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Quantum Effects in General Relativity: Investigating Repulsive Gravity of Black Holes at Large Distances

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Abstract: **The** paper proposes a theoretical study that investigates the quantum effects on the gravity of black holes. The study utilizes a gravitational model that incorporates quantum mechanics derived from the classical-like quantum hydrodynamic representation. The research calculates the mass density distribution of quantum black holes, specifically in the case of central symmetry. The gravity of the quantum black hole shows contributions coming from the quantum potential energy, which is also sensitive to the presence of the background of gravitational noise. The additional energy, stored in the quantum potential fluctuations and constituting a form of dark energy, leads to a repulsive gravity in the weak gravity limit. This repulsive gravity overcomes the attractive classical Newtonian force at large distances of order of the intergalactic length.

Keywords:

1. Introduction

One of the biggest challenges in physics today is the unification of general relativity and quantum theory. General relativity provides a description of gravity as the curvature of spacetime caused by the presence of matter and energy, while quantum theory describes the behavior of matter and energy at the smallest scales.

The problem is that these two theories are fundamentally different in their approach, and attempts to merge them have so far been unsuccessful. One of the key challenges is the existence of singularities, such as those found in black holes, which arise from the application of general relativity at very small scales.

Another issue is the conflict between the principles of quantum mechanics and general relativity, such as the principle of locality, which states that information cannot travel faster than the speed of light, and the principle of unitarity, which requires that the total probability of all possible outcomes of an experiment add up to one.

There have been various proposals for reconciling general relativity and quantum theory, such as string theory [1-3], loop quantum gravity [4-6], and causal dynamical triangulation [7-9], but none have yet been confirmed by experimental evidence, and the search for a unified theory remains an active area of research in theoretical physics.

Since the discovery of the universe's accelerated expansion, scientists have been trying to determine what is driving this acceleration. However, despite the many attempts to explain it, the current observational data cannot conclusively identify the source.

Recently, the author demonstrated [10,11] that by assuming the covariance of quantum field equations and utilizing their classical-like quantum hydrodynamic representation, it is possible to define the geometry of space-time through a gravity equation that incorporate the quantum mechanics. This is achieved through the use of a generalized least action principle, resulting in a system of equations that describes the quantum-gravitational evolution. This system couples the gravity equation with the quantum equation of boson or fermion fields.

The theoretical study proposed in this paper is centered on the quantum mechanical state of black holes and the resulting gravity in spacetime, where a background of gravitational noise is

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present. The findings of this study have the potential to confirm observational evidence such as the existence of dark energy and the repulsive nature of gravity at large distances.

2. Cosmological Scalar Boson Mass under Self-Gravity

During the collapse of a black hole, the mass distribution becomes highly concentrated, but the repulsive force of the quantum potential may become strong enough to counteract the gravitational force and prevent the collapse. This can result in the formation of stationary mass distributions. The uncertainty principle ensures that the repulsive quantum non-local potential grows sufficiently to overcome the gravitational force, thereby preventing a point-like collapse.

When the mass distribution of a scalar uncharged boson becomes extremely concentrated in space, its gravitational force can generate stable self-bonded states. These states are the quantum mechanical analogue of black hole predicted by General Relativity.

In this section, the author investigates whether the quantum potential force can stop the gravitational collapse when the mass distribution approaches the classical point singularity.

In order to obtain quantum mechanical stationary black hole configurations on a cosmological scale with large mass distributions, we make the assumption that the mass field can be represented as a classical scalar variable. This simplified model of scalar black hole mass serves as a "macroscopic" viewpoint that is acceptable for studying the gravitational behavior of black holes on a cosmological scale. T

The distribution of mass in the space-time (ST) is attributed to the formation of vacuum states resulting from the quantization of spinor and massive boson fields. This mass distribution in the ST is not only non-continuous, but also exhibits physical phenomena arising from the other three fundamental interactions. Hence, the focus is solely on the configuration of an eternal black hole, and the process of black hole evaporation and the production of Hawking radiation are out the description.

2.1. Stationary Scalar Mass Distribution

In the case of scalar mass field obeying to the Klein-Gordon equation, the gravitationally coupled system of motion equation reads [11]

$$R_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R_{\alpha}^{\ \alpha} = \frac{8\pi G}{c^4} T_{\nu\mu} \tag{2.1}$$

$$\partial^{\mu} \psi_{;\mu} = \frac{1}{\sqrt{-g}} \partial^{\mu} \sqrt{-g} \left(g^{\mu\nu} \partial_{\nu} \psi \right) = -\frac{m^2 c^2}{\hbar^2} \psi \tag{2.2}$$

with

$$T_{\mu\nu} = -\frac{m c^2 \, |\psi_{\pm}|^2}{\gamma} \left[\begin{array}{l} \left(\sqrt{1 - \frac{V_{qu}}{m c^2}} - 1 \right) g_{\mu\nu} \\ + \sqrt{1 - \frac{V_{qu}}{m c^2}}^{\,-1} \left(\frac{\hbar}{2 m c} \right)^2 \frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial q^{\mu}} \frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial q_{\lambda}} g_{\lambda\nu} \end{array} \right] \tag{2.3}$$

where

$$\gamma = \frac{1}{\sqrt{\frac{g_{\mu\nu} \dot{q}^{\nu} \dot{q}^{\mu}}{c^2}}} = \frac{1}{c\sqrt{\partial_0 S \; g_{\mu\nu} \partial^{\nu} S \partial^{\mu} S}} = \frac{2}{\hbar^{3/2} c \sqrt{i\partial_0 \, ln[\frac{\psi}{\psi^*}] \; g_{\mu\nu} \partial^{\nu} \, ln[\frac{\psi}{\psi^*}] \partial^{\mu} \, ln[\frac{\psi}{\psi^*}]}} \tag{2.4}$$

and where the quantum potential reads

$$V_{qu} = -\frac{\hbar^2}{m} \frac{1}{|\psi|\sqrt{-g}} \partial_\mu \sqrt{-g}\left(g^{\mu\nu}\partial_\nu |\psi|\right) \tag{2.5}$$

The KGE (2.2) in the hydrodynamic notation, as a function of the real variables $|\psi|$ and $\partial_\mu S$, reads

$$g_{\mu\nu}\partial^\nu S\, \partial^\mu S - \hbar^2 \frac{1}{|\psi|\sqrt{-g}} \partial_\mu \sqrt{-g}\left(g^{\mu\nu}\partial_\nu |\psi|\right) - m^2c^2 = 0 \tag{2.6}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g}\left(g^{\mu\nu} |\psi|^2 \partial_\nu S\right) = 0 \tag{2.7}$$

Given the covariant approach of the theory, the motion equation (2.6) for eigenstates, reads

$$D_{q_0} p_{(k)\mu} = -\frac{1}{c}\frac{\partial L_{(k)}}{\partial q^\mu} = \frac{mc}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} \tag{2.8}$$

where cD_{q_0} is the curvilinear covariant total time derivative, and $p_{(k)\mu}$ reads

$$p_{(k)\mu} = -\partial_\mu S_{(k)} = (\pm) m\gamma_{(k)}\dot{q}_{(k)\mu}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} = (\pm) mcu_{(k)\mu}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} \tag{2.9}$$

. (2.9)

Developing equation (2.8) it follows that

$$\begin{aligned} &D_{q_0}\left(u_{(k)\mu}\sqrt{1-\frac{V_{qu(k)}}{mc^2}}\right) \\ &= \sqrt{1-\frac{V_{qu(k)}}{mc^2}} D_{q_0} u_{(k)\mu} + u_{(k)\mu} D_{q_0}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} = (\pm)\frac{1}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\sqrt{1-\frac{V_{qu(k)}}{mc^2}} \end{aligned} \tag{2.10}$$

and, by utilizing the relation

$$D_{q_0} u_\mu = \frac{du_\mu}{dq_0} - \frac{1}{\gamma}\Gamma^\alpha_{\mu\nu} u_\alpha u^\nu \tag{2.11}$$

