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Article

Stochastic Comparisons of Lifetimes of used Standby Systems

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Abstract: In this paper, we first establish upper stochastic bounds on the lifetime of a used cold standby system with arbitrary age, using the likelihood ratio order and the usual stochastic order. Then, stochastic comparisons are made between the lifetime of a used cold standby system with age t and the lifetime of a cold standby system consisting of used components with age t using the likelihood ratio order and the usual stochastic order. We use illustrative examples to explore the results presented.

Keywords: likelihood ratio order; usual stochastic order; cold-standby system; log-concave; log-convex; lifetime.

1. Introduction and Preliminaries

Cold standby systems are backup systems that are not ready for use until they are needed. In other words, they are kept on standby but not actively operated. This type of backup system is commonly used in situations where the primary system has a long lifetime and is unlikely to fail frequently (see, e.g., Kumar and Agarwal [17], Kou and Zuo [18], Yang [31], and Peng et al. [22]). Reliability analysis of cold standby systems involves evaluating the probability of failure of the primary system and the time required to switch to the backup system. The reliability of the primary system is determined by analyzing its failure rate, while the reliability of the backup system is determined by analyzing its startup time and the probability of failure during startup. In the context of replacement strategies and related optimization problems, cold standby systems have been used repeatedly in the literature (see, e.g., Coit [7], Yu et al. [32], Jia and Wu [13], Xing et al. [30], and Ram et al. [24]).

Several methods can be used to analyze the reliability of cold standby systems, including fault tree analysis, reliability block diagrams, and Markov models. These methods allow engineers to identify potential failure modes, estimate the probability of system failure, and evaluate the effectiveness of backup systems. The study of the reliability of complex systems using cold standby systems has been conducted by many researchers for engineering problems (See, for example, Azaron et al. [3], Wang et al. [28], Wang et al. [29], and Behboudi et al. [5]).

Stochastic comparisons between the random lifetimes of various complex systems have been a subject of increasing interest among engineers and system designers. This enables them to have, for example, an optimization problem to solve and, consequently, a plan to prepare a product with greater reliability. The theory of stochastic orderings in applied probability has been recently utilized to compare the lifetime of coherent systems equipped by cold-standby units from a stochastic point of view (cf. Boland and El-Neweihi [6], Li et al. [19], Eryilmaz [8], Eryilmaz [9], Roy and Gupta [26] and Roy and Gupta [25]).

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However, stochastic comparisons of lifetimes of general cold-standby systems with an arbitrary age, have not been considered in the literature thus far. As will be clarified in the sequel, the lifetime of a component in a system with additional $(n-1)$ cold-standby spares is the sum of random variables, thus, in this regard, a few studies may be found in Zhao and Balakrishnan [33], Amiri et al. [2], Khaledi and Amiri [16] and Amiripour et al. [1] among others.

In the current study, we develop some stochastic ordering results using the well-known usual stochastic order and the likelihood ratio order, involving the lifetime of a used cold-standby system with an arbitrary age, and provide some bounds for the rf of the lifetime of such a system. The rf of the lifetime of a cold-standby system with used components of the age t, is used to provide upper bound and lower bound for the rf of the used cold-standby system with age t.

In the sequel we will use some notations. Let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ be a random vector and $\underset{\sim}{x} = (x_1, \ldots, x_n)$ be a vector of observations as a realization of $\underset{\sim}{X}$. Denote $S_{m,\underset{\sim}{X}} = \sum_{i=1}^{m} X_i$ and $S_{m,\underset{\sim}{x}} = \sum_{i=1}^{m} x_i$ with $m = 1, 2, \ldots, n$. Consider the cold standby system consisting of n components. Initially, one component starts working and the other $n-1$ components are in cold standby mode. When the working component fails, the components in standby mode are replaced one by one until all components have failed and the cold standby system fails. The cold standby system means that the components do not fail or degrade in standby mode and that the standby period does not affect the life of the components in future use. When the failed component is replaced by the standby component, the switch is absolutely reliable and transmission is instantaneous. Let $X_1, \ldots, X_n$ be the lifetimes of the n components with cumulative distribution functions $F_{X_1}, \ldots, F_{X_n}$ and corresponding reliability functions $\bar{F}_{X_1}, \ldots, \bar{F}_{X_n}$. We also assume $X_1, \ldots, X_n$ are independent. Then, the lifetime of the cold standby system is

$$S_{n,\underset{\sim}{X}} = X_1 + X_2 + \ldots + X_n. \tag{1}$$

The rf or the survival function (SF) of the lifetime of the cold standby system given in (1), is

$$\bar{F}_{S_{n,\underset{\sim}{X}}}(t) = P(X_1 + X_2 + \ldots + X_n > t) = \bar{F}_{X_1}(t) * \bar{F}_{X_2}(t) * \ldots * \bar{F}_{X_n}(t), \tag{2}$$

where $*$ represents the convolution operator. It is known that when X_i and X_j, for $i \neq j$ are independent, then $\bar{F}_{X_i}(t) * \bar{F}_{X_j}(t) = \int_{-\infty}^{+\infty} \bar{F}_{X_i}(t-x) f_{X_j}(x)\, dx$, where f_{X_j} is the probability density function (pdf) of X_j, which is the rf of the convolution of X_i and X_j, i.e. the rf of $X_i + X_j$. Thus, from (2), we can write

$$\bar{F}_{S_{n,\underset{\sim}{X}}}(t) = \bar{F}_{S_{n-1,\underset{\sim}{X}}}(t) * \bar{F}_{X_n}(t) = \int_{-\infty}^{+\infty} \bar{F}_{S_{n-1,\underset{\sim}{X}}}(t-x) f_{X_n}(x)\, dx. \tag{3}$$

We will need some other preliminaries in the continuing part of the paper. Suppose that X is the lifetime of a fresh item as its life span. We may need to recognize the distribution of the lifetime of that item at the age t. The random variable $X_t = [X - t | X > t]$, which is called the residual lifetime of an item with original life length X at the time t provided that the item is already alive at this time, is ordinarily utilized to represent and model the lifetime of a used component or item. Let X have pdf f_X (whenever it exists) and rf $\bar{F}_X$. Then, X_t has pdf

$$f_{X_t}(x) = \frac{f_X(t+x)}{\bar{F}_X(t)}\, dx, \quad x \geq 0,$$

and it has rf

$$\bar{F}_{X_t}(x) = \frac{\bar{F}_X(t+x)}{\bar{F}_X(t)} : \quad x \geq 0,$$

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which are valid as $t \in \{t \geq 0 : \bar{F}_X(t) > 0\}$. The mean residual lifetime (MRL) function of X is given by

$$m_X(t) = E[X - t|X > t] = \int_t^{+\infty} \frac{\bar{F}_X(x)}{\bar{F}_X(t)}, \quad t \in \{t \geq 0 : \bar{F}_X(t) > 0\}.$$

Stochastic ordering of distributions have been a useful tool for statisticians in the context of testing statistical hypothesises. Two well-know stochastic order will be used throughout this paper. The following definition can be found in Shaked and Shanthikumar [27].

Definition 1. *Let X and Y be two non-negative random variables with pdfs f_X and f_Y, and refs $\bar{F}_X$ and $\bar{F}_Y$, respectively. Then it is said that X is smaller or equal than Y in the*

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) whenever $\frac{f_Y(t)}{f_X(t)}$ is non-decreasing in $t \geq 0$.
(ii) usual stochastic order (denoted by $X \leq_{st} Y$) whenever $\bar{F}_X(t) \leq \bar{F}_Y(t)$, for all $t \geq 0$.

