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The Virial Theorem for Retarded gravity

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Abstract

The general theory of relativity (GR) is symmetric under smooth coordinate transformations also known as diffeomorphisms. The general coordinate transformation group has a linear subgroup denoted the Lorentz group of symmetry which is maintained also in the weak field approximation to GR. The dominant operator in the weak field equation of GR is thus the d'Alembert (wave) operator which has a retarded potential solution. Galaxy Clusters are huge physical systems having dimensions of many hundreds of millions of light years. Thus any change at the cluster center will be noticed at the rim only hundreds of millions of years later. Those retardation effects are neglected in present day cluster modelling and in particular are neglected in virial calculations used to relate mass and velocities on the cluster. The significant differences between the predictions of Newtonian instantaneous action at a distance and observed velocities are usually explained by either assuming dark matter or by modifying the laws of gravity (MOND). In this paper we will show that taking general relativity seriously without neglecting retardation effects one can explain the velocities in a galactic cluster without postulating dark matter. It should be stressed that the current approach does not require that velocities, v are high, in fact the vast majority of cluster bodies are substantially subluminal. In other words, the ratio of $\frac{v}{c} \ll 1$. Typical velocities in galaxies are less than 1000 km/s, which makes this ratio 0.01 or smaller. However, one should consider the fact that every gravitational system even if it is made of subluminal bodies has a

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retardation distance, beyond which the retardation effect cannot be neglected. Every natural system such as stars and galaxies and even galactic clusters exchange mass with its environment. For example, the sun losses mass through the solar wind and galaxies accrete gas from the intergalactic medium. This means that all natural gravitational systems have a finite retardation distance. The question is thus quantitative, how large is the retardation distance? The change of mass of the sun is quite small and thus the retardation distance of the solar system is quite large allowing us to neglect retardation effects within the solar system. However, for galaxies and galaxy clusters the retardation distance is within the system itself and cannot be neglected.

1 Introduction

The importance of gravitational retardation for explaining galaxy rotation curves was discussed in previous papers [1, 2, 3, 4, 5, 6, 7, 8, 9] along with a detailed introduction to the subject and relevant references this will not be repeated here. Moreover, it was shown that the well known Tully-Fisher relation which relates the baryonic mass of galaxies to the fourth power of their asymptotic velocity [10] can be also derived from retarded gravity [11]. Another line of evidence concerning "dark matter" is concerned with gravitational lensing. We have shown [12, 13] that lensing "dark matter" and rotation curves "dark matter" must be the same. This is not a coincidence, as retardation dictates that this should be so.

Here we discuss galaxy clusters which are larger scale structures, the mass of which is inferred using the virial theorem according to the pioneering work of Fritz Zwicky [14, 15, 16, 17] which is discussed in some detail in [18]. This was an early work suggesting "missing matter" or "dark matter" as it became to be known afterwards.

The first work to discuss the virial theorem in the astrophysical context was due to Poincaré [19]. However, while Poincaré can be forgiven for using Newtonian gravity (and the form of the virial theorem derived from it) at 1911, four years before the introduction

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Figure 1: The Coma Cluster.

of general relativity [20] by Einstein, Zwicky cannot be excused for using obsolete physical arguments in 1933, 18 years after general relativity was published. His work led him to a large overestimate of the Coma cluster mass which is depicted in figure 1. We shall point out the correction needed to the virial theorem due to general relativity in the sections to follow. Below we shall discuss the amendments to the virial theorem as dictated by general relativity making the "dark matter" assumption redundant.

The plan of this paper is as follows: we briefly introduce retarded gravity, then we discuss the virial theorem in its Newtonian form which is followed by a discussion of the corrections to the virial theorem dictated by general relativity.

