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Bayes Estimation For The Rayleigh Weibull Distribution Based On Progressive Type-II Censored Samples For Cancer Data In Medicine

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Abstract: The aim of this study is to obtain the Bayes estimators and the maximum likelihood estimators (MLEs) for the unknown parameters of the Rayleigh Weibull (RW) distribution based on progressive type-II censored samples. The approximate Bayes estimators are calculated using the idea of Lindley and Tierney-Kadane's approximation method under the squared-error loss function when the Bayes estimators are not handed in explicit forms. In this study, the approximate Bayes estimates are compared with the maximum likelihood estimates in the aspect of the estimated risks (ERs) using Monte Carlo simulation. The asymptotic confidence intervals for the unknown parameters are obtained using MLEs of parameters. In addition, the coverage probabilities the parametric bootstrap estimates are ccomputed. Real lifetime data sets belonging to the cancer types as bladder cancer, head and neck cancer, and leukemia are used to illustrate the emprical results belonging to the approximate Bayes estimates, the maximum likelihood estimates, and the parametric bootstrap intervals.

Keywords: Rayleigh Weibull distribution; Progressive type-II censored sample; Bayes estimator; Approximate Bayes estimator; Asymptotic confidence intervals; Bootstarp intervals; Lindley's approximation; Tierney-Kadane's approximation; Squared-error loss function; Monte Carlo simulation

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1. Introduction

Probability distributions are often used to model real data especially in the fields of medicine, engineering, biological studies, and etc. In general, medical data such as lifetime has a (right) skewed distribution model. Therefore, statistical analyzes depend on the assumed probability distribution of the skewed medical data. The Rayleigh distribution proposed by Lord Rayleigh [1] in 1880, which is a special form of the Weibull distribution, is one of the most popular distributions in the analysis of skewed data. It has wide applications in life and reliability analysis, especially in modeling real-lifetime data in clinical researches. Extensions and generalizations of the known probability distributions have been suggested in order to obtain the best model that fits the data.

In this study, Bayesian estimators of the Rayleigh Weibull (RW) distribution proposed to give flexibility to the Rayleigh distribution are investigated in details. Since the prior distribution of the parameters is used in the Bayesian estimation method, it is more convenient to use Bayesian estimators of the parameters of the (right) skewed distributions in the decision-making process of the medical studies. In many studies, Bayesian estimation has been investigated based on complete and censored samples for different distributions by Kundu and Gupta [2], Almoghy et al. [3], Xie and Gui [4], Cai and Gui [5], Jiang and Gui [6].

The Rayleigh Weibull (RW) distribution with parameters (α, β) denote by $RW(\alpha, \beta)$, where $\beta > 0$ is the shape parameter and $\alpha > 0$ is the scale parameter, is introduced by Smadi and Alrefaei [7]. The probability density function

(pdf), cumulative distribution function (cdf), survival function and hazard function of the random variable X has Rayleigh Weibull (RW) distribution with α and β parameters can be given as follows;

$$f(x; \alpha, \beta) = 2\alpha\beta^2 x^{2\alpha-1} \exp(-\beta^2 x^{2\alpha}), \quad x \geq 0, \alpha > 0, \beta > 0 \quad (1)$$

$$F(x; \alpha, \beta) = 1 - \exp(-\beta^2 x^{2\alpha}) \quad (2)$$

$$\bar{F}(x; \alpha, \beta) = \exp(-\beta^2 x^{2\alpha}) \quad (3)$$

$$h(x; \alpha, \beta) = 2\alpha\beta^2 x^{2\alpha-1} \quad (4)$$

In the medical studies, since researchers can not observe lifetimes of all subjects in a life test experiment due to the time and cost constraints, censored data are needed. Since the complete data is not always available, there are censoring schemes that reduce time and cost. In the life test experiments, one of the most frequently used censoring scheme is progressive type II (PTR-II) censoring scheme. In the progressive type-II right censoring scheme, the items are removed from the experiment and then a censored sample is created thus saving time and cost. This type of censored scheme is explained as follows. Suppose that n identical items are put to the test and m failures are to be observed. At the time of the first failure, R_1 items from the rest of the surviving $n - R_1 - 1$ items are randomly selected, and then removed. Likewise, at the time of the second failure, R_2 items of the remaining $n - R_2 - 2$ items are randomly selected, and then removed, and so the process continues like this. Lastly, at the time of the m^{th} failure, all the surviving items are censored. Progressive type-II right censoring scheme is visually demonstrated with $\mathbf{R} = (R_1, R_2, \dots, R_m)$ scheme. In this lifetime process, $\mathbf{X}^R = (X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \dots, X_{m:m:n}^{R_m})$ with $X_{1:m:n}^{R_1} < X_{2:m:n}^{R_2} < \dots < X_{m:m:n}^{R_m}$ is called as Progressive Type-II right censored sample with $\mathbf{R} = (R_1, R_2, \dots, R_m)$. In Progressive Type-II right censoring, using $\mathbf{R} = (0, 0, \dots, n - m)$, Type-II right censoring is obtained. The joint probability density function (pdf) of this censored sample is given by ([8], [9]);

$$f_{X_{1:m:n}^R, X_{2:m:n}^R, \dots, X_{m:m:n}^R}(x_1, x_2, \dots, x_m) = c \prod_{i=1}^m f(x_i) [1 - F(x_i)]^{R_i}, \quad -\infty < x_1 < x_2 < \dots < x_m < \infty \quad (5)$$

$$\text{where } c = n(n - R_1 - 1) \times \dots \times (n - R_1 - R_2 - \dots - R_{m-1} - m + 1).$$

There are a lot of studies that refer to the parameter estimation of different distributions under the PTR-II censored samples (Ali Mousa [10], Balakrishnan [11], Ali Mousa and Al-Sagheer [12], Wu et al. [13], Panahi and Asadi [14], Aljua'id [15], Ahmed [16], Singh et al. [17], Liao and Gui [18], Abbas et al. [19], Sultan et al. [20], Alshenawy [21], Mukhtar [22], Wu and Gui [23], Almongy et al. [24], Qiao and Gui [25], Wu [26], El-Morshedy et al. [27], El-Sherpieny et al. [28], Liang et al. [29], Alshenawy et al. [30], Almetwally et al. [31], Muhammed and Almetwally [32]).