The stochastic orders in Definition 1 are connected to each other as it has been proved that $X \leq_{lr} Y$ implies $X \leq_{st} Y$ (see, e.g., Shaked and Shanthikumar [27]). Nonparametric classes of life distributions are usually based on the pattern of aging in some sense. For example by comparing X_{t_1} with X_{t_2} for $t_1 \leq t_2$ as two time points according to the likelihood ratio order, new patterns of aging are produced. The common parametric families of life distributions also feature monotone aging. In this context, the following definition is also applied in the paper.

Definition 2. *The random variable X with pdf f_X is said to have*

(i) Increasing likelihood ratio property (denoted as $X \in ILR$) whenever $f_X(t)$ is log-concave in $t \geq 0$.
(ii) Decreasing likelihood ratio property (denoted as $X \in DLR$) whenever $f_X(t)$ is log-convex in $t \geq 0$.

For example the exponential distribution has both ILR and DLR properties. The gamma distribution with shape parameter α and scale parameter λ has ILR property if $\alpha > 1$ and it has DLR property if $\alpha < 1$. For recognizing ILR and DLR properties in further well-known distributions we refer the readers to Bagnoli and Bergstrom [4]. The following definition is due to Karlin [14].

Definition 3. *Suppose that $h(x, y)$ is a non-negative function for all $x \in \mathfrak{X} \subseteq \mathbb{R}$ and for all $y \in \mathfrak{Y} \subseteq \mathbb{R}$. Then, it is said that $h(x, y)$ is totally positive of order 2 (denoted as TP_2) in $(x, y) \in \mathfrak{X} \times \mathfrak{Y}$, whenever $h(x_1, y_1)h(x_2, y_2) \geq h(x_1, y_2)h(x_2, y_1)$ for all $x_1 \leq x_2 \in \mathfrak{X}$ and for all $y_1 \leq y_2 \in \mathfrak{Y}$.*

According to Definition 3, by using the convention that $a/0 = +\infty$ for any $a > 0$, then $h(x, y)$ is TP_2 in (x, y) in the desired subset of $\mathbb{R}^2$, i.e. over the set $\mathfrak{X} \times \mathfrak{Y}$ if and only if, $\frac{h(x_2,y)}{h(x_1,y)}$ is non-decreasing in $y \in \mathfrak{Y}$, or equivalently if, $\frac{h(x,y_2)}{h(x,y_1)}$ is non-decreasing in $x \in \mathfrak{X}$.

2. Stochastic bounds for the lifetime of a used standby system

Let us consider independent and non-negative rvs $X_1, X_2, \ldots, X_n$, and denote by $X_1(t), X_2(t), \ldots, X_n(t)$ the corresponding conditional rvs, so that $X_i(t) = [X_i|X_i > t]$, for all $t \geq 0$ for which $R_{X_i}(t) > 0$. Note that for the random vector $\underset{\sim}{Z} = (Z_1, Z_2, \ldots, Z_m)$, the conditional rv $[\underset{\sim}{Z}|\underset{\sim}{Z} \in A]$ is a vector-valued rv whose joint distribution is equal with

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conditional join distribution of $\underline{Z}$ give $\underline{Z} \in A$, where A is a subset of $\mathbb{R}^m$ for which $P(\underline{Z} \in A) > 0$. The rv $X_i(t)$ has pdf

$$f_{X_i(t)}(x) = \frac{f_{X_i}(x) I[x > t]}{R_{X_i}(t)}. \tag{4}$$

Consider, now, the vector $\underline{X}(t) = (X_1, X_2, \dots, X_n | X_1 > t, X_2 > t, \dots, X_n > t)$ which, since X_i's are independent, it has joint cdf

$$f_{\underline{X}(t)}(\underline{x}) = \prod_{i=1}^{n} \frac{f_{X_i}(x_i) I[x_i > t]}{R_{X_i}(t)}. \tag{5}$$

Note that the ith marginal distribution of $\underline{X}(t)$ is the distribution of $X_i(t)$ provided that $X_1, X_2, \dots, X_n$ are independent. We can find that for any function $\psi : \mathbb{R}^n \mapsto \mathbb{R}$,

$$E[\psi(\underline{X}(t))] = \frac{E[\psi(\underline{X}) I[\underline{x} > \underline{t}]]}{\bar{F}_{\underline{X}}(\underline{t})} \tag{6}$$

where $\underline{t} = (t, t, \dots, t)$ is a vector with size n and $\bar{F}_{\underline{X}}$ is the joint reliability function of $\underline{X}$. We will utilize $\psi(\underline{x}) = S_{n,\underline{x}}$. Note that $(S_{n,\underline{X}})(t)$ is equal in distribution with the conditional rv $[S_{n,\underline{X}} | S_{n,\underline{X}} > t]$, where $S_{n,\underline{X}} = \sum_{i=1}^{n} X_i$. The following lemma will be used in the sequel.

Lemma 4. *Let $\underline{X} = (X_1, X_2, \dots, X_n)$ be a random vector, having joint pdf $f_{\underline{X}}(\underline{x})$. Consider $\underline{Y} = (Y_1, Y_2, \dots, Y_n)$ as a random vector with joint pdf*

$$f_{\underline{Y}}(\underline{y}) = \frac{I[S_{n,\underline{y}} > t] f_{\underline{X}}(\underline{y})}{P(S_{n,\underline{X}} > t)}, \tag{7}$$

Then, $(S_{n,\underline{X}})(t) =_{st} S_{n,\underline{Y}}$, where $=_{st}$ means equality in distribution.

Proof. It is sufficient to show that $R_{(S_{n,\underline{X}})(t)}(s) = R_{S_{n,\underline{Y}}}(s)$, for all $s \geq 0$. We can write

$$\begin{aligned} R_{(S_{n,\underline{X}})(t)}(s) &= P(S_{n,\underline{X}} > s) \\ &= P(S_{n,\underline{X}} > s | S_{n,\underline{X}} > t) \\ &= \begin{cases} 1 & : s \leq t \\ \frac{R_{S_{n,\underline{X}}}(s)}{R_{S_{n,\underline{X}}}(t)} & : s > t. \end{cases} \end{aligned}$$

On the other hand, from (24), we obtain

$$\begin{aligned} R_{S_{n,\underline{Y}}}(s) &= P(S_{n,\underline{Y}} > s) \\ &= \frac{P(S_{n,\underline{X}} > t \vee s)}{P(S_{n,\underline{X}} > t)} \\ &= \begin{cases} 1 & : s \leq t \\ \frac{R_{S_{n,\underline{X}}}(s)}{R_{S_{n,\underline{X}}}(t)} & : s > t, \end{cases} \end{aligned}$$

in which $t \vee s = \max\{t, s\}$. Thus, we proved the desired identity and, hence, the result follows. ∎

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The following result indicates that the lifetime of a used standby system of age t with n components is dominated in the sense of the usual stochastic order by the lifetime of a standby system composed of used components plus $(n-1)t$. Denote by $X_{i,t} = [X_i - t | X_i > t]$ which is valid for all $t \geq 0$ for which $R_{X_i}(t) > 0$ and note that X_i is the random lifetime of the ith component in the standby system, with $i = 1, 2, \ldots, n$. Denote

$$\underline{X}_t = (X_1 - t, X_2 - t, \ldots, X_n - t | X_1 > t, X_2 > t, \ldots, X_n > t), \quad t : P(\underline{X} > tI) > 0,$$

where $I = (1, 1, \ldots, 1)^\top$ is a vector with n components and $\underline{X} = (X_1, X_2, \ldots, X_n)^\top$. Notice that the marginal distribution of the ith random element in $\underline{X}_t$ corresponds with the distribution of $X_{i,t}$, $i = 1, 2, \ldots, n$, provided that $X_1, X_2, \ldots, X_n$ are independent.

