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Enhancing Decomposition Approach for Solving Multi-Objective Dynamic Non-Linear Programming Problems Involving Fuzziness

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Abstract: In real-life scenario, there are many mathematical tools to handle the incomplete and imprecise data. One of them is the fuzzy approach. This article aims to contribute to the literature of fuzzy multi-objective dynamic programming issues involving the fuzzy objective functions. The piecewise quadratic fuzzy numbers characterize these fuzzy parameters. Some basic notions in the problem under the α −pareto optimal solution concept is redefined and analyzed to study the stability of the problem. Furthermore, a technique, named as enhancing decomposition approach, is presented for achieving a subset for the parametric space that contains the sameα −pareto optimal solution. For a better understanding and comprehension of the suggested concept, a numerical example is provided.

Keywords: Optimization problem Multi-objective optimization problems; Dynamic non-linear programming; Fuzzy set; Piecewise quadratic fuzzy numbers; Close interval approximation; α −pareto optimal solution; Decision- making; Stability

MSC: 90C31; 90C70

1. Introduction

One of the most essential methodologies for solving optimization problems is dynamic programming (DP), where the so-called principle of optimality, as defined by Bellman [1] in 1957, is used to create its methods. The multi-objective dynamic programming (MODP) is a method for resolving problems with competing objective functions that follows the DP properties (Mine and Fukushima [2], 1979; Carraway et al. [3]; Abo- Sinna and Hussein [4]; Abo-Sinna and Hussein [5]).

Osman [6-7] introduces the ideas of solvability set, the stability sets of first-kind and second-kind, as well as the analysis of these terms for parametric convex non-linear programming problem. In order to a certain class of multi-objective convex programming problems, Osman and Dauer [8] dedicated themselves to the discovering of the first-kind stability set. In addition, they provided a technique to compute this set and the related pareto optimum solution.

First and foremost, Zadeh [9] presented the philosophy of fuzziness in literature, which can be applied to deal with the issues in real-life scenario where the information is in the form of ambiguousness and incompleteness. Bellman and Zadeh [10] created a method for solving decision-making problems involving fuzziness that improved and aided managerial decision-making. Linear programming along with fuzzy programming involving numerous objective functions were presented by Zimmermann [11] in 1978. Several people afterwards worked in the field of fuzziness. As a result of the convenience, the piecewise linear fuzzy numbers such as interval, triangular, trapezoidal, pentagonal, hexagonal fuzzy numbers, etc. have been applied in the literature [12-16]. Many authors have investigated the solution methodology as well as their applications involving fuzziness, fuzzy systems, and fuzzy mathematical programming problems [17-18].

In the literature, fuzzy dynamic programming models in particular have gotten a lot of attention (see, Bellman and Zadeh [10]; Zimmermann [19]; Esogbue [20]; Esogbue and Bellman [21]; Hussein and Abo- Sinna [22]). Tanaka and Asai [23] introduced fuzzy parameters to multi-objective linear programming (MOLP) problems. General fuzzy multi-objective non-linear programming (MONLP) models were formulated by Orlovski [24] in 1984. Sakawa and Yano [25-26] developed the idea of pareto optimum optimality and proposed a new interactive fuzzy approach for MOLP and MONLP issues with fuzzy parameters. For fuzzy MONLP situations, Osman and El-Banna [27], in 1993, proposed a qualitative analysis and stability. There are enormous researches, who developed the MODP (for instance, Moghaddam and Ghoseiri [28]; Muruganantham et al. [29]; Li et al. [30]; Deng et al. [31]; Besheli et al. [32]; Peraza et al. [33]; Azevedo et al. [34]; Ni et al. [35]; Wu et al. [36]; Liu et al. [37]; Zou et al. [38]; and Zhang et al. [39]).

Based on aforesaid literature survey, in this article, the parameters of the model are redefined and further studied for MODP problems involving the fuzzy parameters in objective functions, as a result of the above literature.

The main contribution of this article is as follows:

(i) For the core terminology associated with the stability in non-linear programming problem, the parameters are rearranged to study in case of MODP.

(ii) An algorithm for computing the subset of the parametric space that possesses the same associated pareto optimal solution, is developed.

The remainder of this article is organized as in Fig. 1 below:

Fig. 1 Layout of Remaining Paper

2. Preliminaries

In this section, we recall some basic concepts.