The motion equation, to which the stationary condition $\frac{du_\mu}{dt} = 0$ must be applied, reads

$$\begin{aligned} \frac{du_\mu}{dt} - \frac{c}{\gamma}\Gamma^\alpha_{\mu\nu} u_\alpha u^\nu &= -u_\mu \frac{d}{dt}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) \pm \frac{1}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) \\ &= -u_\mu \frac{d}{dt}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2 \frac{1}{|\psi|\sqrt{-g}}\partial_\mu\sqrt{-g}\left(g^{\mu\nu}\partial_\nu|\psi|\right)}\right) \\ &\pm \frac{c}{\gamma_{(k)}}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2 \frac{1}{|\psi|\sqrt{-g}}\partial_\mu\sqrt{-g}\left(g^{\mu\nu}\partial_\nu|\psi|\right)}\right) \end{aligned} \tag{2.12}$$

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2.2. The Mass Distribution in Central Symmetric Scalar Uncharged Black Hole

In classical General Relativity, the collapse in a central gravitational field results in a final point-like mass density, approached with increasing velocity. However, in the quantum case, the quantum potential generates an expansive force

$$\partial_\mu \ln \sqrt{1-\frac{V_{qu}}{mc^2}} \tag{2.13}$$

that counteracts gravity, leading to deceleration and potentially halting the collapse. As a result, stable stationary configurations may exist at an equilibrium point. This suggests that the interplay between quantum effects and gravity can lead to different outcomes than those predicted by classical General Relativity.

From a general standpoint, the stationary mass distribution, as described by equation (2.12), depends on the metric tensor defined by the Quantum Einstein Gravity (QGE) equations (2.1), and vice versa. Although the general solution of these coupled equations is quite complex, the simplifying assumption of central symmetry can be introduced to extract useful information. This assumption leads to the quantum analogue of the Schwarzschild black hole, where the metric tensor satisfies a particular condition [13].

$$ds^2 = e^{\nu} c^2 dt^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^{\lambda} dr^2 \tag{2.14}$$

where $q_\mu = (ct, r, \theta, \varphi)$ and

$$g_{00} = e^{\nu}; g_{11} = -e^{\lambda}; g_{22} = -r^2; g_{33} = -r^2 \sin^2\theta; \ \sqrt{-g} = | e^{\frac{\lambda+\nu}{2}} r^2 \sin^2\theta |^{-1}; \tag{2.15}$$

that inserted into the gravity equation leads to the relations [13]

$$\frac{8\pi G}{c^4} T_1^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{2.16}$$

$$\frac{8\pi G}{c^4} T_0^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \tag{2.17}$$

$$\frac{8\pi G}{c^4} T_0^1 = -e^{-\lambda}\frac{\dot{\lambda}}{r} \tag{2.18}$$

with

$$\gamma = \frac{1}{\sqrt{\frac{g_{\mu\nu}\dot{q}^{\nu}\dot{q}^{\mu}}{c^2}}} \tag{2.19}$$

and

$$V_{qu} = -\frac{\hbar^2}{m}\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right) \tag{2.20}$$

where the apex and the dot over the letters mean derivation respect to r and ct, respectively.

Assuming that in the stationary distributions the mass is enclosed in a sphere of radius R_0, for $r > R_0$ we can use the approximated gravitational relations [13]

$$-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}\cong 0 \tag{2.21}$$

$$-e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2}\cong 0 \tag{2.22}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r}\cong 0 \tag{2.23}$$

whose solutions read

$$\lambda+\nu=0 \tag{2.24}$$

$$g_{11}=-e^{\lambda}=-e^{-\nu}\cong-\left(1-\frac{R_g}{r}\right)^{-1} \qquad \frac{r}{R_g}\gg 1 \tag{2.25}$$

$$g=-\frac{1}{r^4\sin^4\theta} \tag{2.26}$$

$$e^{\lambda}=\frac{r}{r-R_g}=\frac{1}{g_{00}} \tag{2.27}$$

from which the quantum potential reads [14]

$$V_{qu}=-\frac{\hbar^2}{m}\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right) \tag{2.28}$$

By introducing the relations (2.24-28) into the motion equation, it follows that

$$\begin{aligned}\frac{du_\mu}{cdt}-\frac{1}{\gamma}\Gamma^{\alpha}_{\mu\nu}u_\alpha u^\nu &= -u_\mu\frac{1}{c}\frac{d}{dt}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\\ &\quad+\frac{1}{\gamma}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\\ &= -u_\mu\frac{1}{c}\frac{\partial}{\partial q^\nu}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\dot{q}^\nu\\ &\quad+\frac{1}{\gamma}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\\ &= 0+\frac{1}{\gamma}\frac{\partial}{\partial q^\mu}\left(\ln\sqrt{1+\left(\frac{\hbar}{mc}\right)^2\frac{1}{|\psi|\sqrt{-g}}\partial^1\sqrt{-g}\left(e^{-\lambda}\partial_1|\psi|\right)}\right)\end{aligned} \tag{2.29}$$

and, by the stationary condition in the BH system of reference at large distance

$$u_\mu=\lim_{\frac{r}{R_g}\to\infty}(\gamma,0,0,0)=\lim_{\frac{r}{R_g}\to\infty}\left(\frac{1}{\sqrt{g^{00}}},0,0,0\right)=(1,0,0,0) \tag{2.30}$$

and

$$u^\mu=\lim_{\frac{r}{R_g}\to\infty}\left(g^{00}\gamma,0,0,0\right)=\lim_{\frac{r}{R_g}\to\infty}\left(\sqrt{g^{00}},0,0,0\right)\cong(1,0,0,0) \tag{2.31}$$

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we obtain

$$\Gamma^{\alpha}_{\mu\nu} u_{\alpha} u^{\nu} = \Gamma^{0}_{10} u_{0} u^{0} = \frac{1}{2} u_{0} u^{0} g^{00} \partial_{1} g_{00} = u^{0} u^{0} \partial_{1} g_{00} \cong \partial_{1} g_{00}$$

$$= -\partial_{1}\left(\ln\left(1+\left(\frac{\hbar}{mc}\right)^{2} \frac{1}{|\psi|\sqrt{-g}} \partial^{1} \sqrt{-g}\left(e^{-\lambda} \partial_{1} |\psi| \right) \right) \right) \tag{2.32}$$

$$-\partial_{1} \frac{r-R_{g}}{r} = -\partial_{1}\left(\ln\left(1-\left(\frac{\hbar}{mc}\right)^{2} \frac{1}{|\psi|\sqrt{-g}} \partial^{1} \sqrt{-g}\left(\frac{r-R_{g}}{r} \partial_{1} |\psi| \right) \right) \right)$$

$$\partial_{1} \frac{r-R_{g}}{r} = \partial_{1}\left(\ln\left(1-\left(\frac{\hbar}{mc}\right)^{2} \frac{1}{|\psi| r^{-2}} \partial^{1} r^{-2}\left(\frac{r-R_{g}}{r} \partial_{1} |\psi| \right) \right) \right) \tag{2.33}$$

where R_g is the gravitational radius of the BH and R_c the Compton's length, leading to the BH mass density at large distances (see Appendix) following the law

$$\lim_{r\to\infty} |\psi| = G_{0} e^{-\varsigma \frac{r}{R_c}} \tag{2.34}$$

The constants G_0 is defined by the normalization condition.