Theorem 5. *Let $X_1, X_2, \ldots, X_n$ be independent rvs which are non-negative and $X_{1,t}, X_{2,t}, \ldots, X_{n,t}$ be also independent, for a fixed $t > 0$. Then:*

$$(S_{n,\underline{X}})_t \leq_{st} S_{n,\underline{X}_t} + (n-1)t. \tag{8}$$

Proof. For $n = 1$, the result is trivial. Let us assume $n = 2$. From Lemma 4, they can be founds rvs Y_1 and Y_2 with joint pdf

$$f_{\underline{Y}}(\underline{y}) = \frac{I[S_{2,\underline{y}} > t] f_{\underline{X}}(\underline{y})}{R_{S_{2,\underline{X}}}(t)}, \quad y_i \geq 0, i = 1, 2, \tag{9}$$

in which $\underline{y} = (y_1, y_2)$, such that $(S_{2,\underline{X}})(t) =_{st} S_{2,\underline{Y}}$. From (9), since X_1 and X_2 are independent, Y_1 has pdf

$$f_{Y_1}(y_1) = \frac{f_{X_1}(y_1) R_{X_2}(t - y_1)}{R_{S_{2,\underline{X}}}(t)}.$$

Then:

$$\begin{aligned} \frac{f_{Y_1}(y_1)}{f_{X_1(t)}(y_1)} &= \frac{R_{X_1}(t) R_{X_2}(t - y_1)}{R_{S_{2,\underline{X}}}(t) I[y_1 > t]} \\ &= \begin{cases} +\infty & : y_1 \leq t \\ \frac{R_{X_1}(t)}{R_{S_{2,\underline{X}}}(t)} & : y_1 > t. \end{cases} \end{aligned}$$

It is clear that $\frac{f_{Y_1}(y_1)}{f_{X_1(t)}(y_1)}$ is decreasing in y_1, thus, $Y_1 \leq_{lr} X_1(t)$. Since $\leq_{lr}$ implies $\leq_{st}$, thus

$$Y_1 \leq_{st} X_1(t). \tag{10}$$

By Equation (9), the conditional pdf of Y_2 given $Y_1 = y_1$ is derived as:

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{X_2}(y_2) I[S_{2,\underline{y}} > t]}{R_{X_2}(t - y_1)}.$$

For any $y_1 \geq 0$, we show that $\frac{f_{Y_2|Y_1}(y_2|y_1)}{f_{X_2(t)}(y_2)}$ is decreasing in $y_2 \geq 0$, as $y_1 + y_2 > t$. We have

$$\frac{f_{Y_2|Y_1}(y_2|y_1)}{f_{X_2(t)}(y_2)} = \begin{cases} +\infty & : y_2 \leq t, y_1 > t - y_2 \\ \frac{R_{X_2}(t)}{R_{X_2}(t - y_1)} & : y_2 > t, y_1 > t - y_2, \end{cases}$$

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which is decreasing in $y_2 \geq 0$. Therefore, $[Y_2|Y_1 = y_1] \leq_{lr} X_2(t)$. Hence,

$$[Y_2|Y_1 = y_1] \leq_{st} X_2(t). \tag{11}$$

From Equations (10) and (11), using Theorem 6.B.3 in Shaked and Shanthikumar [27], one gets:

$$[X_1 + X_2|X_1 + X_2 > t] = (S_{2,\underline{X}})(t) \leq_{st} X_1(t) + X_2(t).$$

It is easily verified that $S_{2,\underline{X}}(t) \leq_{st} S_{2,\underline{X}(t)}$, holds, if and only if, $S_{2,\underline{X}}(t) - t \leq_{st} S_{2,\underline{X}(t)} - t$. So, $(S_{2,\underline{X}})_t \leq_{st} S_{2,\underline{X}_t} + t$, i.e., (13) holds when $n = 2$. Finally, we prove the result by induction. Suppose that for $n = m \geq 2$, it holds that $(S_{m,\underline{X}})_t \leq_{st} S_{m,\underline{X}_t} + (m-1)t$. From preservation property of $\leq_{st}$ under a change in location of distributions, we have:

$$S_{m,\underline{X}}(t) \leq_{st} S_{m,\underline{X}(t)}. \tag{12}$$

From (12), since for $m = 2$ the result was proved, thus an application of Theorem 1.A.3(b) in Shaked and Shanthikumar [27] yields:

$$\begin{aligned} S_{m+1,\underline{X}}(t) &\leq_{st} S_{m,\underline{X}}(t) + X_{m+1}(t) \\ &\leq_{st} S_{m,\underline{X}(t)} + X_{m+1}(t) \\ &=_{st} S_{m+1,\underline{X}(t)}. \end{aligned}$$

Equivalently, one can write: $(S_{m+1,\underline{X}})_t \leq_{st} S_{m+1,\underline{X}_t} + mt$. Hence, the proof is obtained. ∎

In the context of Theorem 5 one may realize that, if in (13), $t = 0$, then $\leq_{st}$ becomes $=_{st}$. Using Theorem 5, an upper bound for the mean residual lifetime (MRL) function of a standby system can be provided. Since $\leq_{st}$ implies the expectation order, thus, from Theorem 5,

$$\begin{aligned} m_{S_{n,\underline{X}}}(t) &= E[(S_{n,\underline{X}})_t] \\ &\leq E[S_{n,\underline{X}_t}] + (n-1)t \\ &= \sum_{i=1}^{n} m_{X_i}(t) + (n-1)t, \end{aligned}$$

where $m_{S_{n,\underline{X}}}(t) = E[\sum_{i=1}^{n} X_i - t| \sum_{i=1}^{n} X_i > t]$ is the MRL function of a standby system with independent component lifetimes $X_1, X_2, \ldots, X_n$, and $m_{X_i}(t) = E[X_{i,t}]$ is the MRL function of X_i for $i = 1, 2, \ldots, n$. The bound provided for the MRL function of a standby system is valuable because the MRL function of the system which depends on the distribution of convolution of n rvs, which has no closed form in many situation, does not have an explicit formula. The rv X has a gamma distribution with the shape parameter α and the scale parameter λ whenever X has density $f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$ (denote it by $X_i \sim G(\alpha, \lambda)$). Let us suppose that under consideration is a standby system with independent heterogenous exponential component lifetimes $X_1, X_2, \ldots, X_n$, so that X_i has mean $\frac{1}{\lambda_i}$ ($X_i \sim Exp(\lambda_i)$), where $\lambda_i > 0, i = 1, 2, \ldots, n$. It is known that $m_{X_i}(t) = \frac{1}{\lambda_i}$, however, the $S_{n,\underline{X}} = \sum_{i=1}^{n} X_i$ as the random lifetime of the standby system, has a gamma distribution ($S_{n,\underline{X}} \sim G(n, \sum_{i=1}^{n} \lambda_i)$) with an indefinite MRL function. However,

$$m_{S_{n,\underline{X}}}(t) \leq \sum_{i=1}^{n} \frac{n}{\lambda_i} + (n-1)t, \text{ for all } t \geq 0.$$

In the following example, the result of Theorem 5 is applied.