The main purpose of this study is to obtain the approximate Bayes estimators under the square error loss functions and then to check them with maximum likelihood estimators (MLEs) in the aspect of the estimated risk (ER). The sections of this study are given respectively. In the first section, an introduction to the RW distribution and progressive Type-II right censored sample is given. In Section 2, the MLEs and bootstrap confidence intervals for the unknown parameters are obtained. In section 3, the approximate Bayes estimators under the squared error loss function using the Lindley's and Tierney-Kadane's approximations for the unknown parameters are acquired. In section 4, the approximate Bayes estimations are compared with the maximum likelihood (ML) estimations in the aspect of the ER, and then the coverage probabilities of the asymptotic confidence intervals and the bootstrap confidence intervals are observed by using Monte Carlo simulation. In section 5, the real lifetime data sets due to the cancer types as bladder cancer, neck cancer, and leukemia are given to illustrate the empirical results belonging to the approximate Bayes estimates, the maximum likelihood estimates, and the parametric bootstrap intervals. In section 6, conclusion part is given.

1. Maximum Likelihood Estimation (MLE)

Let $\mathbf{X}^R = (X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \dots, X_{m:m:n}^{R_m})$ denotes PTR-II censored sample taken from $RW(\alpha, \beta)$ distribution with the pdf and cdf in Eq. (1) and Eq. (2). Then, the likelihood function $\ell(\alpha, \beta)$ can be written as follows;

$$\ell(\alpha, \beta) = \prod_{i=1}^m 2\alpha\beta^2 x_{i:m:n}^{2\alpha-1} \exp(-\beta^2 x_{i:m:n}^{2\alpha}) \left[\exp(-\beta^2 x_{i:m:n}^{2\alpha}) \right]^{R_i}$$

$$\ell(\alpha, \beta) \propto 2^m \alpha^m \beta^{2m} \exp\left(\sum_{i=1}^m -\beta^2 x_{i:m:n}^{2\alpha}\right) \prod_{i=1}^m x_{i:m:n} \left[\exp\left(\sum_{i=1}^m -\beta^2 x_{i:m:n}^{2\alpha}\right) \right]^{R_i} \quad (6)$$

The log-likelihood function, $L(\alpha, \beta) = \ln \ell(\alpha, \beta)$ can be given as follows;

$$L(\alpha, \beta) \propto m \ln(2) + m \ln(\alpha) + 2m \ln(\beta) + (2\alpha - 1) \sum_{i=1}^m \ln(x_{i:m:n}) - \beta^2 \sum_{i=1}^m (1 + R_i) x_{i:m:n}^{2\alpha} \quad (7)$$

Taking the partial derivatives of $L(\alpha, \beta)$ according to the α and β parameters, and then equalizing them to zero, the following equations can be obtained as follows;

$$L_1 = \frac{\partial L}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^m \ln(x_{i:m:n}) + \beta^2 \sum_{i=1}^m 2(1 + R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n}) = 0 \quad (8)$$

$$L_2 = \frac{\partial L}{\partial \beta} = \frac{2m}{\beta} - 2\beta \sum_{i=1}^m (1 + R_i) x_{i:m:n}^{2\alpha} = 0 \quad (9)$$

The nonlinear equations given by Eq.(8) and Eq.(9) can be solved by using Newton-Raphson (NR) iterative method.

2.1. Asymptotic Confidence Interval (ACI)

Let $\Theta = (\alpha, \beta)$ be the Fisher information matrix of Θ parameter vector given by

$$I_{\Theta} = -E \begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}$$

Since $I(\Theta)$ is difficult to compute, the observed Fisher information $I(\hat{\Theta})$ is used approximate to expect Fisher information matrix. Let $\hat{\Theta} = (\hat{\alpha}, \hat{\beta})$ be the MLEs of the parameters $\Theta = (\alpha, \beta)$. The observed Fisher information matrix is given by

$$I(\hat{\Theta}) = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}_{\Theta=\hat{\Theta}}$$

Therefore, the observed variance–covariance matrix for the MLEs $(\hat{\alpha}, \hat{\beta})$ is the inverse of the observed information matrix given by [33]

$$I^{-1}(\hat{\Theta}) = \begin{bmatrix} \hat{Var}(\hat{\alpha}) & \hat{Cov}(\hat{\alpha}, \hat{\beta}) \\ \hat{Cov}(\hat{\alpha}, \hat{\beta}) & \hat{Var}(\hat{\beta}) \end{bmatrix}$$

Under some regularity conditions, $\hat{\Theta}$ is approximately bivariate normaly distributed with mean Θ and variance-covariance matrix $I(\hat{\Theta})$ as $\left[\hat{\Theta} \sim N(\Theta, I^{-1}(\hat{\Theta})) \right]$, [34]. Thus, the 100(1- α)% confidence interval for α and β can be constructed as $\left(\hat{\alpha} \mp z_{\frac{\alpha}{2}} \times \sqrt{\hat{Var}(\hat{\alpha})} \right)$ and $\left(\hat{\beta} \mp z_{\frac{\alpha}{2}} \times \sqrt{\hat{Var}(\hat{\beta})} \right)$ where z_{α} denotes the upper α -th quantile of the standard normal distribution.

2.2. Bootstrap Confidence Interval

Confidence intervals for the unknown $\Theta = (\alpha, \beta)$ parameters are obtained by using the percentile bootstrap confidence interval (P-BCI) method proposed by Efron [35]. The steps for estimating the bootstrap parametric confidence intervals of the parameters $\Theta = (\alpha, \beta)$ by using the P-BCI method are given as follows [36].

Step 1. Generate the PTR-II censored samples $\mathbf{X}^R = (X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \dots, X_{m:m:n}^{R_m})$ taken from the RW distribution with the $\Theta = (\alpha, \beta)$ parameters.

Step 2. Let ML estimates of the $\Theta = (\alpha, \beta)$ parameters be $\hat{\Theta} = (\hat{\alpha}_{MLE}, \hat{\beta}_{MLE})$.