2 Beyond the Newtonian Approximation

Retarded gravity can be obtained from the weak field approximation to general relativity [5]. The metric perturbation h_{00} can be given in term of a retarded potential ϕ as follows [5, 12]:

$$\phi = -G \int \frac{\rho(\vec{x}', t - \frac{R}{c})}{R} d^3x', \quad \phi \equiv \frac{c^2}{2} h_{00}, \quad h_{00} = \frac{2}{c^2}\phi \tag{1}$$

In the above G is the gravitational universal constant, $\vec{x}$ is where the potential is measured, $\vec{x}'$ is the location of the mass element generating the potential, $\vec{R} \equiv \vec{x} - \vec{x}'$, $R \equiv |\vec{R}|$, and ρ is the mass density. The duration $\frac{R}{c}$ for galaxies may be a few tens of thousands of years, but can be considered short in comparison to the time taken for the galactic density to change significantly. Similarly for clusters of galaxies the duration $\frac{R}{c}$ for galaxies may be a few tens of millions of years, but can be considered short in comparison to the time taken for the galactic cluster density to change significantly. Thus, we can write a Taylor series for the density:

$$\rho(\vec{x}', t - \frac{R}{c}) = \sum_{n=0}^{\infty} \frac{1}{n!} \rho^{(n)}(\vec{x}', t)(-\frac{R}{c})^n, \qquad \rho^{(n)} \equiv \frac{\partial^n \rho}{\partial t^n}. \tag{2}$$

By inserting Equations (2) into Equation (1) and keeping the first three terms, we will obtain:

$$\phi = -G \int \frac{\rho(\vec{x}', t)}{R} d^3x' + \frac{G}{c} \int \rho^{(1)}(\vec{x}', t) d^3x' - \frac{G}{2c^2} \int R \rho^{(2)}(\vec{x}', t) d^3x' \tag{3}$$

The Newtonian potential is the first term:

$$\phi_N = -G \int \frac{\rho(\vec{x}', t)}{R} d^3x' \tag{4}$$

the second term does not contribute to the force affecting subluminal particles as its gradient is null and the third term is the lower order correction to the Newtonian potential:

$$\phi_r = -\frac{G}{2c^2} \int R \rho^{(2)}(\vec{x}', t) d^3x' \tag{5}$$

The geodesic equation for a any "slow" test particle moving under the above space-time metric can be approximated [5] using the force per unit mass as follows:

$$\frac{d\vec{v}}{dt} = \vec{f}, \qquad \vec{f} \equiv -\vec{\nabla}\phi \tag{6}$$

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The total force per unit mass is thus:

$$\begin{aligned}
\vec{f} &= \vec{f}_N + \vec{f}_r \\
\vec{f}_N &\equiv -\vec{\nabla}\phi_N = -G\int \frac{\rho(\vec{x}',t)}{R^2}\hat{R} d^3x', \quad \hat{R} \equiv \frac{\vec{R}}{R}, \\
\vec{f}_r &\equiv -\vec{\nabla}\phi_r = -\frac{G}{2c^2}\int \rho^{(2)}(\vec{x}',t)\hat{R} d^3x' \qquad (7)
\end{aligned}$$

Now consider a point particle which has a mass m_j and is located at $\vec{r}_j(t)$, such a particle will have a mass density of:

$$\rho_j = m_j \delta^{(3)}(\vec{x}' - \vec{r}_j(t)) \qquad (8)$$

in which $\delta^{(3)}$ is a three dimensional Dirac delta function. This particle will cause a Newtonian potential:

$$\phi_{Nj} = -G\frac{m_j}{R_j(t)}, \qquad \vec{R}_j(t) = \vec{x} - \vec{r}_j(t), \qquad R_j(t) = |\vec{R}_j(t)| \qquad (9)$$

and a retardation potential of the form:

$$\begin{aligned}
\phi_{rj} &= -\frac{Gm_j}{2c^2}\frac{\partial^2}{\partial t^2}R_j(t) = \frac{Gm_j}{2c^2}\left(\hat{R}_j \cdot \vec{a}_j - \frac{\vec{v}_j^2 - (\vec{v}_j \cdot \hat{R}_j)^2}{R_j(t)}\right), \\
\hat{R}_j &\equiv \frac{\vec{R}_j}{R_j}, \ \vec{v}_j \equiv \frac{d\vec{r}_j}{dt}, \ \vec{a}_j \equiv \frac{d\vec{v}_j}{dt}. \qquad (10)
\end{aligned}$$