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Example 6. *Suppose that $X_1 \sim G(2,3)$ and $X_2 \sim G(1/2,3)$ are two independent random variables and assume that $X_{1,t}$ and $X_{2,t}$ are also independent. Note that $X_1 \in ILR$ and $X_2 \in DLR$. Consider a two-units standby system. Using Theorem 5, an upper bound for the rf of the used standby system with age $t = 0.1$ is derived. Specifically, it is shown that*

$$(S_{2,\underline{X}})_t = (X_1 + X_2)_t \leq_{st} X_{1,t} + X_{2,t} + t = S_{2,\underline{X}_t} + t.$$

Since it is trivial that for all $x \leq 0.1$, one has $P(S_{2,\underline{X}_t} + t > x) = 1$, thus for all $x \leq 0.1$, $P(S_{2,\underline{X}_t} + t > x) \geq P((S_{2,\underline{X}})_t > x)$. Therefore, it is enough to show that $P(S_{2,\underline{X}_t} + t > x) \geq P((S_{2,\underline{X}})_t > x)$, for all $x > 0.1$. It is seen that for $t = 0.1$

$$P((S_{2,\underline{X}})_t > x) = \frac{\bar{F}_{X_1+X_2}(0.1+x)}{\bar{F}_{X_1+X_2}(0.1)} = \frac{\int_{0.1+x}^{+\infty} u^{\frac{3}{2}} exp(-3u)\, du}{\int_{0.1}^{+\infty} u^{\frac{3}{2}} exp(-3u)\, du}.$$

On the other hand, for all $x > 0.1$, we can get

$$P(S_{2,\underline{X}_t} + t > x) = \int_{0.1}^{x} \bar{F}_{X_{1,0.1}}(x - x_2) f_{X_{2,0.1}}(x_2 - 0.1) dx_2 + \bar{F}_{X_{2,0.1}}(x - 0.1),$$

in which the rf of $X_{1,0.1} = [X_1 - 0.1 | X_1 > 0.1]$ is acquired as

$$\bar{F}_{X_{1,0.1}}(x) = \frac{\bar{F}_{X_1}(0.1+x)}{\bar{F}_{X_1}(0.1)} = \frac{(1 + 3(0.1+x))\exp(-3(0.1+x))}{1.3\exp(-0.3)},$$

and, similarly, the rf of $X_{2,0.1} = [X_2 - 0.1 | X_2 > 0.1]$ is obtained as

$$\bar{F}_{X_{2,0.1}}(x) = \frac{\bar{F}_{X_2}(0.1+x)}{\bar{F}_{X_2}(0.1)} = \frac{\int_{0.1+x}^{+\infty} u^{-\frac{1}{2}} \exp(-3u) du}{\int_{0.1}^{+\infty} u^{-\frac{1}{2}} \exp(-3u) du},$$

and, consequently,

$$f_{X_{2,0.1}}(x) = \frac{(0.1+x)^{-\frac{1}{2}} \exp(-3(x+0.1))}{\int_{0.1}^{+\infty} u^{-\frac{1}{2}} \exp(-3u) du}.$$

In Figure 1, the graph of SFs of $(S_{2,\underline{X}})_t$ and $S_{2,\underline{X}_t} + t$ is plotted, which makes it clear that for $0.1 < x < 5$, $P(S_{2,\underline{X}_t} + t > x) \geq P(S_{2,\underline{X}})_t)$ when $t = 0.1$.

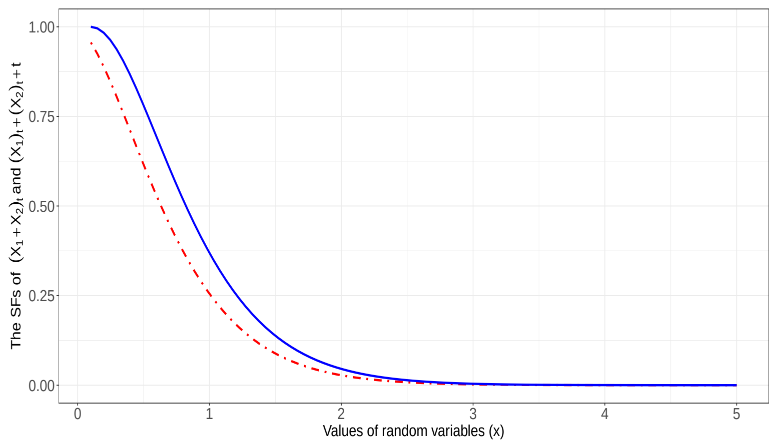

Figure 1. Plot of the survival function of $S_{2,\underline{X}_t} + t$ (solid line) and the survival function of $(S_{2,\underline{X}})_t$ (dot-dashed line) for $t = 0.1$ and for $x \in (0.1, 5)$.

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In the sequel of this section, the result of Theorem 5 is strengthened to the case where the likelihood ratio order is used. However, in this case, the random lifetimes of the components need to to have log-concave density functions which means that the components lifetimes have to fulfill the ILR property. We first give the following technical lemmas. The proof of the following lemmas, being straightforward, are omitted.

Lemma 7. *Suppose that Y and Z are two rvs with pdfs f_Y and f_Z, respectively. The following assertions hold:*

(a) If Y has support $S_Y = (l_Y, +\infty)$ and Z has support $S_Z = (l_Z, +\infty)$, so that $l_Y \geq l_Z$ then, $Y \geq_{lr} Z$ if, and only if, $\frac{f_Y(s)}{f_Z(s)}$ is non-decreasing in $s > l_Y$.

(b) If $Y \geq_{lr} Z$, then $Y - c \geq_{lr} Z - c$, for $c > 0$.

Lemma 8. *The following assertions hold true:*

(a) For any non-negative rv W, for all $t \geq 0$ for which $P(W > t) > 0$, it holds that $W_t =_{st} W(t) - t$ where $W_t = [W - t | W > t]$ and $W(t) = [W | W > t]$.

(b) For all $t \geq 0$, for which $P(X_i > t) > 0, i = 1, 2, \ldots, n$ it holds that $S_{n,\underline{X}(t)} =_{st} S_{n,\underline{X}_t} + nt$.

Theorem 9. *Let $X_1, X_2, \ldots, X_n$ be non-negative independent rvs which are all ILR, and suppose that $X_{1,t}, X_{2,t}, \ldots, X_{n,t}$ are independent, for a fixed $t > 0$. Then:*

$$(S_{n,\underline{X}})_t \leq_{lr} S_{n,\underline{X}_t} + (n-1)t. \tag{13}$$

Proof. Fix $t > 0$. Firstly, we prove that

$$Y = S_{n,\underline{X}(t)} \geq_{lr} (S_{n,\underline{X}})(t) = Z,$$

in which $S_{n,\underline{X}(t)} = \sum_{i=1}^{n} X_i(t)$ and $(S_{n,\underline{X}})(t) = (\sum_{i=1}^{n} X_i)(t)$ with $X_i(t) = [X_i | X_i > t]$. Since $l_Y = nt$ and $l_Z = t$, thus by Lemma 7 it is enough to show that $\frac{f_Y(s)}{f_Z(s)}$ is non-decreasing in s for all $s > nt$. One has

$$\begin{aligned} f_Z(s) &= \frac{f_{S_{n,\underline{X}}}(s) I[s > t]}{R_{S_{n,\underline{X}}}(t)} \\ &= \frac{f_{S_{n,\underline{X}}}(s)}{R_{S_{n,\underline{X}}}(t)}, \quad s > nt. \end{aligned}$$