2.3. The mass Distribution near the Center of Schwarzschild Black Hole

In the classical case the BH mass collapses into a point, in the quantum case, for the uncertainty principle (see 2.44), the maximum concentration is inside a sphere whose radius is of order of magnitude of Compton's length R_c. Thence, for macroscopically massive BH, with condition $R_g \gg R_c$ (for BH of mass $m \sim 10^{35} kg$ $\frac{R_g}{R_c} \sim 10^{85}$,), we can assume with a good approximation that in the limit $\frac{r}{R_g} \to 0$ (at least $\frac{r}{R_g} \gg 10^{-85}$) there is no mass for $R_c \ll r < R_g$. Thus, by observing that

$$\lim_{\frac{r}{R_g}\to 0} \lambda = \lim_{\frac{r}{R_g}\to 0} \ln\frac{r}{r-R_g} \cong -\frac{r}{R_g} = 0 \tag{2.35}$$

and that

$$\frac{8\pi G}{c^4} T_1^1 = -e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2} \cong -\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2} \cong 0 \tag{2.36}$$

$$\frac{8\pi G}{c^4} T_0^0 = -e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2} \cong -\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2} \cong 0 \tag{2.37}$$

$$e^{\lambda} \frac{8\pi G}{c^4} T_0^1 = -\frac{\dot{\lambda}}{r} = 0 \tag{2.38}$$

equation (A.9), in Appendix, can be also retained for $x = \frac{r}{R_g} \to 0$.

Thus, providing that at small distance from the BH center it holds

$$\lim_{x\to 0} |\tilde{y}^2_{(x)}| << |\tilde{y}_{(x)}| \tag{2.39}$$

We obtain the differential equation

$$\lim_{x\to 0} \tilde{y}'_{(x)} = R_g y'_{(r)} = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) + \frac{R_g^{\ 2}}{R_c^{\ 2}} \frac{x}{x-1}\left(1 - C_0 e^{\ln\left(\frac{x-1}{x}\right)}\right) \tag{2.40}$$

whose solution, given in Appendix, reads

$$\lim_{\frac{r}{R_g}\to 0} |\psi| \cong G_0 e^{-\frac{1-\varsigma^2}{e}\frac{R_g}{R_c} z} \qquad r \gg R_c, \quad R_g \gg R_c \tag{2.41}$$

The output (2.41) shows that, for large mass BHs (e.g., $m \sim 10^{35} kg$) the mass density is practically null outside the sphere of Compton's radius at the center of the BH (e.g., for $r = 10R_c$ $|\psi| \sim G_0 e^{-\frac{1-\varsigma^2}{e}10^{86}} \sim G_0 e^{-\frac{1}{e}10^{86}}$ (see Appendix).

This output in agreement with the uncertainty principle leads to a piece of information about the minimum mass for the formation of BHs. In fact, in order that the BH energy due to its localization does not exceed the value mc^2 (otherwise a new BH it is formed), by the uncertainty principle it follows that

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{\mathrm{v}\hbar}{2\Delta x} \leq mc^2 \tag{2.42}$$

where

$$\mathrm{v} \geq \Delta \mathrm{v} = \frac{\Delta p}{m} = \frac{\hbar}{2m\Delta x} \tag{2.43}$$

that leads to $mc^2 > \frac{\hbar^2}{4m\Delta x^2}$ and, finally, to

$$\Delta x > \frac{\hbar}{2mc} = \frac{R_c}{2} \tag{2.44}$$

Besides, since in order to form a BH all the mass must be inside the gravitational radius, we must have that

$$R_g = \frac{2Gm}{c^2} > \frac{\Delta x}{2} = r_{\min} = \frac{R_c}{4} \tag{2.45}$$

and, thence, that

$$\frac{R_c}{4R_g} = \frac{\hbar}{8mcR_g} = \frac{\hbar c}{8m^2 G} = \pi \frac{m_p^{\ 2}}{m^2} < 1 \tag{2.46}$$

leading to the condition for the black hole mass m

$$m > \pi^{1/2} m_p \tag{2.47}$$

where $m_p = \sqrt{\frac{\hbar c}{8\pi G}}$ is the reduced Planck mass.

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For small masses when $m \to 0$ (quantum case) the gravitational radius R_g tends to zero, while the Compton's radius R_c goes to infinity so that it exists the minimum mass (2.47) for the formation of the black hole ($R_g = \frac{R_c}{4}$). This condition is safe for our universe since low energy elementary particles cannot form BHs.

On the other hand, it is noteworthy to observe that for very large mass $m \to \infty$, $V_{qu} \propto \frac{1}{m} \to 0$ (classical limit) the BH Compton's radius R_c goes to zero and the point singularity of the classical general relativity is asymptotically approached.

Additionally, as black holes with a Planck mass cannot be divided into two smaller black holes, they represent the lightest possible configuration of scalar uncharged mass density that can be achieved solely through gravitational interaction. Moreover, since the condition expressed in equation (2.47) also applies to quantized fields, the fundamental lowest state of a quantum black hole is heavier than $\pi^{1/2} m_p$.

It is worth noting that the mass density output of macroscopic black holes enclosed within a sphere with a radius smaller than the Planck length and on the order of the Compton wavelength is typically derived from a continuous (classical) description of the fields and may undergo modifications in the context of quantized fields.

3. Gravitational Field of Black Holes at Large Distance in Spacetime with Background Fluctuations

In this paragraph, we derive the weak gravitational force of black holes over long distances. The large distance approximation is used because the gravitational radius of a black hole is much smaller than the cosmological physical scale, allowing us to treat the mass distribution of the black hole as point-like.

fact, given the mass distribution of BH (2.34) arranged in the form

$$|\psi|^2 = |\psi_0|^2 \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 B_{(r)} \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} \tag{3.1}$$

at large distance reads

$$\begin{aligned}
&\lim_{\frac{r}{R_c} \to \infty} |\psi_0|^2 \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 B_{(r)} \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} \\
&= \lim_{\frac{r}{R_c} \to \infty} = \frac{B_{(r)}}{\int B_{(r)} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} d^3\Omega} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^2 \frac{\varsigma}{2R_c} e^{-\varsigma \frac{r}{R_c}} \\
&= \lim_{R_c \to 0} \frac{B_{(r)}}{\int B_{(r)} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^3 e^{-\varsigma \frac{r^2}{4R_c{}^2}} d^3\Omega} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^3 e^{-\varsigma \frac{r^2}{4R_c{}^2}} \\
&= \frac{B_{(r)}}{\int B_{(r)} \delta^3{}_{(r)} d^3\Omega} \delta^3{}_{(r)} = \frac{B_{(r=0)}}{B_{(r=0)}} \delta^3{}_{(r)} = \delta^3{}_{(r)}
\end{aligned} \tag{3.2}$$

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where the normalization condition $\int |\psi|^2 \, d^3\Omega = 1$ has been used.

It must be noted that at temperature bigger than the absolute zero, the BH quantum potential undergoes fluctuation $\delta \overline{E}_{qu}$ [15-17] that brings the equivalent additional mass density

$$m_{BH} \, | \delta \psi_0 |^2 = \frac{\delta \overline{E}_{qu}}{c^2} \tag{3.3}$$

Being so, for cosmological problems (i.e., $R_c \to 0$), we assume that the total mass density of the BH field in the spacetime with fluctuating background reads

$$\lim_{r \to \infty} |\psi|^2 \simeq \delta^3_{(r)} + |\delta \psi_0|^2 \tag{3.4}$$

Furthermore, as black holes are quantum objects with a significant quantum potential energy (as described in the appendix), we anticipate that their gravity over long distances may result in quantum effects contributing to the Newtonian law.