Step 3. To generate the bootstrap samples $\mathbf{X}^R = (X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \dots, X_{m:m:n}^{R_m})$ with $R_1, R_2, \dots, R_m$ scheme, using the $\hat{\Theta} = (\hat{\alpha}_{MLE}, \hat{\beta}_{MLE})$. Find the bootstrap estimate of the $\Theta = (\alpha, \beta)$ parameters as $\hat{\Theta}^* = (\hat{\alpha}_{MLE}^*, \hat{\beta}_{MLE}^*)$.

Step 4. Repeat Step 3 NBoot times.

Step 5. Let $F^*(x) = P(\hat{\Theta}^* \leq x)$ as the cumulative distribution function of $\hat{\Theta}^*$. Define $\hat{\Theta}^* = F^{*-1}(x)$ for given x . The approximate bootstrap 100(1- α)% confidence interval for Θ is given as $\left(\hat{\Theta}_{\frac{\gamma}{2}}^*, \hat{\Theta}_{1-\frac{\gamma}{2}}^* \right)$.

1. Bayes Estimation

For Bayesian estimation, it is assumed that the α and β parameters of the $RW(\alpha, \beta)$ distribution have the following independent prior $Gamma(a_1, b_1)$, and $Gamma(a_2, b_2)$ densities, respectively as follows;

$$\pi_1(\alpha) = \frac{\alpha^{a_1-1} \exp(-b_1\alpha) b_1^{a_1}}{\Gamma(a_1)} \quad a_1, b_1, \alpha > 0 \quad (10)$$

$$\pi_2(\beta) = \frac{\beta^{a_2-1} \exp(-b_2\beta) b_2^{a_2}}{\Gamma(a_2)} \quad a_2, b_2, \beta > 0 \quad (11)$$

In this case, the joint prior distribution of the α and β parameters can be written as follows;

$$\pi(\alpha, \beta) = \frac{\alpha^{a_1-1} b_1^{a_1} \beta^{a_2-1} b_2^{a_2}}{\Gamma(a_1) \Gamma(a_2)} \exp(-b_1\alpha) \exp(-b_2\beta) \quad a_i, b_i, \alpha, \beta > 0, i = 1, 2 \quad (12)$$

From Eq.(12), the log of the prior density function is given as follows;

$$\begin{aligned} \rho(\alpha, \beta) = & (a_1 - 1) \ln \alpha + (a_2 - 1) \ln \beta - b_1 \alpha - b_2 \beta + \\ & + a_1 \ln(b_1) + a_2 \ln(b_2) - \ln[\Gamma(a_1)] - \ln[\Gamma(a_2)] \end{aligned} \quad (13)$$

By using the $L(\alpha, \beta)$, and $\pi(\alpha, \beta)$, the joint posterior density function of the α and β parameters can be written as follows;

$$P(\alpha, \beta) = \frac{k(\alpha, \beta) \exp\left(\sum_{i=1}^m -\beta^2 x_{i:m:n}^{2\alpha}\right) \prod_{i=1}^m x_{i:m:n} \left[\exp\left(\sum_{i=1}^m -\beta^2 x_{i:m:n}^{2\alpha}\right)\right]^{R_i}}{\int_0^\infty \int_0^\infty k(\alpha, \beta) \exp\left(\sum_{i=1}^m -\beta^2 x_{i:m:n}^{2\alpha}\right) \prod_{i=1}^m x_{i:m:n} \left[\exp\left(\sum_{i=1}^m -\beta^2 x_{i:m:n}^{2\alpha}\right)\right]^{R_i} d\alpha d\beta} \quad (14)$$

where $k(\alpha, \beta) = \alpha^{m+a_1-1} \beta^{2m+a_2-1} \exp(-b_1\alpha) \exp(-b_2\beta)$.

Thus, the Bayes estimate of any function of α and β , say $u(\alpha, \beta)$, under the **squared error** loss function can be written as follows;

$$\hat{u}_B(\alpha, \beta) = E[u(\alpha, \beta)] = \frac{\int_0^\infty \int_0^\infty u(\alpha, \beta) e^{[L(\alpha, \beta) + \rho(\alpha, \beta)]} d\alpha d\beta}{\int_0^\infty \int_0^\infty e^{[L(\alpha, \beta) + \rho(\alpha, \beta)]} d\alpha d\beta} \quad (15)$$

The Bayes estimate of any function of α and β given in Eq. (15), which consists of the ratio of two integrals, can not be obtained in closed-form, and then the Bayes estimators of these parameters using the Lindley's approximation, and Tierney-Kadane's approximation under the squared error loss (quadratic loss) function are computed.

3.1. Lindley's Approximation

Lindley's approximation suggested by Lindley [37] is an approximate Bayes method used to approximate the ratio of two integrals such as given in Eq.(15) that cannot be solved analytically. This method uses third derivatives of the

log-likelihood function, and has an error of order $O(n^{-1})$. Lindley's approximation has been used by many authors such as Ahmad and Jaheen [38], Kundu and Gupta [39], Preda et al. [40] to compute the approximate Bayes estimators of different lifetime distributions based on the censored samples. For the two-parameter case, where θ_1 and θ_2 notations are used for the α and β parameters, the formula with the Lindley's approximation can be written as follows;

$$\begin{aligned} u_{BLindley}(\hat{\theta}_1, \hat{\theta}_2) &= E[u(\theta_1, \theta_2) / X] \approx \\ &\approx \left[u + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 (u_{ij} + 2u_i \rho_j) \sigma_{ij} + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \sum_{l=1}^2 L_{ijkl} \sigma_{ij} \sigma_{kl} u_l \right] \\ &= u_{MLE}(\hat{\theta}_1, \hat{\theta}_2) + \frac{1}{2} [a_{1i} + a_{2i}] + \frac{1}{2} [(u_1 \sigma_{11} + u_2 \sigma_{12})d + (u_1 \sigma_{21} + u_2 \sigma_{22})e] \end{aligned} \quad (16)$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the MLE of the θ_1 and θ_2 parameters, respectively, and let

$$a_{1i} = (u_{1i} + 2u_1 \rho_i) \sigma_{1i}, \quad i = 1, 2$$

$$d = L_{111} \sigma_{11} + L_{121} \sigma_{12} + L_{211} \sigma_{21} + L_{221} \sigma_{22}$$

$$e = L_{112} \sigma_{11} + L_{122} \sigma_{12} + L_{212} \sigma_{21} + L_{222} \sigma_{22}$$

and

$$\rho_i = \frac{\partial \rho(\theta_1, \theta_2)}{\partial \theta_i}, \quad i = 1, 2 \quad u_i = \frac{\partial u(\theta_1, \theta_2)}{\partial \theta_i}, \quad i = 1, 2 \quad u_{ij} = \frac{\partial^2 u(\theta_1, \theta_2)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2$$

$$L_{ij} = \frac{\partial^2 L(\theta_1, \theta_2)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, 2 \quad L_{ijk} = \frac{\partial^3 L(\theta_1, \theta_2)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \quad i, j, k = 1, 2$$

$$[-L_{ij}]^{-1} = [\sigma_{ij}], \quad i, j = 1, 2$$

σ_{ij} is the (i, j) -th element of the matrix $[\sigma_{ij}]$.