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Denote $S_{n-1,\underline{X}(t)} = \sum_{i=1}^{n-1} X_i(t)$, and similarly, $S_{n-1,\underline{X}} = \sum_{i=1}^{n-1} X_i$. Therefore, for all $s > nt$:

$$\begin{aligned}
\frac{f_Y(s)}{f_Z(s)} &= R_{S_{n,\underline{X}}}(t)\frac{\int_0^{+\infty} f_{X_n(t)}(s-y)f_{S_{n-1,\underline{X}(t)}}(y)dy}{\int_0^{+\infty} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy} \\
&= \frac{R_{S_{n,\underline{X}}}(t)}{R_{X_n}(t)}\frac{\int_0^{s-t} f_{X_n}(s-y)f_{S_{n-1,\underline{X}(t)}}(y)dy}{\int_0^{s} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy} \\
&= \frac{R_{S_{n,\underline{X}}}(t)}{R_{X_n}(t)}\int_0^s \frac{f_{S_{n-1,\underline{X}(t)}}(y)}{f_{S_{n-1,\underline{X}}}(y)}\frac{I[y<s-t]f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)}{\int_0^{s} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy}dy \\
&= \frac{R_{S_{n,\underline{X}}}(t)}{R_{X_n}(t)}\int_0^s \frac{f_{S_{n-1,\underline{X}(t)}}(y)}{f_{S_{n-1,\underline{X}}}(y)}\frac{I[y<s-t]f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)}{\int_0^{s-t} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy}dy\ P_n(t|s), \qquad (14)
\end{aligned}$$

where

$$\begin{aligned}
P_n(t|s) &= P(X_n > t|S_{n,\underline{X}} = s) \\
&= P(S_{n-1,\underline{X}} \le s-t|S_{n,\underline{X}} = s) \\
&= \int_0^{s-t} f_{S_{n-1,\underline{X}}|S_{n,\underline{X}}}(y|s)dy \\
&= \frac{\int_0^{s-t} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy}{\int_0^{s} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy}.
\end{aligned}$$

Thus, from Eq. (14), one can write

$$\frac{f_Y(s)}{f_Z(s)} = \frac{R_{S_{n,\underline{X}}}(t)}{R_{X_n}(t)}P_n(t|s)E[\Phi(Y^\star(s))], \qquad (15)$$

where $\Phi(y) = \frac{f_{S_{n-1,\underline{X}(t)}}(y)}{f_{S_{n-1,\underline{X}}}(y)}$ and $Y^\star(s)$ is an rv with pdf

$$f^\star(y|s) = \frac{I[y<s-t]f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)}{\int_0^{s-t} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy}. \qquad (16)$$

Since $X_i \in ILR$, for all $i = 1,2,\ldots,n$ thus from Theorem 1.C.53 in Shaked and Shanthikumar [27], we deduce that $[X_n|S_{n,\underline{X}} = s_1] \le_{lr} [X_n|S_{n,\underline{X}} = s_2]$, for all $s_1 \le s_2$. Hence, $[X_n|S_{n,\underline{X}} = s_1] \le_{st} [X_n|S_{n,\underline{X}} = s_2]$, for all $s_1 \le s_2$ which validates that

$$P_n(t|s) \ \text{ is non-decreasing in } \ s > nt, \qquad (17)$$

for all $t \ge 0$. From 16, we can write $f^\star(y|s) = \phi_1(y)\phi_2(s)\xi(y,s)$, in which

$$\phi_1(y) = f_{S_{n-1,\underline{X}}}(y) \ge 0,\ \ \phi_2(s) = \left[\int_0^{s-t} f_{X_n}(s-y)f_{S_{n-1,\underline{X}}}(y)dy\right]^{-1} > 0$$

and $\xi(y,s) = I[y < s-t]f_{X_n}(s-y)$. Since X_n is ILR, thus $f_{X_n}(s-y)$ is TP_2 in $(y,s) \in (0,+\infty)\times(0,+\infty)$. It is also plain to see that $I[y < s-t]$ is also TP_2 in $(y,s) \in (0,+\infty)\times(nt,+\infty)$. Hence, as the product of two TP_2 functions in a common domain is itself another TP_2 function, thus $\xi(y,s)$ is also TP_2 in $(y,s) \in (0,+\infty)\times(nt,+\infty)$. From Lemma 3 in

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Kayid and Alshagrawi [15], $f^\star(y|s)$ is TP_2 in $(y,s) \in (0,+\infty) \times (nt,+\infty)$. This reveals that $Y^*(s_1) \leq_{lr} Y^*(s_2)$, for all $s_1 \leq s_2 \in (nt,+\infty)$, and consequently,

$$Y^*(s_1) \leq_{st} Y^*(s_2), \text{ for all } s_1 \leq s_2 \in (nt,+\infty). \tag{18}$$

On the other hand, since $X_i \in ILR$, by Theorem 2.10 in Izadkhah et al. [12], the rv $X_i(t) = [X_i|X_i > t]$ having weighted distribution with respect to X_i with the weight function $w(x) = I[x > t]$ which is log-concave in x, for all $t \geq 0$, is also ILR. Now, by Theorem 1.C.9 of Shaked and Shanthikumar [27], since $X_i(t) \geq_{lr} X_i$ for all $i = 1,2,\ldots,n$, thus $S_{n,\underline{X}(t)} \geq_{lr} S_{n,\underline{X}}$. That is

$$\Phi(y) = \frac{f_{S_{n-1,\underline{X}(t)}}(y)}{f_{S_{n-1,\underline{X}}}(y)} \text{ is non-decreasing in } y > 0. \tag{19}$$

So, from (18), $E[\Phi(Y^\star(s))]$ is also non-decreasing in $s \in (nt,+\infty)$. In view of (17) and Eq. (15), it follows that $\frac{f_Y(s)}{f_Z(s)}$ is non-decreasing in $s \in (nt,+\infty)$. Therefore, we demonstrate it that $S_{n,\underline{X}(t)} \geq_{lr} (S_{n,\underline{X}})(t)$ which by Lemma 8(b) it further implies that $S_{n,\underline{X}(t)} + nt \geq_{lr} (S_{n,\underline{X}})(t)$. By applying Lemma 7, by choosing $c = t$, one gets $S_{n,\underline{X}(t)} + (n-1)t \geq_{lr} (S_{n,\underline{X}})(t) - t$, which by Lemma 8(a) yields $S_{n,\underline{X}(t)} + (n-1)t \geq_{lr} (S_{n,\underline{X}})_t$. The proof is completed. ∎

3. Comparison of a used standby system with a standby system composed of used components

In this section, we make stochastic comparisons between lifetimes of a used standby system with age t and another standby system composed of used components each has age t. We establish that when the $(n-1)$ ones of the n component lifetime distribution have a general density function (with absolutely continuous distribution) and the nth component is exponentially distributed, then the standby system with used components is more reliable and has smaller risk than the used standby system.

Theorem 10. *Let $X_1, X_2, \ldots, X_n$ be independent non-negative rvs with pdfs $f_{X_1}, f_{X_2}, \ldots, f_{X_n}$, respectively, so that*

$$S_{n-1,\underline{X}_t} + t \geq_{lr} S_{n-1,\underline{X}}, \tag{20}$$

for a fixed $t \geq 0$, in which $S_{n-1,\underline{X}} = \sum_{i=1}^{n-1} X_i$ and $S_{n-1,\underline{X}_t} = \sum_{i=1}^{n-1} X_{i,t}$ and that X_n follows exponential distribution with parameter λ_n. Suppose that $X_{1,t}, X_{2,t}, \ldots, X_{n,t}$ where $X_{i,t} = [X_i - t|X_i > t], i = 1,2,\ldots,n$ are also independent rvs. Then:

$$(S_{n,\underline{X}})_t \leq_{lr} S_{n,\underline{X}_t} \tag{21}$$

where $S_{n,\underline{X}} = \sum_{i=1}^{n} X_i$ and $S_{n,\underline{X}_t} = \sum_{i=1}^{n} X_{i,t}$.

