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On Orthogonal Fuzzy Iterative Mappings with Applications to Volterra Type Integral Equations and Fractional Differential Equations

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Abstract: In this paper, we report orthogonal fuzzy versions of some celebrated iterative mappings. We provide various concrete conditions on the real valued functions $\mathcal{J}, \mathcal{S} : (0,1] \to (-\infty, \infty)$ for the existence of fixed-points of $(\mathcal{J}, \mathcal{S})$-fuzzy iterative mappings. We obtain many fixed point theorems in orthogonal fuzzy metric spaces. We apply $(\mathcal{J}, \mathcal{S})$-fuzzy version of Banach fixed point theorem to show the existence and uniqueness of the solution. We support these results with several non-trivial examples and applications to Volterra-type integral equations and fractional differential equations.

Keywords: fixed point; fuzzy metric spaces; $(\mathcal{J}, \mathcal{S})$-fuzzy iterative mappings; fractional differential equations

1. Introduction

A self-mapping $L : \mathcal{B} \to \mathcal{B}$ is contained a fixed point if $L(\sigma) = \sigma$ for $\sigma \in \mathcal{B}$. It has a great achievement to attain a unique solution in nonlinear equations. It has increased the domain of mathematics. In 1960, Schweizer and Sklar [1] initiated the concept of continuous t-norm (in short ctn) which is a binary relation. In 1965, Zadeh [2] initiated the concept of a fuzzy set (FS) and its properties. Then in 1975, Kramosil and Michalek [3] initiated the notion of fuzzy metric space (in short, FMS) by using the concepts of ctn and FSs. In 1994, George and Veeramani [4] presented the further modified version of FMSs. After that, Grabeic [5] initiated and improved the well known Banach's fixed point theorem (FPT) in the framework of FMSs in the context of Kramosil and Michalek [3]. By following the concepts of Grabeic [5], Gregori and Sapena [6] provided an addition to Banach's contraction theorem by using FMSs.

In 1968, Kannan [7] provided a new type of contraction and proved some fixed point (in short, FP) results for discontinuous mappings. Karapinar [8] established a new type of contraction via interpolative contraction and proved some FP results on it. So, he provided a new way of research, and many authors worked on it and proved different FP results on it, see [9–14]. Hierro et al. [16] proved the FP result in FMSs. Then, Zhou et al. [15] generalized the result of Hierro et al. [16] in the framework of FMSs. Nazam et al. [17] proved some FP results in orthogonal (Ψ, Φ) complete metric spaces. Hezarjaribi [18] established several FP results in a newly introduced concept named orthogonal fuzzy metric space (in short, OFMS). For several important results and applications, see the following literature [19–24]. Uddin et al. [25] proved several fixed point results for contraction mappings in the context of orthogonal controlled FMSs. Ishtiaq et al. [26] extend the results proved in [25] in a more generalized framework named orthogonal neutrosophic metric spaces. Recently, Uddin

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et al. [27] and Saleem et al. [28] derived several fixed point results and applications in the context of intuitionistic FMSs.

Inspired by the results in [8,15–18], we aim to establish FP results in the framework of an OFMS. We divide this paper into four main parts. The first part is based on the introduction. In the second part, we will revise some basic concepts for understanding our main results. In the third part, we give some FP results in OFMS and some examples to illustrate our results. In the 4th part, we provide an application to Voltera-type integral equations and fractional differential equations.

2. Preliminaries

In this section, we provided several basic definitions and results.

Definition 1. *[15] A binary operation $* : H \times H \to H$ (where $H = [0,1]$) is called a ctn if it varifying the below axioms:*

(1) $\sigma * \theta = \theta * \sigma$ and $\sigma * (\theta * \omega) = (\sigma * \theta) * \omega$ for all $\sigma, \theta, \omega \in H$;
(2) $*$ is continuous;
(3) $\sigma * 1 = \sigma$ for all $\sigma \in H$;
(4) $\sigma * \theta \leq \omega * \varpi$ when $\sigma \leq \omega$ and $\theta \leq \varpi$, with $\sigma, \omega, \omega, \varpi \in H$.

Definition 2. *[15] A triplet $(\mathcal{B}, \vartheta, *)$ is termed as FMS if $*$ is ctn, $\mathcal{B}$ is arbitrary set, and ϑ is FS on $\mathcal{B} \times \mathcal{B} \times (0,\infty)$ fulfilling the accompanying conditions for all $\sigma, \theta, \omega \in \mathcal{B}$ and $\varsigma, \varpi > 0$.*

(i) $\vartheta(\sigma, \theta, \varsigma) > 0$;
(ii) $\vartheta(\sigma, \theta, \varsigma) = 0$ if and only if $\sigma = \theta$;
(iii) $\vartheta(\sigma, \theta, \varsigma) = \vartheta(\theta, \sigma, \varsigma)$;
(iv) $\vartheta(\sigma, \omega, \varsigma + \varpi) \geq \vartheta(\sigma, \theta, \varsigma) * \vartheta(\theta, \omega, \varpi)$;
(v) $\vartheta(\sigma, \theta, .) : (0,\infty) \to [0,1]$.

Example 1. *Let $\mathcal{B} = \mathbb{R}^+$ and $\vartheta(\sigma, \theta, \varsigma) = \frac{\varsigma}{\varsigma + L^*(\sigma,\theta)}$, consider a ctn as $m * n = mn$. Then, $\mathcal{B}$ is FMS.*

Definition 3. *[5] A mapping $L : \mathcal{B} \to \mathcal{B}$ satisfying the following inequality,*

$$\vartheta(L\sigma, L\theta, k\varsigma) \geq \vartheta(\sigma, \theta, \varsigma) \ \forall \sigma, \theta \in \mathcal{B},$$

is called a fuzzy contraction with $k \in [0,1)$.

Definition 4. *[18] Let $(\mathcal{B}, \vartheta, *)$ be a FMS and $\perp \in \mathcal{B} \times \mathcal{B}$ be a binary relation. Suppose $\exists\, \sigma_0 \in \mathcal{B}$ such that $\sigma_0 \perp \sigma$ or $\sigma \perp \sigma_0$ for all $\sigma \in \mathcal{B}$. Then we say that $\mathcal{B}$ is an OFMS. We denote OFMS by $(\mathcal{B}, \vartheta, *, \perp)$.*

Definition 5. *[18] A mapping $L : \mathcal{B} \to \mathcal{B}$ verifying the below inequality,*

$$\vartheta(L\sigma, L\theta, k\varsigma) \geq \vartheta(\sigma, \theta, \varsigma) \ \forall \sigma, \theta \in \mathcal{B}, \text{ with } \sigma \perp \theta,$$

is called an orthogonal fuzzy contraction(in short, OFC) where $(\mathcal{B}, \vartheta, *, \perp)$ is an OFMS, and $k \in [0,1)$.

Theorem 1. *[18] Assume that $(\mathcal{B}, \vartheta, *, \perp)$ is an OFMS. Let a mapping $L : \mathcal{B} \to \mathcal{B}$ be a continuous $\perp$-preserving. Thus L has a unique FP $u \in \mathcal{B}$. Furthermore,*

$$\lim_{n\to\infty} \vartheta(L^n\sigma, L\theta, \varsigma) = 1,$$

for all $u \in \mathcal{B}$.

Remark 1. *The fuzzy contraction is an orthogonal fuzzy contraction but the converse may not be held in general.*

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Example 2. *Suppose $\mathcal{B} = [0, 10)$ with FMS ϑ as defined as in Example 1, then the $(\mathcal{B}, \vartheta, *, \perp)$ represents a FMS. Define $\perp \subseteq \mathcal{B}^2$ by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \sigma \vee \theta.$$

Then $(\mathcal{B}, \vartheta, *, \perp)$ is an OFMS with ctn $\sigma * \theta = \sigma\theta$. Let the mapping $L : \mathcal{B} \to \mathcal{B}$ is given by

$$L(\sigma) = \left\{ \begin{array}{l} \frac{\sigma}{3} \text{ for } \sigma \leq 3 \\ 0 \text{ for } \sigma > 3 \end{array} \right\}.$$

We note that

$$\begin{array}{rcl} \vartheta(L(4), L(3), (0.4)1) & \geq & \vartheta(4, 3, 1) \\ \vartheta(0, 1, (0.4)1) & \geq & \vartheta(4, 3, 1) \\ 0.2857 & \geq & 0.5. \end{array}$$

This is a contradiction, so, L is not a fuzzy contraction. However, L is an orthogonal fuzzy contraction.

Lemma 1. *Let $(\mathcal{B}, \vartheta, *)$ be a FMS and $\{a_n\} \subset \mathcal{B}$ be a sequence satisfying $\lim_{n\to\infty} \vartheta(a_n, a_{n+1}, \varsigma) = 1$. If the sequence $\{a_n\}$ is not Cauchy, then there are $\{a_{nk}\}$,$\{a_{mk}\}$ and $\varepsilon \geq 0$ such that*

$$\lim_{k\to\infty} \vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) = (1+\varepsilon). \tag{2.1}$$

$$\lim_{k\to\infty} \vartheta\left(a_{n_K}, a_{m_K}, \varsigma\right) = \lim_{k\to\infty} \vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right) = \lim_{k\to\infty} \vartheta\left(a_{n_K}, a_{m_K+1}, \varsigma\right) = 1+\varepsilon. \tag{2.2}$$

Proof. Let $(\mathcal{B}, \vartheta, *)$ be a FMS. Given $\{a_n\}$ is not Cauchy and $\lim_{n\to\infty} \vartheta(a_n, a_{n+1}, \varsigma) = 1$. Thus, for every $\varepsilon > 0$. There exists a natural number k_0 such that for smallest $m \geq n$ we have

$$\vartheta(a_{n+1}, a_m, \varsigma) \geq 1+\varepsilon \text{ and } \vartheta(a_{n+1}, a_m, \varsigma) < 1+\varepsilon \ \forall n, m \geq k_0.$$

As a result, we construct two subsequences of $\{a_n\}$; $\{a_{nk}\}$ and $\{a_{mk}\}$ verifying the following inequalities

$$\vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right) \geq 1+\varepsilon \text{ and } \vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) < 1+\varepsilon \ \forall n_k, m_k > k_0.$$

