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Article

Heyting Locally Small Spaces and Esakia Duality

Artur Piękosz

Department of Applied Mathematics, Cracow University of Technology, ul. Warszawska 24, 31-155 Kraków, Poland; apiekosz@pk.edu.pl

Abstract: We develop the theory of Heyting locally small spaces, including Stone-like dualities such as a new version of Esakia duality and a system of concrete isomorphisms and equivalences. In such a way, we continue building tame topology, realising Grothendieck's ideas. We use up-spectral spaces and define the standard up-spectralification of a Kolmogorov locally small space. This research gives more understanding of locally definable spaces over structures with topologies.

Keywords: tame topology; Esakia duality; locally small space; up-spectral space

MSC: primary 54E99; 06D20; 18B30; secondary 54A05; 54B30; 18F10

1. Introduction

Equivalences (including isomorphisms) and dualities between categories are seen as important and fruitful forms of symmetry in pure mathematics. We extend Esakia Duality to Heyting locally small spaces, which we introduce. This duality was originally proved by Leo Esakia in [1] (see also [2]) for spaces that combine topology and order, which he called hybrids (we are especially interested in strict hybrids). Then we provide a system of concrete isomorphisms, equivalences and dual equivalences between our categories that give a deeper understanding of locally small spaces and notions related to them.

Esakia Duality has been studied and used after L. Esakia by such authors as: Guram and Nick Bezhanishvili ([3] and [4]) together with their collaborators ([5,6]) as well as S.A. Celani and R. Jansana ([7]) and many others.

We also see Esakia Duality as a subcase of Stone Duality ([8]) or Priestley Duality ([9]). Many extensions of Stone Duality have been achieved in recent years. For example: the locally compact Hausdorff case in [10], removing the zero-dimensionality and the commutativity assumptions in [11], generalisations of Gelfand–Naimark–Stone Duality to completely regular spaces in [12] and its application to the characterisation of normal, Lindelöf and locally compact Hausdorff spaces in [13]. There exist extensions of Stone Duality that drop compactness completely: the paper [14] considers all zero-dimensional Hausdorff spaces. Stone Duality has been also extended in [15] to (non-distributive) orthomodular lattices (corresponding to spectral presheaves), in [16] to some non-distributive (implicative, residuated, or co-residuated) lattices and applied to the semantics of substructural logics, in [17] to a non-commutative case of left-handed skew Boolean algebras. Applications of Stone Duality have appeared in [18] (canonical extensions of lattice-ordered algebras) and [19] (the semantics of non-distributive propositional logics). Some recent applications of Stone Duality have appeared in [20] in the theory of C^*-algebras.

The main objective of this paper is to extend our results from [21] by presenting a version of Esakia Duality for Heyting locally small spaces (we try to follow the conventions of [22]). The present paper continues the research from [23], [21] on some versions of Stone Duality or Priestley Duality for locally small spaces as well as a version of Esakia Duality for small spaces.

We are developing tame topology, postulated by Alexander Grothendieck in his scientific programme [24]. Such an approach to topology is intended to eliminate pathological

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phenomena of usual topology. We are encouraged by the authors of [25] who concluded that the tame topology suggested by A. Grothendieck had not been defined. Notice that Grothendieck's ideas on the notion of space interest some people concerned with physical and philosophical questions (see Cruz Morales [26]). We develop the theory of tame spaces such as small spaces or locally small spaces, as a way to realise Grothendieck's postulate in a purely topological context.

One important useful tool for us is the theory of up-spectral (called also almost-spectral) spaces developed by M. Hochster ([27]) O. Echi together with his collaborators ([28,29]), L. Acosta and I.M. Rubio Perilla ([30]). Up-spectral spaces can be better understood in the wider context of Balbes-Dwinger spaces, see [31] and [32].

Smopologies have appeared implicitly in real algebraic geometry (see [33] (Definition 7.1.14) or [34] (p. 12)), in o-minimality ([35–37]), and in more general contexts of model theory ([38–40]). They should be helpful in such branches of mathematics as the generalisations of o-minimality, analytic geometry or algebraic geometry. While in the usual topology we have only spectral reflections (see [22] (Chapter 11)), smopologies allow transferring structural information without any losses between algebraic and topological structures using dualities or equivalences. Transferring information between the topological and the algebraic languages in new versions of Esakia Duality should give more understanding of locally definable spaces over structures with (especially definable) topologies.

Regarding the set-theoretical foundations, we use (without mentioning this) Mac Lane's standard Zermelo–Fraenkel axioms with the Axiom of Choice and the existence of a set which is a universe as in [41] (p. 23). Such a setting allows speaking about proper classes of sets or categories while staying formally in the axiomatic system **ZFC**. (See "Axiomatic assumptions" in [42] for the full explanation of our axiomatic system.)

2. Generalities about Locally Small Spaces

Notation. We shall use a special notation for operations on families of sets, for example for family intersection

$$\mathcal{U} \cap_1 \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}, \quad \mathcal{U} \cap_1 V = \mathcal{U} \cap_1 \{V\}$$

or for other operations

$$int_1(\mathcal{A}) = \{int(A) : A \in \mathcal{A}\}, \quad Gen_1(\mathcal{A}) = \{Gen(A) : A \in \mathcal{A}\},$$
$$\mho\mathcal{A} = \{\bigcup \mathcal{B} : \mathcal{B} \subseteq \mathcal{A}\}, \quad \Omega\mathcal{A} = \{\bigcap \mathcal{B} : \mathcal{B} \subseteq \mathcal{A}\}.$$

Let a family $\mathcal{A}$ of subsets of a set X be given. Define

$$ba(\mathcal{A}) = \text{the Boolean algebra generated by } \mathcal{A} \text{ in } \mathcal{P}(X)$$

$$\mathcal{A}^o = \{Y \subseteq X \mid Y \cap_1 \mathcal{A} \subseteq \mathcal{A}\}.$$

Elements of $\mathcal{A}^o$ are called the sets *compatible with* the family $\mathcal{A}$, while elements of $ba(\mathcal{A})$ are the sets *constructible from* $\mathcal{A}$.

Definition 1 ([23,43])**.** *A* locally small space *is a pair* $(X, \mathcal{L}_X)$*, where* X *is any set and* $\mathcal{L}_X \subseteq \mathcal{P}(X)$ *satisfies the following conditions:*
(LS1) $\emptyset \in \mathcal{L}_X$*,*
(LS2) *if* $A, B \in \mathcal{L}_X$*, then* $A \cap B, A \cup B \in \mathcal{L}_X$*,*
(LS3) $\bigcup \mathcal{L}_X = X$.

Elements of $\mathcal{L}_X$ *are also called* smops *(i.e.,* small open *subsets of* X*). The family* $\mathcal{L}_X$ *is called a* smopology*. Assume* $(X, \mathcal{L}_X)$ *and* $(Y, \mathcal{L}_Y)$ *are locally small spaces. Then a mapping* $f : X \to Y$ *is:*
(a) bounded *if* $\mathcal{L}_X$ *refines* $f^{-1}(\mathcal{L}_Y)$*, which means that each* $A \in \mathcal{L}_X$ *admits* $B \in \mathcal{L}_Y$ *such that* $A \subseteq f^{-1}(B)$*,*
(b) continuous *if* $f^{-1}(\mathcal{L}_Y) \cap_1 \mathcal{L}_X \subseteq \mathcal{L}_X$ *(i.e.,* $f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}_X^o$*),*

(c) strongly continuous *if* $f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}_X$.

We have the category **LSS** *of locally small spaces and bounded continuous maps, called sometimes also* strictly continuous *maps.*

Definition 2. *If* $X \in \mathcal{L}_X$ *, then* $(X, \mathcal{L}_X)$ *is called a* small space. *The full subcategory in* **LSS** *generated by the small spaces is denoted by* **SS**.

Remark 1. *The isomorphisms of* **LSS** *are such bijections* $f : X \to Y$ *between locally small spaces that* $f^{-1}(\mathcal{L}_Y^o) = \mathcal{L}_X^o$ *and the families* $\mathcal{L}_X$ *and* $f^{-1}(\mathcal{L}_Y)$ *refine each other. The latter condition means that the bornologies generated by those families are equal. One can easily check that isomorphisms are exactly those mappings that satisfy the condition* $f^{-1}(\mathcal{L}_Y) = \mathcal{L}_X$. *Such mappings are called* strict homeomorphisms.

Definition 3. *We have the following additional families and types of subsets of X:*

1. *The family* $\mathcal{L}'_X = X \setminus_1 \mathcal{L}_X$ *is called a* co-smopology *and its members are called* co-smops.
2. *The family* $Con(X) = ba(\mathcal{L}_X)$ *contains the* constructible *sets, which are the Boolean combination of smops.*
3. *The family* $\mathcal{L}_X^o$ *contains subsets compatible with smops, which will be called* (admissible) open sets, *and their complements that will be called* (admissible) closed sets. *The family of all admissible open sets* $\mathcal{L}_X^o$ *in X will be usually denoted by* $AdOp(X)$, *and the family of all admissible closed sets will be denoted by* $AdCl(X)$ *or* $\mathcal{L}_X^c$.
4. *A subset of a smop that is a constructible set will be called a* small constructible set. *The family of all small constructible sets will be denoted by* $sCon(X)$.
5. *A subset of X whose traces on smops are constructible is a* locally constructible set. *The family of all locally constructible sets will be denoted by* $LCon(X)$.