Thus any point particle moving at the vicinity of particle j will be affected by the following force per unit mass:

$$\begin{aligned}
\vec{f}_j &= \vec{f}_{Nj} + \vec{f}_{rj} \\
\vec{f}_{Nj} &= -\vec{\nabla}\phi_{Nj} = -G\frac{m_j}{R_j^2}\hat{R}_j, \ \vec{f}_{rj} = -\vec{\nabla}\phi_r = \frac{Gm_j}{2R_j^2c^2}\left(R_j\vec{a}_{\perp j} + \hat{R}_j\vec{v}_{\perp j}^2 - 2(\vec{v}_j \cdot \hat{R}_j)\vec{v}_{\perp j}\right) \\
\vec{a}_{\perp j} &\equiv \vec{a}_j - (\vec{a}_j \cdot \hat{R}_j)\hat{R}_j, \qquad \vec{v}_{\perp j} \equiv \vec{v}_j - (\vec{v}_j \cdot \hat{R}_j)\hat{R}_j. \qquad (11)
\end{aligned}$$

Now consider a point particle of mass m_k which is located at $\vec{r}_k(t)$, this particle will feel the force:

$$\begin{aligned}
\vec{F}_{j,k} &= \vec{F}_{Nj,k} + \vec{F}_{rj,k} \\
\vec{F}_{Nj,k} &= -G\frac{m_j m_k}{R^2_{k,j}}\hat{R}_{k,j}, \qquad \vec{R}_{k,j} \equiv \vec{r}_k - \vec{r}_j,\ R_{k,j} \equiv |\vec{R}_{k,j}(t)|,\ \hat{R}_{k,j} \equiv \frac{\vec{R}_{k,j}}{R_{k,j}}, \\
\vec{F}_{rj,k} &= \frac{G m_j m_k}{2R^2_{k,j}c^2}\left(R_{k,j}\vec{a}_{\perp j,k} + \hat{R}_{k,j}\vec{v}^2_{\perp j,k} - 2(\vec{v}_{j,k}\cdot\hat{R}_{k,j})\vec{v}_{\perp j,k}\right) \\
\vec{a}_{\perp j,k} &\equiv \vec{a}_j - (\vec{a}_j\cdot\hat{R}_{k,j})\hat{R}_{k,j}, \qquad \vec{v}_{\perp j,k} \equiv \vec{v}_j - (\vec{v}_j\cdot\hat{R}_{k,j})\hat{R}_{k,j}.
\end{aligned} \tag{12}$$

We notice once again (see [6]) that while Newtonian forces are prominent at "short" distances the retardation forces are the most significant at large distances in which it drops as $\frac{1}{R}$ and this fact is not related to the Taylor series approximation that we have used here. Now let us consider the gravitational effect of particle k on particle j, this is easily calculated by exchanging the indices j and k in the above expression. As $R_{j,k} = R_{k,j}$ but $\hat{R}_{j,k} = -\hat{R}_{k,j}$ it follows that the Newtonian force satisfies Newton's third law: $\vec{F}_{Nk,j} = -\vec{F}_{Nj,k}$, however, since there is no simple relation between the velocity and acceleration of the particle j and k it follows that generally speaking $\vec{F}_{rk,j} \neq -\vec{F}_{rj,k}$ and thus both the retardation force and the total gravitational force do not satisfy Newton's third law. This is well known in electromagnetism and discussed in a series of papers [21, 22, 23, 24, 25, 26].

3 The Virial Theorem

Consider a system of N particles each with mass m_k and location $\vec{r}_k(t)$ we define the quantity $\bar{G}$ [27, 28]:

$$\bar{G} \equiv \sum_{k=1}^{N} \vec{p}_k \cdot \vec{r}_k, \qquad \vec{p}_k \equiv m_k \vec{v}_k. \tag{13}$$

taking the derivative of $\bar{G}$ it follows that:

$$\frac{d\bar{G}}{dt} = 2T + \sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k, \qquad T \equiv \frac{1}{2}\sum_{k=1}^{N} m_k \vec{v}_k^2. \tag{14}$$