By axiom (iv) of the FMS, we have the following information:

$$\begin{array}{rcl} 1+\varepsilon & > & \vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) \\ & \geq & \vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right).\vartheta\left(a_{m_K}, a_{m_K+1}, \varsigma\right) \\ & \geq & (1+\varepsilon).\vartheta\left(a_{m_K}, a_{m_K+1}, \varsigma\right). \end{array}$$

This implies that,

$$\lim_{k\to\infty} \vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) = (1+\varepsilon).$$

Again by utilizing axiom (iv) of the FMS, we have

$$\frac{\vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right)}{\vartheta\left(a_{m_K}, a_{m_K+1}, \varsigma\right)} \geq \vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right) \geq 1+\varepsilon.$$

We get

$$\lim_{k\to\infty} \vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right) = 1+\varepsilon.$$

Since,

$$\vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right) \geq \vartheta\left(a_{n_K+1}, a_{n_K}, \varsigma\right).\vartheta\left(a_{n_K}, a_{m_K}, \varsigma\right),$$

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we have the following inequality:

$$\begin{aligned} \frac{\vartheta\left(a_{n_K+1}, a_{m_K}, \varsigma\right)}{\vartheta\left(a_{n_K+1}, a_{n_K}, \varsigma\right)} &\geq \vartheta\left(a_{n_K}, a_{m_K}, \varsigma\right) \\ &\geq \vartheta\left(a_{m_K}, a_{n_K+1}, \varsigma\right).\vartheta\left(a_{n_K+1}, a_{n_K}, \varsigma\right). \end{aligned}$$

That is

$$\lim_{k\to\infty} \vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) = 1+\varepsilon.$$

Since,

$$1+\varepsilon > \vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) \geq \vartheta\left(a_{n_K+1}, a_{n_K}, \varsigma\right).\vartheta\left(a_{n_K}, a_{m_K+1}, \varsigma\right)$$

$$\begin{aligned} \frac{1+\varepsilon}{\vartheta\left(a_{n_K+1}, a_{n_K}, \varsigma\right)} &\geq \vartheta\left(a_{n_K}, a_{m_K+1}, \varsigma\right) \\ &\geq \vartheta\left(a_{n_K}, a_{n_K+1}, \varsigma\right).\vartheta\left(a_{n_K+1}, a_{m_K+1}, \varsigma\right) \end{aligned}$$

That is

$$\vartheta\left(a_{n_K}, a_{m_K}, \varsigma\right) = (1+\varepsilon).$$

This completes the proof. □

Definition 6. *[18] The OFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying the property (R) is called $\perp$- regular.*

(R) For any O- sequence $\{\sigma_n\} \subseteq \mathcal{B}$ converging to σ, we have either $\sigma \perp \sigma_n$, or $\sigma_n \perp \sigma$ for all $n \in \mathbb{N}$.

3. Main Results

3.1. Banach Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

In this section, we present the new results for orthogonal fuzzy interpolative contractions (OFIPC) involving the functions $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$.

Definition 7. *Let $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ will be called a Banach type $(\mathcal{J}, \mathcal{S})$-OFIPC, if there exists $v \in (0,1]$ verifying*

$$\mathcal{J}\left(\vartheta\left(L\sigma, L\theta, \varsigma\right)\right) \geq \mathcal{S}\left(\vartheta\left(\sigma, \theta, \varsigma\right)\right)^{v}, \tag{3.3}$$

for all $(\sigma, \theta) \in \mathcal{B}$, $\vartheta\left(L\sigma, L\theta, \varsigma\right) > 0$.

Example 3. *Let $\mathcal{B}$ =[1,7) and define the FMS $\vartheta(\sigma, \theta, \varsigma) = e^{-\frac{|\sigma-\theta|}{\varsigma}}$, Let $\perp \subset \mathcal{B}^2$ defined by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \{\sigma, \theta\}.$$

Then $(\mathcal{B}, \vartheta, *, \perp)$ is OFMS with $m * n = mn$. Define $L : \mathcal{B} \to \mathcal{B}$ by

$$L(\sigma) = \left\{ \begin{array}{c} 5 \text{ if } 1 \leq \sigma < 2, \\ 3.1 \text{ if } 2 \leq \sigma < 3, \\ 1.8 \text{ if } 3 \leq \sigma < 7. \end{array} \right\}$$

Define $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{ and } \mathcal{S} = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

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Case 1: Here, L is a Banach type $(\mathcal{J}, \mathcal{S})$-OFIPC. But,

$$\begin{aligned}
(\vartheta(L1, L2, k1)) &\geq (\vartheta(1,2,1))^{\frac{1}{2}} \\
\vartheta\left(5, 3.1, \left(\frac{1}{2}\right)1\right) &\geq (\vartheta(1,2,1))^{\frac{1}{2}} \\
e^{-\frac{|5-3.1|}{0.5}} &\geq \left(e^{-\frac{|1-2|}{1}}\right)^{\frac{1}{2}} \\
0.0224 &\geq 0.6065.
\end{aligned}$$

This is a contradiction. Hence, L is not Banach-type FIPC.

Case 2: Here, L is a Banach type $(\mathcal{J}, \mathcal{S})$-OFIPC. But,

$$\begin{aligned}
\vartheta(L1, L3, 1) &\geq (\vartheta(1,3,1))^{\frac{1}{2}} \\
\vartheta\left(5, 1.8, \frac{1}{2}1\right) &\geq (\vartheta(1,3,1))^{\frac{1}{2}} \\
e^{-\frac{|5-1.8|}{0.5}} &\geq \left(e^{-\frac{|1-3|}{1}}\right)^{\frac{1}{2}} \\
0.0017 &\geq 0.3678.
\end{aligned}$$

This is a contradiction. Hence, L is not Banach type OFIPC.

Case 3: Here, L is a Banach type $(\mathcal{J}, \mathcal{S})$-OFIPC. But,

$$\begin{aligned}
\vartheta(L1, L4, k1) &\geq (\vartheta(1,4,1))^{\frac{1}{2}} \\
\vartheta\left(5, 1.8, \frac{1}{2}1\right) &\geq (\vartheta(1,4,1))^{\frac{1}{2}} \\
e^{-\frac{|5-1.8|}{0.5}} &\geq \left(e^{-\frac{|1-4|}{1}}\right)^{\frac{1}{2}} \\
0.0017 &\geq 0.2231.
\end{aligned}$$

This is a contradiction. Hence, L is not Banach type OFIPC.

Hence in general, let $\sigma, \theta \in \mathcal{B}$ such that $\sigma \perp \theta$ or $\sigma \perp \theta$

$$\begin{aligned}
\mathcal{J}(\vartheta(L\sigma, L\theta, \varsigma)) &= -\frac{\varsigma}{\mid L\sigma - L\theta \mid} = -\frac{\varsigma}{L \mid \sigma - \theta \mid} \\
&\geq -\frac{\varsigma}{\mid \sigma - \theta \mid} = \mathcal{S}(\vartheta(\sigma, \theta, \varsigma))^{\frac{1}{2}}
\end{aligned}$$

Therefore, the Banach contraction is fulfilled.

For $\perp$ (orthogonal relation), two functions $(\mathcal{J}, \mathcal{S}) : (0,1] \to \mathbb{R}$, and self-mapping L, we write the below properties:

(i) for every $\sigma_0 \in \mathcal{B}$, there is $\sigma_1 = L(\sigma_0)$ such that $\sigma_1 \perp \sigma_0$ or $\sigma_0 \perp \sigma_1$;
(ii) $\mathcal{J}$ is non-decreasing and for every $1 > r \geq t > 0$, one has $\mathcal{S}(r) > \mathcal{J}(t)$;
(iii) $\lim_{s \to L^-} \inf \mathcal{S}(s) > \lim_{s \to L^-} \sup(\mathcal{J}(s))$;
(iv) if $t \in (0,1]$ such that $\mathcal{J}(t) \geq \mathcal{S}(1)$;
(v) $\sup_{\sigma_a > \varepsilon} \mathcal{J}(\sigma_a) > -\infty$;
(vi) $\lim_{\sigma_a \to \delta} \inf \mathcal{S}(\sigma_a) > \mathcal{J}(\delta)\ \forall \delta \in (0,1)$;
(vii) if $\{\mathcal{J}(y_n)\}$ and $\{\mathcal{S}(y_n)\}$ are converging to same limit and $\{\mathcal{J}(y_n)\}$ is strictly increasing, then $\lim_{n \to \infty} y_n = 1$;
(viii) $\mathcal{J}\left(\sigma_a^{v}\sigma_b^{\eta}\right) \geq \mathcal{J}(\sigma_a)$ and $\mathcal{S}(\sigma_a) > \mathcal{J}(\sigma_a)\ \forall \sigma_a \in (0,1)$.

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The next two theorem deals with Banach type $(\mathcal{J},\mathcal{S})$-OFIPC.

Theorem 2. *Suppose $\perp$ be a transitive orthogonal relation (in short, TOR), then, each $\perp$- preserving self-mapping (in short, PSM) on a $\perp$- regular OCFMS $(\mathcal{B},\vartheta,*,\perp)$ satisfying (3.3) and (i)-(iv), have a FP in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0 \in \mathcal{B}$ s.t. $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$, then by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ s.t $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$ then σ_n is FP of L $\forall\, n \geq 0$. We let that $\sigma_n \neq \sigma_{n+1}$ $\forall\, n \in \mathbb{N}\cup\{0\}$. Let $y_n = \vartheta(\sigma_n,\sigma_{n+1},\varsigma)$ $\forall\, n \geq 0$. By the first part of (ii) and (3.3), we have

$$\mathcal{J}(y_n) \geq \mathcal{J}(\vartheta(L\sigma_{n-1},L\sigma_n,\varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma_{n-1},\sigma_n,\varsigma))^v\right).$$

By utilizing (ii), we have

$$\mathcal{J}(y_n) \geq \mathcal{S}\left((y_{n-1})^v\right) > \mathcal{J}\left((y_{n-1})^v\right). \tag{3.4}$$

Since, $\mathcal{J}$ is non decreasing, one gets $y_n > y_{n-1}$for every $n \geq 1$, we have $L < 1$, that is $\lim_{n\to\infty} y_n = L+$. If $L < 1$, by (3.4), we get the following information:

$$\mathcal{J}(L+) = \lim_{n\to\infty}\mathcal{J}(y_n) \geq \lim_{n\to\infty}\inf\mathcal{S}\left((y_{n-1})^v\right) \geq \lim_{\sigma_a\to L+}\inf\mathcal{S}(\sigma_a).$$

So this contradicts (iii), so $L = 1$.