Remark 2. *Notice that for a small space (i.e., if* $X \in \mathcal{L}_X$*) we have*

$$\mathcal{L}_X = AdOp(X) \text{ and } \mathcal{L}'_X = AdCl(X).$$

In this case $\mathcal{L}'_X$ *is also a smopology and the small space* $(X, \mathcal{L}'_X)$ *is called the* inverse space *of* $(X, \mathcal{L}_X)$ *and denoted by* X_{inv} *([21]).*

Fact 1 ([43])**.** *We have the functor of* smallification sm : **LSS** $\to$ **SS** *given by* $sm : (X, \mathcal{L}_X) \to (X, \mathcal{L}_X^o)$ *and* $sm(f) = f$, *which is a concrete reflector (see [44]).*

Definition 4. *We have the following important topologies in a locally small space:*

1) *The topology* $\tau(\mathcal{L}_X)$ *generated by the smops will be called the* original topology *and denoted* τ_{orig}. *The closure, the interior and the exterior operations in the original topology will be denoted by* $\overline{\cdot}$ *(or by* $cl(\cdot)$*),* $int(\cdot), ext(\cdot)$, *respectively.*
2) *The topology* $\tau_{inv} = \tau(AdCl(X))$ *generated by the admissible closed sets will be called the* inverse topology. *The closure and the interior operations in the inverse topology will be denoted by* $\overline{\cdot}^{inv}, int_{inv}(\cdot)$, *respectively.*
3) *The topology* $\tau_{tinv} = \tau(\mathcal{L}'_X)$, *generated by the co-smops, which can be called the* tiny inverse topology.
4) *The topology* $\tau(\mathcal{L}_X \setminus_1 \mathcal{L}_X)$ *generated by the differences of smops will be called the* constructible topology. *The closure and the interior operations in the constructible topology will be denoted by* $\overline{\cdot}^{con}, int_{con}(\cdot)$, *respectively.*

Remark 3. *While the family* $AdOp(X)$ *usually properly contains* $\mathcal{L}_X$, *they generate the same original topology* $\tau(\mathcal{L}_X) = \tau(AdOp(X))$.

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Definition 5. Weakly open sets *are the sets beloning to the original topology and their complements (=intersections of admissible closed sets) are the* weakly closed sets. *We can express this in symbols as follows:*

$$\tau_{orig} = \mho\mathcal{L}_X = \mho AdOp(X), \quad X \setminus_1 \tau_{orig} = \Omega\mathcal{L}'_X = \Omega AdCl(X).$$

Similarly, we have constructibly weakly open *and* constructibly weakly closed *sets,* inversely weakly open *and* inversely weakly closed *sets, etc. A mapping that is continuous in the topologies of weakly open sets is called* weakly continuous.

Proposition 1. *For a locally small space $(X, \mathcal{L}_X)$, we have*

$$\overline{A} = \bigcup_{V \in \mathcal{L}_X} \overline{A \cap V}, \qquad int(A) = \bigcup_{V \in \mathcal{L}_X} int(A \cap V).$$

The topological operations on the right-hand side may be taken in X or in V.

Proof. We always have $\bigcup_{V \in \mathcal{L}_X} \overline{A \cap V} \subseteq \overline{A}$. If $x \in \overline{A}$, then there exists $V \in \mathcal{L}_X$ such that $x \in \overline{A} \cap V$. For any $W \in \mathcal{L}_X$ if $x \in W$ then $W \cap V \in \mathcal{L}_X$ and $W \cap V \cap A \neq \emptyset$. That is why $x \in \overline{A \cap V} \cap V$, so x belongs to the set on the right-hand side of the first claim. For the non-trivial inclusion $int(A) \subseteq \bigcup_{V \in \mathcal{L}_X} int(A \cap V)$ in the second claim, assume $x \in int(A) \cap V$ for some $V \in \mathcal{L}_X$. Then $x \in W \subseteq A$ for some $W \in \mathcal{L}_X$. Hence $x \in W \cap V \subseteq A \cap V$ and $x \in int(A \cap V)$. □

Example 1. *The tiny inverse topology is usually strictly smaller than the inverse topology. In symbols: we usually have $\tau_{tinv} = \mho\mathcal{L}'_X \subset \tau_{inv} = \mho AdCl(X)$.*

Indeed, consider an infinite set X and define $\mathcal{L}_X = \mathrm{Fin}(X)$. Then $\tau(\mathcal{L}'_X) = \mathrm{Cofin}(X) \cup \{\emptyset\}$ but the inverse topology equals $\tau(AdCl(X)) = AdCl(X) = \mathcal{P}(X)$.

Example 2. *The family $ba(AdOp(X))$ may be a proper subfamily of $LCon(X)$. Indeed, take as X the disjoint union $\bigsqcup_{n \in \mathbb{N}} \mathbb{R}^{2n}_{om}$ of $2n$-th powers of the o-minimal real line $\mathbb{R}_{om}$ ([43]) indexed by $n \in \mathbb{N}$. Then the smops of $\mathbb{R}^{2n}_{om}$ are the finite unions of cartesian products of open intervals. Smops in the disjoint union are the finite unions of smops in particular cartesian powers $\mathbb{R}^{2n}_{om}$. The locally constructible set $A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_n = \{x \in \mathbb{R}^{2n} : \#\{k : x_k = 0\}$ is even$\}$ is not a Boolean combination of admissible open sets. To see this, consider the closure algebra $(LCon(X), \overline{\cdot})$ in the sense of Definition 2.2.1 of [2]. If A were a Boolean combination of m admissible open sets, then A would be a union of at most 2^m locally closed sets in this closure algebra, hence, by Proposition 2.5.27 of [2], we would have $\rho^{2^m}(A) = \emptyset$ where $\rho(Z) = \partial(\partial Z)$ and $\partial Z = \overline{Z} \setminus Z$. For $n > 2^m$, we have $\rho^n(A_n) = \{0\}^{2n} \neq \emptyset$, so A_n is not a Boolean combination of m smops in $\mathbb{R}^{2n}_{om}$. Consequently, $ba(AdOp(X)) \subset LCon(X)$.*

Example 3. *Usually also the Boolean algebra $Con(X) = ba(\mathcal{L}_X)$ is strictly smaller than the Boolean algebra $ba(AdOp(X))$. Indeed, for the space from Example 1, we have $Con(X) = \mathrm{Fin}(X) \cup \mathrm{Cofin}(X)$ and $ba(AdOp(X)) = \mathcal{P}(X)$.*

Proposition 2. *We have the following equality between topologies:*

$$\tau(\mathcal{L}_X \setminus_1 \mathcal{L}_X) = \tau(ba(\mathcal{L}_X)) = \tau(ba(AdOp(X))) = \tau(sCon(X)) = \tau(LCon(X)).$$

Hence we have the following descriptions of the constructible topology:

$$\mho LCon(X) = \mho sCon(X) = \mho ba(AdOp(X)) = \mho ba(\mathcal{L}_X) = \mho(\mathcal{L}_X \setminus_1 \mathcal{L}_X).$$

Proof. The inclusions $\mathcal{L}_X \setminus_1 \mathcal{L}_X \subseteq ba(\mathcal{L}_X) \subseteq ba(AdOp(X)) \subseteq LCon(X)$ are obvious. Each locally constructible set is a union of a family of smop differences. □

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Definition 6. *For a locally small space* $(X, \mathcal{L}_X)$ *and a subset* $Y \subseteq X$*, we have the induced* subspace $(Y, \mathcal{L}_X \cap_1 Y)$ *of* $(X, \mathcal{L}_X)$*. This subspace is called:* open *if* $Y \in AdOp(X)$*,* closed *if* $Y \in AdCl(X)$*,* decent *if* Y *is constructibly dense,* small *if* Y *is a subset of a smop.*

Lemma 1. *Assume that X is a locally small space.*

1. *If* $V \in AdOp(X)$*, then for the induced subspace* $(V, \mathcal{L}_X \cap_1 V)$ *we have* $AdOp(V) = AdOp(X) \cap_1 V$.
2. *If* $F \in AdCl(X)$*, then for the induced subspace* $(F, \mathcal{L}_X \cap_1 F)$ *we have* $AdOp(F) = AdOp(X) \cap_1 F$.
3. *If* S *is a small subset of* X*, then for the induced subspace* $(S, \mathcal{L}_X \cap_1 S)$ *we have* $AdOp(S) = AdOp(X) \cap_1 S$.
4. *If* $Y \subseteq X$ *is decent, then for the induced subspace* $(Y, \mathcal{L}_X \cap_1 Y)$ *we have* $AdOp(Y) = AdOp(X) \cap_1 Y$.

Proof. 1. We check $AdOp(X) \cap_1 V \subseteq AdOp(V)$ first. For each $W \in AdOp(X)$ we have $W \cap V \in AdOp(X) \cap \mathcal{P}(V)$. For any $A \in \mathcal{L}_X \cap_1 V$, we have $W \cap V \cap A \in \mathcal{L}_X \cap \mathcal{P}(V) = \mathcal{L}_X \cap_1 V$. This proves $W \cap V \in AdOp(V)$. For the other inclusion assume $U \in AdOp(V)$. For each $L \in \mathcal{L}_X$, we have $U \cap L = U \cap L \cap V \in \mathcal{L}_X \cap_1 V \subseteq \mathcal{L}_X$. This means $U \in AdOp(X) \cap \mathcal{P}(V)$. Since $U = U \cap V \in AdOp(X) \cap_1 V$, we have $AdOp(V) \subseteq AdOp(X) \cap_1 V$.

2. Assume $W \in AdOp(X)$. For any $L \in \mathcal{L}_X$, the set $W \cap (L \cap F) = (W \cap L) \cap F \in \mathcal{L}_X \cap_1 F$. Since $W \cap F \in AdOp(F)$, the inclusions $AdOp(X) \cap_1 F \subseteq AdOp(F)$ and $AdCl(X) \cap_1 F \subseteq AdCl(F)$ are clear. We now prove $AdCl(F) \subseteq AdCl(X) \cap_1 F$. For $M \in AdCl(F)$, we have $(F \setminus M) \cap_1 (\mathcal{L}_X \cap_1 F) \subseteq \mathcal{L}_X \cap_1 F$. Take $L \in \mathcal{L}_X$. Then $(X \setminus M) \cap L = ((X \setminus F) \cap L) \cup ((F \setminus M) \cap L)$. But $(F \setminus M) \cap L = F \cap K_L$ for some $K_L \in \mathcal{L}_X$ and we may assume $K_L \subseteq L$. We get $(X \setminus M) \cap L = ((X \setminus F) \cap L) \cup K_L \in \mathcal{L}_X$. This proves $X \setminus M \in AdOp(X)$, so $M = M \cap F \in AdCl(X) \cap_1 F$.