From Eq.(13), we get

$$\rho_1 = \frac{a_1 - 1}{\alpha} - b_1, \quad \rho_2 = \frac{a_2 - 1}{\beta} - b_2$$

and then, the following values of L_{ij} for $i, j = 1, 2$ and L_{ijk} for $i, j, k = 1, 2$ are handed as follows,

$$L_{11} = -\frac{m}{\alpha^2} - \beta^2 \sum_{i=1}^m 4(1+R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})^2$$

$$L_{12} = L_{21} = -2\beta^2 \sum_{i=1}^m 2(1+R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})$$

$$L_{22} = -\frac{2m}{\beta^2} - 2 \sum_{i=1}^m (1+R_i) x_{i:m:n}^{2\alpha}$$

$$L_{111} = \frac{2m}{\alpha^3} - \beta^2 \sum_{i=1}^m 8(1+R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})^3$$

$$L_{112} = L_{112} = L_{121} = L_{211} = -2\beta \sum_{i=1}^m 4(1+R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})^2$$

$$L_{122} = L_{122} = L_{221} = L_{212} = -2 \sum_{i=1}^m 2(1+R_i) x_{i:m:n}^{2\alpha} \ln(x_{i:m:n})$$

$$L_{222} = \frac{4m}{\beta^3}$$

Finally, the approximate Bayes estimators for the α and β parameter of the $RW(\alpha, \beta)$ distribution based on progressive type-II censored samples under the squared error loss function are obtained as follows,

$$\hat{\alpha}_{BL} = \hat{\alpha}_{MLE} + \left(\frac{a_1 - 1}{\hat{\alpha}_{MLE}} - b_1 \right) \sigma_{11} + \left(\frac{a_2 - 1}{\hat{\beta}_{MLE}} - b_2 \right) \sigma_{12} + \frac{1}{2} [\sigma_{11}d + \sigma_{21}e] \quad (17)$$

$$\hat{\beta}_{BL} = \hat{\beta}_{MLE} + \left(\frac{a_1 - 1}{\hat{\alpha}_{MLE}} - b_1 \right) \sigma_{21} + \left(\frac{a_2 - 1}{\hat{\beta}_{MLE}} - b_2 \right) \sigma_{22} + \frac{1}{2} [\sigma_{12}d + \sigma_{22}e] \quad (18)$$

respectively.

3.2. Tierney-Kadane's Approximation

Tierney-Kadane's Approximation proposed by Tierney and Kadane [41] is a method as an alternative to the Lindley's approximation. This method uses second derivatives of a function composed of the log-likelihood function and the log-prior function, and has an error of order $O(n^{-2})$. Therefore, Tierney-Kadane's Approximation is more advantageous than Lindley's approximation. Tierney-Kadane's approximation has been used by many authors such as Gencer

and Gencer [42], Kim and Han [43], Elshahhat and Rastogi [44], Singh et al. [45], to compute the approximate Bayes estimators of different lifetime distributions based on the censored samples. This approximation can be defined as follows;

$$\eta(\alpha, \beta) = \frac{1}{n} [L(\alpha, \beta) + \rho(\alpha, \beta)]$$

$$\eta^*(\alpha, \beta) = \frac{1}{n} \ln u(\alpha, \beta) + \eta(\alpha, \beta)$$

where $L(\alpha, \beta)$ denotes the log-likelihood function, and $\rho(\alpha, \beta)$ denotes the log of the joint prior density. Thus, by means of the Tierney-Kadane's approximation given in Eq. (15) can be written as follows;

$$\hat{u}_{BTK}(\alpha, \beta) = E[u(\alpha, \beta)] = \frac{\int e^{n\eta^*(\alpha, \beta)} d(\alpha, \beta)}{\int e^{n\eta(\alpha, \beta)} d(\alpha, \beta)}$$

$$\approx \left[\frac{\det \Sigma^*}{\det \Sigma} \right]^{1/2} \exp \left\{ n \left[\eta^*(\hat{\alpha}^*, \hat{\beta}^*) - \eta(\hat{\alpha}, \hat{\beta}) \right] \right\}$$

where $(\hat{\alpha}^*, \hat{\beta}^*)$ and $(\hat{\alpha}, \hat{\beta})$ maximize $\eta^*(\alpha, \beta)$ and $\eta(\alpha, \beta)$, respectively. Σ^* and Σ are minus the inverse Hessians of $\eta^*(\alpha, \beta)$ and $\eta(\alpha, \beta)$ at $(\hat{\alpha}^*, \hat{\beta}^*)$ and $(\hat{\alpha}, \hat{\beta})$, respectively.