The contribution coming from the quantum potential, contained into the energy density tensor of the QGE, reads

$$\begin{aligned} R_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R_\alpha^{\ \alpha} &= \frac{8\pi G}{c^4} \frac{m c^2 |\psi_\pm|^2}{\gamma} \left(\left(\sqrt{1 - \frac{V_{qu}}{mc^2}} - 1 \right) g_{\mu\nu} + \sqrt{1 - \frac{V_{qu}}{mc^2}}^{\,-1} \left(\frac{\hbar}{2mc} \right)^2 \frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial q^\mu} \frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial q^\nu} \right) \\ &= \frac{8\pi G}{c^4} \frac{m c^2 |\psi_\pm|^2}{\gamma} \left(\left(\sqrt{1 - \frac{V_{qu}}{mc^2}} - 1 \right) g_{\mu\nu} + \sqrt{1 - \frac{V_{qu}}{mc^2}} u_\mu u_\nu \right) \end{aligned} \tag{3.5}$$

where it has been used the identity $\left(\frac{1}{mc} \right)^2 p_\mu p^\lambda = u_\mu u_\nu \left(1 - \frac{V_{qu}}{mc^2} \right)$. Moreover, given that

$$V_{qu} = -\frac{\hbar^2}{m} \frac{1}{|\psi| \sqrt{-g}} \partial^1 \sqrt{-g} \left(e^{-\lambda} \partial_1 |\psi| \right) \tag{3.6}$$

$$g = -\frac{1}{r^4 \sin^4 \theta} \tag{3.7}$$

$$e^\lambda = \frac{r}{r - R_g} = \frac{1}{g_{00}} = -g_{11} \tag{3.8}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} = \frac{1}{\sqrt{\frac{g^{\mu\nu} \dot{q}_\nu \dot{q}_\mu}{c^2}}} = \frac{1}{\sqrt{g^{00}}}$$

$$\lim_{r \to \infty} |\psi| = G_0 e^{-\varsigma \frac{r}{R_c}} \tag{3.9}$$

it follows that

$$\lim_{r\gg R_g} V_{qu} = \lim_{r\gg R_g} -\frac{\hbar^2}{m}\frac{1}{e^{-\varsigma x}} r^2 \partial^r r^{-2}\left(e^{-\lambda}\partial_r e^{-\varsigma z}\right)$$
$$= \lim_{r\gg R_g} \frac{\hbar^2}{m}\left(\frac{\varsigma}{R_c}\right)\frac{1}{e^{-\varsigma x}} r^2 \partial^r r^{-2}\left(\left(1-\frac{R_g}{r}\right)e^{-\varsigma x}\right)$$
$$= \lim_{r\gg R_g} \frac{\hbar^2}{m}\frac{\varsigma}{R_c}\left(-2r^{-1}\left(1-\frac{R_g}{r}\right)+\left(\frac{\varsigma}{R_c}\left(1-\frac{R_g}{r}\right)+\frac{R_g}{r^2}\right)\right) \quad (3.10)$$
$$= \lim_{r\gg R_g} -\frac{\hbar^2}{m}\frac{\varsigma}{R_c}\left(2r^{-1}-\frac{\varsigma}{R_c}\right) = \frac{\hbar^2}{m}\frac{\varsigma^2}{R_c{}^2} = mc^2\varsigma^2$$

Thence, the gravitational equation (3.5) in the mixed form reads

$$R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R_\alpha{}^\alpha = \frac{8\pi G}{c^4}\frac{mc^2\,|\psi_\pm|^2}{\gamma}\begin{pmatrix}\left(\sqrt{1+\varsigma^2}-1\right)\delta_\mu{}^\nu \\ +\sqrt{1+\varsigma^2}\,u_\mu u^\nu\end{pmatrix} \quad (3.11)$$

leading to the equation for the trace of the Ricci tensor

$$R_\alpha{}^\alpha = -\frac{8\pi G}{c^4}\frac{mc^2\,|\psi_\pm|^2}{\gamma}\begin{pmatrix}4\left(\sqrt{1+\varsigma^2}-1\right) \\ +\sqrt{1+\varsigma^2}\,u_\alpha u^\alpha\end{pmatrix} \quad (3.12)$$

and to

$$R_0{}^0 = \frac{8\pi G}{c^4}\frac{mc^2\left(\delta_{(r-r_{BH})}+|\delta\psi_0|^2\right)}{\gamma}\begin{pmatrix}\left(\sqrt{1+\varsigma^2}-1\right)\left(\delta_0{}^0-2\right) \\ +\frac{\sqrt{1+\varsigma^2}}{2}u_0 u^0\end{pmatrix} \quad (3.13)$$

Given that at large distance we can use the approximations

$$lim_{r\to\infty} u_0 = 1 \quad (3.14)$$

$$g_{00} = \frac{r-R_g}{r} = \left(1-\frac{R_g}{r}\right) \quad (3.15)$$

(3.13) reads

$$R_0^{\ 0} = \frac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})} + |\,\delta\psi_0\,|^2\right)}{\gamma}\lim_{r\to\infty}\left(\begin{array}{l}\left(\sqrt{1+\varsigma^2}-1\right)\left(\delta_0^{\ 0}-2\right)\\ +\dfrac{\sqrt{1+\varsigma^2}}{2g_{00}}\end{array}\right)$$

$$= \frac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})} + |\,\delta\psi_0\,|^2\right)}{\gamma}\lim_{r\to\infty}\left(1+\left(\frac{1}{g_{00}}-2\right)\frac{\sqrt{1+\varsigma^2}}{2}\right)$$

$$= \frac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})} + |\,\delta\psi_0\,|^2\right)}{\gamma}\lim_{r\to\infty}\left(1+\left(\frac{\frac{r}{R_g}+2}{1-\frac{r}{R_g}}\right)\frac{\sqrt{1+\varsigma^2}}{2}\right) \tag{3.16}$$

$$\cong \frac{8\pi G}{c^2}\frac{m\left(\delta_{(r-r_{BH})} + |\,\delta\psi_0\,|^2\right)}{\gamma}\left(1-\frac{\sqrt{1+\varsigma^2}}{2}\right)$$

leading to the identity

$$R_0^{\ 0} = R_{00} = \frac{1}{c^2}\frac{\partial}{\partial q^\alpha}\frac{\partial\phi}{\partial q^\alpha} \cong \frac{4\pi G}{c^2} m\left(\delta_{(r-r_{BH})} + |\,\psi_0\,|^2\right)\left(1-\frac{\sqrt{1+\varsigma^2}}{2}\right) \tag{3.17}$$

By integrating the flux of the gravitational force $\frac{\partial\phi}{\partial q^\alpha}$ on the sphere of radius $r-r_{BH}$ it follows that

$$\iiint \frac{\partial}{\partial q^\alpha}\frac{\partial\phi}{\partial q^\alpha}dV = \oiint \frac{\partial\phi}{\partial q^\alpha}\cdot dS^\alpha = \frac{\partial\phi}{\partial r}4\pi\left(r-r_{BH}\right)^2 = 4\pi Gm\iiint\left(\delta_{(r-r_{BH})} + |\,\delta\psi_0\,|^2\right)\left(1-\frac{\sqrt{1+\varsigma^2}}{2}\right)dV \tag{3.18}$$

By the δ-shape approximation of the BH mass distribution it can be posed $V_{qu(r-r_{BH}=0)} = 0$ so that the BH gravitational field at large distance reads

$$\frac{\partial\phi}{\partial r} = Gm\left(\frac{1-\pi\sqrt{1+\varsigma^2}\,|\,\delta\psi_0\,|^2\iiint\left(r-r_{BH}\right)^2\frac{dr}{R_c}}{\left(r-r_{BH}\right)^2}\right)$$

$$= Gm\left(\frac{1}{\left(r-r_{BH}\right)^2} - \frac{1}{3}\frac{\pi\sqrt{1+\varsigma^2}\,|\,\delta\psi_0\,|^2}{R_c}\left(r-r_{BH}\right)\right) \tag{3.19}$$

where the repulsive force

$$-Gm\frac{1}{3}\frac{\pi\sqrt{1+\varsigma^2}\,|\,\delta\psi_0\,|^2}{R_c}\left(r-r_{BH}\right) \tag{3.20}$$

overcomes the attractive one when

$$\left(r-r_{BH}\right)^3 \geq \frac{3R_c}{\pi\sqrt{1+\varsigma^2}\,|\,\delta\psi_0\,|^2} \tag{3.21}$$

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From (3.11) we can observe that the cosmological pressure density originating by the BH at large distance is constant and reads

$$\lim_{r\to\infty} \Lambda_Q^{BH} = \left(\sqrt{1+\varsigma^2}-1\right) \cong \frac{\varsigma^2}{2} \tag{3.22}$$

and that the repulsive gravity is generated by the presence of the dark energy/mass density $|\delta\psi_0|$ of the background fluctuations.