In the above $\vec{F}_k$ is the total force acting on particle k by all the other particles:

$$\vec{F}_k \equiv \sum_{j=1}^{N} F_{j,k} \;\Rightarrow\; \sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k = \sum_{k=1}^{N}\sum_{j=1}^{N} \vec{F}_{j,k} \cdot \vec{r}_k = \sum_{k=2}^{N}\sum_{j=1}^{k-1} (\vec{F}_{j,k} \cdot \vec{r}_k + \vec{F}_{k,j} \cdot \vec{r}_j), \tag{15}$$

In which we split the sum in terms below and above this diagonal and we add them together in pairs. Now **if** $\vec{F}_{j,k}$ satisfies Newton's third laws (and this cannot be assumed in huge gravitational systems) it follows that $\vec{F}_{j,k} = -\vec{F}_{k,j}$ it follows that:

$$\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k = \sum_{k=2}^{N}\sum_{j=1}^{k-1} \vec{F}_{j,k} \cdot \vec{R}_{k,j}. \tag{16}$$

now if we assume only Newtonian forces we may write the Newtonian potential energy associated with the interaction of two particles j and k as:

$$V_{Njk} = m_k \phi_{Nj} = m_j \phi_{Nk} = -\frac{G m_j m_k}{R_{k,j}} \;\Rightarrow\; \vec{F}_{j,k} = -\vec{\nabla}_{\vec{r}_k} V_{Njk} = -\frac{dV_{Njk}}{dR_{k,j}} \hat{R}_{k,j} \tag{17}$$

Hence:

$$\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k = -\sum_{k=2}^{N}\sum_{j=1}^{k-1} \frac{dV_{Njk}}{dR_{k,j}} R_{k,j} = \sum_{k=2}^{N}\sum_{j=1}^{k-1} V_{Njk} = V_{NT} \tag{18}$$

V_{NT} is the total gravitational energy which is negative. Thus:

$$\frac{d\bar{G}}{dt} = 2T + V_{NT}. \tag{19}$$

Taking a temporal average of the above quantity and making the reasonable assumption that for a bounded system $\bar{G}$ is always finite we arrive at the result:

$$\left\langle \frac{d\bar{G}}{dt} \right\rangle \equiv \lim_{\tau->\infty} \frac{1}{\tau} \int_0^\tau \frac{d\bar{G}}{dt} = \lim_{\tau->\infty} \frac{G(\tau)-G(0)}{\tau} = 0. \Rightarrow 2\langle T \rangle = -\langle V_{NT} \rangle = \langle |V_{NT}| \rangle . \tag{20}$$

Now one may define a "typical" squared velocity:

$$\left\langle v^2 \right\rangle_t \equiv \frac{1}{M} \left\langle \sum_{k=1}^{N} m_k \vec{v}_k^2 \right\rangle = 2\frac{\langle T \rangle}{M}, \qquad M = \sum_{k=1}^{N} m_k. \tag{21}$$

M is the total mass of the system. In practice one evaluates $\langle v^2 \rangle_t$ by making an "ergodic" assumption that is that the temporal average may be replaced by a statistical average. Moreover, one may define the gravitational radius of a system by:

$$r_g = \frac{GM^2}{|V_{NT}|} \tag{22}$$

Thus we obtain using equation (20) and equation (21) the result:

$$\left\langle v^2 \right\rangle_t = \frac{\langle |V_{NT}| \rangle}{M} = \frac{GM}{r_g}, \tag{23}$$

The gravitational radius was found to be closely relate to the median radius r_h in which half the systems mass is contained such that $r_h \simeq 0.4 r_g$ [27, 29], thus we may estimate the typical velocity which is what Zwicky did. However, for large gravitational system retardation cannot be neglected which implies a modification of the analysis to be described below.