In this case, $\eta(\alpha, \beta)$, $\eta^*(\alpha, \beta)$ and Σ are given as follows;

$$\eta(\alpha, \beta) = \frac{1}{n} \left[m \ln(2) + m \ln(\alpha) + 2m \ln(\beta) + (2\alpha - 1) \sum_{i=1}^m \ln(x_{i:m:n}) - \beta^2 \sum_{i=1}^m (1 + R_i) x_{i:m:n}^{2\alpha} + (a_1 - 1) \ln \alpha + (a_2 - 1) \ln \beta - b_1 \alpha - b_2 \beta + a_1 \ln(b_1) + a_2 \ln(b_2) - \ln[\Gamma(a_1)] - \ln[\Gamma(a_2)] \right]$$

$$\eta^*(\alpha, \beta) = \frac{1}{n} \left[\ln u(\alpha, \beta) + m \ln(2) + m \ln(\alpha) + 2m \ln(\beta) + (2\alpha - 1) \sum_{i=1}^m \ln(x_{i:m:n}) - \beta^2 \sum_{i=1}^m (1 + R_i) x_{i:m:n}^{2\alpha} + (a_1 - 1) \ln \alpha + (a_2 - 1) \ln \beta - b_1 \alpha - b_2 \beta + a_1 \ln(b_1) + a_2 \ln(b_2) - \ln[\Gamma(a_1)] - \ln[\Gamma(a_2)] \right]$$

and

$$\Sigma = \begin{bmatrix} -\frac{\partial^2 \eta(\alpha, \beta)}{\partial \alpha^2} & -\frac{\partial^2 \eta(\alpha, \beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \eta(\alpha, \beta)}{\partial \alpha \partial \beta} & -\frac{\partial^2 \eta(\alpha, \beta)}{\partial \beta^2} \end{bmatrix}^{-1}$$

respectively.

Through the Tierney-Kadane's approximation, the approximate Bayes estimators of the α and β parameters of the $RW(\alpha, \beta)$ distribution based on the progressive type-II censored samples under the squared error loss function are obtained as follows;

$$\eta_{\alpha}^*(\alpha, \beta) = \frac{\ln \alpha}{n} + \eta(\alpha, \beta)$$

and

$$\Sigma_{\alpha}^* = \begin{bmatrix} -\frac{\partial^2 \eta_{\alpha}^*(\alpha, \beta)}{\partial \alpha^2} & -\frac{\partial^2 \eta_{\alpha}^*(\alpha, \beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \eta_{\alpha}^*(\alpha, \beta)}{\partial \alpha \partial \beta} & -\frac{\partial^2 \eta_{\alpha}^*(\alpha, \beta)}{\partial \beta^2} \end{bmatrix}^{-1}$$

$$\hat{\alpha}_{BTK} = \left[\frac{\det \Sigma_{\alpha}^*}{\det \Sigma} \right]^{1/2} \exp \left\{ n \left[\eta_{\alpha}^*(\hat{\alpha}^*, \hat{\beta}^*) - \eta(\hat{\alpha}, \hat{\beta}) \right] \right\} \quad (19)$$

$$\eta_{\beta}^*(\alpha, \beta) = \frac{\ln \beta}{n} + \eta(\alpha, \beta)$$

and

$$\Sigma_{\beta}^* = \begin{bmatrix} -\frac{\partial^2 \eta_{\beta}^*(\alpha, \beta)}{\partial \alpha^2} & -\frac{\partial^2 \eta_{\beta}^*(\alpha, \beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \eta_{\beta}^*(\alpha, \beta)}{\partial \alpha \partial \beta} & -\frac{\partial^2 \eta_{\beta}^*(\alpha, \beta)}{\partial \beta^2} \end{bmatrix}^{-1}$$

$$\hat{\beta}_{BTK} = \left[\frac{\det \Sigma_{\beta}^*}{\det \Sigma} \right]^{1/2} \exp \left\{ n \left[\eta_{\beta}^*(\hat{\alpha}^*, \hat{\beta}^*) - \eta(\hat{\alpha}, \hat{\beta}) \right] \right\} \quad (20)$$

1. Simulation Study

In this section, Monte Carlo simulation studies for different sample sizes (n and m) and different censoring schemes are done. In the aspect of the estimated risks, the performances of the approximate Bayes estimates computed with Lindley and Tierney-Kadane's approximation method under the squared error loss function for the α and β parameters of $RW(\alpha, \beta)$ based on progressive type II censored sample are compared with those of the MLE. Informative priors for $a_1=3, b_1=1, a_2=3, b_2=1$ are used while computing the approximate Bayes estimates. Estimated risk for the estimate of the α parameter can be computed with the $\hat{\alpha}_{ER} = E(\hat{\alpha}_i - \alpha)^2, i = 1, 2, \dots, 10000$, where $\hat{\alpha}$ is the MLE or the approximate Bayes estimation, and α is generated from the Gamma distribution with parameter (a_1, b_1) . In

addition, the estimated risk for the estimate of the β parameter is computed in the same way. All the computations are based on 10000 replications.

In this simulation study, in order to produce the progressive Type-II censored samples from the $RW(\alpha, \beta)$ distribution, we have benefited from the algorithm presented in Balakrishnan and Sandhu [46]. The algorithm for the $RW(\alpha, \beta)$ distribution is given as follows;

1. Let $W_1, W_2, \dots, W_m$ be m -sized samples generated from the $\text{Uniform}(0,1)$ distribution.
2. $V_i = W_i^{\left(i + \sum_{j=m-i+1}^m R_j\right)^{-1}}$ is defined by replacing $i = 1, 2, \dots, m$.
3. $U_{i:m:n}^R = 1 - V_m V_{m-1} \cdots V_{m-i+1}$ is obtained by replacing $i = 1, 2, \dots, m$.

Thus $U_{1:m:n}^R < U_{2:m:n}^R < \cdots < U_{m:m:n}^R$ are progressively Type-II censored samples with the censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$ taken from the $\text{Uniform}(0,1)$ distribution. And then finally,

$$X_{i:m:n}^R = \left[-\frac{\ln(1 - \ln U_{i:m:n}^R)}{\beta^2} \right]^{\left(\frac{1}{2\alpha}\right)}, \quad i = 1, 2, \dots, m$$
 is the progressively Type-II censored i^{th} order statistics with the censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$ taken from the $RW(\alpha, \beta)$ distribution. The estimated risks of the approximate Bayes estimates computed with Lindley and Tierney-Kadane's approximation method under the squared error loss function and ML estimates for the α and β parameters of $RW(\alpha, \beta)$ based on progressive type II censored sample are tabulated in Table1.

In Table 2 and Table 3, coverage probabilities, lengths, lower and upper bounds for the asymptotic confidence intervals (ACI) and bootstrap confidence intervals for the α and β parameters.