The sequence $\{\sigma_n\}$is Cauchy: Let $\{\sigma_n\}$ is not OCS, so that the following lemma 1, there exist two subsequences $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ of $\{\sigma_n\}$ and $\varepsilon > 0$ such that (2.1) and (2.2) satisfied. From (2.1), we deduce

$$\vartheta\left(\sigma_{n_K+1},\sigma_{m_K+1},\varsigma\right) > (1+\varepsilon).$$

Since, $\sigma_n \perp \sigma_{n+1}$ $\forall\, n \geq 0$, so by transitive of $\perp$, we have $\sigma_{n_k} \perp \sigma_{m_k}$ $\forall\, k \geq 1$,

$$\mathcal{J}\left(\vartheta\left(\sigma_{n_K+1},\sigma_{m_K+1},\varsigma\right)\right) \geq \mathcal{J}\left(\vartheta\left(L\sigma_{n_K},L\sigma_{m_K},\varsigma\right)\right) \geq \mathcal{S}\left(\left(\vartheta\left(\sigma_{n_K},\sigma_{m_K},\varsigma\right)\right)^v\right)$$

If $\sigma_{a_k} = \vartheta\left(\sigma_{n_K+1},\sigma_{m_K+1},\varsigma\right)$, $\sigma_{b_k} = \vartheta\left(\sigma_{n_K},\sigma_{m_K},\varsigma\right)$, we have

$$\mathcal{J}\left(\sigma_{a_k}\right) \geq \mathcal{S}\left(\left(\sigma_{b_k}\right)^v\right), \text{ for all } k \geq 1. \tag{3.5}$$

By (2.1), we have $\lim_{k\to\infty}\sigma_{a_k} = (1+\varepsilon)\times$ and (3.5) implies

$$\lim_{\sigma_a\to(1+\varepsilon)}\sup\mathcal{J}\left(\sigma_{a_k}\right) \geq \lim_{k\to\infty}\sup\mathcal{J}\left(\sigma_{a_k}\right) \geq \lim_{k\to\infty}\inf\mathcal{S}\left(\left(\sigma_{b_k}\right)^v\right) \geq \lim_{\sigma_a\to 0}\inf\mathcal{S}(\sigma_a). \tag{3.6}$$

The information obtained in (3.6), contradicts the assumption (iii) and thus stamping the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B},\vartheta,*,\perp)$ hence there is $\sigma_a \in \mathcal{B}$ so that $\sigma_n \to \sigma_a$ as $n \to \infty$. Since, $(\mathcal{B},\vartheta,*,\perp)$ is a $\perp$-regular space, so, we write $\sigma_a \perp \sigma_n$ or $\sigma_n \perp \sigma_a$. We claim that $\vartheta(\sigma_a,L\sigma_a,\varsigma) = 1$. If $\vartheta(\sigma_{n+1},L\sigma_a,\varsigma) > 1$, then we have (3.3)

$$\begin{aligned}\mathcal{J}\left(\vartheta\left(\sigma_{n+1},L\sigma_a,\varsigma\right)\right) &\geq \mathcal{J}\left(\vartheta\left(L\sigma_n,L\sigma_a,\varsigma\right)\right) \geq \mathcal{S}\left(\left(\vartheta\left(\sigma_n,\sigma_a,\varsigma\right)\right)^v\right)\\ &> \mathcal{J}\left(\left(\vartheta\left(\sigma_n,\sigma_a,\varsigma\right)\right)^v\right).\end{aligned}$$

By the first part of (ii), we get

$$\vartheta\left(\sigma_{n+1},L\sigma_a,\varsigma^k\right) > \left(\vartheta\left(\sigma_n,\sigma_a,\varsigma\right)\right)^v.$$

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Applying limit $n \to \infty$, we obtain $\vartheta(\sigma_a, L\sigma_a, \varsigma) \geq 1$. This implies that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. Hence, $\sigma_{\sigma_a} = L\sigma_a$. □

Theorem 3. *Let $\perp$ be a TOR, then, every $\perp$- PSM defined on a $\perp$- regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (3.3) and (i), (iii), (v)-(viii), admits a fixed point in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0 \in \mathcal{B}$ s.t. $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for each $\sigma_1 \in \mathcal{B}$, then by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ s.t $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$ then σ_n is FP of L $\forall\, n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ $\forall\, n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ $\forall\, n \geq 0$. By the first part of (ii) and (3.3), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\sigma_n, \sigma_{n+1}, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v\right) \\ &> \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v \\ &\geq \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)). \end{aligned} \tag{3.7}$$

The inequality shows that (3.7) shows that $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is strictly increasing. If it is not bounded above, then from (v), we obtain $\sup_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) > \varepsilon} \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty$. This implies that

$$\lim_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) \to \varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty.$$

Thus, $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$, otherwise, we have

$$\lim_{\vartheta(\sigma_{n-1}, \sigma_n, \varsigma) \to \varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) = -\infty$$

(i.e., a contradiction (v)). If it is bounded above, then $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is a convergent sequence and by (3.7), $\{\mathcal{S}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ also converges to the same limit point. By using (iii), we have $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$. Hence, L is asymptotically regular (in short, AR).

Now, we assert that $\{\sigma_n\}$ is CS, So by Lemma 1 $\exists$ $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ and $\varepsilon > 0$ such that (2.1) and (2.2), we deduce $\vartheta\left(\sigma_{n_K+1}, \sigma_{m_K+1}, \varsigma\right) > (1+\varepsilon)$. Since $\sigma_n \perp \sigma_{n+1}$ $\forall\, n \geq 0$ so by transitivity of $\perp$, we obtain $\sigma_{n_k} \perp \sigma_{m_k}$. Letting $g = \sigma_{n_k}$ and $e = \sigma_{m_k}$ in (3.3), one writes for all $k \geq 1$,

$$\mathcal{J}\left(\vartheta\left(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma\right)\right) \geq \mathcal{J}\left(\vartheta\left(L\sigma_{n_k}, L\sigma_{m_k}, \varsigma\right)\right) \geq \mathcal{S}\left(\left(\vartheta\left(\sigma_{n_k}, \sigma_{m_k}, \varsigma\right)\right)^v\right).$$

If $\sigma_k = \vartheta\left(\sigma_{n_K+1}, \sigma_{m_K+1}, \varsigma\right)$, $\sigma_{bk} = \vartheta\left(\sigma_{n_K}, \sigma_{m_K}, \varsigma\right)$, we have

$$\mathcal{J}(\sigma_k) \geq \mathcal{S}\left((\sigma_{bk})^v\right), \text{ for all } k \geq 1. \tag{3.8}$$

By (2.1), we have $\lim_{k\to\infty} \sigma_k = (1+\varepsilon)$ and (3.8) implies

$$\lim_{\sigma_a \to (1+\varepsilon)} \sup \mathcal{J}(\sigma_a) \geq \lim_{k\to\infty} \sup \mathcal{J}(\sigma_k) \geq \lim_{k\to\infty} \inf \mathcal{S}\left((\sigma_{bk})^v\right) \geq \lim_{\sigma_a \to 0} \inf \mathcal{S}(\sigma_a). \tag{3.9}$$

The information obtained in (3.9), contradicts the assumption (viii) and thus stamping the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B}, \vartheta, *, \perp)$. The completeness of the space ensures the convergence of $\{\sigma_n\}$, let it converges to $i \in \mathcal{B}$.

Case 1. if $\vartheta(\sigma_{n+1}, Li, \varsigma) = 1$ for some $n \geq 0$, Then

$$\vartheta(i, Li, \varsigma) \geq \vartheta(i, \sigma_{n+1}, \varsigma) . \vartheta(\sigma_{n+1}, Li, \varsigma)$$

taking limit $n \to \infty$ on both sides, we have $\vartheta(i, Li, \varsigma) \geq 1$. This implies that $\vartheta(i, Li, \varsigma) = 1$. Hence, $i = Li$.

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Case 2. for all $n \geq 0$, $\vartheta(\sigma_{n+1}, Li, \varsigma) < 1$, then by $\perp$- regularity of $\mathcal{B}$, we find $\sigma_n \perp i$ or $i \perp \sigma_n$. By (3.3), one writes

$$\mathcal{J}(\vartheta(\sigma_{n+1}, Li, \varsigma)) \geq \mathcal{J}(\vartheta(L\sigma_n, Li, \varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma_n, i, \varsigma))^v\right) \text{ for all } n \geq 0.$$

By taking $\sigma_n = \vartheta(\sigma_{n+1}, Li, \varsigma)$ and $b_n = \vartheta(\sigma_n, i, \varsigma)$, one writes

$$\mathcal{J}(\sigma_n) \geq \mathcal{S}\left((b_n)^v\right) \text{ for all } n \geq 0. \tag{3.10}$$

Note that $\sigma_n \to \delta$ and $b_n \to 1$ as $n \to \infty$. Applying limits on (3.10), we have

$$\limsup_{i\to\delta} \mathcal{J}(i) \geq \limsup_{n\to\infty} \mathcal{J}(\sigma_n) \geq \liminf_{n\to\infty} \mathcal{S}\left((b_n)^v\right) \geq \limsup_{i\to 0} \mathcal{S}(i).$$

This contradicts (v) if $\delta > 1$. Thus, we have $\vartheta(i, Li, \varsigma) = 1$, that is i is a FP of L.

□

Example 4. *Let $\mathcal{B} = \{1,2,3,4,5,6,7,8\}$ and define the FMS $\vartheta(\sigma,\theta,\varsigma) = \frac{\varsigma}{\varsigma+|\sigma-\theta|}$. Let $\perp \subset \mathcal{B}^2$ defined by*

$$\sigma \perp \theta \text{ if } \sigma\theta \leq \sigma \vee \theta \text{ for } \sigma \neq \theta.$$

Then $(\mathcal{B}, \vartheta, *, \perp)$ is OFMS with $\sigma * \theta = \sigma\theta$. Define $L : \mathcal{B} \to \mathcal{B}$ by

$$L(\sigma) = \left\{ \begin{array}{c} 5 \text{ if } \sigma = 5 \\ \sigma - 1 \text{ otherwise} \end{array} \right\}.$$

Define $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{ and } \mathcal{S} = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

Case 1: Here, L is a Banach type $(\mathcal{J}, \mathcal{S})$-OFIPC. But,

$$\begin{array}{rcl} \vartheta(L2, L1, 1) & \geq & (\vartheta(2,1,1))^{\frac{1}{2}} \\ \vartheta(1,5,1) & \geq & (\vartheta(2,1,1))^{\frac{1}{2}} \\ \left(\frac{1}{1+|1-5|}\right) & \geq & \left(\frac{1}{1+|2-1|}\right)^{\frac{1}{2}} \\ 0.2 & \geq & 0.7071. \end{array}$$

Which is a contradiction. Hence, L is not Banach-type FIPC.