From (3.22) it is also interesting to note that the large distance mass density of the BH (2.34) acquire the form

$$\lim_{r\to\infty} |\psi|^2 = G_0 e^{-\sqrt{8\Lambda_Q^{BH}}\left(r/R_c\right)} \tag{3.23}$$

3.1. Quantum Potential Fluctuations Generated by the Background Fluctuations

To determine the parameter $|\delta\psi_0|^2$, we must move beyond the static vacuum solution and consider that the vacuum is filled with stochastic gravitational waves. These waves originate from various sources, including relic gravitational waves from the Big Bang and other sources [18].

Taking into account the vacuum fluctuations in the background, it becomes possible to define the stochastic generalization of the quantum-hydrodynamic equations [19] that for the wave function $\psi = |\psi| e^{-\frac{iS}{\hbar}}$, in the low velocity limit, are given by the equations

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\partial}{\partial q_i}(|\psi|^2 \dot{q}_i) = 0 \tag{3.24}$$

$$\dot{q}_i = \frac{p_i}{m} = \frac{1}{m}\frac{\partial S_{(q,t)}}{\partial q_i} \tag{3.25}$$

$$\dot{p}_i = -\frac{\partial(H + V_{qu})}{\partial q_i} \tag{3.26}$$

where $S_{(q,t)} = -\frac{\hbar}{2} ln \frac{\psi}{\psi *}$, where H is the Hamiltonian of the system and where V_{qu} is given by the low velocity limit of (2.5).

The ripples of the vacuum curvature are assumed to manifest themselves by an additional fluctuating mass density δn_{vac} into the vacuum so that

$$n_{tot} \equiv \overline{n} + \delta n_{vac} \tag{3.27}$$

where $\overline{n}$ is linked to n by the relation $lim_{\delta n_{vac}\to 0} \overline{n} = n$, that, introduced into the quantum potential

$$V_{qu(n_{tot})} = -\frac{\hbar^2}{2m} n_{tot}^{-1/2} \frac{\partial^2 n_{tot}^{1/2}}{\partial q_i \partial q_i} \tag{3.28}$$

leads to the quantum fluctuating force [19] $-\frac{\partial V_{qu(n)}}{\partial q_i}$ we are going to determine.

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Being the energy/mass density δn_{vac} defined positive, the mean vacuum fluctuations give rise to non-zero additional (dark) energy density into the vacuum.

Being the energy/mass density δn_{vac} defined positive, this paragraph describes the assumption that the mean vacuum fluctuations $<\delta n_{vac}>$ give rise to an additional dark energy density into the vacuum. The assumption is made that this vacuum dark energy/matter does not interact with the physical system under consideration, and therefore, the gravity interaction is disregarded in the Hamiltonian H in (3.26). The evolution of the total dark energy is assumed to depend on cosmological dynamics and that it has reached an equilibrium configuration.

Thence, we assume $<\delta n_{vac}>$ locally uniformly distributed with zero mean fluctuations $\delta n_{(q,t)}$ such as

$$\delta n_{vac} \cong <\delta n_{vac}> + \delta n_{(q,t)} \tag{3.29}$$

3.2. Spectrum and Correlation Function of Mass Density Noise in Quantum Spacetime with Curvature Fluctuations

When determining the features of the fluctuations of the quantum potential, which consequently produce the force noise, we employ the stipulation that:

The fluctuations of the vacuum curvature are described by the wave function ψ_{vac} with density $\delta n_{vac} = |\psi_{vac}|^2$

They do not have Hamiltonian interaction with the physical system (gravitational interaction is disregarded).

In this case the wave function of the overall system ψ_{tot} reads

$$\psi_{tot} \cong \psi\psi_{vac} \tag{3.30}$$

Moreover, by assuming that, the equivalent mass of the dark energy is much smaller than the mass of the system (i.e., $m_{tot} = m_{dark} + m \cong m$), the overall quantum potential (2.5) reads

$$\begin{aligned} V_{qu(n_{tot})} &= -\frac{\hbar^2}{2m_{tot}} |\psi|^{-1} |\psi_{vac}|^{-1} \frac{\partial^2 |\psi||\psi_{vac}|}{\partial q_i \partial q_i} = \\ &= -\frac{\hbar^2}{2m} \left(|\psi|^{-1} \frac{\partial^2 |\psi|}{\partial q_i \partial q_i} + |\psi_{vac}|^{-1} \frac{\partial^2 |\psi_{vac}|}{\partial q_i \partial q_i} + |\psi|^{-1} |\psi_{vac}|^{-1} \frac{\partial |\psi_{vac}|}{\partial q_i} \frac{\partial |\psi|}{\partial q_i} \right) \end{aligned} \tag{3.31}$$

Moreover, given the vacuum mass density noise of wave-length λ

$$\delta n_{vac(\lambda)} = |\psi_{vac(\lambda)}|^2 \propto cos^2 \frac{2\pi}{\lambda} q \tag{3.32}$$

associated to the fluctuation wave-function

$$\psi_{vac} \propto \pm cos \frac{2\pi}{\lambda} q \tag{3.33}$$

It follows that the quantum potential energy fluctuations read

$$\delta \overline{E}_{qu} = \int_V n_{tot(q,t)} \delta V_{qu(q,t)} dV \tag{3.34}$$

where

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$$\begin{aligned}\delta V_{qu(q,t)} &= -\frac{\hbar^2}{2m}\left(|\psi_{vac}|^{-1}\frac{\partial^2 |\psi_{vac}|}{\partial q_i \partial q_i} + |\psi|^{-1}|\psi_{vac}|^{-1}\frac{\partial |\psi_{vac}|}{\partial q_i}\frac{\partial |\psi|}{\partial q_i}\right) \\ &= \frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial |\psi|}{\partial q_i}\left(\pm cos\frac{2\pi}{\lambda}q\right)^{-1}\left(\pm sin\frac{2\pi}{\lambda}q\right)\right) \\ &= \frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial |\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)\end{aligned} \quad (3.35)$$

For $V \to \infty$, the unidimensional case leads to

$$\begin{aligned}\delta\overline{E}_{qu(\lambda)} &= \frac{1}{\overline{n}_{tot}V}\frac{\hbar^2}{2m}\int_V n_{tot(q,t)}\left(\left(\frac{2\pi}{\lambda}\right)^2 + |\psi|^{-1}\frac{\partial |\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)dq \\ &= \frac{1}{\overline{n}_{tot}V}\frac{\hbar^2}{2m}\left(\left(\frac{2\pi}{\lambda}\right)^2\int_V n_{tot(q,t)}dq + \int_V n_{tot(q,t)}\left(|\psi|^{-1}\frac{\partial |\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)dq\right) \cong \frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2\end{aligned} \quad (3.36)$$

In (3.36) it has been used the normalization condition $\int_V n_{tot(q,t)}dq = \overline{n}_{tot}V$ and, on large volume ($V \gg \lambda_c^3$ see (3.39)), it has been used the approximation

$$lim_{\lambda\to 0}\int_{-\infty}^{\infty} n_{tot(q,t)}\left(|\psi|^{-1}\frac{\partial |\psi|}{\partial q_i} tan\frac{2\pi}{\lambda}q\right)dq \ll \overline{n}_{tot}V\left(\frac{2\pi}{\lambda}\right)^2$$

For the three-dimensional case (3.36) leads to

$$\delta\overline{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m}\sum_i (k_i)^2 = \frac{\hbar^2}{2m}|k|^2 \quad (3.37)$$

The equation (3.37) reveals that the energy arising from the mass density fluctuations of the vacuum becomes greater as the square of the inverse of λ. Thus, the corresponding fluctuations in the quantum potential may produce extremely large energy fluctuations, even for very small noise amplitudes (i.e., $T \to 0$) when λ approaches zero), at very short distances.

To ensure that the convergence to the deterministic limit of quantum mechanic (5.24-26) is achieved for $T \to 0$, a further condition is required on the spatial correlation function of the quantum potential noise as $\lambda \to 0$.