Definition 1 (Jain, 2010). **A piecewise quadratic fuzzy number** (PQFN) is designated by $\widetilde{A}_{PQ} = (b_1, b_2, b_3, b_4, b_5)$, where $b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$ are real numbers, and its membership function $\mu_{\widetilde{A}_{PQ}}$ is given by (see, Fig. 2)

$$\mu_{\tilde{A}_{PQ}} = \begin{cases} 0, & x < b_1; \\ \frac{1}{2}\frac{1}{(b_2-b_1)^2}(x-b_1)^2, & b_1 \le x \le b_2; \\ \frac{1}{2}\frac{1}{(b_3-b_2)^2}(x-b_3)^2+1, & b_2 \le x \le b_3; \\ \frac{1}{2}\frac{1}{(b_4-b_3)^2}(x-b_3)^2+1, & b_3 \le x \le b_4; \\ \frac{1}{2}\frac{1}{(b_5-b_4)^2}(x-b_5)^2, & b_4 \le x \le b_5; \\ 0, & x > b_5. \end{cases}$$

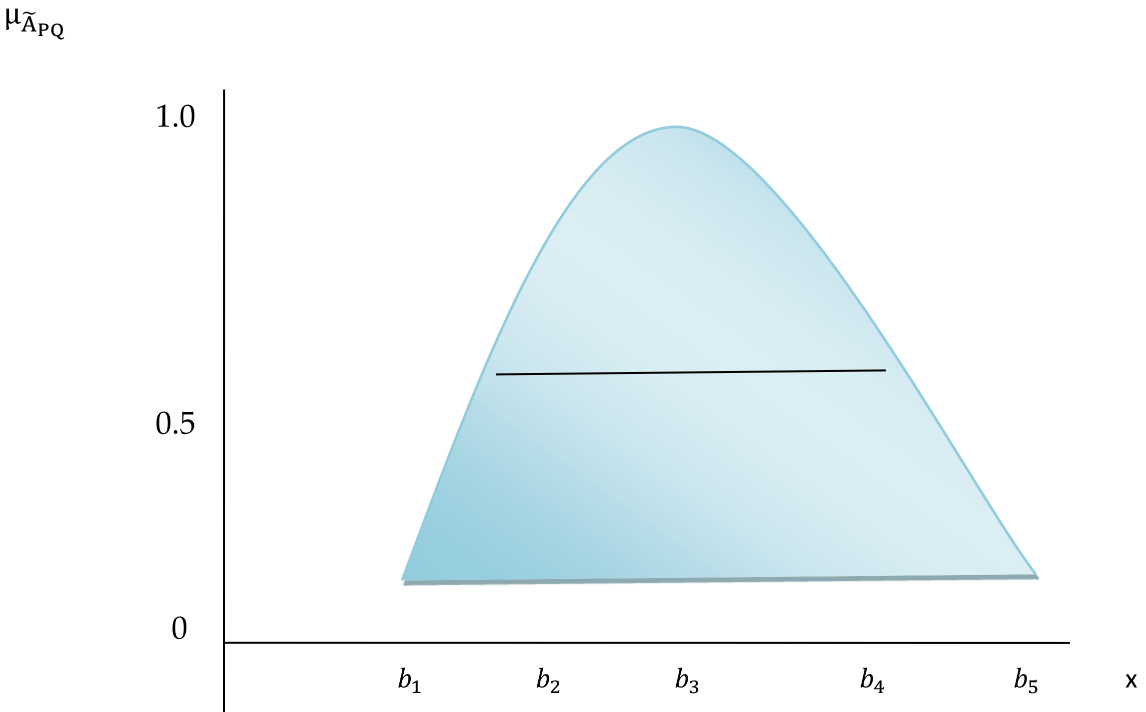

Fig.2. Graph Illustration of a PQFN

Definition 2. (Jain [41]). For a given PQFN $\tilde{A}$, the interval approximation, denoted by $[\tilde{A}_{PQ}] = [p_\alpha^L, p_\alpha^U]$, is called the closed interval approximation, when the below mentioned condition is satisfied:

$$p_\alpha^L = \inf\left\{y \in \mathfrak{R}:\ \mu_{\tilde{A}_{PQ}} \ge 0.5\right\}, \text{ and } p_\alpha^U = \sup\left\{y \in \mathfrak{R}:\ \mu_{\tilde{A}_{PQ}} \ge 0.5\right\}.$$

Definition 3. (Jain [41]). Suppose that $\tilde{a}_{PQ} = (p_1, p_2, p_3, p_4, p_5)$ and $\tilde{b}_{PQ} = (q_1, q_2, q_3, q_4, q_5)$ be two P.Q.F.N.s. Then

(i) Addition: $\tilde{a}_{PQ} \oplus \tilde{b}_{PQ} = (p_1 + q_1, p_2 + q_2, p_3 + q_3, p_4 + q_4, p_5 + q_5)$.

(ii) Subtraction: $\tilde{a}_{PQ} \ominus \tilde{b}_{PQ} = (p_1 - q_5, p_2 - q_4, p_3 - q_3, p_4 - q_2, p_5 - q_1)$.