Case 2: Here, L is a Banach type $(\mathcal{J}, \mathcal{S})$-OFIPC. But,

$$\begin{array}{rcl} \vartheta(L3, L1, 1) & \geq & (\vartheta(3,1,1))^{\frac{1}{2}} \\ \vartheta(2,5,1) & \geq & (\vartheta(3,1,1))^{\frac{1}{2}} \\ \left(\frac{1}{1+|2-5|}\right) & \geq & \left(\frac{1}{1+|3-1|}\right)^{\frac{1}{2}} \\ 0.25 & \geq & 0.57771. \end{array}$$

Which is a contradiction. Hence, L is not Banach-type FIPC.

Since, condition of Theorem 2 (ii) is hold because for every $t \in (0,1)$ $\mathcal{S}(t) > \mathcal{J}(t)$ also all the remaining conditions of Theorem 2 are hold.

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3.2. Kannan Type $(\mathcal{J},\mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 8. *Let $\mathcal{J},\mathcal{S}:(0,1]\to\mathbb{R}$ be two functions. A mapping $L:\mathcal{B}\to\mathcal{B}$ defined on OFMS $(\mathcal{B},\vartheta,*,\perp)$ will be called a Kannan type $(\mathcal{J},\mathcal{S})$-OFIPC, if there exists $v\in(0,1)$ verifying*

$$\mathcal{J}\left(\vartheta\left(L\sigma,L\theta,\varsigma\right)\right)\geq\mathcal{S}\left(\left(\vartheta\left(\sigma,L\sigma,\varsigma\right)\right)^{v}\left(\vartheta\left(\theta,L\theta,\varsigma\right)\right)^{1-v}\right). \tag{3.11}$$

for all $(\sigma,\theta)\in\mathcal{B}$, $\vartheta\left(L\sigma,L\theta,\varsigma\right)>0$.

Theorem 4. *Let $\perp$ be a TOR, then, every $\perp$- PSM defined on a $\perp$- regular OCFMMS $(\mathcal{B},\vartheta,*,\perp)$ satisfying (3.11) and (i)-(iv), have a fixed point in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0\in\mathcal{B}$ s.t. $\sigma_0\perp\sigma_1$ or $\sigma_1\perp\sigma_0$ for every $\sigma_1\in\mathcal{B}$, then by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ such that $\sigma_n=L\left(\sigma_{n-1}\right)=L^n\left(\sigma_0\right)$ and $\sigma_{n-1}\perp\sigma_n$ for every $n\in\mathbb{N}$. Observe that, if $\sigma_n=L\left(\sigma_n\right)$ then σ_n is FP of L $\forall\ n\geq 0$. Let $\sigma_n\neq\sigma_{n+1}$ $\forall\ n\in\mathbb{N}\cup\{0\}$. Let $y_n=\vartheta\left(\sigma_n,\sigma_{n+1},\varsigma\right)$ $\forall\ n\geq 0$. By the first part of (ii) and (3.11), we have

$$\mathcal{J}\left(y_n\right)\geq\mathcal{J}\left(\vartheta\left(L\sigma_{n-1},L\sigma_n,\varsigma\right)\right)\geq\mathcal{S}\left(\left(\vartheta\left(\sigma_{n-1},L\sigma_{n-1},\varsigma\right)\right)^{v}\left(\vartheta\left(\sigma_n,L\sigma_n,\varsigma\right)\right)^{1-v}\right).$$

By utilizing (ii), we have

$$\mathcal{J}\left(y_n\right)\geq\mathcal{S}\left(\left(y_{n-1}\right)^{v}\left(y_n\right)^{1-v}\right)>\mathcal{J}\left(\left(y_{n-1}\right)^{v}\left(y_n\right)^{1-v}\right). \tag{3.12}$$

Since, $\mathcal{J}$ is non decreasing, one gets $y_n>y_{n-1}$for every $n\geq 1$, we have $L<1$, that is $\lim_{n\to\infty}y_n=L+$. If $L<1$, by (3.12), we obtain the following information:

$$\mathcal{J}\left(L+\right)=\lim_{n\to\infty}\mathcal{J}\left(y_n\right)\geq\lim_{n\to\infty}\inf\mathcal{S}\left(\left(y_{n-1}\right)^{v}\left(y_n\right)^{1-v}\right)\geq\lim_{\sigma_a\to L+}\inf\mathcal{S}\left(\sigma_a\right).$$

So this contradicts (iii), hence $L=1$.

The sequence $\{\sigma_n\}$is Cauchy: Assume that $\{\sigma_n\}$ is not CS, so that the following lemma 1, there exist two subsequences $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ of $\{\sigma_n\}$ and $\varepsilon>0$ such that (2.1) and (2.2) satisfied. From (2.1), we deduce

$$\vartheta\left(\sigma_{n_K+1},\sigma_{m_K+1},\varsigma\right)>(1+\varepsilon).$$

Since, $\sigma_n\perp\sigma_{n+1}$ $\forall\ n\geq 0$, so by transitive of $\perp$, we get $\sigma_{n_k}\perp\sigma_{m_k}$ $\forall\ k\geq 1$,

$$\begin{aligned}\mathcal{J}\left(\vartheta\left(\sigma_{n_K+1},\sigma_{m_K+1},\varsigma\right)\right)&\geq\mathcal{J}\left(\vartheta\left(L\sigma_{n_K},L\sigma_{m_K},\varsigma\right)\right)\geq\mathcal{S}\left(\left(\vartheta\left(\sigma_{n_K},L\sigma_{n_K},\varsigma\right)\right)^{v}\left(\vartheta\left(\sigma_{m_K},L\sigma_{m_K},\varsigma\right)\right)^{1-v}\right)\\&\geq\mathcal{S}\left(\left(\vartheta\left(\sigma_{n_K},\sigma_{n_K+1},\varsigma\right)\right)^{v}\left(\vartheta\left(\sigma_{m_K},\sigma_{m_{K+1}},\varsigma\right)\right)^{1-v}\right).\end{aligned}$$

If $\sigma_k=\vartheta\left(\sigma_{n_K+1},\sigma_{m_K+1},\varsigma\right)$, $\sigma_{b_k}=\vartheta\left(\sigma_{n_K},\sigma_{n_K+1},\varsigma\right)$, $\sigma_{c_k}=\vartheta\left(\sigma_{m_K},\sigma_{m_K+1},\varsigma\right)$, we have

$$\mathcal{J}\left(\sigma_k\right)\geq\mathcal{S}\left(\left(\sigma_{b_k}\right)^{v}\left(\sigma_{c_k}\right)^{1-v}\right),\forall\ k\geq 1. \tag{3.13}$$

By (2.1), we have $\lim_{k\to\infty}\sigma_k=(1+\varepsilon)$ and (3.13) implies

$$\lim_{\sigma_k\to(1+\varepsilon)}\sup\mathcal{J}\left(\sigma_k\right)\geq\lim_{k\to\infty}\sup\mathcal{J}\left(\sigma_k\right)\geq\lim_{k\to\infty}\inf\mathcal{S}\left(\left(\sigma_{b_k}\right)^{v}\left(\sigma_{c_k}\right)^{1-v}\right)\geq\lim_{\sigma_k\to 0}\inf\mathcal{S}\left(\sigma_k\right). \tag{3.14}$$

The information obtained in (3.14), contradicts the assumption (iii) and thus stamping the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B},\vartheta,*,\perp)$ hence, there is $\sigma_{\sigma_a}\in\mathcal{B}$ so that $\sigma_n\to\sigma_a$ as $n\to\infty$. Since,

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$(\mathcal{B},\vartheta,*,\perp)$ is a $\perp$-regular space, so, we write $\sigma_a \perp \sigma_n$ or $\sigma_n \perp \sigma_a$. We claim that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. If $\vartheta(\sigma_{n+1}, L\sigma_a, \varsigma) > 1$, then (3.11)

$$\begin{aligned}\mathcal{J}(\vartheta(\sigma_{n+1}, L\sigma_a, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n, L\sigma_a, \varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma_n, L\sigma_n, \varsigma))^v (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}\right) \\ &> \mathcal{J}\left((\vartheta(\sigma_n, L\sigma_n, \varsigma))^v (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}\right) \\ &\geq \mathcal{J}\left((\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^v (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}\right).\end{aligned}$$

By the first part of (ii), we get

$$\vartheta(\sigma_{n+1}, L\sigma_a, k\varsigma) > (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^v (\vartheta(\sigma_a, L\sigma_a, \varsigma))^{1-v}.$$

Applying limit $n \to \infty$, we obtain $\vartheta(\sigma_a, L\sigma_a, k\varsigma) \geq 1$. This implies that $\vartheta(\sigma_a, L\sigma_a, \varsigma) = 1$. Hence, $\sigma_a = L\sigma_a$. □

Theorem 5. *Let $\perp$ be a TOR, then, every $\perp$-PSM defined on a $\perp$- regular OCFMS $(\mathcal{B},\vartheta,*,\perp)$ satisfying (3.11) and (i), (iii), (v)-(viii), have a fixed point in $\mathcal{B}$.*

Proof. Choose an initial guess $\sigma_0 \in \mathcal{B}$ such that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$, then by using the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ s.t $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every $n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$ then σ_n is FP of L $\forall\, n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ $\forall\, n \in \mathbb{N}\cup\{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ $\forall\, n \geq 0$. By the first part of (ii) and (3.11), we have

$$\begin{aligned}\mathcal{J}(\vartheta(\sigma_n, \sigma_{n+1}, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma_{n-1}, L\sigma_{n-1}, \varsigma))^v (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{1-v}\right) \\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v}\right) \\ &> \mathcal{J}\left((\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^v (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{1-v}\right) \\ &\geq \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)).\end{aligned} \tag{3.15}$$

The inequality shows that (3.15) shows that $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is strictly increasing. If it is not bounded above, by (v), we obtain $\sup_{\vartheta(\sigma_{n-1},\sigma_n,\varsigma)>\varepsilon} \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty$. This implies that

$$\lim_{\vartheta(\sigma_{n-1},\sigma_n,\varsigma)\to\varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) > -\infty.$$

Thus, $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$, otherwise, we have

$$\lim_{\vartheta(\sigma_{n-1},\sigma_n,\varsigma)\to\varepsilon+} \sup \mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma)) = -\infty$$

(i.e., a contradiction (v)). If it is bounded above, then $\{\mathcal{J}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ is a CS and by (3.15), $\{\mathcal{S}(\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))\}$ also converges to the same limit point. Thus, by (iii), we obtain $\lim_{n\to\infty} \vartheta(\sigma_{n-1}, \sigma_n, \varsigma) = 1$. Hence, L is AR.