One way to obtain the shape of the spatial correlation function $G(\lambda)$ is through a stochastic calculation, which can be quite complex [20]. However, a simpler approach to obtain $G(\lambda)$ this function can be achieved by considering the spectrum of the fluctuations, as described in [19].

Since each component of spatial frequency $k = \frac{2\pi}{\lambda}$ brings the quantum potential energy contribution (3.36), its probability of happening, reads

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$$p_{(\lambda)} \propto exp\left[-\frac{\delta \bar{E}_{qu}}{kT}\right] = exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right] = exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right] \quad (3.38)$$

where

$$\lambda_c = 2\frac{\hbar}{(2mkT)^{1/2}} \quad (3.39)$$

is the De Broglie length.

From (3.38) it comes out that the spectrum $S(k)$ of the spatial frequency

$$S(k) \propto p(\frac{2\pi}{\lambda}) = exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right] = exp\left[-\left(\frac{k\lambda_c}{2}\right)^2\right] \quad (3.40)$$

is not white and the components with wave-length λ smaller than λ_c go quickly to zero.

Besides, from (3.40) the spatial shape $G(\lambda)$ reads

$$G_{(\lambda)} \propto \int_{-\infty}^{+\infty} exp[ik\lambda] S_{(k)} dk \propto \int_{-\infty}^{+\infty} exp[ik\lambda]\, exp\left[-\left(k\frac{\lambda_c}{2}\right)^2\right] dk \propto \frac{\pi^{1/2}}{\lambda_c} exp\left[-\left(\frac{\lambda}{\lambda_c}\right)^2\right] \quad (3.41)$$

One can see from equation (3.41) that the quantum potential progressively suppresses uncorrelated MDD fluctuations on shorter and shorter distance scales, which in turn allows for the realization of deterministic quantum mechanics in systems whose physical length is much smaller than the De Broglie length.

The assumption for sufficiently general case is that the mass density noise correlation function is Gaussian, with zero correlation time, isotropic in space, and independent among different coordinates. Under these assumptions, it can be expressed as:

$$< \delta n_{(q_\alpha,t)}, \delta n_{(q_\beta+\lambda,t+\tau)} > = < \delta n_{(q_\alpha)}, \delta n_{(q_\beta)} >_{(T)} \ G(\lambda)\delta(\tau)\delta_{\alpha\beta} \quad (3.42)$$

3.3. The (Dark) Energy Density of Quantum Potential Fluctuations

The energy associated with the quantum potential noise of a body with mass m can be evaluated using the probability energy fluctuation function

$$p_{(E_{(\lambda)})} = A\, exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right] \quad (3.43)$$

where

$$A=\frac{1}{\int_0^{\lambda_{max}} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda}=\frac{1}{\int_0^{N\lambda_c} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda+\int_{N\lambda_c}^{\lambda_{max}} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda}$$
$$=\frac{1}{\int_0^{N\lambda_c} exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda+\int_{N\lambda_c}^{\lambda_{max}} d\lambda}=\frac{1}{\int_0^{N\lambda_c} exp\left[-\left(\frac{\pi\lambda_c}{\lambda}\right)^2\right]d\lambda+\left(\lambda_{max}-N\lambda_c\right)} \quad (3.44)$$

where $\lambda_c=\sqrt{2}\frac{\hbar}{\sqrt{mkT}}$ and $N\gg1$.

In this case, the energy density of quantum potential fluctuation reads

$$mc^2\,|\,\delta\psi_0\,|^2{}_{(r)}=\frac{\int\delta\overline{E}_{qu(\lambda)}p_{(\lambda)}d\lambda}{\int p_{(\lambda)}d\lambda}=\frac{\int_0^{\lambda_{max}}\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 p_{(\lambda)}d\lambda}{\int_0^{\lambda_{max}} p_{(\lambda)}d\lambda}\cong\int_0^{\lambda_{max}}\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 p_{(\lambda)}d\lambda$$
$$=\frac{1}{\lambda_{max}}\int_0^{\lambda\,max}\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda \quad (3.45)$$

where for (3.37) in the three-dimensional case $\lambda=|\,k\,|$ and where $m\,|\,\delta\psi_0\,|^2$ is the additional mass density in the vacuum that the body of mass m acquires due to the background fluctuations. Besides, it can be posed

$$\lambda_{max}=l_u\approx10^{27}m \quad (3.46)$$

where l_u is the diameter of the universe.

Moreover, since, for SMBHs (of order of Sagittarius A* of mass of about of 10^{38} kg) at $1°K$

$$\lambda_c=2\frac{\hbar}{(2mkT)^{1/2}}\approx\frac{1,41\times10^{-34}}{(3\times10^{38}10^{-23})^{1/2}}\approx3\times10^{-41}m\approx0 \quad (3.47)$$

it follows that

$$A=\frac{1}{l_u} \quad (3.48)$$

and that

$$mc^2 \mid \delta\psi_0 \mid^2 = \frac{1}{l_u}\frac{\hbar^2}{2m}\int_0^{l_u}\left(\frac{2\pi}{\lambda}\right)^2 exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right]d\lambda$$

$$= \frac{1}{l_u}\frac{\hbar^2}{2m}\int_{\infty}^{2\pi l_u^{-1}} exp\left[-\frac{\frac{\hbar^2}{2m}}{kT}\mid k\mid^2\right]d\mid k\mid = \frac{\hbar}{2l_u}\sqrt{\frac{\pi kT}{2m}} \tag{3.49}$$

leading to

$$\mid \delta\psi_0 \mid^2_{(r)} \approx \frac{1}{l_u}\frac{\hbar}{2mc}\sqrt{\frac{\pi kT}{2mc^2}} \tag{3.50}$$

3.4. Repulsive Gravity at Large Distance

By introducing (3.50) into (3.21), the repulsive Newtonian gravity overcomes the attractive one at the distance

$$(r - r_{BH})_{rep} \approx \left(\frac{3R_c}{\pi\sqrt{1+\varsigma^2}\mid \delta\psi_0 \mid^2}\right)^{1/3} = \left(\frac{3R_c}{\pi\left(1+\Lambda_Q^{BH}\right)\mid \delta\psi_0 \mid^2}\right)^{1/3} \sim \left(l_u\sqrt{\frac{2mc^2}{\pi kT}}\right)^{1/3} \tag{3.51}$$

that for SMBHs mass of order of $10^{38\div41} Kg$, $\varsigma \ll 1$, reads

$$(r - r_{BH})_{rep} >\approx 10^{21} \quad m \tag{3.52}$$

The gravitational force between galaxies becomes repulsive at intergalactic distances, which is on the same order of magnitude as the typical radius of galaxies ($\sim 10^{20} m$). This may affect the external part of the galactic disc. However, the amplitude of the background fluctuations decreases with the expansion of the radius l_u of the universe since $lim_{t\to\infty} \mid \delta\psi_0 \mid^2_{(r)} \sim \frac{1}{l_u}\frac{\hbar}{2mc}\sqrt{\frac{\pi kT}{2mc^2}} \to 0$), and the repulsive force asymptotically approaches zero, leading to a final static universe. The decrease in temperature with $T \to 0$ due to expansion results in a perfectly quantum final stationary state with $\lambda_c \to \infty$ and $\lambda_{qu} \to \infty$.

3.5. The Energy Density of Quantum Potential Fluctuations of a Classical Body

On the other hand, it is interesting to calculate the dark energy associated to the quantum potential fluctuations of ordinary macroscopic (classical) bodies.