Table 1. ER values of the MLEs and the Approximate Bayes Estimates of the α and β parameters for $a_1=3, b_1=1, a_2=3, b_2=1$.

n	m	Censoring scheme	MLE		BAYES _{Lindley}		BAYES _{Tierney-Kadane}	
		$\mathbf{R}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$
20	10	A	0.0211	0.0135	0.0196	0.0125	0.0190	0.0125
		B	0.0124	0.0160	0.0113	0.0148	0.0111	0.0147
		C	0.0139	0.0133	0.0122	0.0123	0.0121	0.0123
	20	D	0.0082	0.0097	0.0074	0.0090	0.0073	0.0090
	30	A	0.0218	0.0134	0.0206	0.0130	0.0202	0.0129
		B	0.0110	0.0149	0.0106	0.0141	0.0103	0.0141
		C	0.0229	0.0144	0.0220	0.0137	0.0214	0.0137

30	15	A	0.0140	0.0085	0.0129	0.0081	0.0128	0.0081
		B	0.0085	0.0112	0.0078	0.0106	0.0078	0.0106
		C	0.0141	0.0088	0.0134	0.0085	0.0133	0.0084
	20	A	0.0106	0.0077	0.0099	0.0073	0.0099	0.0073
		B	0.0070	0.0091	0.0066	0.0086	0.0066	0.0086
		C	0.0071	0.0077	0.0065	0.0073	0.0065	0.0072
	25	A	0.0072	0.0069	0.0068	0.0065	0.0068	0.0065
		B	0.0059	0.0077	0.0055	0.0073	0.0055	0.0073
		C	0.0065	0.0075	0.0063	0.0072	0.0061	0.0072
50	30	D	0.0054	0.0068	0.0051	0.0064	0.0051	0.0064
	20	A	0.0110	0.0068	0.0104	0.0066	0.0104	0.0066
		B	0.0062	0.0079	0.0059	0.0076	0.0059	0.0076
		C	0.0061	0.0060	0.0057	0.0059	0.0057	0.0059
	30	A	0.0072	0.0048	0.0068	0.0046	0.0068	0.0046
		B	0.0046	0.0063	0.0045	0.0061	0.0045	0.0061
		C	0.0045	0.0048	0.0043	0.0047	0.0043	0.0047
	40	A	0.0048	0.0041	0.0046	0.0039	0.0046	0.0039
		B	0.0038	0.0048	0.0036	0.0047	0.0036	0.0047

Table 1. Continued

n	m	Censoring scheme	MLE		BAYES _{Lindley}		BAYES _{Tierney-Kadane}	
		R	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$	$\hat{\alpha}_{ER}$	$\hat{\beta}_{ER}$

50	40	C	0.0036	0.0042	0.0034	0.0041	0.0034	0.0041
	50	D	0.0030	0.0039	0.0028	0.0038	0.0028	0.0038
70	30	A	0.0076	0.0045	0.0074	0.0044	0.0073	0.0043
		B	0.0043	0.0056	0.0041	0.0055	0.0040	0.0054
		C	0.0039	0.0042	0.0037	0.0041	0.0037	0.0040
40	A	A	0.0057	0.0032	0.0056	0.0031	0.0055	0.0030
		B	0.0034	0.0031	0.0032	0.0030	0.0031	0.0029
		C	0.0043	0.0028	0.0042	0.0027	0.0041	0.0027
50	A	A	0.0038	0.0031	0.0036	0.0030	0.0035	0.0029
		B	0.0030	0.0038	0.0029	0.0037	0.0028	0.0036
		C	0.0028	0.0030	0.0027	0.0029	0.0026	0.0028
70	D	D	0.0022	0.0029	0.0021	0.0028	0.0020	0.0026
100	25	A	0.0095	0.0070	0.0092	0.0068	0.0091	0.0067
		B	0.0049	0.0065	0.0048	0.0063	0.046	0.0062
		C	0.0038	0.0046	0.0037	0.0045	0.0036	0.0044
40	A	A	0.0057	0.0032	0.0056	0.0032	0.0056	0.0032
		B	0.0034	0.0043	0.0032	0.0042	0.0031	0.0041
		C	0.0028	0.0031	0.0027	0.0030	0.0027	0.0030
50	A	A	0.0043	0.0025	0.0042	0.0025	0.0042	0.0025
		B	0.0027	0.0037	0.0026	0.0036	0.0026	0.0036
		C	0.0025	0.0027	0.0024	0.0026	0.0024	0.0026
70	A	A	0.0028	0.0024	0.0026	0.0022	0.0025	0.0021
		B	0.0022	0.0027	0.0021	0.0026	0.0021	0.0025
		C	0.0020	0.0024	0.0019	0.0021	0.0019	0.0020
90	A	A	0.0019	0.0020	0.0018	0.0019	0.0018	0.0019
		B	0.0018	0.0022	0.0017	0.0021	0.0016	0.0020
		C	0.0017	0.0021	0.0016	0.0020	0.0016	0.0020
100	D	D	0.0015	0.0019	0.0014	0.0018	0.0013	0.0018

where

A: The censoring at the end of the experiment $R = 0, 0, \dots, n - m$,

B:The censoring at the beginning of the experiment $R = n - m, 0, \dots, 0$,

C: Other the censoring schemes $R = 0, 0, \dots, n - m, \dots, 0$,

D: Complete sample $R = 0, 0, \dots, 0$.

Table 2. Confidence average width and coverage probability for the asymptotic confidence interval and the bootstrap confidence interval of the parameter α ($\alpha=0.5$).