(iii) Scalar multiplication: $\alpha.\tilde{b}_{PQ} = \begin{cases} (\alpha q_1, \alpha q_2, \alpha q_3, \alpha q_4, \alpha q_5), & \alpha > 0, \\ (\alpha q_5, \alpha q_4, \alpha q_3, \alpha q_2, \alpha q_1), & \alpha < 0. \end{cases}$

Definition 4. (Jain [41]). Suppose $[A] = [p_\alpha^L,\ p_\alpha^U]$, and $[B] = [q_\alpha^L,\ q_\alpha^U]$ are two inexact interval for the PQFN. Then, the arithmetic rules are presented as follows:

(i) $[A] \oplus [B] = [p_\alpha^L + q_\alpha^L,\ p_\alpha^U + q_\alpha^U]$,

(ii) $[A] \ominus [B] = [p_\alpha^L - q_\alpha^U,\ p_\alpha^U - q_\alpha^L]$,

(iii) $\alpha[A] = \begin{cases} [\alpha p_\alpha^L,\ \alpha p_\alpha^U], & \alpha > 0, \\ [\alpha p_\alpha^U, \alpha p_\alpha^L], & \alpha < 0. \end{cases}$

(iv) $[A] \odot [B] = \left[\frac{p_\alpha^U q_\alpha^L + p_\alpha^L q_\alpha^U}{2}, \frac{p_\alpha^L q_\alpha^L + p_\alpha^U q_\alpha^U}{2}\right]$.

(v) $\frac{[A]}{[B]} = \begin{cases} \left[\frac{2p_\alpha^L}{q_\alpha^L + q_\alpha^U}, \frac{2p_\alpha^U}{q_\alpha^L + q_\alpha^U}\right], & [B] > 0 \text{ and } q_\alpha^L + q_\alpha^U \neq 0, \\ \left[\frac{2q_\alpha^U}{q_\alpha^L + q_\alpha^U}, \frac{2q_\alpha^L}{q_\alpha^L + q_\alpha^U}\right], & [B] < 0 \text{ and } q_\alpha^L + q_\alpha^U \neq 0. \end{cases}$

Definition 5. (Jain [41]). The order relations $\{=_{LU}, \leq_{LU,}, \geq_{LU}\}$ for the intervals [A] and [B] is designated as follows:

(i) $[A] =_{(L,U)} [B]$ iff $p_\alpha^L = q_\alpha^L$ and $p_\alpha^U = q_\alpha^U$.

(ii) $[A](\leq_{(L,U)})[B]$ iff $p_\alpha^L(\leq_{(L,U)})q_\alpha^L$ and $p_\alpha^U(\leq_{(L,U)})q_\alpha^U$ or

$$p_\alpha^L + p_\alpha^U(\leq_{(L,U)})q_\alpha^L + q_\alpha^U.$$

(iii) $[A](\geq_{(L,U)})[B]$ iff $p_\alpha^L(\geq_{(L,U)})q_\alpha^U$ and $p_\alpha^U(\geq_{(L,U)})q_\alpha^L$ or

$$p_\alpha^L + p_\alpha^U(\geq_{(L,U)})q_\alpha^U + q_\alpha^L.$$

3. PROBLEM STATEMENT

A minimization type problem involving the fuzzy parameters within the objective functions is formulated as follows

$$\text{(PQF-VMP)}\ \min G_l\left(g_{l1}(x_1,\tilde{a}_1^{PQ}), g_{l2}(x_2,\tilde{a}_2^{PQ}), \dots, g_{lN}(x_N,\tilde{a}_N^{PQ})\right), l=\overline{1,L}, L\geq 2$$

s. t.

$$H_q\left(h_{q1}(x_1), h_{q2}(x_2), \dots, h_{QN}(x_N)\right)\leq 0, q=\overline{1,Q},$$

$$x_n\in X_n, n=\overline{1,N}.$$

Here, $X_n\subset\Re^{r_n}, n=\overline{1,N}$; x_n is a r_n vector, $G_l, l=\overline{1,L}$ and $H_q, q=\overline{1,Q}$ are convex real function of class $C^{(1)}$ on $\Re^N$ and $g_{ln}, h_{qn}. l=\overline{1,L}; q=\overline{1,Q}; n=\overline{1,N}$ are real-valued functions on X_n, and $\tilde{a}^{PQ}=(\tilde{a}_{11}^{PQ},\tilde{a}_{22}^{PQ},\dots,\tilde{a}_{ln}^{PQ}), l=\overline{1,L}; n=\overline{1,N}$ represent the fuzzy parameters in vector form, $\inf_{ln}(x_n,\tilde{a}_{ln}^{PQ})$. It is assumed that the aforesaid fuzzy parameters are designated as per the reference Jain [41], as well as the PQF-VMP is stable (Rockafellar [44]).