Proof. We prove that $\frac{f_{S_{n,\underline{X}_t}}(s)}{f_{(S_{n,\underline{X}})_t}(s)}$ is non-decreasing in $s \geq 0$. Let us write

$$\begin{aligned}
\frac{f_{S_{n,\underline{X}_t}}(s)}{f_{(S_{n,\underline{X}})_t}(s)} &= \frac{R_{S_{n,\underline{X}}}(t)}{R_{X_n}(t)} \frac{\int_0^{+\infty} f_{X_{n,t}}(s-y) f_{S_{n-1,\underline{X}_t}}(y)dy}{\int_0^{+\infty} f_{X_n}(s+t-y) f_{S_{n-1,\underline{X}}}(y)dy} \\
&= \frac{R_{S_{n,\underline{X}}}(t)}{R_{X_n}(t)} \frac{\int_0^{+\infty} I[y < s] f_{X_n}(s+t-y) f_{S_{n-1,\underline{X}_t}}(y)dy}{\int_0^{+\infty} f_{X_n}(s+t-y) f_{S_{n-1,\underline{X}}}(y)dy}.
\end{aligned}$$

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Hence, it suffices to prove that

$$\frac{\int_0^{+\infty} I[y<s] f_{X_n}(s+t-y) dF_{S_{n-1,\underline{X}_t}}(y)}{\int_0^{+\infty} f_{X_n}(s+t-y) dF_{S_{n-1,\underline{X}}}(y)} \text{ is non-decreasing in } s \geq 0. \tag{22}$$

By changing the variable y into $y-t$ in the numerator in (22) one has

$$\begin{aligned}\frac{\int_0^{+\infty} I[y<s] f_{X_n}(s+t-y) dF_{S_{n-1,\underline{X}_t}}(y)}{\int_0^{+\infty} f_{X_n}(s+t-y) dF_{S_{n-1,\underline{X}}}(y)} &= \frac{\int_0^{+\infty} I[y<s+t] f_{X_n}(s+2t-y) dF_{S_{n-1,\underline{X}_t}}(y-t)}{\int_0^{+\infty} f_{X_n}(s+t-y) dF_{S_{n-1,\underline{X}}}(y)} \\ &= \int_0^{+\infty} I[y>t] \frac{f_{X_n}(s+2t-y)}{f_{X_n}(s+t-y)} \frac{f_{S_{n-1,\underline{X}_t}}(y-t)}{f_{S_{n-1,\underline{X}}}(y)} dF^\star(y|s) \\ &= E[\Phi(s, Y^\star(s))],\end{aligned}$$

where $\Phi(s,y) = I[y>t]\frac{f_{X_n}(s+2t-y)}{f_{X_n}(s+t-y)}\frac{f_{S_{n-1,\underline{X}_t}}(y-t)}{f_{S_{n-1,\underline{X}}}(y)}$, and $Y^\star(s)$ is an rv with support $[0, s+t]$ having pdf

$$\begin{aligned} f^\star(y|s) &= \frac{f_{X_n}(s+t-y) f_{S_{n-1,\underline{X}}}(y)}{\int_0^{+\infty} f_{X_n}(s+t-y) f_{S_{n-1,\underline{X}}}(y) dy} \\ &= \frac{\exp(\lambda_n y) f_{S_{n-1,\underline{X}}}(y) I[y<s+t]}{\int_0^{+\infty} \exp(\lambda_n y) f_{S_{n-1,\underline{X}}}(y) I[y<s+t] dy}.\end{aligned}$$

Note that, since $X_n \sim Exp(\lambda_n)$, thus for all $y \in [0, s+t]$ we have:

$$\frac{f_{X_n}(s+2t-y)}{f_{X_n}(s+t-y)} = \exp(-\lambda_n t)\frac{I[y<s+2t]}{I[y<s+t]} = \exp(-\lambda_n t).$$

Therefore, for all $y \in [0, s+t]$

$$\Phi(s,y) = \exp(-\lambda_n t) I[y>t] \frac{f_{S_{n-1,\underline{X}_t}}(y-t)}{f_{S_{n-1,\underline{X}}}(y)} \text{ is non-decreasing in } y \geq 0. \tag{23}$$

Notice that $f^\star(y|s)$ is TP_2 in $(y,s) \in (0,+\infty) \times (0,+\infty)$, which means that $Y^\star(s_1) \leq_{lr} Y^\star(s_2)$, for all $s_1 \leq s_2 \in [0,+\infty)$ and, consequently, $Y^\star(s_1) \leq_{st} Y^\star(s_2)$, for all $s_1 \leq s_2 \in [0,+\infty)$. This is enough together with the non-parenthetical part of Lemma 2.2(i) in Misra and van der Meulen (2003) to obtain $E[\Phi(s_1, Y^\star(s_1))] \leq E[\Phi(s_2, Y^\star(s_2))]$, for all $s_1 \leq s_2 \in [0,+\infty)$ which fulfills (22) as a correct statement and, thus, the stochastic order relation given is (21) stands valid. ∎

The following example of a single unit system equipped with a cold-standby unit which has an exponential lifetime distribution, fulfills the result of Theorem 10.

Example 11. *Let we have a cold-standby system with size $n=2$ heterogenous components with lifetimes X_1 and X_2 so that X_1 has an arbitrary lifetime distribution ($F_{X_1}(0^-)=0$) and X_2 has an exponential lifetime distribution with parameter λ_2 and we, further, assume that X_1 and X_2 are independent. Then, since*

$$S_{1,\underline{X}_t} + t =_{lr} X_{1,t} + t =_{lr} X_1(t) \geq_{lr} X_1$$

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thus the likelihood ratio ordering in Equation (20) holds true for $n = 2$, and hence by Theorem 10, one concludes $X_{1,t} + X_{2,t} \geq_{lr} (X_1 + X_2)_t$, for all $t \geq 0$. Let us suppose that X_1 has a gamma distribution with $\alpha_1 = 2$ and $\lambda_1 = 3$ and X_2 has exponential distribution with $\lambda_2 = 3$ as the age $t = 5$ is chosen. By routine calculation, for all $x \geq 0$, one has

$$f_{X_{1,t}+X_{2,t}}(x) = \frac{27x}{16}(5 + \frac{x}{2})e^{-3x} \text{ and } f_{(X_1+X_2)_t}(x) = \frac{27}{257}(5 + x)^2 e^{-3x}.$$

In Figure 2, we plot the graph of the function LR given by

$$LR(x) := \frac{f_{X_{1,t}+X_{2,t}}(x)}{f_{(X_1+X_2)_t}(x)} = \frac{257x(10 + x)}{32(5 + x)^2}$$

The function $LR(x)$ is increasing with respect to $x \geq 0$, according to Theorem 10 as Figure 2 confirms it.