(n, m)	R	ML estimates	Lower Limit	Upper Limit	ACI width	Probability of coverage	Boot ML estimates	Boot Lower Limit	Boot Upper Limit	Boot ACI width	Boot Probability of coverage
20,10	A	0.6108	0.2598	0.9618	0.7020	0.9618	0.7301	0.0403	1.0654	1.0251	0.9700
20,10	B	0.5618	0.3117	0.8118	0.5001	0.9540	0.4987	0.0403	0.8309	0.7906	0.9560
20,10	C	0.5740	0.3184	0.8297	0.5112	0.9460	0.8986	0.0446	0.8945	0.8498	0.9490
20,20	D	0.5318	0.3478	0.7159	0.3681	0.9300	0.6347	0.4034	0.8236	0.4203	0.9100
50,30	A	0.5414	0.3656	0.7172	0.3515	0.9400	0.5393	0.4112	0.7922	0.3811	0.9000
50,30	B	0.5316	0.3822	0.6810	0.2988	0.9200	0.5465	0.4209	0.7432	0.3223	0.9000
50,30	C	0.5217	0.3885	0.6549	0.2664	0.9600	0.5138	0.4223	0.7106	0.2882	0.9100
50,50	D	0.5129	0.4012	0.6246	0.2234	0.9400	0.6025	0.4269	0.6568	0.2299	0.9300
100,50	A	0.5170	0.3844	0.6501	0.2657	0.9420	0.7353	0.4174	0.8799	0.4625	0.9100
100,50	B	0.5126	0.4068	0.6183	0.2114	0.9560	0.6427	0.4300	0.7583	0.3283	0.9220
100,50	C	0.5149	0.4174	0.6124	0.1950	0.9480	0.7634	0.4666	0.8495	0.3829	0.9590
100,70	A	0.5107	0.4046	0.6167	0.2121	0.9420	0.7674	0.4575	0.8715	0.4140	0.9270
100,70	B	0.5110	0.4195	0.6025	0.1830	0.9510	0.7105	0.4637	0.7769	0.3132	0.9380
100,70	C	0.5110	0.4228	0.5992	0.1765	0.9600	0.7092	0.4658	0.7935	0.3277	0.9520
100,100	D	0.5059	0.4285	0.5834	0.1549	0.9520	0.6822	0.4650	0.7321	0.2671	0.9380

Table 3. Confidence average width and coverage probability for the asymptotic confidence interval and the bootstrap confidence interval of the parameter β ($\beta=0.8$).

(n, m)	R	ML estimates	Lower Limit	Upper Limit	ACI width	Probability of coverage	Boot ML estimates	Boot Lower Limit	Boot Upper Limit	Boot ACI width	Boot Probability of coverage
20,10	A	0.8456	0.5594	1.1317	0.5723	0.9500	0.6020	0.0628	1.2287	1.1659	0.9090
20,10	B	0.8161	0.5293	1.1028	0.5735	0.9430	1.2557	0.0588	1.0514	0.9926	0.8790
20,10	C	0.8320	0.5624	1.1017	0.5394	0.9230	0.6366	0.0620	1.1208	1.0588	0.8780
20,20	D	0.8215	0.6064	1.0365	0.4301	0.9500	0.6937	0.5947	1.0367	0.4690	0.9500
50,30	A	0.8082	0.5238	0.3592	0.3293	0.9600	0.5375	0.4021	0.7620	0.3599	0.9100
50,30	B	0.7983	0.6248	0.9718	0.3470	0.9300	0.8135	0.6215	0.9815	0.3600	0.9300
50,30	C	0.8046	0.6531	0.9562	0.3031	0.9500	0.7046	0.6641	0.9831	0.3190	0.9400
50,50	D	0.7952	0.6612	0.9293	0.2681	0.9800	0.7776	0.6571	0.9312	0.2741	0.9700
100,50	A	0.8087	0.6947	0.9227	0.2280	0.9300	1.0515	0.7127	1.1498	0.4371	0.9220
100,50	B	0.8005	0.6701	0.9309	0.2608	0.9480	1.0057	0.6774	1.0668	0.3894	0.9440
100,50	C	0.8072	0.6926	0.9219	0.2293	0.9410	1.0665	0.7409	1.1613	0.4204	0.9220

100,70	A	0.8022	0.7043	0.9002	0.1959	0.9590	0.9923	0.7421	1.0638	0.3216	0.9540
100,70	B	0.7979	0.6861	0.9096	0.2285	0.9490	1.0143	0.7277	1.0827	0.3550	0.9210
100,70	C	0.7987	0.6974	0.9000	0.2026	0.9530	0.9865	0.7361	1.0494	0.3133	0.9480
100,100	D	0.8000	0.7052	0.8948	0.1896	0.9480	0.9645	0.7395	1.0215	0.2820	0.9410

As shown in Table 1, for all censoring schemes, the performances of the Tierney-Kadane approximate Bayes estimates outdo those of both the ML estimates and the Lindley approximate Bayes estimates. For all the estimation methods, it is observed that for the same n and all censoring schemes as $\frac{m}{n} \rightarrow 1$, the estimated risk values of the ML and the approximate Bayes estimates tend to decrease. Also, in complete sample case ($n = m$), the estimated risk values of the ML and the approximate Bayes estimates are the smallest as expected. In addition, as seen from Table 2 and Table 3, when the n and m values increase, the coverage probabilities reach the desired level as expected. In different n and m values, the coverage probabilities of the ACIs and the Bootstrap confidence intervals are approximately $1 - \alpha = 0.95$.

1. Real Data Analysis

In this section, parameter estimates for the three estimation methods are obtained and then the performances of ML and Bayes estimation methods are compared using three different real datasets. We applied the goodness-of-fit of censored data for the RW distribution using approximate KS test statistics proposed by Pakyari and Balakrishnan [47]. The test statistics KS and the corresponding p-value are calculated via R software using parametric bootstrap for censored data sets.

Real Data-1: The Real data-1 set represent the remission times (in months) of a random sample of 128 bladder cancer patients [19].

Real Data-1, $n = 128$:

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46	1.76	2.02	2.02
2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31
3.36	3.36	3.48	3.52	3.57	3.64	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5
4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49	5.62	5.71	5.85	6.25
6.31	6.54	6.76	6.93	6.94	6.97	7.09	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66	7.87
7.93	8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66	10.75	11.25	11.64
11.79	11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.8	14.24	14.76	14.77	14.83	15.96	16.62	17.12
17.14	17.36	18.1	19.13	20.28	21.73	22.69	23.63	25.74	25.82	32.15	34.26	36.66	43.01	46.12	79.05

Censored Data-1 based on real data-1 are obtained according to the censoring schemes- (19*0,108).

Censored Data-1, $m = 20$:

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46
1.76	2.02	2.02	2.07	2.09	2.23	2.26						

The approximate KS and the corresponding p-value (in parentheses) for censored data-1 set are 0.4276 (1.000). These results are displayed in Table 4. Accordingly, it is seen that censored data-1 set fit the RW distribution.

Table 4. Results of the KS test for the censored data-1.