Definition 6. ([17]). The α −level set of the fuzzy numbers $\tilde{a}_{ln}^{PQ}$ refers the usual set $L_\alpha(\tilde{a}_{ln}^{PQ})$ where the degree of the membership function is grater that the level α as described below:

$$L_\alpha(\tilde{a}_{ln}^{PQ})=\left(a_{ln}: \mu_{\tilde{a}_{ln}^{PQ}}(a_{ln})\geq\alpha, l=1,2,\dots,L; n=\overline{1,N}\right)$$

For a certain value of α, the aforesaid PQF-VMP problem is converted into the following problem (Sakawa and Yano [25])

$$(\alpha\text{-VMP})\quad \min G_l(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2),\dots,g_{lN}(x_N,a_N)), l=\overline{1,L}, L\geq 2$$

s. t.

$$H_q\left(h_{q1}(x_1), h_{q2}(x_2),\dots,h_{QN}(x_N)\right)\leq 0, q=\overline{1,Q}, a_{ln}\in L_\alpha(\tilde{a}_{ln}^{PQ}),$$

$$x_n\in X_n, n=\overline{1,N}.$$

Since PQF-VMP problem becomes stable, therefore the α-VMP would also be stable.

Definition 7 (Mine and Fukushima [2]). The objective function G_l is called separable provided that

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there would exist functions $G_l^n, n = \overline{1,N}$ defined on $\Re^n$ and functions Ω_l^n designated on $\Re^2$ satisfies, for $n = \overline{2,N}$.

$$G_l^n(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{ln}(x_n,a_n))$$

$$= \phi_l^n\left(G_l^{n-1}(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{ln-1}(x_{n-1},a_{n-1})), g_{ln}(x_n,a_n)\right), \text{ and}$$

$$G_l^N(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{lN}(x_N,a_N)) = G_l(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{ln}(x_n,a_n))$$

Similarly, it can be illustrated that

$$H_q^n\left(h_{q1}(x_1), h_{q2}(x_2), \dots, h_{QN}(x_N)\right)$$

$$= \chi_q^n\left(H_q^{n-1}\left(h_{q1}(x_2), h_{q2}(x_2), \dots, h_{qn-1}(x_{n-1}), h_{Qn}(x_n)\right)\right), \text{ and}$$

$$H_q^n\left(h_{q1}(x_1), h_{q2}(x_2), \dots, h_{QN}(x_N)\right) = H_q\left(h_{q1}(x_1), h_{q2}(x_2), \dots, h_{QN}(x_N)\right)$$

In a situation when all the objectives as well as the constraints become separable, we claim that the α-VMP problem would also be separable. In addition, the functions, designated by ϕ_l^n and χ_q^n, are referred as the separating functions for the set G as well as for the set H.

Consequently, the separation of the α-VMP is termed as monotone provided that all ϕ_l^n and χ_q^n are strictly increasing functions relative to the first argument for each constant second argument for each $y \in \Re$,

$$\phi_l^n(r,y) > \phi_l^n(\hat{r},y) \text{ iff } r > \hat{r}, \text{ and } \chi_l^n(r,y) > \chi_l^n(\hat{r},y) \Leftrightarrow r > \hat{r}, \text{ for each } l = \overline{1,L};$$

$$q = \overline{1,Q} \text{and} , n = \overline{1,N}.$$

Definition 8. ($\alpha -$ pareto optimal solution)**.** The feasible solution $x^* = (x_1^*, x_2^*, \dots, x_N^*), a^* = (a_1^*, a_2^*, \dots, a_N^*)$ to the α–VMP, is referred as α –pareto optimal solution provided that we do not find the feasible $(x_1', x_2', \dots, x_N'), a^* = (a_1', a_2', \dots, a_N') \in L_\alpha(\tilde{a})$ such that

$$G_l(g_{l1}(x_1', a_1'), g_{l2}(x_2', a_2'), \dots, g_{ln}(x_N', a_N')) \le G_l(g_{l1}(x_1^*, a_1^*), g_{l2}(x_2^*, a_2^*), \dots, g_{ln}(x_N^*, a_N^*)) \text{ for all } l$$

and

$$G_r(g_{s1}(x_1', a_1'), g_{s2}(x_2', a_2'), \dots, g_{sn}(x_N', a_N')) < G_r(g_{s1}(x_1^*, a_1^*), g_{s2}(x_2^*, a_2^*), \dots, g_{sn}(x_N^*, a_N^*)), \quad \text{for}$$

minimum one index $s \in \{1, 2, \dots, L\}$.

Assumption 1: The α-VMP problem possesses the reparability property. In addition, the separation

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property refers to monotonicity.