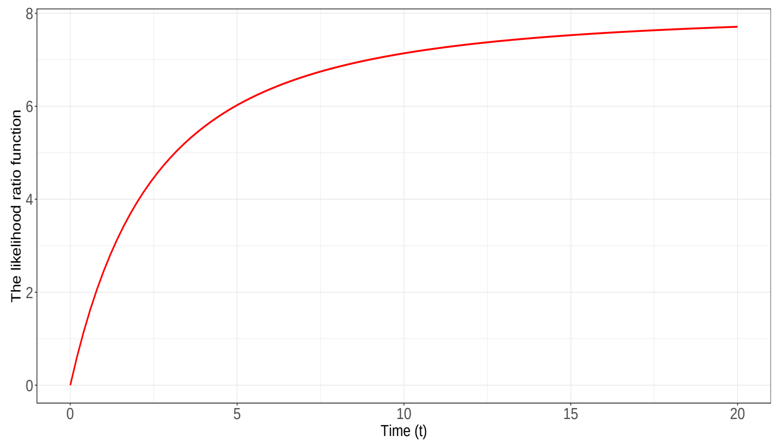

Figure 2. Plot of the likelihood ratio $LR(x) := \frac{f_{X_{1,t}+X_{2,t}}(x)}{f_{(X_1+X_2)_t}(x)}$ for $t = 5$

Now, we prove another result to compare the lifetime of a used standby system with the lifetime of another system composed of used components with respect to the usual stochastic order. The following lemma is essential to our development.

Lemma 12. *Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with non-negative random components with joint pdf $f_{\underline{X}}(\underline{x})$. Fix $l \in \{1, 2, \ldots, n\}$. Let $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ be another random vector with joint pdf*

$$f_{\underline{Y}}(\underline{y}) = \frac{f_{X_l}(t + S_{n,\underline{y}})}{f_{X_l}(S_{n,\underline{y}})c(t,l)} f_{\underline{X}}(\underline{y}), \tag{24}$$

where $c(t,l) = E\left(\frac{f_{X_l}(t+S_{n,\underline{X}})}{f_{X_l}(S_{n,\underline{X}})}\right) < +\infty$. Then, $S^l_{n,\underline{X}} =_{st} S_{n,\underline{Y}}$, where $S^l_{n,\underline{X}}$ is an rv with pdf $f_{S^l_{n,\underline{X}}}(s) = \frac{f_{X_l}(t+s)}{f_{X_l}(s)c(t,l)} f_{S_{n,\underline{X}}}(s)$.

Proof. We show that $S^l_{n,\underline{X}}$ and $S_{n,\underline{Y}}$ have the same cdf. It is notable that for any function $\beta : \mathbb{R} \mapsto \mathbb{R}$, one has

$$E[\beta(S^l_{n,\underline{X}})] = \frac{1}{c(t,l)} E\left(\beta(S_{n,\underline{X}}) \frac{f_{X_l}(t + S_{n,\underline{X}})}{f_{X_l}(S_{n,\underline{X}})}\right). \tag{25}$$

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For all $s \geq 0$, one one hand we can derive

$$P(S_{n,\underline{Y}} \leq s) = \frac{F_{S_{n,\underline{X}}}(s)}{c(t,l)} E\left(\frac{f_{X_l}(t + S_{n,\underline{X}})}{f_{X_l}(S_{n,\underline{X}})} \mid S_{n,\underline{X}} \leq s \right).$$

On the other hand, for all $s \geq 0$, one has

$$\begin{aligned} P(S^l_{n,\underline{X}} \leq s) &= \frac{1}{c(t,l)} \int_0^s \frac{f_{X_l}(t+s')}{f_{X_l}(s')} f_{S_{n,\underline{X}}}(s')\, ds' \\ &= \frac{F_{S_{n,\underline{X}}}(s)}{c(t,l)} \int_0^s \frac{f_{X_l}(t+s')}{f_{X_l}(s')} \frac{f_{S_{n,\underline{X}}}(s')}{F_{S_{n,\underline{X}}}(s)}\, ds' \\ &= \frac{F_{S_{n,\underline{X}}}(s)}{c(t,l)} E\left(\frac{f_{X_l}(t + S_{n,\underline{X}})}{f_{X_l}(S_{n,\underline{X}})} \mid S_{n,\underline{X}} \leq s \right). \end{aligned}$$

The proof of the result is complete. ∎

We introduce some notation before stating the result. Let $f_Z(t)$ be the density of the rv Z which is differentiable. The function $\eta_Z(t) = -\frac{f'_Z(t)}{f_Z(t)}$ is well-known as the Glaser's eta function which is very useful in the study of the shape of the hazard rate function and the mean residual life function (see, e.g., Glaser [10] and Gupta and Viles [11]).

Theorem 13. *Let $X_1, X_2, \ldots, X_n$ be independent rvs which are non-negative whose densities are differentiable and $X_{1,t}, X_{2,t}, \ldots, X_{n,t}$ be also independent, for a fixed $t > 0$. Then if there exists an $l \in \{1, 2, \ldots, n\}$ such that*

(i) $\eta_{S_{n,\underline{X}}}(t + x) - \eta_{S_{n,\underline{X}}}(x) \geq \eta_{X_l}(t + x) - \eta_{X_l}(x)$, *for all* $x \geq 0$;
(ii) $\eta_{X_l}(t + x) - \eta_{X_l}(x)$ *is increasing in* $x \geq 0$;
(iii) $\eta_{X_i}(t + x) - \eta_{X_i}(x) \leq \eta_{X_l}(t + x) - \eta_{X_l}(x)$, *for every* $i = 1, 2, \ldots n$ *and for all* $x \geq 0$;

we have

$$(S_{n,\underline{X}})_t \leq_{st} S_{n,\underline{X}_t}. \tag{26}$$

Proof. Under the assumption (i), it is found that $\frac{f_{S_{n,\underline{X}}}(t+x) f_{X_l}(x)}{f_{S_{n,\underline{X}}}(x) f_{X_l}(t+x)}$ is decreasing in $x \geq 0$. Thus from Theorem 3.2.(a) of Misra et al. [21] , $(S_{n,\underline{X}})_t \leq_{lr} S^l_{n,\underline{X}}$, and, therefore, $(S_{n,\underline{X}})_t \leq_{st} S^l_{n,\underline{X}}$. Suppose now that $n = 2$. Note that the assumption (ii) is equivalent to $\frac{f_{X_l}(t+x)}{f_{X_l}(x)}$ being a log-concave function in $x \geq 0$. By Lemma 12 and the Equation (24), there exist non-negative rvs Y_1 and Y_2 with joint pdf

$$f_{(Y_1,Y_2)}(y_1, y_2) = \frac{f_{X_l}(t + S_{2,\underline{y}})}{f_{X_l}(S_{2,\underline{y}}) c(t,l)} f_{(X_1,X_2)}(y_1, y_2), \tag{27}$$

so that $S^l_{2,\underline{X}} =_{st} Y_1 + Y_2$. By 27, since X_1 and X_2 are independent, thus Y_1 follow pdf

$$f_{Y_1}(y) = C_0 f_{X_1}(y) E\left(\frac{f_{X_l}(t + y_1 + X_2)}{f_{X_l}(y_1 + X_2)} \right), y_1 \geq 0;$$

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where C_0 is the normalizing constant. Suppose that X_i^l is an rv with pdf $f_{X_i^l}(x) = C_1\frac{f_{X_l}(t+x)}{f_{X_l}(x)}f_{X_i}(x)$ for every $i = 1,2$. We can then write

$$\frac{f_{Y_1}(y_1)}{f_{X_1^l}(y_1)} = C_2 E\left(\frac{f_{X_l}(t+y_1+X_2)f_{X_l}(y_1)}{f_{X_l}(y_1+X_2)f_{X_l}(t+y_1)}\right),$$

in which C_2 is the normalizing constant. This ratio is decreasing in $y \geq 0$ since $\frac{f_{X_l}(t+x)}{f_{X_l}(x)}$ is a log-concave function in $x \geq 0$. This is equivalent to saying that $Y_1 \leq_{lr} X_1^l$, and therefore,

$$Y_1 \leq_{st} X_1^l. \tag{28}$$

Once again, in spirit of (27), given that $Y_1 = y$, the conditional pdf of Y_2 is derived as