Now, we assert that $\{\sigma_n\}$ is CS, So by Lemma 1 there exist $\{\sigma_{n_k}\}$, $\{\sigma_{m_k}\}$ and $\varepsilon > 0$ such that (2.1) and (2.2), we examine that $\vartheta(\sigma_{n_K+1}, \sigma_{m_K+1}, \varsigma) > (1+\varepsilon)$. Since $\sigma_n \perp \sigma_{n+1}$ for all $n \geq 0$ so by transitivity of $\perp$, we have $\sigma_{n_k} \perp \sigma_{m_k}$. Letting $g = \sigma_{n_k}$ and $e = \sigma_{m_k}$ in (3.11), one writes for all $k \geq 1$,

$$\begin{aligned}\mathcal{J}(\vartheta(\sigma_{n_k+1}, \sigma_{m_k+1}, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_{n_k}, L\sigma_{m_k}, \varsigma)) \geq \mathcal{S}\left((\vartheta(\sigma_{n_k}, L\sigma_{n_k}, \varsigma))^v (\vartheta(\sigma_{m_k}, L\sigma_{m_k}, \varsigma))^{1-v}\right) \\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n_k}, \sigma_{n_k+1}, \varsigma))^v (\vartheta(\sigma_{m_k}, \sigma_{m_k+1}, \varsigma))^{1-v}\right).\end{aligned}$$

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If $\sigma_k = \vartheta\left(\sigma_{n_K+1}, \sigma_{m_K+1}, \varsigma\right), \sigma_{b_k} = \vartheta\left(\sigma_{n_K}, \sigma_{n_K+1}, \varsigma\right), \sigma_{c_k} = \vartheta\left(\sigma_{m_K}, \sigma_{m_K+1}, \varsigma\right)$, we have

$$\mathcal{J}\left(\sigma_k\right) \geq \mathcal{S}\left(\left(\sigma_{b_k}\right)^{v}\left(\sigma_{c_k}\right)^{1-v}\right), \forall\, k \geq 1. \tag{3.16}$$

By (2.1), we have $\lim_{k\to\infty} \sigma_k = (1+\varepsilon) \times$ and (3.16) implies

$$\lim_{\sigma_a \to (1+\varepsilon)} \sup \mathcal{J}\left(\sigma_a\right) \geq \lim_{k\to\infty} \sup \mathcal{J}\left(\sigma_k\right) \geq \lim_{k\to\infty} \inf \mathcal{S}\left(\left(\sigma_{b_k}\right)^{v}\left(\sigma_{c_k}\right)^{1-v}\right) \geq \lim_{\sigma_a\to 0} \inf \mathcal{S}\left(\sigma_a\right). \tag{3.17}$$

The information got in (3.17), contradicts the assumption (viii) and thus stamping the sequence $\{\sigma_n\}$ as OC in the OCFMS $(\mathcal{B}, \vartheta, *, \perp)$. The completeness of the space ensures the convergence of $\{\sigma_n\}$, let it converges to $i \in \mathcal{B}$.

Case 1. if $\vartheta\left(\sigma_{n+1}, Li, \varsigma\right) = 1$ for some $n \geq 0$, Then

$$\vartheta\left(i, Li, \varsigma\right) \geq \vartheta\left(i, \sigma_{n+1}, \varsigma\right).\vartheta\left(\sigma_{n+1}, Li, \varsigma\right)$$

taking limit $n \to \infty$ on both sides, we have $\vartheta\left(i, Li, \varsigma\right) \geq 1$. This implies that $\vartheta\left(i, Li, \varsigma\right) = 1$. Hence, $i = Li$.

Case 2. for all $n \geq 0$, $\vartheta\left(\sigma_{n+1}, Li, \kappa\right) < 1$, then by $\perp$- regularity of $\mathcal{A}$, we find $\sigma_n \perp i$ or $i \perp \sigma_n$. By (3.11), one writes

$$\mathcal{J}\left(\vartheta\left(\sigma_{n+1}, Li, \kappa\right)\right) \geq \mathcal{J}\left(\vartheta\left(L\sigma_n, Li, \kappa\right)\right) \geq \mathcal{S}\left(\left(\vartheta\left(\sigma_n, L\sigma_n, \kappa\right)\right)^{v}\left(\vartheta\left(i, Li, \kappa\right)\right)^{1-v}\right) \text{ for all } n \geq 0.$$

By taking $\sigma_n = \vartheta\left(\sigma_{n+1}, Li, \kappa\right)$ and $b_n = \vartheta\left(\sigma_n, \sigma_{n+1}, \kappa\right)$, one writes

$$\mathcal{J}\left(\sigma_n\right) \geq \mathcal{S}\left(\left(b_n\right)^{v}\left(\vartheta\left(i, Li, \kappa\right)\right)^{1-v}\right) \text{ for all } n \geq 0. \tag{3.18}$$

Take $\delta = \vartheta\left(i, Li, \kappa\right)$. Note that $\sigma_n \to \delta$ and $b_n \to 1$ as $n \to \infty$. Applying limits on (3.18), we have

$$\limsup_{i\to\delta} \mathcal{J}\left(i\right) \geq \lim_{n\to\infty} \sup \mathcal{J}\left(\sigma_n\right) \geq \lim_{n\to\infty} \inf \mathcal{S}\left(\left(b_n\right)^{v}\left(\delta\right)^{1-v}\right) \geq \limsup_{i\to 0} \mathcal{S}\left(i\right).$$

This contradicts (v) if $\delta > 1$. Thus, we have $\vartheta\left(i, Li, \kappa\right) = 1$, that is i is a fixed point of L.

□

3.3. Chatarjea Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 9. *Let $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ will be called a Chatarjea type $(\mathcal{J}, \mathcal{S})$-OFIPC, verifying*

$$\mathcal{J}\left(\vartheta\left(L\sigma, L\theta, \varsigma\right)\right) \geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma, L\theta, \varsigma\right)\right)\left(\vartheta\left(\theta, L\sigma, \varsigma\right)\right)}\right), \tag{3.19}$$

for all $(\sigma, \theta) \in \mathcal{B}$, $\vartheta\left(L\sigma, L\theta, \varsigma\right) > 0$.

Theorem 6. *Let $\perp$ be a TOR, then, every $\perp$- PSM defined on a $\perp$- regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (3.19) and (i)-(iv), have a fixed point in $\mathcal{B}$.*

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Proof. Chasing the starting steps taken in proof of Theorem 4, we have

$$\begin{aligned}
\mathcal{J}\left(y_n\right) &\geq \mathcal{J}\left(\vartheta\left(L\sigma_{n-1}, L\sigma_n, \varsigma\right)\right) \geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma_{n-1}, L\sigma_n, \varsigma\right)\right)\left(\vartheta\left(\sigma_n, L\sigma_{n-1}, \varsigma\right)\right)}\right) \\
&\geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma_{n-1}, L\sigma_n, \varsigma\right)\right)\left(\vartheta\left(\sigma_n, \sigma_n, \varsigma\right)\right)}\right) \\
&\geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma_{n-1}, L\sigma_n, \varsigma\right)\right)}\right) \\
&\geq \mathcal{S}\left(\sqrt{\vartheta\left(\sigma_{n-1}, \sigma_{n+1}, \varsigma\right)}\right) \qquad (3.20) \\
&\geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma_{n-1}, \sigma_n, \varsigma\right)\right)\left(\vartheta\left(\sigma_n, \sigma_{n+1}, \varsigma\right)\right)}\right). \qquad (3.21)
\end{aligned}$$

Suppose that $\vartheta\left(\sigma_{n-1}, \sigma_n, \varsigma\right) > \vartheta\left(\sigma_n, \sigma_{n+1}, \varsigma\right)$ for some $n \geq 1$, then by (3.21) and (ii), we have

$$\mathcal{J}\left(y_n\right) \geq \mathcal{S}\left(y_n\right) > \mathcal{J}\left(y_n\right). \tag{3.22}$$

The information obtained in (3.22) contradicts the definition of $\mathcal{J}$, therefore, we go with

$$\mathcal{J}\left(y_n\right) \geq \mathcal{S}\left(y_n\right) > \mathcal{J}\left(y_n\right), \forall n \geq 1.$$

Now crawling through the proof of Theorem 4, we reach to the statement $\sigma_n \to o$ as $n \to \infty$, and then taking the support of $\perp$-regularity of the space $(\mathcal{B}, \vartheta, *, \perp)$, we achieve that $\sigma_n \perp o$ or $o \perp \sigma_n$. We need to have $\vartheta\left(o, Lo, \varsigma\right) = 1$. Letting $\vartheta\left(\sigma_{n+1}, Lo, \varsigma\right) < 1$ and using (3.19),

$$\begin{aligned}
\mathcal{J}\left(\vartheta\left(\sigma_{n+1}, Lo, \varsigma\right)\right) &\geq \vartheta\left(L\sigma_n, Lo, \varsigma\right) \geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma_n, Lo, \varsigma\right)\right)\left(\vartheta\left(o, L\sigma_n, \varsigma\right)\right)}\right) \\
&\geq \mathcal{S}\left(\sqrt{\left(\vartheta\left(\sigma_n, Lo, \varsigma\right)\right)\left(\vartheta\left(o, \sigma_{n+1}, \varsigma\right)\right)}\right) \\
&> \mathcal{J}\left(\sqrt{\left(\vartheta\left(\sigma_n, Lo, \varsigma\right)\right)\left(\vartheta\left(o, \sigma_{n+1}, \varsigma\right)\right)}\right).
\end{aligned}$$