Assumption 2. For each n, the set $X_n \cap L_\alpha(\tilde{a})$ is assumed to be compact and $G_l^n(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{ln}(x_n,a_n))$, $l=\overline{1,L}$. Also, we assume that $H_q^n\left(h_{q1}(x_1), h_{q2}(x_2), \dots, h_{QN}(x_N)\right)$, $q=\overline{1,Q}$ are continuous functions of $(x_1, x_2, \dots, x_n)$ and $(a_1, a_2, \dots, a_n)$.

Based on the weighting method (Chankong and Haimes [40]), α-VMP problem can be treated as presented below:

$$(\alpha\text{-VMP}_w) \qquad \min \sum_{l=1}^{L} w_l G_l\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{lN}(x_N,a_N)\right)$$

s. t.

$$H_q\left(h_{q1}(x_1), h_{q2}(x_2), \dots, h_{QN}(x_N)\right) \le 0, q=\overline{1,Q}, a \in L_\alpha(\tilde{a}),$$

$$x_n \in X_n, n=\overline{1,N}, w \in W = \left\{w \in \mathfrak{R}^L : \sum_{l=1}^{L} w_l = 1,\ w_l \ge 0\right\}.$$

Consider that each G_l follows the addition rule, i. e., for $l=\overline{1,L}$, we have

$$G_l\left(g_{l1}(x_1,a_1), g_{l2}(x_2,a_2), \dots, g_{lN}(x_N,a_{nN})\right) = \bar{g}_{l1}(x_1,a_1), \bar{g}_{l2}(x_2,a_2), \dots, \bar{g}_{LN}(x_N,a_{nN}).$$

So, the objective function in α-VMP$_w$ attaints the following form:

$$\sum_{n=1}^{N} \sum_{l=1}^{L} w_l\, \bar{g}_{Ln}(x_n,a_{nn}) = \sum_{n=1}^{N} w g_n(x_n,a_n).$$

Now, let us define:

$$A_n(w,u) = \min \left\{ \begin{array}{c} \sum_{i=1}^{n} w g_i(x_i,a_i) : G_l^l\left(g_{l1}(x_1), g_{l2}(x_2), \dots, g_{ln}(x_n)\right) \le u_q, \\ q=\overline{1,Q}, x_1 \in X_1, \dots,\ x_n \in X_n, n=\overline{1,N}, a \in L_\alpha(\tilde{a}) \end{array} \right\}. \qquad (1)$$

$$w = (w_1, \dots,\ w_L) > 0, u = (u_1, \dots, u_Q)$$

The recursive relation, for $n=\overline{1,N}$ is presented as follows:

$$A_n(w,u) = \min_{x_n \in X_n, a \in L_\alpha(\tilde{a})} \left(A_{n-1}\left(w, u^{n-1}(x_n,u)\right) + w g_n(x_n,a_n)\right) \qquad (2)$$

where, $u^{n-1}(x_n,u) = \left(u_1^{n-1}(x_n,u), \dots, u_Q^{n-1}(x_n,u)\right)$.

Assuming the monotonicity of G_l^l, let u_Q^{n-1} be defined as

$$u_Q^{n-1}(x_n, u) = \sup\{v \in \Re: u_Q^{n-1}(v, h_{qn}(x_n) \leq u_q, q = \overline{1, Q}\}.$$

Theorem 1. Let the assumptions 1 and 2 be satisfied. Also, suppose $(x_1^*, x_2^*, \ldots, x_n^*)$, $(a_1^*, a_2^*, \ldots, a_n^*) \in L_\alpha(\tilde{a})$ is an α −pareto optimal solution of $A_n(w^*, u)$ for some $w^* \in W$. Then $(x_1^*, x_2^*, \ldots, x_{n-1}^*), (a_1^*, a_2^*, \ldots, a_{n-1}^*) \in L_\alpha(\tilde{a})$ would be an α −pareto optimal solution for $A_{n-1}(w^*, u^{n-1}(x_n, u))$.

Proof (see, Mine and Fukushima [2]).

4. Stability Set of the First Kind

Definition 9. Given a particular w^* containing the corresponding α −pareto optimal solution(x^*, a^*). The stability set of first kind of $(\alpha - \text{VMP})$ relative to (x^*, a^*) is designated as follows:

$$S(x^*, a^*) = \{(w^*, a^*) \in \Re^{6l}: \text{is an } \alpha - \text{pareto optimal solution of } (\alpha - \text{VMP})\}.$$

Here, $\Re^{6l} = \Re^{l+5l}, w \in \Re^l, P = (p_1, p_2, p_3, p_4, p_5) \in \Re^{5l}, l = \overline{1, L}, L \geq 2$.