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{X_2}(t+y_1+y_2)f_{X_2}(y_2)}{f_{X_2}(y_1+y_2)E\left(\frac{f_{X_2}(t+y_1+X_2)}{f_{X_2}(y_1+X_2)}\right)}.$$

For any fixed $y_1 \geq 0$, the log-concavity of $\frac{f_{X_2}(t+x)}{f_{X_2}(x)}$ in $x \geq 0$, implies that $\frac{f_{Y_2|Y_1}(y_2|y_1)}{f_{X_2^l}(y_2)}$ is decreasing in $y_2 \geq 0$; that is $[Y_2|Y_1 = y_1] \leq_{lr} X_2^l$ Thus,

$$[Y_2|Y_1 = y_1] \leq_{st} X_2^l. \tag{29}$$

Now, taking into accounts the ordering relations in (28) and (29) and using Theorem 6.B.3 in Shaked and Shanthikumar [27], we get $S^l_{2,\underline{X}} \leq_{st} S_{2,\underline{X}^l}$, where $S_{n,\underline{X}^l} = \sum_{i=1}^n X_i^l$, for every $n = 1,2,\ldots$. We use the induction method and assume that

$$S^l_{n,\underline{X}} \leq_{st} S_{n,\underline{X}^l} \tag{30}$$

holds true for $n = m \geq 2$, i.e., $S^l_{m,\underline{X}} \leq_{st} S_{m,\underline{X}^l}$. From known properties of the usual stochastic order

$$S^l_{m+1,\underline{X}} \leq_{st} S^l_{m,\underline{X}} + X^l_{m+1} \leq_{st} S_{m,\underline{X}^l} + X^l_{m+1} =_{st} S_{m+1,\underline{X}^l},$$

which means that (30) stands valid for $n = m+1$. Thus, we proved that $S^l_{n,\underline{X}} \leq_{st} S_{n,\underline{X}^l}$. Since the assumption (iii) means that $f_{X_l}(t+x)f_{X_i}(x)/f_{X_l}(x)f_{X_i}(t+x)$ is decreasing in $x \geq 0$, thus an application of Theorem 3.2.(a) of Misra et al. [21] implies that $X_i^l \leq_{lr} X_i^t$, so $X_i^l \leq_{st} X_i^t$ for all $i = 1,2,\ldots,n$. Note that $X_i^t =_{st} X_{i,t}, i = 1,2,\ldots,n$ where $X_{i,t} =_{st} [X_i - t|X_i > t]$ is the residual lifetime after age $t \geq 0$. From Theorem 1.A.3.(b) in Shaked and Shanthikumar [27] it follows that $S_{n,\underline{X}^l} \leq_{st} S_{n,\underline{X}_t}$. Hence, the result is proved. ∎

The next example clarifies that the property that a standby system composed of used components has a greater reliability than a used standby system is fulfilled in the context of *DLR* lifetime components distribution.

Example 14. *We consider components with heterogenous independent lifetime distributions. Suppose that* $X_i \sim G(\alpha_i, \lambda), i = 1,2,\ldots,n$. *Note that as* $X_1, X_2, \ldots, X_n$ *are independent rvs, consequently,* $S_{n,\underline{X}} \sim G(\sum_{i=1}^n \alpha_i, \lambda)$, *thus* $f_{S_{n,\underline{X}}}(x) = \frac{\lambda^{\sum_{i=1}^n \alpha_i} x^{\sum_{i=1}^n \alpha_i - 1} e^{-\lambda x}}{\Gamma(\sum_{i=1}^n \alpha_i)}$. *We assume that* $\alpha_i \leq 1$ *for all* $i = 1,2,\ldots,n$ *and that* $\alpha_l = \max_{1\leq i\leq n}\{\alpha_i\}$. *Therefore,*

$$\eta_{X_i}(x) = \frac{1-\alpha_i}{x} + \lambda, \quad \eta_{S_{n,\underline{X}}}(x) = \frac{1-\sum_{i=1}^n \alpha_i}{x} + \lambda.$$

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For all $x \geq 0$, it is seen that

$$\eta_{S_{n,\underline{X}}}(t+x) - \eta_{S_{n,\underline{X}}}(x) = \frac{t(\sum_{i=1}^{n} \alpha_i - 1)}{x(t+x)} \tag{31}$$

$$\geq \frac{t(\alpha_l - 1)}{x(t+x)} = \eta_{X_l}(t+x) - \eta_{X_l}(x). \tag{32}$$

Thus, the assumption (i) in Theorem 13 holds true. It is further observed that $\eta_{X_l}(t+x) - \eta_{X_l}(x) = \frac{(\alpha_l - 1)t}{x(t+x)}$ which is an increasing function in $x \geq 0$, that is, the assumption (ii) in Theorem 13 is satisfied. Since $\alpha_i \leq \alpha_l$, for all $i = 1, 2, \ldots, n$, thus the assumption (iii) in Theorem 13 is also valid, and consequently, $(\sum_{i=1}^{n} X_i)_t \leq_{st} \sum_{i=1}^{n} X_{i,t}$.

4. Concluding Remarks

With this work we have achieved two goals. The first is to develop some stochastic upper bounds on the random lifetime of a cold standby system that is not fresh or new and has age t, having been in operation and still functioning by time t. Two well-known stochastic orders, namely the likelihood ratio order and a weaker stochastic order, the usual stochastic order, were applied to obtain the stochastic upper bound. The interesting point is that the reliability function of the lifetime of the used cold standby system with n units is always dominated (without any further assumptions) by the reliability function of the lifetime of a cold standby system with $(n-1)$ units consisting of used components with common age t, provided that the lifetimes of the used components are shifted t times, as is the case, for example, in the burn-in process. For example, may be the case in the burn-in process, where a product is put into use for a time interval of length t before being handed over to the customer, is a realistic situation. However, the domination of this stochastic upper bound over the lifetime of the cold standby system used in terms of the likelihood ratio order requires the further assumption that the components have the ILR property. The second objective was to find conditions under which the lifetime of a used cold standby system with an age of t is dominated by the lifetime of a cold standby unit with used components, each with an age of t, in terms of the likelihood ratio order and the usual stochastic order. In general, and as confirmed by our research as a whole, it is found that the use of cold standby units that were previously in use for an equally long period of time (e.g., t) is preferable to a used cold standby system with age t because it satisfies larger stochastic lifetimes. Therefore, a cold standby system with components $X_{1,t}, X_{2,t}, \ldots, X_{n,t}$ that has random lifetime $X_{1,t} + X_{2,t} + \ldots + X_{n,t}$ is more reliable than a used cold standby system with random lifetime $(X_1 + X_2 + \ldots + X_n)_t$ in most situations. We hope that the research conducted in this study will be useful to engineers and system designers.

In the future study, we will use the hazard rate order and the reversed hazard rate order to determine new bounds on the reliability function and the cumulative distribution function, respectively, of a second-hand cold standby system. We will look for conditions under which the lifetime of a used cold standby system of age t is dominated by the lifetime of a cold standby unit with used components each of age t, in terms of new stochastic orders. Stochastic comparisons between the inactivity times of cold standby systems according to known standard stochastic orders are another problem that can be studied in future work. The importance of the loss caused by further inactivity of engineering systems may motivate us to conduct the above study to design an optimal cold standby unit with less stochastic inactivity.

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