3. We know that $S \subseteq L$ for some $L \in \mathcal{L}_X$. Since $(S, \mathcal{L}_X \cap_1 S)$ is a small space, we have $AdOp(S) = \mathcal{L}_S = \mathcal{L}_X \cap_1 S = (\mathcal{L}_X \cap_1 L) \cap_1 S = (AdOp(X) \cap_1 L) \cap_1 S = AdOp(X) \cap_1 S$.

4. For the non-trivial inclusion $AdOp(Y) \subseteq AdOp(X) \cap_1 Y$, assume $W \in AdOp(Y)$. Then for each $L \in \mathcal{L}_X$ there exists $W_L \in \mathcal{L}_X \cap_1 L$ such that $W \cap L = W_L \cap Y$. We check if the set $\tilde{W} = \bigcup_{L \in \mathcal{L}_X} W_L$ is admissible open in X. For any $M \in \mathcal{L}_X$, we conclude $\tilde{W} \cap M = \bigcup_{L \in \mathcal{L}_X} W_L \cap M$ is equal to $W_M \in \mathcal{L}_X$ since $\mathcal{L}_X \ni V \mapsto V \cap Y \in \mathcal{L}_Y$ is an isomorphism of lattices and, consequently, for each $L \in \mathcal{L}_X$ we have $M \cap W_L \subseteq W_M$. Finally, $W = \tilde{W} \cap Y$.

□

3. Specialisation

Now we extend the facts from Section 4 of [21] about the relation of specialisation to the case of locally small spaces. While we have not defined the inverse smopology in this case, we have the inverse topology as well as the constructible topology which do not change if we pass to the smallification $(X, \mathcal{L}_X^o)$ of the locally small space $(X, \mathcal{L}_X)$. That is why a series of facts passes to the locally small case.

Recall that for a topological space (X, τ_X) the point x *specialises to* y (we write $x \rightsquigarrow y$) if each neighbourhood containing y also contains x. In this situation x is a *generalisation* of y, and y is a *specialisation* of x. For subsets $A \subseteq X$, we write:

$Gen(A) = \{x \in X : x \in Gen(a) \text{ for some } a \in A\}$,

$Spez(A) = \{x \in X : x \in Spez(a) \text{ for some } a \in A\}$.

A specialisation relation is always a preorder (called also a quasi-order) on X and does not depend on the ambient topological space (if $Y \subseteq X$ and $x, y \in Y$, then $x \rightsquigarrow y$ in X iff $x \rightsquigarrow y$ in Y).

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Fact 2. *For a locally small space $(X, \mathcal{L}_X)$ and $x, y \in X$, the following conditions are equivalent for the original topology on X:*

1. *x specialises to y ($x \rightsquigarrow y$),*
2. *$y \in \overline{\{x\}}$,*
3. *$y \notin ext\{x\}$,*
4. *each smop containing y also contains x,*
5. *each admissible open set containing y also contains x,*
6. *each co-smop containing x also contains y,*
7. *$y \rightsquigarrow_{inv} x$ (read: y specialises to x in τ_{inv}.)*

Fact 3. *For each locally small space $(X, \mathcal{L}_X)$ and $A \subseteq X$, we have:*
$Spez(A) \subseteq \overline{A}$ and $Gen(A) \subseteq \overline{A}^{inv}$.

Fact 4. *In a pre-Boolean locally small space, $x \rightsquigarrow y$ iff $x = y$.*

Fact 5. *In any locally small space $(X, \mathcal{L}_X)$, we have the following:*
(1) If $A \subseteq X$ is weakly closed ($A \in \Omega AdCl(X)$), then $Spez(A) = A$.
(2) If $A \subseteq X$ is weakly open ($A \in \mho\mathcal{L}_X$), then $Gen(A) = A$.

Recall that a subset $Q \subseteq X$ is called *saturated* if it is an intersection of weakly open sets, i.e., $Q \in \Omega\mho\mathcal{L}_X$. Moreover, a locally small space $(X, \mathcal{L}_X)$ is called T_0 (or *Kolmogorov*) if the family $\mathcal{L}_X$ separates points ([22, Reminder 1.1.4]), which means that for $x, y \in X$ the following condition is satisfied:

$$\text{if } x \in A \iff y \in A \text{ for each } A \in \mathcal{L}_X, \text{ then } x = y.$$

By $\mathbf{LSS}_0$ we denote the full subcategory in **LSS** generated by T_0 objects, while by $\mathbf{SS}_0$ we denote the full subcategory in **LSS** generated by small T_0 objects.

Fact 6. *If $(X, \mathcal{L}_X)$ is T_0, then the specialisation relation $\rightsquigarrow$ is a partial order.*

Fact 7. *If $(X, \mathcal{L}_X)$ is T_0 and $Q \subseteq X$, then the following are equivalent:*

1. *Q is saturated,*
2. *$Gen(Q) = Q$.*

4. Up-spectral Locally Small Spaces

Definition 7. *For a topological space (X, τ_X), we have the following notation:*

1. *$CO(X) = CO(X, \tau_X)$ is the family of all compact open subsets of X,*
2. *$ICO(X) = ICO(X, \tau_X)$ is the family of all open subsets of X such that any intersection with a compact open set is compact,*
3. *co-$ICO(X)$ is the family of complements of sets from $ICO(X)$,*
4. *$ClOp(X) = ClOp(X, \tau_X)$ is the family of clopen subsets of X.*

Moreover, (X, τ_X) is called

1. semi-spectral *if $CO(X) \cap_1 CO(X) \subseteq CO(X)$,*
2. coherent *if $CO(X)$ forms a basis of the topology and X is semi-spectral.*

Definition 8 ([30], [23], [29])**.** *A topological space (X, τ_X) is* up-spectral *if it is (T_0) sober and coherent but not necessarily compact.*

Fact 8 (compare [22, Corollary 1.5.5])**.** *In an up-spectral topological space, the following holds for any $A \subseteq X$:*

1. *$\overline{A} = Spez(\overline{A}^{con})$ where $\tau_{con} = \mho(CO(X) \setminus_1 CO(X))$,*
2. *$\overline{A}^{inv} = Gen(\overline{A}^{con})$ where $\tau_{inv} = \mho$ co-$ICO(X)$.*

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The importance of this class of spaces can be seen in the following theorem.

Theorem 1 ([27,28,30], characterisation of up-spectral spaces)**.** *For any topological space (X, τ_X), the following conditions are equivalent:*

1. *X is almost-spectral,*
2. *X is open dense in a spectral space,*
3. *X is open in a spectral space,*
4. *X is a sober BD-space,*
5. *X is homeomorphic to the prime spectrum of a distributive lattice with minimum,*
6. *X is up-spectral,*
7. *X admits a trivial one-point spectralification,*
8. *X is semi-spectral and locally spectral,*
9. *X is the underlying topological space of a scheme,*
10. *X is the underlying topological space of an open subscheme in an affine scheme.*

Proof. This is Theorems 7 and 8 in [30], Theorem 2.1 in [28] and Proposition 16 in [27]. □

Definition 9 ([23])**.** *A mapping $g : (X, \tau_X) \to (Y, \tau_Y)$ between up-spectral spaces is called* spectral *if the following conditions are satisfied:*

(1) *g is* bounded*: $g(CO(X))$ refines $CO(Y)$,*
(2) *g is* s-continuous*: $g^{-1}(ICO(Y)) \subseteq ICO(X)$.*

Following [23], we shall denote by **uSpec** *the category of up-spectral topological spaces and spectral mappings between them.*

Definition 10. *An* up-spectral locally small space *is a locally small space $(X, \mathcal{L}_X)$ where $(X, \mho\mathcal{L}_X)$ is an up-spectral topological space and $\mathcal{L}_X = CO(X)$. We get the category* **uSpLSS** *of up-spectral locally small spaces and bounded continuous mappings.*

Remark 4. *For objects of* **uSpLSS** *we have (see [29] and [22]):*

$$AdOp(X) = ICO(X),\ AdCl(X) = \text{co-}ICO(X),\ LCon(X) = ClOp(X, \tau_{con}).$$

Definition 11. *An* up-Priestley space *is a system $(X, \sigma_X, \leq_X)$ where (X, σ_X) is a Boolean (i.e. zero-dimensional Hausdorff locally compact, [45]) topological space and $\leq_X$ is a partial order on X satisfying the* Priestley separation axiom

$$\text{if } x \not\leq_X y, \text{ then } \exists\ V \in CO(X) \text{ such that } V = \uparrow V, x \in V, y \notin V.$$

A Priestley mapping *between up-Priestley spaces is a continuous non-decreasing mapping. We have the category* **uPri** *of up-Priestley spaces and Priestley mappings.*

Definition 12. *A* Priestley locally small space *is a system $(X, \mathcal{L}^\sigma_X, \leq_X)$ where $(X, \mathcal{L}^\sigma_X)$ is a Boolean locally small space (Definition 16) and $\leq_X$ is a partial order on X satisfying the* Priestley separation axiom

$$\text{if } x \not\leq_X y, \text{ then } \exists\ V \in \mathcal{L}^\sigma_X \text{ such that } V = \uparrow V, x \in V, y \notin V.$$

A Priestley morphism *between Priestley locally small spaces is a strictly (equivalently: weakly) continuous non-decreasing mapping. We have the category* **PLSS** *of Priestley locally small spaces and Priestley morphisms.*

Theorem 2. *All the following categories are concretely isomorphic:*

$$\mathbf{uSpec},\quad \mathbf{uPri},\quad \mathbf{PLSS},\quad \mathbf{uSpLSS}.$$

Proof. We have four concrete functors:

1. the functor $k_1 : \mathbf{uSpec} \to \mathbf{uPri}$, $\quad k_1(X, \tau_X) = (X, \tau_{con}, \rightsquigarrow_{inv})$
For an up-spectral space (X, τ_X), the constructible topology τ_{con} is Boolean and the generalisation relation $\rightsquigarrow_{inv}$ (as well as the specialisation relation $\rightsquigarrow$) clearly satisfies the Priestley separation axiom.
Assume $f : X \to Y$ is bounded s-continuous. Then it is continuous in the constructible topologies as well as in the original topologies, so non-decreasing in the generization relation.
2. the functor $k_2 : \mathbf{uPri} \to \mathbf{PLSS}$, $\quad k_2(X, \sigma_X, \leq) = (X, CO(X, \sigma_X), \leq)$
For an object $(X, \sigma_X, \leq)$ of $\mathbf{uPri}$, the space $(X, CO(X, \sigma_X))$ is clearly locally small ($CO(X, \sigma_X)$ is a smopology) and Boolean since $AdOp(X) = ClOp(X, \sigma_X) = AdCl(X)$. Obviously, $\leq$ satisfies the Priestley separation axiom.
Assume $f : X \to Y$ is non-decreasing and continuous. Then
$$f^{-1}(ICO(Y)) = f^{-1}(ClOp(Y)) \subseteq ClOp(X) = ICO(X).$$
Hence f is continuous between the locally small spaces.
Since the image of a Hausdorff compact set is compact, it is also a subset of a compact open set by local compactness. That is why $f(CO(X))$ is a refinement of $CO(Y)$. Hence f is bounded between the locally small spaces.
3. the functor $k_3 : \mathbf{PLSS} \to \mathbf{uSpLSS}$, $\quad k_3(X, \mathcal{L}^\sigma_X, \leq) = (X, Up_\leq \cap \mathcal{L}^\sigma_X)$
The family $Up_\leq \cap \mathcal{L}^\sigma_X$ is the compact-open basis of an up-spectral topology τ_X (compare the characterisation of spectral topologies in Theorem 1.5.11 of [22]). This compact-open basis in the up-spectral topology is, in particular, a smopology. Hence $(X, Up_\leq \cap \mathcal{L}^\sigma_X)$ is an up-spectral locally small space.
Assume $f : X \to Y$ is non-decreasing and bounded continuous in the constructible smopologies $\mathcal{L}^\sigma_X, \mathcal{L}^\sigma_Y$. Each compact-open in τ_X is compact-open in σ_X so its image under f is contained in a small constructible in τ_Y, a subset of a compact-open in τ_Y. We also have $f^{-1}(ICO(Y, \tau_Y)) \subseteq ICO(X, \tau_X)$ since the preimage of a clopen upset is a clopen upset. Hence f is bounded continuous as the mapping between the up-spectral locally small spaces.
4. the functor $k_4 : \mathbf{uSpLSS} \to \mathbf{uSpec}$, $\quad k_4(X, \mathcal{L}_X) = (X, \mho\mathcal{L}_X)$
For an up-spectral locally small space $(X, \mathcal{L}_X)$, the topological space $(X, \mho\mathcal{L}_X)$ is up-spectral by Definition 10.
Assume $f : X \to Y$ is bounded continuous. Then f is spectral as a mapping between the up-spectral topological spaces by Definition 9.

Moreover, $k_4k_3k_2k_1$, $k_1k_4k_3k_2$, $k_2k_1k_4k_3$, $k_3k_2k_1k_4$ are the identity functors since
$$\tau_X = \mho(Up_{\rightsquigarrow_{inv}} \cap CO(X, \sigma_X)), \quad \sigma_X = \tau(Up_\leq \cap CO(X, \sigma_X))_{con},$$
$$\mathcal{L}^\sigma_X = CO(X, \tau(Up_\leq \cap \mathcal{L}^\sigma_X)_{con}), \quad \mathcal{L}_X = Up_{\rightsquigarrow_{inv}} \cap CO(X, \tau(\mathcal{L}_X)_{con})$$
and $\leq = \rightsquigarrow_{inv}$ is the generalisation relation of the up-spectral topology τ_X. □

We give an analogue of (1.3) in [22] and Example 3 in [21].

Proposition 3. *For an up-spectral topological space (X, τ_X), the corresponding up-spectral locally small space $(X, CO(X))$ has the property, that the small constructible weakly open sets are exactly the compact open sets, so smops, i.e., $\mathcal{L}_X = CO(X)$ and $sCon(X) \cap \mho\mathcal{L}_X = \mathcal{L}_X$.*

Proof. Small constructible sets are clopen in the constructible topology, so compact. If they are additionally open in the original topology, then they are smops. □

Remark 5. *From [21], Examples 4–7, we know that:*

1. *A weakly open constructible set in a locally small space may not be a smop.*

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2. *A T_0 locally small space may have all smops (topologically) compact but not be sober.*
3. *A T_0 locally small space may be (topologically) sober but not compact.*
4. *A T_0 locally small space may be sober and compact with not all smops compact.*

Proposition 4. *If X is a T_0 sober locally small space with all smops compact, then X is an up-spectral locally small space.*

Proof. Since $\mathcal{L}_X \subseteq CO(X)$, we have $CO(X) = \mathcal{L}_X$. The other conditions for up-spectrality are obvious. □

Proposition 5. *If a locally small space has all smops compact, then:*

(1) *the ideals in $\mathcal{L}_X$ are in a bijective correspondence with the weakly open sets, so also with the weakly closed sets,*
(2) *the prime ideals of $\mathcal{L}_X$ are in a bijective correspondence with the non-empty, irreducible, weakly closed sets.*

Proof. The proof is similar to that of Proposition 3 of [21]. □

Recall that if not all smops are compact, then by Example 8 from [21] it may happen that $I \subset i(s(I))$, using notation from the above mentioned proof.

5. Stone Duality for Up-spectral Spaces

Corollary 1 (Stone Duality for spectral spaces)**.** *The categories* **Spec**, **SpSS**, **Pri**, **PSS** *are dually equivalent to* **Lat**.

Proof. Follows from Theorem 2 and the classical Stone Duality. □

Definition 13. *A* bornology *in a bounded lattice $(L, \vee, \wedge, 0, 1)$ is an ideal $B \subseteq L$ such that*

$$\bigvee B = 1.$$

Recall that each distributive bounded lattice L corresponds to the spectral space $(\mathcal{PF}(L), \tau(\widetilde{L}))$ where $\widetilde{L} = \{\widetilde{a} : a \in L\}$. Similarly, we define $\widetilde{B} = \{\widetilde{a} : a \in B\}$. A bornology B will be called special *if $(\widetilde{B} \cap_1 \bigcup \widetilde{B})^o = \widetilde{L} \cap_1 \bigcup \widetilde{B}$ as subsets of $\mathcal{P}(\bigcup \widetilde{B})$ and $\bigcup \widetilde{B}$ is constructibly dense in $\mathcal{PF}(L)$.*

Example 4. *Let $\widetilde{\mathbb{R}}$ be the real spectrum $\mathrm{Spec}_r \mathbb{R}[Y]$ of the ring $\mathbb{R}[Y]$. Consider the following families of compact open sets in this spectral space:*

$$CO(\widetilde{\mathbb{R}}) = \textit{ finite unions of intervals of type } [r^+, s^-] \textit{ where } r < s \in \mathbb{R}$$

$$\textit{or } [-\infty, s^-] \textit{ or } [r^+, +\infty],$$

$$\mathcal{B} = \textit{ finite unions of intervals } [r^+, s^-] \textit{ where } r < s \in \mathbb{R}.$$

Then $\mathcal{B}$ is a bornology in the distributive bounded lattice $CO(\widetilde{\mathbb{R}})$. Notice that $CO(\widetilde{\mathbb{R}}) \subset \mathcal{B}^o =$

$$= \textit{ locally finite unions of intervals } [r^+, s^-], r < s \in \mathbb{R} \textit{ and singletons } \{-\infty\}, \{+\infty\}.$$

Define $X_s = \bigcup \widetilde{\mathcal{B}} = \widetilde{\mathbb{R}} \setminus \{-\infty, +\infty\}$. Still

$$CO(\widetilde{\mathbb{R}}) \cap_1 X_s \subset (\mathcal{B}^o) \cap_1 X_s = (\mathcal{B} \cap_1 X_s)^o =$$

$$= \textit{the family of all locally finite unions of intervals } [r^+, s^-], r < s \in \mathbb{R}.$$

Hence the bornology $\mathcal{B}$ is not special.

Definition 14 ([23])**.** *The category* **SpecB** *has*

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1. *as objects: systems* $((X, \tau_X), CO_s(X), X_s)$ *where* (X, τ_X) *is a spectral topological space,* $CO_s(X)$ *is a special bornology in* $CO(X)$ *and* $X_s = \bigcup CO_s(X)$*,*
2. *as morphisms: spectral mappings* $g : X \to Y$ *such that* $g(X_s) \subseteq Y_s$ *and satisfying the* condition of boundedness*:* $CO_s(X)$ *is a refinement of* $g^{-1}(CO_s(Y))$*.*

Definition 15. 1. *An* object of **LatB** *is a system* (L, L_s) *with* $L = (L, \vee, \wedge, 0, 1)$ *a bounded distributive lattice and* L_s *a special bornology in* L*.*
2. *A* morphism of **LatB** *from* (L, L_s) *to* (M, M_s) *is such a homomorphism of bounded lattices* $h : L \to M$ *that is* dominating*:* $\forall a \in M_s \, \exists b \in L_s \quad a \leq h(b)$*.*

Corollary 2 (Stone Duality for up-spectral spaces)**.** *The concretely isomorphic categories* **uSpec**, **uSpLSS**, **uPri**, **PLSS** *are equivalent to* **SpecB** *and dually equivalent to* **LatB**.

Proof. Follows from Theorem 2 and a special case ($X_d = X_s$) of the Stone Duality for locally small spaces ([23], Theorem 1). □

6. Boolean Locally Small Spaces

Proposition 6. *For a locally small space, the following are equivalent:*
1. $AdOp(X) = AdCl(X)$*,*
2. $AdOp(X) = LCon(X)$*,*
3. $AdCl(X) = LCon(X)$
4. $AdOp(X)$ *is a Boolean subalgebra of* $\mathcal{P}(X)$*.*

Proof. Obviously, (2) or (3) are equivalent and imply (1). We prove that (1) implies (2). If $A \subseteq X$ is locally constructible, then it is locally a finite union of finite intersections of smops due to (1). That is why it is admissible open. Moreover, (2) implies (4) implies (1) is clear. □

Definition 16.
1. *If a locally small space satisfies the above conditions, then we will call it a* pre-Boolean locally small space*.*
2. *A Hausdorff pre-Boolean locally small space with all smops compact will be called a* Boolean locally small space*.*
3. *The category of Boolean locally small spaces and bounded (weakly) continuous maps will be denoted by* **BLSS**.