4.1. Computation of first kind stability set

Let a point (x^*, a^*) be an α −pareto optimal solution for $(\alpha - \text{VMP})$. Therefore, we can find a point $w^* \in W$ so that (x^*, a^*) becomes an α −pareto optimal solution of (α-VMP$_w$). Based on the stability for (α-VMP$_w$), it refers that we can find a point $w \in \Re^l, w \geq 0, E \in \Re^Q, E \geq 0$ and $F \in \Re^l, F \geq 0$ so that the below mentioned Kuhn- Tucker conditions are hold (Mangasarian [42]; Khalifa and Kumar [43]).

$$w^T \frac{\partial G}{\partial x}(x^*, a^*) + E^T \frac{\partial H}{\partial x}(x^*) = 0, \quad (3)$$

$$w^T \frac{\partial G}{\partial a}(x^*, a^*) - F^T \frac{\partial \mu_{\tilde{a}}}{\partial a}(a^*) = 0, \quad (4)$$

$$\sum_{i=1}^n w_i = 1, \quad (5)$$

$$H(x^*) \leq 0, \quad (6)$$

$$\alpha - \mu_{\tilde{a}}(a^*) \leq 0, \quad (7)$$

$$E^T H(x^*) = 0, \quad (8)$$

$$F^T(\alpha - \mu_{\tilde{a}}(a^*)) = 0, \quad (9)$$

$$w \geq 0, E, \text{and } F \geq 0. \quad (10)$$

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Let the two sets $B(x^*)$ and I be defined by

$$B(x^*) = \left\{q: H_q\left(h_{q1}(x_1^*), h_{q2}(x_2^*), \dots, h_{QN}(x_N^*)\right) = 0\right\}, \text{ and}$$

$$I = \left\{l \in \{1,2,\dots,L\}: \mu_{\tilde{a}_l}(a_l) = \alpha\right\}.$$

As a result, we get the two linear independent systems of equations below.

$$w^T \frac{\partial G}{\partial x}(x^*, a^*) + \sum_{q \in B(x^*)} \gamma_q \frac{\partial H_q}{\partial x}(x^*) = 0, \quad (11)$$

$$\sum_{l=1}^{L} w_l \frac{\partial G_l}{\partial a_\beta}(x^*, a^*) - F_l \frac{\partial \mu_{\tilde{a}_l}}{\partial a_l}(a_l^*) = 0, \quad (12)$$

$$\sum_{l=1}^{L} w_l = 1, w_l \geq 0, l = \overline{1, L}, \gamma_q \geq 0, q \in B(x^*), F_l \geq 0, l \in I, F_l = 0, l \neq I. \quad (13)$$

The system (13) can be rewritten as presented below

$$[M' \quad V'] \begin{bmatrix} w \\ \mu \end{bmatrix} = 0 \quad (14)$$

where, $M' = [c'_{ij}]$ is $r \times L$ matrix, $V' = [v'_{ij}]$ is an $h \times k$ matrix, $w \in \mathfrak{R}^L, \mu \in \mathfrak{R}^K, w \geq 0, w \neq 0$ and $\mu \geq 0, E \in \mathfrak{R}^r$, and $F \in \mathfrak{R}^h$. Here, r is the cardinality of $B(x^*)$, and h is the cardinality of I.

Consider that $v'_{ij} = 0,\ j = \overline{1, m},\ i \in J \subset \{1,2,\dots,r\}$. Here, the cardinality of J is $(r-m)$. Therefore, let us consider the below mentioned system in matrix form:

$$[M \quad V] \begin{bmatrix} w \\ \mu \end{bmatrix} = 0 \quad (15)$$

Here, M is a matrix of order $m \times L$.

V is a matrix of order $m \times K$.

Consequently, system (11) along with the equation $\sum_{j=1}^{L} M'_{ij} w_j = 0, i \in J$, provides another system (14) that, in turn, becomes equivalent to the previous system (11).

Proposition 1. (Zeleny [45]). If $K \geq m$, then

$$S(x^*, a^*) = \left\{(w, p) \in \mathfrak{R}^{6l}: w^T M^T (V_1^T)_j^{-1} \leq 0, j = \overline{1, m},\ \sum_{j=1}^{L} M'_{ij} w_j = 0, i \in J\right\} \quad (16)$$

where, $V = [V_1 \quad V_2]$. Also V_1 and V_2 are matrices of order $m \times m$ and $m \times (k-m)$, respectively.

Proposition 2. (Zeleny [45]). If $K \geq m$, then we have

$$(x^*, a^*) = \left\{(w, p) \in \mathfrak{R}^{6l}: (w^T M_2^T - M_1^T (V_1^T)^{-1} V_2^T) = 0, j = \overline{1, k-m}, w^T M^T (V_1^T)_j^{-1} \leq\right.$$

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$0, \sum_{j=1}^{L} M'_{ij} w_j = 0, i \in J\}$ (17)

5. An Algorithm

In this section, an algorithm for determining the $S(x^*, a^*)$ is presented in the below steps:

Step 1: Start at $\alpha = 0$.