In the quasi-Minkowskian space-time, constituted by atoms interacting by Lennard Jones potential [15, 21] for which the quantum non-local range of interaction λ_{qu} is limited to microscopic distance of order of λ_c so that

$$\lambda_{qu} \cong \lambda_c \tag{3.54}$$

it follows that

$$mc^2 \,|\, \delta\psi_0 \,|^2 \cong \frac{\int_0^{\lambda_c} \delta \overline{E}_{qu(\lambda)} p_{(\lambda)} d\lambda}{\int_0^{\lambda_c} p_{(\lambda)} d\lambda} \cong \frac{\frac{\hbar^2}{2m}}{l_u} \int_0^{\lambda_c} \left(\frac{2\pi}{\lambda}\right)^2 exp\left[-\frac{\frac{\hbar^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2}{kT}\right] d\lambda \tag{3.55}$$

$$\cong \frac{1}{l_u} \hbar \sqrt{\frac{kT}{2m}} \int_1^{\infty} exp\left[-x^2\right] dx$$

$$|\, \delta\psi_0 \,|^2 \approx \frac{1}{l_u} \hbar \sqrt{\frac{kT}{2m}} \frac{1}{mc^2} \int_1^{\infty} exp\left[-x^2\right] dx \qquad 0 < r < \lambda_c \tag{3.56}$$

$$|\, \delta\psi_0 \,|^2 = 0 \qquad r > \lambda_c \tag{3.57}$$

The output (3.56) leads to the Newtonian force (3.19)

$$\frac{\partial\phi}{\partial r} = Gm \left(\frac{1 + \frac{|\,\delta\psi_0\,|^2}{R_c} \lambda_c{}^4}{\left(r - r_{BH}\right)^2} \right) \cong -Gm \left(\frac{1 + 4 \frac{|\,\delta\psi_0\,|^2}{mc} \frac{\hbar^3}{(kT)^2}}{\left(r - r_{BH}\right)^2} \right) \tag{3.58}$$

In the case of the proton at 1°K (3.58) reads

$$\frac{\partial\phi}{\partial r} = \frac{Gm}{\left(r - r_{BH}\right)^2} \left(1 + \sim 10^{-76}\right) \tag{3.59}$$

The equation (3.59) implies that the dark energy produced by the background fluctuations for classical macroscopic bodies is negligible because the range of the quantum potential is short.

This suggests that the repulsive gravitational force causing the expansion of the universe is mainly attributed to black holes and supermassive black holes due to the quantum nature of space-time with a fluctuating background metric.

The correction to the Newtonian gravity that arises from the energy of the quantum potential in massive bodies such as BHs and SMBHs is similar in some way to the concept of Modified Newtonian Dynamics (MOND) theories [22], which suggest a modification of the Newtonian gravity for very low accelerations in order to account for the observed motion of galaxies.

The current model proposes that the modification of Newtonian gravity at large distances is explained as coming from the gravitational effects of the energy of the quantum potential in huge massive bodies such as BHs and SMBHs.

4. Conclusions

The work shows that quantum black holes with central symmetry have a mass density distribution that is not point-like but concentrated in a sphere of radius approximately equal to its Compton wavelength.

Due to the significant quantum potential energy, there exists an isotropic pressure density term in the gravity equation, resulting in an additional contribution to the gravity force at weak limit.

In the presence of fluctuations of the space-time background metric, this contribution leads to a repulsive gravitational force that dominates over the Newtonian force at distances typical of intergalactic space.

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Appendix

Schwarzschild black hole mass distribution

The differential equation (2.33) can be solved by posing

$$\left(\frac{\hbar}{mc}\right)^2 \frac{r^2}{|\psi|}\partial^1 r^{-2}\left(\frac{r-R_g}{r}\partial_1|\psi|\right)=1-A_0e^{f(r)} \tag{A.1}$$

from which it follows that

$$\left(\partial_1\frac{r-R_g}{r}\right)A_0e^{f(r)}=\left(\partial_1 f_{(r)}\right)A_0e^{f(r)} \tag{A.2}$$

leading to the solution

$$f_{(r)}=\left(\frac{r-R_g}{r}+C\right) \tag{A.3}$$

Furthermore, by posing

$$|\psi|=G_0e^{g(r)} \tag{A.4}$$

it follows that

$$\left(\frac{\hbar}{mc}\right)^2 r^2G_0^{-1}e^{-g(r)}\partial^1 r^{-2}\left(\frac{r-R_g}{r}g'_{(r)}G_0e^{g(r)}\right)=1-A_0e^{f(r)} \tag{A.5}$$

and thence,

$$\left(\frac{\hbar}{mc}\right)^2 r^2G_0^{-1}e^{-g(r)}\left(g'_{(r)}G_0e^{g(r)}\partial^1\left(r^{-2}\frac{r-R_g}{r}\right)+r^{-2}\frac{r-R_g}{r}\partial^1\left(g'_{(r)}G_0e^{g(r)}\right)\right)=$$

$$\left(\frac{\hbar}{mc}\right)^2\left(r^2g'_{(r)}\partial^1 r^{-3}\left(r-R_g\right)+\frac{r-R_g}{r}\left(g'^2_{(r)}+g''_{(r)}\right)\right)= \tag{A.6}$$

$$\left(\frac{\hbar}{mc}\right)^2\left(g'_{(r)}\left(\frac{1}{r}-\frac{3\left(r-R_g\right)}{r^2}\right)+\frac{r-R_g}{r}\left(g'^2_{(r)}+g''_{(r)}\right)\right)=1-A_0e^{f(r)}$$

that by posing

$$g'_{(r)}=y_{(r)} \tag{A.7}$$

leads to the Riccati's differential equation

$$y'_{(r)}=y_{(r)}\left(\frac{3}{r}-\frac{1}{r-R_g}\right)-y^2_{(r)}+\left(\frac{mc}{\hbar}\right)^2\frac{r}{r-R_g}\left(1-C_0e^{\frac{r-R_g}{r}}\right) \tag{A.8}$$

where $C_0=A_0e^{-C}$.

Moreover, by using the adimensional variable $x=\frac{r}{R_g}$ it follows that

$$\tilde{y}'_{(x)}=R_g y'_{(r)}=\tilde{y}_{(x)}\left(\frac{3}{x}-\frac{1}{x-1}\right)-R_g\tilde{y}^2_{(x)}+\frac{R_g}{R_c^{\,2}}\frac{x}{x-1}\left(1-C_0e^{\frac{x-1}{x}}\right) \tag{A.9}$$

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where $R_c = \frac{\hbar}{mc}$.

The condition of mass density at infinity

$$\lim_{r\to\infty} |\psi| = \lim_{r\to\infty} G_0 e^{g(r)} = 0 \tag{A.10}$$

$$\lim_{r\to\infty} |\psi|' = \lim_{r\to\infty} G_0 g'_{(r)} e^{g(r)} = 0 \tag{A.11}$$

leads to the condition on $g(r)$

$$\lim_{r\to\infty} g_{(r)} = -\infty \tag{A.12}$$

Large distance BH mass density distribution

On the condition that $\lim_{x\to\infty} |\tilde{y}^2_{(x)}| \gg \frac{|\tilde{y}_{(x)}|}{x}$ (to be checked at the end) and by choosing $c = 1$ so that $C_0 = \frac{A_0}{e}$, equation (A.9) simplifies to

$$\lim_{x\to\infty} \tilde{y}^{|}_{(x)} \cong -R_g \tilde{y}^2_{(x)} + \frac{R_g}{R_c^{\,2}}(1 - A_0) \tag{A.13}$$

that by posing

$$\tilde{y}_{(x)} = \frac{u'}{R_g u} \tag{A.14}$$

leads to

$$\lim_{x\to\infty}\left(\frac{u'}{R_g u}\right)' = \lim_{x\to\infty}\left(\frac{u''}{R_g u} - \frac{u'^2}{R_g u^2}\right) \cong -\frac{u'^2}{R_g u^2} + \frac{R_g}{R_c^{\,2}}(1 - A_0) \tag{A.15}$$

$$\lim_{x\to\infty} u'' \cong u\frac{R_g^{\,2}}{R_c^{\,2}}(1 - A_0) \tag{A.16}$$

owing the solution

$$u = u_0 e^{\pm\frac{R_g}{R_c}(1-A_0)^{1/2} x} \tag{A.17}$$

leading to

$$\tilde{y}_{(x)} = \pm\frac{R_g}{R_c}(1 - A_0)^{1/2} \tag{A.18}$$

and to

$$\lim_{x\to\infty} |\psi| = \lim_{x\to\infty} G_0 e^{\int g'_{(r)} dr} = G_0 e^{\int R_g \tilde{y}_{(x)} dx} = G_0 e^{-\frac{R_g}{R_c}(1-A_0)^{1/2} x} = G_0 e^{-\varsigma\frac{r}{R_c}} \tag{A.19}$$

where $\varsigma = (1 - A_0)^{1/2}$ and where the condition $\lim_{r \to \infty} g_{(r)} = -\infty$ requires to consider the negative solution of $\tilde{y}_{(x)}$.