Definition 17. *We have the following categories:*
1) **BoolSp** *is the category of zero-dimensional Hausdorff locally compact spaces and continuous mappings (see Dimov [45]),*
2) **BAB** *is the category of Boolean algebras with special bornologies and dominating Boolean homomorphisms.*

Proposition 7. **BLSS** *and* **BoolSp** *are concretely isomorphic categories. In particular, Boolean locally small spaces are up-spectral.*

Proof. For a Boolean topological space (X, σ) the family $\mathcal{L}^\sigma = CO(X, \sigma)$ is a smopology for which $\mathcal{L}^\sigma \subseteq ClOp(X, \sigma) \subseteq AdOp(X, \mathcal{L}^\sigma)$. On the other hand, $AdCl(X) \subseteq AdOp(X) \subseteq ClOp(X)$, hence $AdOp(X) = AdCl(X) = ClOp(X)$ and $(X, \mathcal{L}^\sigma)$ is a Boolean locally small space.

For a Boolean locally small space $(X, \mathcal{L}^\sigma)$, each smop is compact, so $\mathcal{L}^\sigma \subseteq CO(X, \mathfrak{U}\mathcal{L}^\sigma)$ and $AdOp(X) = AdCl(X) \subseteq ClOp(X)$. The topology $\mathfrak{U}\mathcal{L}^\sigma$ is Hausdorff, locally compact and has a clopen basis $AdOp(X) = AdCl(X)$, hence it is a Boolean topology.

The morphisms are the (weakly) continuous mappings in both categories.

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In particular, each smop in a Boolean locally small space is spectral. The whole space is semispectral and locally spectral, so up-spectral ([23]). □

Theorem 3. **BAB** *is dually equivalent to* **BoolSp**, *so also to* **BLSS**.

Proof. The Dimov's version of Stone Duality [45] says that **BoolSp** is dually equivalent to his category $ZLBA$. One needs to notice that $ZLBA$ is our category **BAB** since in Boolean algebras the special bornologies are exactly the *distinguished dense ideals* (i.e., the dense ideals I satisfying the condition: for each ideal J in I the supremum of J in the Boolean algebra exists). We check this in four steps:

1. Each bornology in a Boolean algebra is a dense ideal.
   Let A be a Boolean algebra and B a bornology in A. Take $a \in A \setminus \{0\}$. Then $a' \neq 1 = \bigvee_{b \in B} b$. There exists $b \in B$ such that $b \not\leq a'$, so $a \not\leq b'$. We have $a = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$. Since $a \wedge b' < a$, we have $a \wedge b \neq 0$ and B is dense.
2. Each distinguished dense ideal is a bornology.
   If $\bigvee_A I = a \neq 1$ then $a' \neq 0$. Take $i \in I$ such that $0 \neq i \leq a'$. Then $i = (i \vee a) \wedge (i \vee a') = a \wedge a' = 0$. Contradiction proves that $\bigvee_A I = 1$.
3. Special bornologies are distinguished.
   If B is a special bornology, then each clopen set in the constructibly dense open $\bigcup \widetilde{B}$ is of the form $\tilde{a}^s = \tilde{a} \cap \bigcup \widetilde{B}$ where $a \in A$. It follows from Proposition 2.6 of [45] that B is distinguished.
4. Each distinguished dense ideal is special.
   If I is a distinguished dense ideal in a Boolean algebra A, then the dense set $\bigcup \tilde{I}$ is constructibly dense. The condition $(\widetilde{I} \cap_1 \bigcup \widetilde{I})^o = AdOp(\bigcup \widetilde{I}) = LCon(\bigcup \widetilde{I}) = ClOp(\bigcup \widetilde{I}) = \widetilde{L} \cap_1 \bigcup \widetilde{I}$ is satisfied by Proposition 2.6 of [45].

In both cases the morphisms are the dominating Boolean homomorphisms. □

7. The Standard Up-spectralification

We extend the facts from Section 5 of [21] about the standard spectralification of T_0 small spaces to the locally small case.

Definition 18. *An* embedding *of locally small spaces is an injective map* $e : X \to Y$ *such that* $e(\mathcal{L}_X) = \mathcal{L}_Y \cap_1 e(X)$.

Definition 19 ([23]). *An* up-spectralification *of a locally small space* X *is the pair* (e, Y) *where:* Y *is an up-spectral locally small space and* $e : X \to Y$ *is an embedding between locally small spaces with the image* $e(X)$ *dense in the constructible topology of* Y.

Definition 20. *The* standard up-spectralification *of a* T_0 *locally small space* $(X, \mathcal{L}_X)$ *is the locally small space (using notations from Theorem 1 of [23])*

$$X^{usp} = (\bigcup \widetilde{\mathcal{L}_X}, \widetilde{\mathcal{L}_X})$$

with the embedding $X \ni x \mapsto \hat{x} \in \hat{X} \subseteq \bigcup \widetilde{\mathcal{L}_X}$.

Remark 6. *Identifying* X *with* $\hat{X}$, *we have:*

1. $CO(X^{usp}) \cap_1 X = \mathcal{L}_{X^{usp}} \cap_1 X = \mathcal{L}_X$,
2. $ICO(X^{usp}) \cap_1 X = AdOp(X^{usp}) \cap_1 X = AdOp(X)$,
3. $ClOp(X^{usp}_{con}) \cap_1 X = LCon(X^{usp}) \cap_1 X = LCon(X) = ClOp(X_{con})$.

Since X *is constructibly dense in* X^{usp}, *we have the following bijection:*

$$LCon(X) \ni A \mapsto A^{usp} \in LCon(X^{usp}),$$

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where A^{usp} is the only member of $LCon(X^{usp})$ such that $A^{usp} \cap X = A$. One can see that $A^{usp} = \overline{A}^{con}$ taken in X^{usp} and $\overline{A^{usp}} \cap X = \overline{A}^X$. By bijectivity, we have:

i) $V \in AdOp(X)$ *iff* $V^{usp} \in AdOp(X^{usp})$,

ii) $F \in AdCl(X)$ *iff* $F^{usp} \in AdCl(X^{usp})$.

Fact 9. *If X is small, then $X^{usp} = X^{sp}$ is an object of* **SpSS**.

The following fact is purely topological.

Fact 10. *If C is a non-empty, weakly closed set in the original topology of a locally small space X, then the following conditions are equivalent:*

1. *C is irreducible in X,*
2. *the closure $\overline{C}$ in the standard up-spectralification X^{usp} is irreducible.*

Remark 7. *We have the functor $usp : \mathbf{LSS}_0 \to \mathbf{uSpLSS}$ given by formulas*

$$usp(X, \mathcal{L}_X) = (\bigcup \widetilde{\mathcal{L}_X}, \widetilde{\mathcal{L}_X}), \quad usp(f) = ((\mathcal{L}^o f)^\bullet)_s$$

where $((\mathcal{L}^o f)^\bullet)_s$ is the restriction of $(\mathcal{L}^o f)^\bullet$ to the standard up-spectralifications of the domain and the codomain of f.

8. Heyting Locally Small Spaces

Definition 21. *(1) A locally small space will be called* pre-semi-Heyting *if any of the following equivalent conditions are satisfied:*

1. *the interior in the original topology of any admissible closed set is an admissible open set, i.e., $int_1 AdCl(X) \subseteq AdOp(X)$,*
2. *the closure in the original topology of any admissible open set is an admissible closed set, i.e., $cl_1 AdOp(X) \subseteq AdCl(X)$.*

(2) A locally small space will be called pre-Heyting *if any of the following equivalent conditions are satisfied:*

1. *the interior in the original topology of any locally constructible set is an admissible open set, i.e., we have $int_1 LCon(X) \subseteq AdOp(X)$.*
2. *the closure in the original topology of any locally constructible set is an admissible closed set, i.e., we have $cl_1 LCon(X) \subseteq AdCl(X)$.*

(3) A locally small space will be called Heyting *if it is T_0 and pre-Heyting.*

Inspired by [22] and [21], we have the following propositions.

Proposition 8. *Assume that X is a pre-semi-Heyting locally small space. Then:*

1. *the following maps are well defined:*
   (a) *the open regularisation map $N : AdOp(X) \to AdOp(X)$ given by $N(A) = int\overline{A}$,*
   (b) *the closed regularisation map $\overline{N} : AdCl(X) \to AdCl(X)$ given by $\overline{N}(F) = \overline{intF}$,*
2. *for each $V \in AdOp(X)$, the subspace $(V, \mathcal{L}_X \cap_1 V)$ is pre-semi-Heyting.*

Proof. (1) Obvious by definition.
(2) For $A, V \in AdOp(X)$, we have that the set $\overline{A \cap V}^V = V \cap (\overline{A \cap V}) \in V \cap_1 AdCl(X) = AdCl(V)$ is a local co-smop in V. Notice that $(V, \tau(\mathcal{L}_X \cap_1 V))$ is a topological subspace of $(X, \tau(\mathcal{L}_X))$.
□

Proposition 9 (characterisation of pre-Heyting spaces)**.** *For a locally small space $X = (X, \mathcal{L}_X)$, the following conditions are equivalent:*

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1. *X is pre-Heyting,*
2. *for each $B \in \mathcal{L}_X \setminus_1 \mathcal{L}_X$, $\overline{B} \in AdCl(X)$,*
3. *for each $A \in LCon(X)$, the subspace $(A, \mathcal{L}_X \cap_1 A)$ is pre-Heyting,*
4. *for each $A \in LCon(X)$, the subspace $(A, \mathcal{L}_X \cap_1 A)$ is pre-semi-Heyting,*
5. *for each $F \in AdCl(X)$, the subspace $(F, \mathcal{L}_X \cap_1 F)$ is pre-semi-Heyting.*