Step 2: Define the membership grades of the fuzzy number $\tilde{a}$ as per definition 2.

Step 3: Formulate the piecewise quadratic fuzzy dynamic multi-objective problem,

i. e., (PQF-VMP)

Step 4: Choose a point $w^* \in W$, that is by using the relation (2) to achieve the α −pareto optimal solution (x^*, a^*) of (α-VMP$_w$).

Step 5: Putting the value of the (x^*, a^*) in the Kuhn- Tucker conditions, we have systems (11) as well as (15). In addition, we can use the Gauss Elimination method for solving system (12).

Step 6: Based on the Lagrange multipliers values, we obtain

(i) When $r = q + k - m$, we have $S(x^*, a^*) = \{\epsilon w^* : \epsilon > 0\}$,

(ii) When $k \geq m$, we have that $S(x^*, a^*)$ is provided by (16),

(iii) When $k < m$, we have that $S(x^*, a^*)$ is provided by (17).

Step 7: Set $\alpha = (\alpha + \varepsilon) \in [0, 1]$. Then, move to step 1.

Step 8: Repeat the interval at the steps of the proposed algorithm until the $[0, 1]$ is completely nullified.

6. A Numerical Example

Take into account the following (PQF-VMP)

$$min \begin{pmatrix} g_1(x, \tilde{a}_1^{PQ}) = \left(x_1 - \tilde{a}_{11}^{pq}\right)^2 + x_2^2 + x_3^2, \\ g_2(x, \tilde{a}_2^{PQ}) = (x_1 - 1)^2 + \left(x_2 + \tilde{a}_{22}^{PQ}\right) + (x_3 - 2)^2, \\ g_3(x, \tilde{a}_3^{PQ}) = 2x_1 + x_2^2 + \left(x_3 - \tilde{a}_{33}^{PQ}\right)^2 \end{pmatrix} \quad (18)$$

s. t.

$$Q = \{x \in \Re^3 : x_1 + x_2 + x_3 \leq 3,\ x_j \geq 0,\ j = 1, 2, 3\}. \quad (19)$$

Here, $\tilde{a}_{11}^{pq} = (0, 0.2, 3, 4,6, 6)$, $\tilde{a}_{22}^{PQ} = (0.2, 0.4, 4, 5.6, 7)$, $\tilde{a}_{33}^{PQ} = (2, 3.4, 6, 9.8, 11)$

The close intervals approximation for $\tilde{a}_{11}^{pq}, \tilde{a}_{22}^{PQ}$, and $\tilde{a}_{33}^{PQ}$ are as follows:

$$\tilde{a}_{11}^{pq} \in [0.2, 4.6], \quad \tilde{a}_{22}^{PQ} \in [0.4, 5.6] \text{ and } \quad \tilde{a}_{33}^{PQ} \in [3.4, 9.8]. \tag{20}$$

The (0.5-VMP) can be written as

$$\min\begin{pmatrix} g_1(x, a_1) = (x_1 - a_{11})^2 + x_2^2 + x_3^2, \\ g_2(x, a_2) = (x_1 - 1)^2 + (x_2 + a_{22}) + (x_3 - 2)^2, \\ g_3(x, a_3) = 2x_1 + x_2^2 + (x_3 - a_{33})^2 \end{pmatrix} \tag{21}$$

s. t.

$$x_1 + x_2 + x_3 \leq 3,$$

$$\tilde{a}_{11}^{pq} \in \ [0.2, 4.6], \ \ \tilde{a}_{22}^{PQ} \in [0.4, 5.6]$$

$$\text{and } \ \tilde{a}_{33}^{PQ} \in [3.4, 9.8],$$

$$x_1, x_2, x_2 \geq 0. \tag{22}$$

By applying the weighting method (Chankong and Haimes [40]), we have

$$\sum_{l=1}^{3} w_l g_1\,(x, a)$$

s. t. (23)

Constraints in (22).

At the point $w_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, the dynamic programming approach steps arise.

Firstly,

$$\mathrm{A}_1(\mathrm{w}^*, 0) = \min\left\{\sum_{l=1}^{3} w_l^0 g_{l1}\,(x_1,\ a_{11}): x_1 \leq 3, 0.2 \leq a_{11} \leq 4.6, x_1 \geq 0\right\}$$

$$= \min\left\{\frac{1}{3}(x_1 - a_{11})^2 + \frac{1}{3}(x_1 - 1)^2 + \frac{2}{3}x_1 : x_1 \leq 3, 0.2 \leq a_{11} \leq 4.6, x_1 \geq 0\right\}.$$

The 0.5 −pareto optimal solution is $(x_1^*,\ \mathrm{a}_{11}^*) = (0.1, 0.2)$.