In order to evaluate the numerical constant ς, we observe that the ratio between the total mass of the BH and the part outside the gravitational radius due to the quantum mass distribution (with unitary normalization $\int_0^\infty |\psi|^2 = 1$) reads

$$\frac{\Delta m_{out}}{m} = \frac{\int_{r=R_g}^{\infty} |\psi|^2}{\int_0^\infty |\psi|^2} = \int_{r=R_g}^{\infty} |\psi|^2 \tag{A.20}$$

and it is vanishing small for cosmological BHs (e.g., for BHs of mass of order $10^{38}\,kg$,assuming unitary outside mass (1 kg) it results $\int_{r=R_g}^{\infty} |\psi|^2 \approx 10^{-38}$). Moreover, by utilizing the expression (2.2)

$$\lim_{\frac{r}{R_c} \to \infty} |\psi|^2 = \delta^3{}_{(r)} \cong \lim_{R_c \to 0} \left(\frac{\varsigma}{2\sqrt{\pi} R_c} \right)^3 e^{-\varsigma \frac{r^2}{4R_c^2}} \tag{A.21}$$

It follows that

$$\int_{r=R_g}^{\infty} |\psi|^2 dV = \frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 \int_{r=R_g}^{\infty} \left(\frac{r}{R_c} \right)^2 e^{-\varsigma \frac{r^2}{4R_c^2}} d\frac{r}{R_c}$$
$$= \frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 \int_{t=\frac{R_g}{R_c}}^{\infty} t^2 e^{-\varsigma \frac{t^2}{4}} dt = \frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 \int_{t=\frac{R_g}{R_c}}^{\frac{l_u}{R_c}} t^2 e^{-\varsigma \frac{t^2}{4}} dt = \frac{\Delta m_{out}}{m} \tag{A.22}$$

F that for BH of $10^{38}\,kg$, a kg of Δm_{out} gives the contribution

$$\frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 \int_{t=\frac{R_g}{R_c}}^{\infty} t^2 e^{-\varsigma \frac{t^2}{4}} dt = \frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 \int_{10^{85}}^{\infty} t^2 e^{-\varsigma \frac{t^2}{4}} dt = 10^{-38} < \frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 10^{170} e^{-\varsigma \frac{10^{170}}{4}} 10^{102} \tag{A.23}$$

and that

$$\frac{1}{\pi^{3/2}} \left(\frac{\varsigma}{2} \right)^3 10^{170} e^{-\varsigma \frac{10^{170}}{4}} 10^{102} > 10^{-38} \tag{A.24}$$

$$\left(\frac{\varsigma}{2} \right)^3 e^{-\varsigma \frac{10^{170}}{4}} = \left(\frac{\varsigma}{2} \right)^3 10^{-\varsigma \frac{10^{170}}{4} \lg_{10} e} >\sim 10^{-310} \tag{A.25}$$

that by posing $\varsigma = 10^{-n}$ leads to

$$10^{-\frac{10^{170-3n}}{4}\lg_{10} e} >\sim 10^{-309+3n} \quad (A.26)$$

to

$$10^{170-3n} > 4\frac{309-3n}{\lg_{10} e} \sim 10^{2} \quad (A.27)$$

and to

$$0 < \zeta <\sim 10^{-56} \quad (A.28)$$

Even for larger values of $\frac{\Delta m_{out}}{m} \sim 10^{-9}$ the order of magnitude (A.28) remains practically the same. This because the ratio $\frac{R_c}{R_g}$ determines ζ.

Thus, BHs with small mass, close to the Planck one with $R_c \sim R_g$, can lead to higher values of ζ. As a consequence of this, smaller the BHs larger the contribution to the repulsive gravity (2.20) and to the cosmological constant (2.22).

Mass distribution at short distance ($r \ll R_g$)

Near the center for $x \ll 1$ we must use the relation

$$\Gamma^{\alpha}_{\mu\nu} u_\alpha u^\nu = u^0 u^0 \partial_1 g_{00} = g^{00}\partial_1 g_{00} = \frac{1}{g_{00}}\partial_1 g_{00} \cong \partial_1 \ln g_{00} \quad (A.29)$$

That leads to the equation

$$\left(\partial_1 \ln\left(\frac{r-R_g}{r}\right)\right) = \left(\partial_1 f_{(r)}\right) \quad (A.30)$$

with the solution

$$f_{(r)} = \left(\ln\left(\frac{r-R_g}{r}\right) + C\right) \quad (A.31)$$

and the equation

$$\tilde{y}^{|}_{(x)} = R_g y'_{(r)} = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) - R_g \tilde{y}^2_{(x)} + \frac{R_g}{R_c^{\,2}}\frac{x}{x-1}\left(1 - C_0 e^{\ln\left(\frac{x-1}{x}\right)}\right) \quad (A.32)$$

Furthermore, on the condition that $\lim_{x\to 0} |\tilde{y}^2_{(x)}| << \frac{|\tilde{y}_{(x)}|}{x}$ (to be checked at the end) Equation (A.9) simplifies to

$$\tilde{y}^{|}_{(x)} = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) + \frac{R_g}{R_c^{\,2}}\frac{x}{x-1}\left(1 - C_0 \frac{x-1}{x}\right) = \tilde{y}_{(x)}\left(\frac{3}{x} - \frac{1}{x-1}\right) - C_0\frac{R_g}{R_c^{\,2}} \quad (A.33)$$

where $x \ll 1$ and $\tilde{y}_{(x)} = \frac{u'}{R_g u}$, leading to the solution

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$$
\begin{aligned}
y_{(x)} &= \left(e^{\int \frac{R_g}{R_c{}^2} C_0 e^{-\int \left(\frac{3}{x} - \frac{1}{x-1}\right) dx} dr} \right) e^{\int \left(\frac{3}{x} - \frac{1}{x-1}\right) dx} \\
&= \left(e^{\int \frac{R_g c^2}{R_c{}^2} C_0 e^{-\int \frac{2}{x} dx} dx} + C \right) e^{\int \frac{2}{x} dx} \\
&\cong \left(e^{-C_0 \frac{R_g{}^2}{R_c{}^2} \frac{1}{x^2}} + C \right) x^2 = g'_{(r)} = \frac{1}{R_g} g'_{(x)}
\end{aligned}
\tag{A.34}
$$

and, thence, for $R_c \ll r \ll R_g$

$$
\begin{aligned}
\lim_{x \to 0} |\psi| &= \lim_{x \to 0} G_0 e^{\int \left(e^{-C_0 \frac{R_g{}^2}{R_c{}^2} \frac{1}{x^2}} + C \right) x^2 dx} = \lim_{x \to 0} G_0 e^{\int \left(1 - C_0 \frac{R_g{}^2}{R_c{}^2} \frac{1}{x^2} + C \right) x^2 dx} \\
&= \lim_{x \to 0} G_0 e^{\frac{(1+C)}{3} x^3 - C_0 \frac{R_g{}^2}{R_c{}^2} x} \cong G_0 e^{-C_0 \frac{R_g{}^2}{R_c{}^2} x} \cong G_0 e^{-\frac{1-\varsigma^2}{e} \frac{R_g}{R_c} z}
\end{aligned}
\tag{A.35}
$$

where it has been used the identity $C_0 = \frac{1-\varsigma^2}{e} \cong e^{-1}$.

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