Proof. (1) $\Leftrightarrow$ (2) Of course, (2) follows from (1). Assume $C \in LCon(X)$. Then, for each $A \in \mathcal{L}_X$, we have $\overline{C} \cap A = \overline{C \cap A} \cap A$. Since $C \cap A \in Con(A)$, this set is a finite union of sets of the form $B_1 \setminus B_2$ with $B_1, B_2 \in \mathcal{L}_A$. Thus $\overline{C \cap A} \cap A \in AdCl(A)$ and $\overline{C} \in AdCl(X)$.
(1) $\Rightarrow$ (3) For $A \in LCon(X)$ and $D \in LCon(A) = LCon(X) \cap_1 A \subseteq LCon(X)$, we have $\overline{D} \in AdCl(X)$. Now $\overline{D}^A = \overline{D} \cap A \in AdCl(A)$.
(3) $\Rightarrow$ (4) Trivial.
(4) $\Rightarrow$ (5) Trivial.
(5) $\Rightarrow$ (2) Each element of $\mathcal{L}_X \setminus_1 \mathcal{L}_X$ is of the form $A \cap F$ with $A \in \mathcal{L}_X, F \in AdCl(X)$. Since $\overline{A \cap F} = \overline{A \cap F} \cap F \in AdCl(X) \cap_1 F$ by (5), we have $\overline{A \cap F} \in AdCl(X) \cap_1 AdCl(X) \subseteq AdCl(X)$. □

Remark 8. *There exist pre-semi-Heyting spectral small spaces that are not pre-Heyting. See Example 8.3.11(iii) in [22].*

Since the smallification does not change the original topology, we have

Fact 11. *If the locally small space $(X, \mathcal{L}_X)$ is (pre-)Heyting, then its smallification $(X, \mathcal{L}_X^o)$ is (pre-)Heyting as a small space or a locally small space.*

Definition 22 (Heyting maps)**.** *A map between pre-Heyting locally small spaces $f : X \to Y$ will be called a* Heyting (bounded continuous) map *if it is bounded continuous and the interior operation (equivalently: the closure operation) commutes with the preimage on the locally constructible sets, that is, any of the equivalent conditions is satisfied*

1. *$f^{-1}(int(C)) = int(f^{-1}(C))$ for $C \in LCon(Y)$,*
2. *$f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ for $C \in LCon(Y)$.*

Definition 23. *By* **HLSS** *we denote the category of Heyting locally small spaces and Heyting bounded continuous maps.*

Since **HLSS** is a full subcategory of **LSS**, we have

Fact 12. *Strict homeomorphisms between Heyting locally small spaces are isomorphisms of* **HLSS**.

9. Heyting Up-spectral Spaces

Definition 24 ([22, Section 8.3])**.** *A topological up-spectral space X will be called* Heyting *if the closure of any constructibly clopen [i.e., locally constructible] set is a co-ICO set. A map between Heyting up-spectral spaces $g : X \to Y$ is called a* Heyting spectral *map if g is spectral and any of the equivalent the conditions holds:*

1. *$f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ for $C \in LCon(Y) = ClOp_{con}(Y)$,*
2. *$f^{-1}(int(C)) = int(f^{-1}(C))$ for $C \in LCon(Y) = ClOp_{con}(Y)$.*

We have the category **HuSpec** *of Heyting up-spectral (topological) spaces and Heyting spectral mappings.*

Corollary 3. *Each homeomorphism between Heyting up-spectral spaces is an isomorphism in* **HuSpec**.

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Definition 25. *We have the category* **HuSpLSS** *of Heyting up-spectral locally small spaces and Heyting strictly continuous maps.*

Proposition 10. *If the standard up-spectralification X^{usp} of a locally small space X is Heyting, then X is Heyting.*

Proof. Assume $\forall A \in LCon(X^{usp})\ \overline{A} \in AdCl(X^{usp})$. Then by taking traces with X, we have $\forall B \in LCon(X)\ \overline{B}^X \in AdCl(X)$ (see Remark 6). □

Proposition 11. *If X is a Heyting locally small space (object of* **HLSS***), then:*

1. *$AdOp(X)$ is a Heyting algebra and $(\widetilde{\mathcal{L}^o_X}, \widetilde{\mathcal{L}_X}, \hat{X})$ is an object of* **LatBD***,*
2. *X^{usp} is a Heyting up-spectral locally small space (object of* **HuSpLSS***).*

Proof. (1) For $U, V \in AdOp(X)$ define $U \to V = int(V \cup (X \setminus U))$. Then, by the Heyting assumption on the locally small space X, we have $U \to V \in AdOp(X)$ and this set is the largest element $Z \in AdOp(X)$ with the property $V \cap Z \subseteq W$. That $(\widetilde{\mathcal{L}^o_X}, \widetilde{\mathcal{L}_X}, \hat{X})$ is an object of **LatBD** was proved in Step 3 of the proof of Theorem 1 of [23].
(2) Since X is Heyting and the lattices $AdOp(X)$ and $ICO(X^{usp})$ are isomorphic Heyting algebras, X^{usp} is Heyting by Remark 6. □

Remark 9. *The functor usp from Remark 7 has a restriction that will be denoted by usp :* **HLSS** $\to$ **HuSpLSS***.*

Definition 26. *An* up-Esakia space $(X, \sigma_X, \leq_X)$ *is an up-Priestley space such that*

$$\text{for all } C \in ClOp(X, \sigma_X) \text{ we have } \downarrow C \in ClOp(X, \sigma_X).$$

An Esakia mapping *between up-Esakia spaces is a Priestley mapping $f : X \to Y$ such that $f(\uparrow x) = \uparrow f(x)$. We have the category* **uEsa** *of up-Esakia spaces and Esakia mappings.*

Definition 27. *An* Esakia locally small space *is such a Priestley locally small space $(X, \mathcal{L}^\sigma_X, \leq_X)$ that satisfies the condition*

$$\text{for all } A \in LCon(X, \mathcal{L}^\sigma_X) \text{ we have } \downarrow A \in LCon(X, \mathcal{L}^\sigma_X).$$

An Esakia morphism *between Esakia locally small spaces is a Priestley morphism $f : X \to Y$ that is a* p-morphism, *i.e., satisfies the condition $f(\uparrow x) = \uparrow f(x)$. We have the category* **ELSS** *of Esakia (locally small) spaces and Esakia morphisms.*

Corollary 4. *The categories* **HuSpec**, **HuSpLSS**, **uEsa**, **ELSS** *are concretely isomorphic.*

Proof. Follows from Theorem 2. □

Definition 28. *The category* **HSpecB** *has:*

1. *systems $((X, \tau_X), CO_s(X), X_s)$ where (X, τ_X) is a Heyting spectral topological space, $CO_s(X)$ is a special bornology in $CO(X)$ and $X_s = \bigcup CO_s(X)$ (called the* decent lump*) as objects,*
2. *Heyting spectral mappings between spectral spaces satisfying the condition of boundedness and respecting the decent lump as morphisms.*

Definition 29. *The category* **HAB** *has*

1. *systems (L, L_s) with $L = (L, \vee, \wedge, \to, 0, 1)$ a Heyting algebra, L_s a special bornology in L as objects,*

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2. *homomorphisms of Heyting algebras $h : L \to M$ that are* dominating *(i.e., $\forall a \in M_s\ \exists b \in L_s\quad a \leq h(b)$) and respect the decent lump (i.e., $h^\bullet(\bigcup \widetilde{M_s}) \subseteq \bigcup \widetilde{L_s}$) as morphisms from (L, L_s) to (M, M_s).*

10. Categories of Spaces with Decent Subsets

While the category **SpecD** was defined in [23] and the category **ESSD** was defined in [21], we introduce similarly:

Definition 30. *We have the following categories:*

1. *the category* **PriD** *whose objects are Priestley spaces with distinguished dense sets (called* decent sets*) and whose morphisms are Priestley mappings respecting the decent sets,*
2. *the category* **EsaD** *whose objects are Esakia spaces with distinguished dense sets (called* decent sets*) and whose morphisms are Esakia mappings respecting the decent sets,*
3. *the category* **PSSD** *whose objects are Priestley small spaces with decent sets and whose morphisms are Priestley morphisms respecting the decent sets.*

We have two consequences of Theorem 2.

Corollary 5. **SpecD**, **PriD**, **PSSD** *are concretely isomorphic.*

Corollary 6. **HSpecD**, **EsaD**, **ESSD** *are concretely isomorphic.*

Two versions of dualities follow.

Corollary 7 (Stone Duality for small spaces)**.** *The categories* **SpecD**, $\mathbf{SS}_0$, **PriD**, **PSSD** *are dually equivalent to* **LatD**.

Corollary 8 (Esakia Duality for small spaces)**.** *The categories* **HSpecD**, **HSS**, **EsaD**, **ESSD**, *are dually equivalent to* **HAD**.

Definition 31. *We have the following categories:*

1. *the category* **uSpecD** *whose objects are up-spectral spaces with distinguished decent sets that are dense in the constructible topology and whose morphisms are spectral mappings respecting the decent sets,*
2. *the category* **uPriD** *whose objects are up-Priestley spaces with distinguished dense sets (called* decent sets*) and whose morphisms are Priestley mappings respecting the decent sets,*
3. *the category* **uEsaD** *whose objects are up-Esakia spaces with distinguished dense sets (called* decent sets*) and whose morphisms are Esakia mappings respecting the decent sets,*
4. *the category* **PLSSD** *whose objects are Priestley locally small spaces with decent sets and whose morphisms are Priestley morphisms respecting the decent sets,*
5. *the category* **ELSSD** *whose objects are Esakia locally small spaces with decent sets and whose morphisms are Esakia morphisms respecting the decent sets.*

Corollary 9 (Stone Duality for locally small spaces)**.** *The categories* $\mathbf{LSS}_0$, **uSpecD**, **uPriD**, **PLSSD**, **SpecBD** *are dually equivalent to* **LatBD**.