Secondly,

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$$\mathrm{A}_2(\mathrm{w}^*,0) = \min\left\{ \begin{array}{c} \sum_{l=1}^{3} w_l^0 g_{l1}\,(x_1^*,\ \mathrm{a}_{11}^*) + g_{l2}(x_2, a_{22}) : x_1^* + x_2 \le 3, x_1^*, x_2 \ge 0, \\ 0.4 \le a_{22} \le 5.6 \end{array} \right\}$$

$$=\min\left\{ \begin{array}{c} 0.5 + \frac{1}{3}x_2^2 + \frac{1}{3}(x_2 + a_{22})^2 + \frac{1}{3}x_2^2 : 0.1 + x_2 \le 3, \\ 0.4 \le a_{22} \le 5.6, x_2 \ge 0 \end{array} \right\}.$$

The 0.5 −pareto optimal solution is$(x_1^*, x_2^*,\ \mathrm{a}_{11}^*, a_{22}^*) = (0.1, 0, 0.2\ ,0.4)$.

Thirdly,

$$\mathrm{A}_3(\mathrm{w}^*,0) = \min\left\{ \begin{array}{c} \sum_{l=1}^{3} w_l^0 g_{l1}\,(x_1^*,\ \mathrm{a}_{11}^*) + g_{l2}(x_2^*,\ a_{22}^*) + g_{l3}(x_3, a_{33}) : \\ x_1^* + x_2^* + x_3 \le 3, x_1^*, x_2^*, x_3 \ge 0, \\ 3.4 \le a_{33} \le 9.8 \end{array} \right\}$$

$$=\min\left\{ \begin{array}{c} 0.39 + \frac{1}{3}x_3^2 + \frac{1}{3}(x_3 - 2)^2 + \frac{1}{3}(x_3 - a_{33})^2 : 0.1 + x_3 \le 3, \\ 3.4 \le a_{33} \le 9.8, x_3 \ge 0 \end{array} \right\}.$$

The 0.5 −pareto optimal solution is summarized as follows:

$$(x_1^*, x_2^*, \mathrm{x}_3^*\ \ \mathrm{a}_{11}^*, a_{22}^*, a_{33}^*) = (0.1, 0, 1.8, \quad 0.2, 0.4, 0.34).$$

Now, let us determine $\mathrm{S}(\mathrm{x}^*, \mathrm{a}^*)$ as described below

Systems (11) and (12) allowed

$$-0.2w_1 - 1.8w_2 + 2w_3 + \gamma_1 = 0,$$

$$0w_1 + 0.8w_2 + 0w_3 + \gamma_2 = 0,$$

$$3.6w_1 - 0.4w_2 - 3.2w_3 + \gamma_3 = 0,$$

$$0.2w_1 + 0w_2 + 0w_3 - \omega_2 = 0,$$

$$0w_1 + 0.8w_2 + 0w_3 - \omega_3 = 0,$$

$$0w_1 + 0w_2 + 3.2w_3 - \omega_4 = 0,$$

Then

$$M = \begin{bmatrix} -0.2 & -1.8 & 2 \\ 0 & 0.8 & 0 \\ 3.6 & -0.4 & -3.2 \end{bmatrix}, \ V = V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M^T(V_1^T)^{-1} = \begin{bmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{bmatrix}$$

$$w^T M^T (V_1^T)^{-1} = [w_1 \quad w_2 \quad w_3] \begin{bmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{bmatrix}$$

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$$= (-0.2w_1 - 1.8w_2 + 2w_3 \quad 0.8w_2 \quad 3.6w_1 - 0.4w_2 - 3.2w_3).$$

Thus, we obtain

$$S(0.1, 0,1.8, \qquad 0.2, 0.4, 0.34) = \left\{ \begin{array}{c} (w,p,s,h): w_1 \geq 9w_2 - 10w_3,\ w_2 \geq 8w_3 - 9w_1, \\ w_1 + w_2 + w_3 = 1, p_1 < p_2 = 1 - 4p_1 < p_3 < p_4, \\ 0 < p_1 < 0.2, s_1 < s_2 = 1 - 1.5s_1 < s_3 < s_4, \\ 0 < s_1 < 0.4,\ h_1 < h_2 = 17 - 4h_1 < h_3 < h_4, \\ 000 < h < 3.4 \end{array} \right\}$$

7. Conclusions and Future Works

The dynamic multi-objective programming issue with piecewise quadratic fuzzy parameters has been investigated in this study. The first kind stability set has been identified, and the algorithm allows the problem solver for the decomposition of the parametric w- space. There are various future research directions to work on the proposed paper. One of them is to extend further this study to other fuzzy-type uncertainties, for example, Intuitionistic fuzzy sets, Pythagorean fuzzy sets, etc. Another possible scope is to include the spherical fuzzy sets, and neutrosophic sets considering wide coverage of decision-making problems in real-life situations.

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