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4 The Virial Theorem for Retarded Gravity

We shall start our analysis from equation (14) but we split the gravitational force acting on particle k to Newtonian and retardation forces:

$$\frac{d\bar{G}}{dt} = 2T + \sum_{k=1}^{N} \vec{F}_{Nk} \cdot \vec{r}_k + \sum_{k=1}^{N} \vec{F}_{rk} \cdot \vec{r}_k, \tag{24}$$

the analysis of the Newtonian term is the same as before so we only need to analyze the retardation term:

$$V_{rT} = \sum_{k=1}^{N} \vec{F}_{rk} \cdot \vec{r}_k = -\sum_{k=1}^{N} m_k \vec{\nabla}_{\vec{r}_k} \phi_r \cdot \vec{r}_k, \tag{25}$$

In the above ϕ_r is the retardation potential generated by all particles which is defined in equation (5) in which the potential of a particle acting on itself is null. Also notice that the density for a system of particles is:

$$\begin{aligned} \rho(\vec{x}) &= \sum_{k=1}^{N} m_k \delta^{(3)}(\vec{x} - \vec{r}_k) \;\Rightarrow \\ V_{rT} &= \int \rho(\vec{x}) \vec{\nabla} \phi_r \cdot \vec{x} d^3x = -\frac{G}{2c^2} \int \rho(\vec{x}) \vec{\nabla} \left(\int R \rho^{(2)}(\vec{x}') d^3x' \right) \cdot \vec{x} d^3x \\ &= -\frac{G}{2c^2} \int \int d^3x d^3x' \rho(\vec{x}) \frac{\vec{R} \cdot \vec{x}}{R} \rho^{(2)}(\vec{x}') = -G \int \int d^3x d^3x' \frac{\rho(\vec{x}) \rho_d(\vec{x}')}{R}. \end{aligned} \tag{26}$$

In which we defined "dark matter density" as follows:

$$\rho_d \equiv \rho^{(2)} \frac{\vec{R} \cdot \vec{x}}{2c^2} \tag{27}$$

Thus:

$$2 \langle T \rangle = \langle |V_{NT}| \rangle - \langle V_{rT} \rangle, \tag{28}$$

If $V_{rT} < 0$ which does not imply that $\rho_d > 0$ is positive everywhere but only "mostly" positive we may define a "dark energy mass" M_d such that:

$$M_d \equiv \frac{|V_{rT}| r_g}{GM} \;\Rightarrow\; \left\langle v^2 \right\rangle_t = \frac{G(M + M_d)}{r_g} \;\Rightarrow\; M_d = \frac{r_g \langle v^2 \rangle_t}{G} - M. \tag{29}$$

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This result of course does not allude to some mysterious "dark matter" but rather tells us something on the dynamics of the galaxy cluster itself.

5 Conclusion

The general theory of relativity (GR) is symmetric under smooth coordinate transformations also known as diffeomorphisms. The general coordinate transformation group has a linear subgroup denoted the Lorentz group of symmetry which is maintained also in the weak field approximation to GR. The dominant operator in the weak field equation of GR is thus the d'Alembert (wave) operator which has a retarded potential solution. Galaxy Clusters are huge physical systems having dimensions of many hundreds of millions of light years. Thus any change at the cluster center will be noticed at the rim only hundreds of millions of years later. Those retardation effects are neglected in present day cluster modelling and in particular are neglected in virial calculations used to relate mass and velocities on the cluster. The significant differences between the predictions of Newtonian instantaneous action at a distance and observed velocities are usually explained by either assuming dark matter or by modifying the laws of gravity (MOND). In this paper we will show that taking general relativity seriously without neglecting retardation effects one can explain the velocities in a galactic cluster without postulating dark matter. It should be stressed that the current approach does not require that velocities, v are high, in fact the vast majority of cluster bodies are substantially subluminal. In other words, the ratio of $\frac{v}{c} \ll 1$. Typical velocities in galaxies are less than 1000 km/s, which makes this ratio 0.01 or smaller. However, one should consider the fact that every gravitational system even if it is made of subluminal bodies has a retardation distance, beyond which the retardation effect cannot be neglected. Every natural system such as stars and galaxies and even galactic clusters exchange mass with its environment. For example, the sun losses mass through the solar wind and galaxies accrete gas from the intergalactic medium. This means that all natural gravitational systems have a finite retardation distance. The question is thus quantitative, how large is the retardation distance? The change of mass of the sun is quite small and thus the retardation distance of the solar system is quite large allowing us to neglect retardation effects within the solar system. However, for galaxies and galaxy clusters the retardation distance is within the system itself and cannot be neglected. The retardation corrections must be taken into account in the virial theorem and will lead to considerable corrections to the virial mass calculated.

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