Model	ML estimates	KS	p-value
RW	Censored Data-1 $\hat{\alpha} = 0.9007$ $\hat{\beta} = 0.7275$	0.4276	1.000

Then, the following ML and approximate Bayes estimates for α and β parameters under PTR-II censoring are acquired. In Table 5, ML, BAYES_{Lindley} and BAYES_{Tierney-Kadane} estimate are given. Besides, in Table 6 bootstrap confidence intervals for α and β parameters are given (0.6771-1.3656) and (0.5046-0.9459), respectively.

Table 5. The ML and approximate Bayes estimates for α and β parameters in real data-1 set.

(n, m)	Censoring scheme	MLE		BAYES _{Lindley}		BAYES _{Tierney-Kadane}	
	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
(128, 20)	(19*0,108)	0.9007	0.7275	0.8878	0.7363	0.8880	0.7355

Table 6. The bootstrap confidence intervals for α and β parameters in real data-1.

(n, m)	Censoring scheme	α			β		
	R	Boot ML Estimate	Boot Lower Limit	Boot Upper Limit	Boot ML Estimate	Boot Lower Limit	Boot Upper Limit
(128, 20)	(19*0,108)	0.9598	0.6771	1.3656	0.7242	0.5046	0.9459

Real Data-2: The Real data-2 set represent the remission times (in days) of 51 leukaemia patients [48].

Real Data-2, $n = 51$:

24	46	57	57	64	65	82	89	90	90	111	117	128	143	148	152
166	171	186	191	197	209	223	230	239	247	254	264	269	273	284	294
304	304	332	341	393	395	487	510	516	518	518	534	608	642	697	955
1160															

Censored data-2 based on real data-2 are obtained according to the censoring schemes- (19*0,31).

Censored Data-2, $m = 20$:

24	46	57	57	64	65	82	89	90	90	111	117	128
143	148	152	166	171	186	191						

The approximate KS and the corresponding p-value (in parentheses) for censored data-2 are 0.4939 (1.000). These results are displayed in Table 7. Accordingly, it is seen that censored data-2 set fit the RW distribution.

Table 7. Results of the KS test for the censored data-2.

Model		ML estimates	KS	p-value
RW	Censored Data-2	$\hat{\alpha} = 1.2166$ $\hat{\beta} = 0.0029$	0.4939	1.000

Then, the following ML and approximate Bayes estimates for α and β parameters under PTR-II censoring are acquired. In Table 8, , ML, BAYES_{Lindley} and BAYES_{Tierney-Kadane} estimate are given. Besides, in Table 8 bootstrap confidence intervals for α and β parameters are given (0.9278-1.8922) and (0.0001-0.0124), respectively.

Table 8. The ML and approximate Bayes estimates for α and β parameters in real data-2 set.

Censoring scheme	MLE	BAYES _{Lindley}	BAYES _{Tierney-Kadane}
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(n, m)	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
(51, 20)	(19*0,31)	1.2166	0.0029	0.9139	0.0089	0.9599	0.0150

Table 9. The bootstrap confidence intervals for α and β parameters in real data-2 set

Censoring scheme		α			β		
(n, m)	R	Boot ML Estimate	Boot Lower Limit	Boot Upper Limit	Boot ML Estimate	Boot Lower Limit	Boot Upper Limit
(51, 20)	(19*0,31)	1.3196	0.9278	1.8922	0.0031	0.0001	0.0124

Real Data-3: The Real data-3 set represent survival times of 45 patients suffering from head and neck cancer treated with combined radiotherapy and chemotherapy [49].

Real Data-3, $n = 45$:

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47	68.46	78.26	74.47	81	43
84	92	94	110	112	119	127	130	133	140	146	155	159	173	179	194
195	209	249	281	319	339	432	469	519	633	725	817	1776			

Censored data-3 based on real data-3 are obtained according to the censoring schemes- (19*0,25).

Censored Data-3, $m = 20$:

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47
68.46	78.26	74.47	81	83	84	92	94	110		

The approximate KS and the corresponding p-value (in parentheses) for censored data-3 are 0.3852 (1.000). These results are displayed in Table 10. Accordingly, it is seen that censored dataset fit the RW distribution.

Table 10. Results of the KS test for the real data- 3.

Model		ML estimates	KS	p-value
RW	Censored Data-3	$\hat{\alpha} = 1.1476$ $\hat{\beta} = 0.0083$	0.3852	1.000

Then, the following ML and approximate Bayes estimates for α and β parameters under PTR-II censoring are acquired. In Table 11, ML, BAYES_{Lindley} and BAYES_{Tierney-Kadane} estimate are given. Besides, in Table 12 bootstrap confidence intervals for α and β parameters are given (0.8764-1.7795) and (0.0006-0.0287), respectively.

Table 11. The ML and approximate Bayes estimates for α and β parameters in real data-3 set.

Censoring scheme		MLE		BAYES _{Lindley}		BAYES _{Tierney-Kadane}	
(n, m)	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
(45, 20)	(19*0,25)	1.1476	0.0083	0.9062	0.0205	0.9420	0.0269

Table 12. The bootstrap confidence intervals for α and β parameters in real data-3 set

(n, m)	Censoring scheme	α			β		
	R	Boot ML Estimate	Boot Lower Limit	Boot Upper Limit	Boot ML Estimate	Boot Lower Limit	Boot Upper Limit
(45,20)	(19*0,25)	1.2359	0.8764	1.7795	0.0086	0.0006	0.0287

6. Conclusions

In this article, the MLE and approximate Bayes estimators for unknown parameters of RW distribution based on progressive type-II censored samples are evaluated. The maximum likelihood estimators of the parameters are obtained by using Newton-Raphson method. Because the Bayes estimators of the parameters cannot be obtained in explicit forms, we have obtained the approximate Bayes estimators using Lindley and Tierney-Kadane’s approximation method under squared-error loss function. We have compared the performance of the approximate Bayes estimates with the MLEs by means of Monte Carlo simulations, and it has been observed that the performances of approximate Bayes estimates are better than those of MLEs. Further, the estimated risk values of the estimates of α and β parameters obtained by using Tierney and Kadane's approximation method are lower than those obtained by using both Lindley's approximation method and MLE. It is also seen that the width of the asymptotic confidence intervals and the bootstrap confidence intervals decreases and the coverage possibilities approach to 0.95 when (n, m) values increase.

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