Proof. Follows from Theorem 1 in [23] and Theorem 2. □

Definition 32. 1. *An* object *of* **HuSpecD** *is a system $((X, \tau_X), X_d)$ where $(X, \mathcal{L}_X)$ is a Heyting up-spectral space and $X_d \subseteq X$ is a constructibly dense subset.*

2. *A* morphism *from $((X, \tau_X), X_d)$ to $((Y, \tau_Y), Y_d)$* in **HuSpecD** *is such a Heyting spectral mapping between Heyting up-spectral spaces $g : (X, \tau_X) \to (Y, \tau_Y)$ that $g(X_d) \subseteq Y_d$.*

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Definition 33. *1. An* object of **HSpecBD** *is a system* $((X, \tau_X), CO_s(X), X_d)$ *where* (X, τ_X) *is a Heyting spectral space,* $CO_s(X)$ *is a bornology in the Heyting algebra* $CO(X)$ *and* $X_d \subseteq \bigcup CO_s(X)$ *(a* decent lump *of X) is constructibly dense and such that* $CO(X)_d = (CO_s(X)_d)^o \subseteq \mathcal{P}(X_d)$*.*

2. *A* morphism *from* $((X, \tau_X), CO_s(X), X_d)$ *to* $((Y, \tau_Y), CO_s(Y), Y_d)$ *in* **HSpecBD** *is such a Heyting spectral mapping* $g : (X, \tau_X) \to (Y, \tau_Y)$ *between Heyting spectral spaces that satisfies the* condition of boundedness $\forall A \in CO_s(X)\ \exists B \in CO_s(Y) \quad g(A) \subseteq B$*, and respects the decent lump:* $g(X_d) \subseteq Y_d$*.*

Similarly to Lemma 3, we have

Lemma 2. *The categories* **HuSpecD** *and* **HSpecBD** *are equivalent.*

Proof. Theorem 5 of [23] gives us equivalence between **uSpec** and **uSpLSS**. Similarly, **uSpecD** is equivalent to **uSpLSSD**, which is clearly equivalent to $\mathbf{LSS}_0$. Now Theorem 1 of [23] gives us equivalence between $\mathbf{LSS}_0$ and **SpecBD**. Restricting to Heyting objects and Heyting morphisms, we get the equivalence between **HuSpecD** and **HSpecBD**. □

Definition 34. *1. An* object of **HABD** *is a system* $(L, L_s, \mathbf{D}_L)$ *with* $L = (L, \vee, \wedge, \to, 0, 1)$ *a Heyting algebra,* L_s *a bornology in L and* $\mathbf{D}_L \subseteq \bigcup \widetilde{L_s} \subseteq \mathcal{PF}(L)$ *a constructibly dense set satisfying* $\widetilde{L} \cap_1 \mathbf{D}_L = (\widetilde{L_s} \cap_1 \mathbf{D}_L)^o \subseteq \mathcal{P}(\mathbf{D}_L)$*.*

2. *A* morphism of **HABD** *from* $(L, L_s, \mathbf{D}_L)$ *to* $(M, M_s, \mathbf{D}_M)$ *is such a homomorphism of Heyting algebras* $h : L \to M$ *that is* dominating *(i.e.,* $\forall a \in M_s\ \exists b \in L_s \quad a \leq h(b)$*) and* $h^\bullet(\mathbf{D}_M) \subseteq \mathbf{D}_L$*.*

Fact 13 ([21], Theorem 2)**.** *If* $h : A \to B$ *is a Heyting algebra homomorphism, then* $h^\bullet : \mathcal{PF}(B) \to \mathcal{PF}(A)$ *is a Heyting spectral map between Heyting spectral spaces.*

Proposition 12. *If* $f : X \to Y$ *is a Heyting bounded continuous map between Heyting locally small spaces, then* $\mathcal{L}^o f : \mathcal{L}^o_Y \to \mathcal{L}_X$ *is a homomorphism of Heyting algebras.*

Proof. We only check up the condition with the intuitionistic implication. For $V, W \in \mathcal{L}^o_Y$, we have $(\mathcal{L}^o f)(W \to V) = f^{-1}(ext(W \setminus V)) = ext(f^{-1}(W) \setminus f^{-1}(V)) = (\mathcal{L}^o f)(W) \to (\mathcal{L}^o f)(V)$. □

11. Esakia Duality for Locally Small Spaces

Corollary 10. *Concretely isomorphic categories* **HuSpecD**, **uEsaD**, **ELSSD** *are equivalent to* **HLSS**.

Proof. Follows from Theorem 2. □

Theorem 4 (Esakia Duality for locally small spaces)**.** *The categories* **HLSS**, **HuSpecD**, **uEsaD**, **ELSSD** *and* **HSpecBD** *are dually equivalent to* **HABD**.

Proof. By the corollary above, we need only to prove that **HLSS** and **HSpecBD** are dually equivalent to **HABD**. Since our proof is about the restrictions of functors from the proof of Theorem 1 in [23], we concentrate on objects and morphisms being Heyting. We shall use the notations from that proof.

Step 1: The restriced functor $\bar{R}$: **HSpecBD** $\to$ **HLSS**.

Assume $((X, \tau_X), CO_s(X), X_d)$ is an object of **HSpecBD**. Then $CO_s(X)$ is a basis of the induced topology on X_s and the original topology of $(X_d, CO_s(X)_d)$ is the topology induced from (X, τ_X). We are to check the Heyting closure condition for the locally small space $(X_d, CO_s(X)_d)$. Assume $B \in LCon(X_d, CO_s(X)_d)$. For $V \in CO_s(X)$, we have $B \cap V \in sCon(X_d, CO_s(X)_d) \subseteq Con(X_d, CO(X)_d)$, so $int(B \cap V) \in CO(X)_d \cap_1 CO_s(X)_d = CO_s(X)_d$.

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Since $CO_s(X)_d$ is a basis of the induced topology of X_d stable under finite intersections, $int(B) = \bigcup_{V \in CO_s(X)} int(B \cap V) \in AdOp(X_d, CO_s(X)_d)$.

Assume $f : X \to Y$ is a morphism of **HSpecBD**. Since

$$\forall C \in Con(Y, CO(Y)) \quad f^{-1}(\overline{C}) = \overline{f^{-1}(C)},$$

we also have

$$\forall D \in Con(Y_d, CO(Y)_d) \quad f_d^{-1}(\overline{D}^d) = \overline{f_d^{-1}(D)}^d.$$

Indeed, for any $D = C \cap Y_d \in Con(Y_d, CO(Y)_d)$, we have $\overline{D}^d = \overline{C \cap Y_d} \cap Y_d = \overline{C} \cap Y_d$. Applying this to the Heyting mapping condition, we have $f_d^{-1}(\overline{D}^d) = f_d^{-1}(\overline{C} \cap Y_d) = f^{-1}(\overline{C}) \cap X_d = \overline{f^{-1}(C)} \cap X_d = \overline{f^{-1}(C) \cap X_d} \cap X_d = \overline{f^{-1}(D) \cap X_d} \cap X_d = \overline{f_d^{-1}(D)}^d$.

Take now $K \in LCon(Y_d, CO_s(Y)_d)$. We can express its topological closure in Y_d (omitting ^{d}) as $\overline{K} = \bigcup\{\overline{K \cap V} : V \in CO_s(Y)_d\}$. That is why $f_d^{-1}(\overline{K}) = \bigcup\{f_d^{-1}(\overline{K \cap V}) : V \in CO_s(Y)_d\}$. Each $K \cap V$ belongs to $Con(Y_d, CO_s(Y)_d) \subseteq Con(Y_d, CO(Y)_d)$. Hence $f_d^{-1}(\overline{K \cap V}) = \overline{f_d^{-1}(K \cap V)}$. We get $\bigcup\{f_d^{-1}(\overline{K \cap V}) : V \in CO_s(Y)_d\} = \bigcup\{\overline{f_d^{-1}(K \cap V)} : V \in CO_s(Y)_d\} = \bigcup\{\overline{f_d^{-1}(K) \cap W} : W \in CO_s(X)_d\} = \overline{f_d^{-1}(K)}$ and f_d is a Heyting bounded continuous mapping.

Step 2: The restricted functor $\bar{S} : \mathbf{HABD}^{op} \to \mathbf{HSpecBD}$.

By the classical Esakia Duality (see Remark 10 in [21]), the topological space $\mathcal{PF}(A)$ is Heyting spectral for a Heyting algebra A. Moreover, $h^\bullet : \mathcal{PF}(M) \to \mathcal{PF}(L)$ a Heyting spectral map for a morphism $h : L \to M$ of **HABD**.

Step 3: The restricted functor $\bar{A} : \mathbf{HLSS} \to \mathbf{HABD}^{op}$.

For a Heyting locally small space $(X, \mathcal{L}_X)$, the intuitionistic implication in the bounded lattice $\mathcal{L}_X^o$ can be introduced by the formula $U \to V = ext(U \setminus V)$, making it a Heyting algebra by Proposition 11. For a morphism f in **HLSS**, the mapping $\mathcal{L}^o f$ is a homomorphism of Heyting algebras by Proposition 12.

Step 4: The functor $\bar{R}\bar{S}\bar{A}$ is naturally isomorphic to $Id_{\mathbf{HLSS}}$.

The mapping $\eta_X : (\hat{X}, \widetilde{\mathcal{L}_X}^d) \to (X, \mathcal{L}_X)$ is a strict homeomorphism by Proposition 2 of [23] and an isomorphism of **HLSS** by Fact 12.

Step 5: The functor $\bar{S}\bar{A}\bar{R}$ is naturally isomorphic to $Id_{\mathbf{HSpecBD}}$.

Each θ_X is an isomorphism in **SpecBD** as well as an isomorphism of **HSpec** by Fact 11 in [21], so an isomorphism of **HSpecBD**.

Step 6: The functor $\bar{A}\bar{R}\bar{S}$ is naturally isomorphic to $Id_{\mathbf{HABD}^{op}}$.

Each $\kappa_L : L \to \widetilde{L}^d$ is an order isomorphism hence, by [32, Exercise IX.4.3], an isomorphism of Heyting algebras. □

Corollary 11. *Restrictions of Heyting bounded continuous maps between Heyting up-spectral locally small spaces to decent locally small subspaces are Heyting bounded continuous.*

Proof. Follows from Step 1 in the proof of Theorem 4. □

Lemma 3. *The categories* **HuSpec** *and* **HSpecB** *are equivalent and dually equivalent to* **HAB**.

