal{L}(\epsilon)|\leq\frac{1}{[3]_{\langle q\rangle}},\\[2ex] \frac{\lambda_q-\beta_q}{2(1+q)}\left|\mathcal{L}(\epsilon)\right|, & |\mathcal{L}(\epsilon)|\geq\frac{1}{[3]_{\langle q\rangle}}.\end{cases}$$

Which completes the proof of Theorem 4. □

6. Corollaries

The theorems presented in this study, namely Theorems 1 and 2, lead to the derivation of several corollaries that closely resemble the illustrative examples provided in Examples 1 and 2. These corollaries serve to further demonstrate and validate the results obtained from the theorems. By applying the principles and findings established in Theorems 1 and 2, these corollaries offer additional concrete instances that align with the examples previously discussed, providing further insight and supporting the overall conclusions of this study.

Corollary 1. *If Φ is an element of Σ defined by (1), it can be said that Φ is a member of the class $\mathcal{S}_{\Sigma}^{*}(\lambda,\beta)$, as per the following statement:*

$$|\alpha_2|\leq\frac{\lambda-\beta}{\sqrt{2\,(1+\lambda)}},\quad |\alpha_3|\leq\frac{(\lambda-\beta)^2}{4}+\frac{\lambda-\beta}{4},$$

and

$$\left|\alpha_3-\epsilon_1\,\alpha_2^2\right|\leq\begin{cases}\frac{\lambda-\beta}{8}, & |1-\epsilon_1|\leq\frac{(1+\lambda)}{2(\lambda-\beta)},\\[2ex] \frac{(\lambda-\beta)^2\left|1-\epsilon_1\right|}{2(1+\lambda)}, & |1-\epsilon_1|\geq\frac{(1+\lambda)}{2(\lambda-\beta)}.\end{cases}$$

Corollary 2. *If Φ is an element of Σ defined by (1), it can be said that Φ is a member of the class $\mathcal{C}_\Sigma(A,B)$, as per the following statement:*

$$|\alpha_2| \le \frac{\lambda-\beta}{\sqrt{2\left[(\lambda-\beta)+2(1+\beta)\right]}}, \quad |\alpha_3| \le \frac{(\lambda-\beta)^2}{16} + \frac{\lambda-\beta}{12}.$$

and

$$\left|\alpha_3 - \epsilon_2\,\alpha_2^2\right| \le \begin{cases} \frac{\lambda-\beta}{24}, & |1-\epsilon_2| \le \frac{(\lambda-\beta)+2(1+\beta)}{3(\lambda-\beta)}, \\ \frac{(\lambda-\beta)^2|1-\epsilon_2|}{4\left((\lambda-\beta)+2(1+\beta)\right)}, & |1-\epsilon_2| \ge \frac{(\lambda-\beta)+2(1+\beta)}{3(\lambda-\beta)}. \end{cases}$$

Conclusion: The present investigation pertains to the examination of the coefficient problems that arise in the context of the newly introduced subclasses of bi-univalent functions using q-Janowski Function, as defined in Definitions 5 and 6, over the disk $\mho$. The subclasses under consideration are denoted by $\mathcal{S}^*_\Sigma(\lambda,\beta;q)$, $\mathcal{C}_\Sigma(\lambda,\beta;q)$, $\mathcal{S}^*_\Sigma(\lambda,\beta)$, and $\mathcal{C}_\Sigma(\lambda,\beta)$. The determination of the Taylor-Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$, as well as the evaluation of the Fekete-Szegö functional problem, has been performed on functions belonging to the aforementioned subclasses of Σ. Through the specialization of parameters in our primary discoveries, we have discerned a number of supplementary novel outcomes. The utilization of fractional q-derivative operators is anticipated to have extensive implications in diverse scientific domains, encompassing mathematics and technology.

Author Contributions: Conceptualization, A.A. (Abdullah Alsoboh) and F.M.S.; methodology, A.A. (Ala Amourah) and A.A. (Abdullah Alsoboh); validation, F.M.S. and D.B.; formal analysis, A. Alsoboh; investigation, A. Amourah, A.A. (Abdullah Alsoboh) and D.B.; writing—original draft preparation, A.A. (Ala Amourah); writing—review and editing, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); supervision, A.A. (Abdullah Alsoboh) All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

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