Given that the function $\mathcal{J}$ satisfies assumption (ii), thus

$$\vartheta\left(\sigma_{n+1}, Lo, \varsigma\right) > \sqrt{\left(\vartheta\left(\sigma_n, Lo, \varsigma\right)\right)\left(\vartheta\left(o, \sigma_{n+1}, \varsigma\right)\right)}.$$

The last inequality implies that $\vartheta\left(o, Lo, \varsigma\right) \geq \sqrt{\vartheta\left(o, Lo, \varsigma\right)}$ (for large n). Hence, $\vartheta\left(o, Lo, \varsigma\right) = 1$, or $o = Lo$. □

Theorem 7. *Let $\perp$ be a TOR, then, every $\perp$- PSM defined on a $\perp$- regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (3.19), (i), (iii), and (v)-(viii), have a fixed point in $\mathcal{B}$.*

Proof. Chasing the steps taken in the proof of Theorem 5 and Theorem 6, we achieve the objective. □

3.4. Ciric-Reich-Rus Type $(\mathcal{J}, \mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 10. *Let $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ be two functions. A mapping $L : \mathcal{B} \to \mathcal{B}$ defined on OFMS $(\mathcal{B}, \vartheta, *, \perp)$ will be called a Ciric-Reich-Rus type $(\mathcal{J}, \mathcal{S})$- OFIPC, if there exists $v, \eta \in [0, 1)$ verifying*

$$\mathcal{J}\left(\vartheta\left(L\sigma, L\theta, \varsigma\right)\right) \geq \mathcal{S}\left(\left(\vartheta\left(\sigma, \theta, \varsigma\right)\right)^{v}\left(\vartheta\left(\sigma, L\sigma, \varsigma\right)\right)^{\eta}\left(\vartheta\left(\theta, L\theta, \varsigma\right)\right)^{1-v-\eta}\right), \tag{3.23}$$

for all $(\sigma, \theta) \in \mathcal{B}$, $\vartheta\left(\sigma, L\sigma, \varsigma\right) > 0$ where $v + \eta < 1$.

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The requirements for the presence of a fixed-point of Ciric-Reich-Rus type $(\mathcal{J},\mathcal{S})$-OFIPC are stated in the following two theorems.

Theorem 8. *Let $\perp$ be a TOR, then, every $\perp$-PSM defined on a $\perp$- regular OCFMS $(\mathcal{B},\vartheta,*,\perp)$ verifying (3.23) and (i)-(iv), admits a fixed point in $\mathcal{B}$.*

Proof. Chasing the starting steps taken in the proof of Theorem 4, we have

$$\begin{aligned}\mathcal{J}(y_n) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1},L\sigma_n,\varsigma))\\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1},\sigma_n,\varsigma))^{v}(\vartheta(\sigma_{n-1},L\sigma_{n-1},\varsigma))^{\eta}(\vartheta(\sigma_n,L\sigma_n,\varsigma))^{1-v-\eta}\right)\\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1},\sigma_n,\varsigma))^{v}(\vartheta(\sigma_{n-1},\sigma_n,\varsigma))^{\eta}(\vartheta(\sigma_n,\sigma_{n+1},\varsigma))^{1-v-\eta}\right)\\ &\geq \mathcal{S}\left((\vartheta(\sigma_{n-1},\sigma_n,\varsigma))^{v-\eta}(\vartheta(\sigma_n,\sigma_{n+1},\varsigma))^{1-v-\eta}\right)\\ &> \mathcal{J}\left((\vartheta(\sigma_{n-1},\sigma_n,\varsigma))^{v-\eta}(\vartheta(\sigma_n,\sigma_{n+1},\varsigma))^{1-v-\eta}\right).\end{aligned} \tag{3.24}$$

By (3.24) and monotonicity of $\mathcal{J}$ implies

$$(y_n)^{v+\eta} \geq (y_{n-1})^{v+\eta}, \forall n \geq 1.$$

Now taking steps as in Theorem 4, we get $\sigma_n \to t$ as $n \to \infty$, and with the support of $\perp$- regularity of $(\mathcal{B},\vartheta,*,\perp)$, we have $\sigma_n \perp t$ or $t \perp \sigma_n$. We need to prove $\vartheta(t,Lt,\varsigma)=1$. Letting $\vartheta(\sigma_{n+1},Lt,\varsigma)<1$ and using (3.23), we have

$$\begin{aligned}\mathcal{J}(\vartheta(\sigma_{n+1},Lt,\varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n,Lt,\varsigma))\\ &\geq \mathcal{S}\left((\vartheta(\sigma_n,t,\varsigma))^{v}(\vartheta(\sigma_n,L\sigma_n,\varsigma))^{\eta}(\vartheta(t,Lt,\varsigma))^{1-v-\eta}\right)\\ &\geq \mathcal{S}\left((\vartheta(\sigma_n,t,\varsigma))^{v}(\vartheta(\sigma_n,\sigma_{n+1},\varsigma))^{\eta}(\vartheta(t,Lt,\varsigma))^{1-v-\eta}\right)\\ &> \mathcal{J}\left((\vartheta(\sigma_n,t,\varsigma))^{v}(\vartheta(\sigma_n,\sigma_{n+1},\varsigma))^{\eta}(\vartheta(t,Lt,\varsigma))^{1-v-\eta}\right).\end{aligned}$$

Using (ii), we get

$$\vartheta(\sigma_{n+1},Lt,\varsigma) > (\vartheta(\sigma_n,t,\varsigma))^{v}(\vartheta(\sigma_n,\sigma_{n+1},\varsigma))^{\eta}(\vartheta(t,Lt,\varsigma))^{1-v-\eta}.$$

Now for large n, the last inequality implies that $\vartheta(t,Lt,\varsigma) \geq 1$. Hence, $\vartheta(t,Lt,\varsigma) = 1$, or $t = Lt$. □

Theorem 9. *Suppose $\perp$ be a TOR, then, every $\perp$- PSM defined on a $\perp$- regular OCFMS $(\mathcal{B},\vartheta,*,\perp)$ verifying (3.23), (i), (iii), and (v)-(viii), have a fixed point in $\mathcal{B}$.*

Proof. Chasing the steps taken in the proof of Theorem 5 and Theorem 8, we complete the proof of Theorem 9. □

3.5. Hardy-Rogers Type $(\mathcal{J},\mathcal{S})$-Orthogonal Fuzzy Interpolative Contraction

Definition 11. *Let $\mathcal{J},\mathcal{S}:(0,1]\to\mathbb{R}$ be two functions. A mapping $L:\mathcal{B}\to\mathcal{B}$ defined on OFMS $(\mathcal{B},\vartheta,*,\perp)$ will be called a Hardy-Rogers type $(\mathcal{J},\mathcal{S})$-OFIPC, if there exists $v,\eta,\gamma,\delta\in[0,1)$ verifying*

$$\mathcal{J}(\vartheta(L\sigma_a,L\sigma_b,\varsigma)) \geq \mathcal{S}\left(\begin{array}{c}(\vartheta(\sigma,\theta,\varsigma))^{v}(\vartheta(\sigma,L\sigma,\varsigma))^{\eta}(\vartheta(\theta,L\theta,\varsigma))^{\gamma}\\ (\vartheta(\sigma,L\theta,\varsigma))^{\delta}(\vartheta(\theta,L\sigma,\varsigma))^{1-v-\eta-\gamma-\delta}\end{array}\right), \tag{3.25}$$

for all $(\sigma,\theta)\in\mathcal{B}$, $\vartheta(L\sigma,L\theta,\varsigma)>0$ where $v+\eta+\gamma+\delta<1$.

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Example 5. *Let* $\mathcal{B} = [1,7)$ *and define the FMS* $\vartheta(\sigma,\theta,\varsigma) = e^{-\frac{|\sigma-\theta|}{\varsigma}}$ *,where Let* $\perp \subset \mathcal{B}^2$ *defined by*

$$\sigma \perp \theta \text{ if } \sigma\theta \le \sigma \vee \theta \text{ for } \sigma \ne \theta.$$

Then $(\mathcal{B},\vartheta,*,\perp)$ *is OFMS with* $m * n = mn$*. Define* $L : \mathcal{B} \to \mathcal{B}$ *by*

$$L(\sigma_a) = \left\{ \begin{array}{c} 5, \text{if } \sigma = 1 \\ \sigma - 1 \text{ otherwise} \end{array} \right\}.$$

Define $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ *by*

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{and } \mathcal{S} = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}$$

Case 1: Here, L is a Hardy-Rogers type $(\mathcal{J},\mathcal{S})$-OFIPC. But,

$$\begin{aligned}
\vartheta(L2, L1, k1) &\ge \begin{array}{c} (\vartheta(2,1,1))^{0.01} (\vartheta(2,L2,1))^{0.02} (\vartheta(1,L1,1))^{0.03} \\ (\vartheta(2,L1,1))^{0.04} (\vartheta(1,L2,1))^{1-0.01-0.02-0.03-0.04} \end{array} \\
\vartheta\left(1,5,1\frac{1}{2}\right) &\ge \begin{array}{c} (\vartheta(2,1,1))^{0.01} (\vartheta(2,1,1))^{0.02} (\vartheta(1,5,1))^{0.03} \\ (\vartheta(2,5,1))^{0.04} (\vartheta(1,1,1))^{0.9} \end{array} \\
\left(e^{-\frac{|1-5|}{0.5}}\right) &\ge \begin{array}{c} \left(e^{-\frac{|1-2|}{1}}\right)^{0.01} \left(e^{-\frac{|1-2|}{1}}\right)^{0.02} \left(e^{-\frac{|1-5|}{1}}\right)^{0.03} \\ \left(e^{-\frac{|2-5|}{1}}\right)^{0.04} (1)^{0.9} \end{array} \\
0.0003 &\ge 0.7632
\end{aligned}$$

This is a contradiction. Hence, L is not Hardy-Rogers type OFIPC.