Proof. That **HuSpec** and **HSpecB** are equivalent follows from Theorem 5 in [23] by restricting to Heyting objects and Heyting morphisms.

That **HSpecB** is dually equivalent to **HAB** follows from Theorem 4 by restricting to the case $X_d = X_s$. □

Corollary 12 (Esakia Duality for up-spectral spaces)**.** *The categories* **HuSpec**, **HuSpLSS**, **uEsa**, **ELSS**, **HSpecB** *are dually equivalent to* **HAB**.

Proof. Follows from Lemma 3 and Corollary 4. □

Corollary 13 (Esakia Duality for spectral spaces)**.** *The categories* **HSpec**, **HSpSS**, **Esa**, **ESS** *are dually equivalent to* **HA**.

12. Conclusions

We have proved many facts about concrete isomorphisms, equivalences and dual equivalences between categories, summarised in the following two tables. (The main new theorem is Theorem 4, followed by Theorem 2.) In each row of the first table, the first four categories are concretely isomorphic, equivalent to the category in the fifth column and dually equivalent to the category in the last column.

SpSS	**Spec**	**Pri**	**PSS**		**Lat**
HSpSS	**HSpec**	**Esa**	**ESS**		**HA**
uSpLSS	**uSpec**	**uPri**	**PLSS**	**SpecB**	**LatB**
HuSpLSS	**HuSpec**	**uEsa**	**ELSS**	**HSpecB**	**HAB**

In each row of the second table, the categories in second, third and fourth columns are concretely isomorphic, equivalent to the category in the first and fifth columns and dually equivalent to the category in the last column.

$\mathbf{SS}_0$	**SpecD**	**PriD**	**PSSD**		**LatD**
HSS	**HSpecD**	**EsaD**	**ESSD**		**HAD**
$\mathbf{LSS}_0$	**uSpecD**	**uPriD**	**PLSSD**	**SpecBD**	**LatBD**
HLSS	**HuSpecD**	**uEsaD**	**ELSSD**	**HSpecBD**	**HABD**

This makes another step, after [23] and [21], in developing tame topology. To do this, we have introduced new notions and notations as well as proved many auxiliary facts about locally small spaces and presented many examples illustrating the introduced notions. Among the things worth mentioning are the notion of a special bornology (Definition 13) in a distributive bounded lattice and Example 2 using the theory of closure algebras. Moreover, we have shown (Theorem 3) how Dimov's version (from [45]) of Stone Duality for Boolean spaces with continuous mappings agrees with our versions of Stone Duality. We have also introduced the standard up-spectralification (Definition 20) of a Kolmogorov locally small space and extended many features of the theory of Heyting small spaces from our previous paper [21] to the locally small context.

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References

1. Esakia, L. Topological Kripke models. *Dokl. Akad. Nauk SSSR* **1974**, *214*, 298–301.
2. Esakia, L. *Heyting Algebras. Duality Theory*; Springer Nature: Cham, Switzerland, 2019.
3. Bezhanishvili, G. *Leo Esakia on Duality in Modal and Intuitionistic Logics*; Springer: Dordrecht, The Netherland, 2014.
4. Bezhanishvili, N. Lattices of intermediate and cylindric modal logics. PhD thesis, Universiteit van Amsterdam, Amsterdam, 2006.
5. Bezhanishvili, G.; Jansana, R. Esakia Style Duality for Implicative Semilattices. *Appl. Categor. Struct.* **2013**, *21*, 181–208.
6. Bezhanishvili, G.; Bezhanishvili, N.; Moraschini, T.; Stronkowski, M. Profiniteness and representability of spectra of Heyting algebras. *Adv. Math.* **2021**, *391*, 107959.
7. Celani, S.A.; Jansana, R. Esakia Duality and Its Extensions. In *Leo Esakia on Duality in Modal and Intuitionistic Logics*; Springer: Dordrecht, The Netherland, 2014; pp. 63–98.
8. Stone, M.H. Topological representations of distributive lattices and Brouwerian logics. *Časopis Pěst. Mat. Fiz.* **1938**, *67*, 1–25.
9. Priestley, H. Representation of distributive lattices by means of ordered Stone spaces. *Bull. Lond. Math. Soc.* **1970**, *2*, 186–190.
10. Bice, T.; Starling, C. Locally Compact Stone Duality. *J. Log. Anal.* **2018**, *10*, 1–36.
11. Bice, T.; Starling, C. General non-commutative locally compact locally Hausdorff Stone duality. *Adv. Math.* **2019**, *341*, 40–91.

12. Bezhanishvili, G.; Morandi, P.J.; Olberding, B. A generalization of Gelfand-Naimark-Stone duality to completely regular spaces. *Topol. Appl.* **2020**, *274*, 107123.
13. Bezhanishvili, G.; Morandi, P.J.; Olberding, B. Gelfand-Naimark-Stone duality for normal spaces and insertion theorems. *Topol. Appl.* **2020**, *280*, 107256.
14. Dimov, G.; Ivanova-Dimova, E. Two Extensions of the Stone Duality to the Category of Zero-Dimensional Hausdorff Spaces. *Filomat* **2021**, *35*, 1851–1878.
15. Cannon, S.; Döring, A. A Generalisation of Stone Duality to Orthomodular Lattices. In *Reality and Measurement in Algebraic Quantum Theory*; Springer Nature: Singapore, 2018; pp. 3–65.
16. Hartonas, C. Duality Results for (Co)Residuated Lattices. *Log. Universalis* **2019**, *13*, 77–99.
17. Kudryavtseva, G. A refinement of Stone duality to skew Boolean algebras. *Algebra Universalis* **2012**, *67*, 397–416.
18. Goldblatt, R. Canonical extensions and ultraproducts of polarities. *Algebra Universalis* **2018**, *79*, 80.
19. Hartonas, C. Canonical Extensions and Kripke-Galois Semantics for Non-distributive Logics. *Log. Universalis* **2018**, *12*, 397–422.
20. Kwaśniewski, B.K.; Meyer, R. Stone duality and quasi-orbit spaces for generalised C^*-inclusions. *Proc. Lond. Math. Soc.* **2020**, *121*, 788–827.
21. Piękosz, A. Esakia duality for Heyting small spaces, *Symmetry*, **14**(12) (2022), 2567.
22. Dickmann, M.; Schwartz, N.; Tressl, M. *Spectral Spaces*; Cambridge University Press: Cambridge, UK, 2019.
23. Piękosz, A. Stone duality for Kolmogorov locally small spaces, *Symmetry*, **13**(10) (2021), 1791.
24. Grothendieck, A. Esquisse d'un Programme [Sketch of a Program]. In *Geometric Galois Actions 1*; Cambridge University Press: Cambridge, UK, 1997; pp. 5–48. (In French, with an English translation on pp. 243–283.)
25. A'Campo, N.; Ji, L.; Papadopoulos, A. On Grothendieck's tame topology. In *Handbook of Teichmüller Theory, Volume VI*; EMS Press: Berlin, Germany, 2016; Volume 27, pp. 521-533.
26. Cruz Morales, J.A. The Notion of Space in Grothendieck. In *Handbook of the History and Philosophy of Mathematical Practice*; Springer Nature: Cham, Switzerland, 2020; pp. 1–21.
27. Hochster, M. Prime ideal structure in commutative rings. *Trans. Am. Math. Soc.* **1969**, *142*, 43–60.
28. Belaid, K.; Echi, O. On a conjecture about spectral sets, *Top. Appl.* **2004**, *139*, 1–15.
29. Echi, O.; Abdallahi, M.O. On the spectralification of a hemispectral space. *J. Algebra Appl.* **2011**, *10*, 687–699.
30. Acosta, L.; Rubio Perilla, I.M. Nearly spectral spaces. *Bol. Soc. Mat. Mex.* **2019**, *25*, 687–700.
31. Acosta, L. *Temas de teoría de retículos*; Universidad Nacional de Colombia: Bogota, Colombia, 2016.
32. Balbes, R.; Dwinger P. *Distributive Lattices*; University of Missouri Press: Columbia, MO, USA, 1974.
33. Bochnak, J.; Coste, M.; Roy, M.-F. *Real Algebraic Geometry*; Springer: Berlin/Heidelberg, Germany, 1998.
34. Andradas, C.; Bröcker, L.; Ruiz, J. *Constructible Sets in Real Geometry*; Springer: Berlin/Heidelberg, Germany, 1996.
35. Delfs, H.; Knebusch, M. *Locally Semialgebraic Spaces*; Springer: Berlin/Heidelberg, Germany, 1985.
36. van den Dries, L. *Tame Topology and O-minimal Structures*; Cambridge University Press: Cambridge, UK, 1998.
37. Piękosz, A. O-minimal homotopy and generalized (co)homology. *Rocky Mt. J. Math.* **2013**, *43*, 573–617.
38. Piękosz, A. On generalized topological spaces I. *Ann. Polon. Math.* **2013**, *107*, 217–241.
39. Piękosz, A. On generalized topological spaces II. *Ann. Polon. Math.* **2013**, *108*, 185–214.
40. Pillay, A. On groups and fields definable in o-minimal structures. *J. Pure Appl. Algebra* **1988**, *53*, 239–255.
41. Mac Lane, S. *Categories for the Working Mathematician*; Springer: New York, NY, USA, 1998.
42. Özel, C.; Piękosz, A.; Al Shumrani, M.; Wajch, E. Partially paratopological groups. *Topol. Appl.* **2017**, *228*, 68–78.
43. Piękosz, A. Locally small spaces with an application. *Acta Math. Hung.* **2020**, *160*, 197–216.
44. Adámek, J.; Herrlich, H.; Strecker, G.E. *Abstract and Concrete Categories. The Joy of Cats*; Wiley: New York, NY, USA, 1990.
45. Dimov, G. Some generalizations of the Stone Duality Theorem. *Publ. Math. Debrecen* **2012**, *80*, 255–293.