Case 2: Here, L is a Hardy-Rogers type $(\mathcal{J},\mathcal{S})$-OFIPC. But,

$$\begin{aligned}
\vartheta(L3, L1, k1) &\ge \left(\begin{array}{c} (\vartheta(3,1,1))^{0.01} (\vartheta(3,L3,1))^{0.02} (\vartheta(1,L1,1))^{0.03} \\ (\vartheta(3,L1,1))^{0.04} (\vartheta(1,L3,1))^{1-0.01-0.02-0.03-0.04} \end{array} \right) \\
e^{-\frac{|2-5|}{0.5}} &\ge \begin{array}{c} \left(e^{-\frac{|1-3|}{1}}\right)^{0.01} \left(e^{-\frac{|1-5|}{1}}\right)^{0.02} \left(e^{-\frac{|2-5|}{1}}\right)^{0.03} \\ \left(e^{-\frac{|3-5|}{1}}\right)^{0.04} \left(e^{-\frac{|1-2|}{1}}\right)^{0.9} \end{array} \\
0.0025 &\ge 0.3104.
\end{aligned}$$

This is a contradiction. Hence, L is not Hardy Rogers type OFIPC.

The requirements for the presence of a fixed-point of the Hardy-Rogers type $(\mathcal{J},\mathcal{S})$-OFIPC is stated in the following two theorems.

Theorem 10. *Let* $\perp$ *be a TOR, then, every* $\perp$*- PSM defined on a* $\perp$*- regular OCFMS* $(\mathcal{B},\vartheta,*,\perp)$ *verifying (3.25) and (i)-(iv), have a fixed point in* $\mathcal{B}$.

Proof. Assume $\sigma_0 \in \mathcal{B}$ such that $\sigma_0 \perp \sigma_1$ or $\sigma_1 \perp \sigma_0$ for every $\sigma_1 \in \mathcal{B}$, then by utilizing the $\perp$-preservation of L, we build an OS $\{\sigma_n\}$ s.t $\sigma_n = L(\sigma_{n-1}) = L^n(\sigma_0)$ and $\sigma_{n-1} \perp \sigma_n$ for every

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$n \in \mathbb{N}$. Note that, if $\sigma_n = L(\sigma_n)$ then σ_n is FP of L for all $n \geq 0$. Let $\sigma_n \neq \sigma_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $y_n = \vartheta(\sigma_n, \sigma_{n+1}, \varsigma)$ $\forall\, n \geq 0$. By the first part of (ii) and (3.25), we have

$$\begin{aligned}
\mathcal{J}(y_n) &\geq \mathcal{J}(\vartheta(L\sigma_{n-1}, L\sigma_n, \varsigma)) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} (\vartheta(\sigma_{n-1}, L\sigma_{n-1}, \varsigma))^{\eta} (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{\gamma} \\ (\vartheta(\sigma_{n-1}, L\sigma_n, \varsigma))^{\delta} (\vartheta(\sigma_n, L\sigma_{n-1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array}\right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\gamma} \\ (\vartheta(\sigma_{n-1}, \sigma_{n+1}, \varsigma))^{\delta} (\vartheta(\sigma_n, \sigma_n, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array}\right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\gamma} \\ (\vartheta(\sigma_{n-1}, \sigma_{n+1}, \varsigma))^{\delta} (1)^{1-v-\eta-\gamma-\delta} \end{array}\right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{v} (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\eta} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\gamma} \\ (\vartheta(\sigma_{n-1}, \sigma_n, \varsigma))^{\delta} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\delta} \end{array}\right) \\
&\geq \mathcal{S}\left((y_{n-1})^{v+\eta+\delta} (y_n)^{\gamma+\delta}\right) \\
&> \mathcal{J}\left((y_{n-1})^{v+\eta+\delta} (y_n)^{\gamma+\delta}\right).
\end{aligned} \tag{3.26}$$

Suppose that $y_n > y_{n-1}$ for some $n \geq 1$. By monotonicity of $\mathcal{J}$ and (3.26), we have $(y_n)^{\gamma+\delta} > (y_n)^{\gamma+\delta}$. This is not possible. Consequently, we obtain $y_n > y_{n-1}$ $\forall\, n \geq 1$. Now taking steps as taken in Theorem 4, we deduce $\sigma_n \to u$ as $n \to \infty$, and with the support of $\perp$-regularity of $(\mathcal{B}, \vartheta, *, \perp)$, we have $\sigma_n \perp u$ or $u \perp \sigma_n$. we need to prove that $\vartheta(u, Lu, \varsigma) = 1$. Letting $\vartheta(\sigma_{n+1}, Lu, \varsigma) < 1$ and using (3.25), we have

$$\begin{aligned}
\mathcal{J}(\vartheta(\sigma_{n+1}, Lu, \varsigma)) &\geq \mathcal{J}(\vartheta(L\sigma_n, Lu, \varsigma)) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} (\vartheta(\sigma_n, L\sigma_n, \varsigma))^{\eta} (\vartheta(u, Lu, \varsigma))^{\gamma} \\ (\vartheta(\sigma_n, Lu, \varsigma))^{\delta} (\vartheta(u, L\sigma_n, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array}\right) \\
&\geq \mathcal{S}\left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} (\vartheta(u, Lu, \varsigma))^{\gamma} \\ (\vartheta(\sigma_n, Lu, \varsigma))^{\delta} (\vartheta(u, \sigma_{n+1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array}\right) \\
&> \mathcal{J}\left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} (\vartheta(u, Lu, \varsigma))^{\gamma} \\ (\vartheta(\sigma_n, Lu, \varsigma))^{\delta} (\vartheta(u, \sigma_{n+1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array}\right).
\end{aligned}$$

Using (ii), we get

$$\vartheta(\sigma_{n+1}, Lu, \varsigma) > \left(\begin{array}{c} (\vartheta(\sigma_n, u, \varsigma))^{v} (\vartheta(\sigma_n, \sigma_{n+1}, \varsigma))^{\eta} (\vartheta(u, Lu, \varsigma))^{\gamma} \\ (\vartheta(\sigma_n, Lu, \varsigma))^{\delta} (\vartheta(u, \sigma_{n+1}, \varsigma))^{1-v-\eta-\gamma-\delta} \end{array}\right).$$

Now for large n, the last inequality implies that $\vartheta(u, Lu, \varsigma) \geq 1$. Hence, $\vartheta(u, Lu, \varsigma) = 1$, or $u = Lu$. □

Theorem 11. *Let $\perp$ be a TOR, then, every $\perp$-PSM defined on a $\perp$- regular OCFMS $(\mathcal{B}, \vartheta, *, \perp)$ verifying (3.25) and (i), (iii), (v)-(viii), have a fixed point in $\mathcal{B}$.*

Proof. Following the steps as taken in Theorem 5 and Theorem 10, the proof is obvious. □

4. Applications

In this section, we discuss the applications of fractional differential equations and Volterra-type Fredholm integral equations.

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4.1. An Application to Fractional Differential Equation

A variety of useful fractional differential features were postulated and searched by Lacroix (1819). Caputo and Fabrizio announced [23] a new fractional technique, in 2015. The need to characterize a class of non-local systems that cannot be properly represented by traditional local theories or fractional models with singular kernel [23] sparked interest in this description. The different kernels that can be selected to satisfy the requirements of different applications are the fundamental difference among fractional derivatives. The Caputo fractional derivative [24], the Cauto -Fabrizio derivative derivative [23], and the Atangana-Baleanu fractional derivative [20], for example, are determined by power laws, the Caputo-Fabrizio derivative by an exponential decay law, and the Atangana_Baleanu derivative by Mittag- Leffler law. A variety of new Caputo-Fabrizio (CFD) models were lately investigated in [19,21,22].

In OFMSs, we will look at one of these models. (represent $C_{(I,\mathbb{R})}$ by $\Bbbk$)

Let $\vartheta : \Bbbk^2 \to [1, \infty)$ be defined by

$$\vartheta(u, v, \varsigma) = e^{-\frac{\|u-v\|}{\varsigma}} = e^{-\sup_{l \in I} \frac{|u(l)-v(l)|}{\varsigma}}, \text{ for all } u, v \in C_{(I,\mathbb{R})}.$$

Then $(\Bbbk, \vartheta, \varsigma)$ is a complete fuzzy metric space, where $I = [0, 1]$ and

$$\Bbbk = \{u | u : I \to \mathbb{R} \text{ and } u \text{ is continuous}\}.$$

The relation $\perp$ on $\Bbbk$ given as follows:

$$u \perp v \text{ iff } u(l) v(l) \geq u(l) \vee v(l), \text{ for all } u, v \in C_{(I,\mathbb{R})},$$

is an orthogonal relation and $(\Bbbk, \vartheta, *, \perp)$ is an OCFMS. Let the function $K_1 : I \times \mathbb{R} \to \mathbb{R}$ be taken as $K_1(s, r) \geq 0$ for all $s \in I$ and $\tau \geq 0$. we shall apply Theorem 2 to resolve the following CFDE:

$$^{C}D^{v}w(s) = K_1(s, w(s)); w \in C_{(I,\mathbb{R})} : \tag{4.27}$$

$$W(0) = 0, Iw(1) = w'(0).$$

We denote CFD of order v by $^{C}D^{v}$ and for $v \in (m-1, m)$; $m = [v] + 1$, we have

$$^{C}D^{v}w(s) = \frac{1}{\Gamma(m-v)} \int_0^s (s-z)^{m-v-1} w(z) \sigma_d z.$$

The notation $I^v w$ is interpreted as follows :

$$w(s) = \frac{1}{\Gamma(m-v)} \int_0^s (s-z)^{v-1} K_1(z, w(z)) \sigma_d z + \frac{s}{\Gamma(m-v)} \int_0^1 \int_0^z (z-p)^{v-1} K_1(p, w(p)) \sigma_d p \sigma_d z.$$

For the mapping $K_1 : I \times \mathbb{R} \to \mathbb{R}$ and $u_0 \in \Bbbk$ we state the following conditions:

(A) For $\tau \geq 0$, let

$$| K_1(s, w(s)) - K_1(s, u(s)) | \leq \frac{\Gamma(v+1)}{\Gamma(v)} | w(s) - u(s) |,$$

for all $w, u \in \Bbbk$ following the order $w \perp u$.

(B) there exists $u_0 \in \Bbbk$ such that

$$\begin{aligned} u_0(s) &\leq \frac{1}{\Gamma(v)} \int_0^s (s-z)^{v-1} K_1(z, w_0(z)) \sigma_d z \\ &+ \frac{l}{\Gamma(v)} \int_0^1 \int_0^z (z-p)^{v-1} K_1(p, u_0(p)) \sigma_d p \sigma_d z. \end{aligned}$$

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We noticed that $K_1 : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz continuous.

For instant, K_1 given by

$$K_1(s, w(s)) = sw(s) \text{ if } w(s) \le \frac{1}{2}, 0 \text{ if } w(s) > \frac{1}{2}.$$

Following (A), K_1 is not continuous and monotone. Moreover, for $s = e^{-\tau}\Gamma(v+1)$,

$$\vartheta(K_1(s, w(s)), K_1(t, u(t)), \varsigma) = e^{-\frac{|K_1(\iota,\frac{1}{2}) - K_1(\iota,\frac{2}{3})|}{\varsigma}} = e^{\frac{s}{2\varsigma}} \ge e^{\frac{s}{6\varsigma}} = e^{-s\frac{|\frac{1}{2}-\frac{1}{3}|}{\varsigma}} = \vartheta(w, u, \varsigma)$$

Theorem 12. *Let the mappings $K_1 : I \times \mathbb{R} \to \mathbb{R}$ and $u_0 \in C_{(I,\mathbb{R})}$ satisfies the conditions (A)-(B), the the equation (23) admits a solution in $\Bbbk$.*

Proof. Let $X = \left\{ J \in C_{(I,\mathbb{R})} : J(s) \ge 0 \text{ for all } s \in I \right\}$ and define $\Psi : X \to X$ by

$$(\Psi J)(s) = \frac{1}{\Gamma(v)} \int_0^s (s-z)^{v-1} K_1(z, J(z))\, \sigma_d z + \frac{s}{\Gamma(v)} \int_0^1 \int_0^z (z-p)^{v-1} K_1(p, J(p))\, \sigma_d p \sigma_d z.$$

We define an orthogonal relation $\perp$ on X by

$$u \perp v \text{ iff } u(s)\, v(s) \ge u(s)\, v(s), \forall\, u, v \in X.$$

According to above conditions ,Ψ is preserving and there is $u_0 \in \Bbbk$ verifying (B) such that $u_n = \mathbb{R}^n(u_0)$ with $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for all $n \ge 0$. we work on the validation of (3.3) in the next lines.

$$\vartheta((\Psi J)(s), \Psi(U)(s), \varsigma) = \exp\begin{pmatrix} \sup |\, \frac{1}{\Gamma(v)} \int_0^s (s-z)^{v-1} K_1(z, J(z))\, \sigma_d z \\ -\frac{1}{\Gamma(v)} \int_0^s (s-z)^{v-1} K_1(z, U(z))\, \sigma_d z \\ +\frac{s}{\Gamma(v)} \int_0^1 \int_0^z (p-z)^{v-1} K_1(p, J(p))\, \sigma_d p \sigma_d z \\ -\frac{s}{\Gamma(v)} \int_0^1 \int_0^z (p-z)^{v-1} K_1(p, U(p))\, \sigma_d p \sigma_d z \end{pmatrix}$$

$$\ge \exp\left(\sup_{s,z \in I} \left\{ \begin{array}{c} \frac{1}{\Gamma(v)}\Gamma(v+1) \int_0^s (s-z)^{v-1} \frac{|J(z)-U(z)|}{\varsigma} \sigma_d z \\ -\frac{s}{\Gamma(v)}\Gamma(v+1) \int_0^1 \int_0^z (p-z)^{v-1} \frac{|J(z)-U(z)|}{\varsigma} \sigma_d z \sigma_d p \end{array} \right\} \right)$$

$$\ge \exp\left(\begin{array}{c} \frac{1}{\Gamma(v)}\Gamma(v+1) \sup_{z \in I} \frac{|J(z)-U(z)|}{\varsigma} \\ \sup_{s \in I} \left\{ \int_0^s (s-z)^{v-1} \mid \sigma_d z - s \int_0^1 \int_0^z (p-z)^{v-1} \sigma_d z \sigma_d p \right\} \end{array} \right)$$

$$\ge \exp\left(\begin{array}{c} \frac{\Gamma(v)\Gamma(v+1)}{\Gamma(v)\Gamma(v+1)} \sup_{z \in I} \frac{|J(z)-U(z)|}{\varsigma} \\ -sB(v+1, 1) \frac{\Gamma(v)\Gamma(v+1)}{\Gamma(v)\Gamma(v+1)} \sup_{s,z \in I} \frac{|J(z)-U(z)|}{\varsigma} \end{array} \right)$$

$$\ge \exp(1 - sB(v+1, 1)) \sup_{s,z \in I} \frac{|\, J(z) - U(z)\, |}{\varsigma}$$

$$\ge \exp\left((1 - sB(v+1)) \sup_{s,z \in I} \frac{|\, J(z) - U(z)\, |}{\varsigma} \right)$$

$$= \left(\exp\left(\sup_{s,z \in I} \frac{|\, J(z) - U(z)\, |}{\varsigma} \right) \right)^{1-sB(v+1,1)}$$

$$= (\vartheta(J(z, U(z), \varsigma))^{1-sB(v+1,1)}; \text{ where } B \text{ is a beta function.}$$

By defining $\mathcal{J}(w) = \ln(w)$ and $\mathcal{S}(w) = D\mathcal{J}(w)$; $w > 0, \tau > 0$, and putting $1 - sB(v+1,1) = D < 1$, the last inequality gets the form:

$$\mathcal{J}(\vartheta(\Psi(J)(s), \Psi(U)(s)), \tau) \geq \mathcal{S}(\vartheta(J, U, \tau)).$$

□

4.2. Application to Volterra Type Integral Equation

There are several types of integral equations but they are only used the "model scientific process" in which the value, or the rate of change of the change of value, of some quantity (or quantities) depends on past history. This opposes in which the present value can obtain the rate at which a quantity evolving. Just as for differential equations, integral equation need to be "solved" to describe and predict how a physical quantity is going to behave as time passes. For solving integral equations, there are things like Fredholm theorems, fixed point methods, boundary element methods, and Nystrom methods. In this paper, we apply Theorem 2 to show the existence of multiplicative Volterra type integral equation given below;

$$f(k) = \int_0^k L(k, h, f, \varsigma)\,\sigma_d h \tag{4.28}$$

for all $k \in H$ and $L : H \times H \times \Bbbk \to \mathbb{R}$. We show the existence of the solution to (4.27).

Let $\vartheta : \Bbbk \times \Bbbk \times (0, \infty) \to \mathbb{R}$ be defined as

$$\vartheta(u, v, \varsigma) = e^{-\frac{|u(l)-v(l)|}{\varsigma}}, \text{ for all } u, v \in C_{(I,\mathbb{R})}.$$

Then $(\Bbbk, \vartheta, *)$ is a CFMS where $I = [0, 1]$ and

$$\Bbbk = \{u | u : I \to \mathbb{R} \text{ and } u \text{ is a continuous}\}.$$

The relation $\perp$ on $\Bbbk$ given as follows

$$u \perp v \text{ iff } u(l)\, v(l) \geq u(l) \vee v(l), \text{ for all } u, v \in C_{(I,\mathbb{R})},$$

is an orthogonal relation and $(\Bbbk, \vartheta, *, \perp)$ is an OCFMS.

The following is the existence theorem for integral equation (4.28).

Theorem 13. *Assume that the following conditions are satisfied.*

(a) Assume that $L : H \times H \times \Bbbk \to \mathbb{R}$ is continuous.
(b) Suppose there exists $\tau > 0$, such that

$$e^{-\frac{|L(k,h,f)-L(k,h,q)|m}{\varsigma}} \geq e^{-\frac{|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2|}{\varsigma}} \tag{4.29}$$

for all $k, h \in [0, 1]$ and $f, q \in C_{(I,\mathbb{R}^+)}$. Then, integral equation (4.28) admits a solution in $C_{(I,\mathbb{R}^+)}$.

Proof. Let $\mathbb{R} = \Bbbk$ and endow it with the relation $\perp$ and fuzzy metric space ϑ. Define the mapping $\Psi : \mathbb{R} \to \mathbb{R}$ by

$$(\Psi f)(k) = \int_0^k L(k, h, f, \varsigma)\,\sigma_d h \tag{4.30}$$

so that the fixed point of Ψ is a solution of integral equation (4.28). According to above definitions, ψ is $\perp$-preserving and there is $u_0 \in \Bbbk$ verifying $u_n = \mathbb{R}^n(u_0)$ with $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for all $n \geq 0$. We work on the validation of (3.3) in the next lines. By assumption (b), we have

$$\begin{aligned}
\vartheta(\Psi(f), \Psi(q), \varsigma) &= e^{-\frac{|(\Psi f)(k)-(\Psi q)(k)|}{\varsigma}} \\
&\geq \int_0^k e^{-\frac{|(\Psi f)(k)-(\Psi q)(k)|}{\varsigma}} \sigma_d h \\
&\geq \int_0^k e^{-\frac{\left|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2\right|}{\varsigma}} \sigma_d h \\
&= e^{-\frac{\left|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2\right|}{\varsigma}} \int_0^k \sigma_d h \\
&= k e^{-\frac{\left|\vartheta(f,q)-\left(\tau\left(\sqrt{\vartheta(f,q)}\right)+1\right)^2\right|}{\varsigma}} \\
&= \vartheta(f, q, \varsigma)
\end{aligned}$$

Hence, by defining $\mathcal{J}(w) = \ln(w)$ and $\mathcal{S}(w) = D\mathcal{J}(w)$;

$$\mathcal{J}(\vartheta(\Psi(f), \Psi(q), \varsigma)) \geq \mathcal{S}(\vartheta(f, q, \varsigma)).$$

So all the conditions of Theorem 2 are satisfied and $v = k$. Hence, the integral equation (4.28) admits at most one solution. □

5. Conclusions

The study of $(\mathcal{J}, \mathcal{S})$-OFIPC proved to be a source of generalization of many well-known contractions. The methodology applied for investigation of fixed point of $(\mathcal{J}, \mathcal{S})$-OFIPC encapsulated existing corresponding methodologies. The results will extend earlier results of [8,15–18].

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