Article

Not peer-reviewed version

A Proof of Collatz Conjecture

HUANG Qingxue *

Posted Date: 31 May 2023

doi: 10.20944/preprints202305.2161.v1

Keywords: pre odd number; post odd number; classification of odd numbers; sequence classification of odd numbers,1-1 mapping between odd numbers and odd number sequences

Article

A Proof of Collatz Conjecture

Huang Qingxue

School of Mathematical Sciences, Zhejiang University, Hangzhou, China; qxhuang@zju.edu.cn

Abstract: In 1937, German mathematician L. Collatz proposed a conjecture: for any positive integer, if it is even, divide it by 2; if it is odd, multiply it by 3 and add 1 to get an even number; keep going on and on, and the final result is 1. This paper gives a sequence classification of odd numbers, and gives 1-1 mapping of odd numbers and odd number sequences,and uses these results to give a proof of this conjecture.

Keywords: pre odd number; post odd number; classification of odd numbers; sequence classification of odd numbers,1-1 mapping between odd numbers and odd number sequences

MR Subject Classification: 11B30

---

1. Introduction

In 1937, German mathematician L. Collatz posed a problem that later became known as Collatz conjecture: for a definite positive integer p, if p is even, divide it by 2; if p is odd, multiply it by 3 and add 1 to get an even number; then continue according to the above rule, the final result will be 1.

Collatz conjecture has been studied by many people and has long been regarded as an unsolved problem.See [1,2,3,4].This paper gives a sequence classification of odd numbers, and an 1-1 mapping between odd numbers and odd number sequences,and gives a proof of this conjecture.

Definition 1.1: Starting from a positive integer p, the above process is called a Collatz sequence; p is said to be successful to 1 if 1 is finally obtained； otherwise,p is not successful to 1.

Remark: Any positive integer p can be written as: $p = 2^k (2m-1)$,here m is a positive integer and k is a positive integer or 0; when $k > 0$, p is even; when $k = 0$, $p=2m-1$ is odd. If any odd number is successful to 1, then since the even number $p = 2^k (2m-1)$ is divided by 2 k times to get the odd number 2m-1, the even number $p = 2^k(2m-1)$ is also successful to 1. In other words, to prove that Collatz conjecture holds, it is sufficient to show that any odd number is successful to 1.

In this paper, the following discussion focuses on the Collatz sequence for odd numbers, and the even numbers in the Collatz sequence are omitted.

Example: For a positive integer p = 44, its Collatz sequence is: 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1， 4， 2,1， .... By removing all the even numbers, it becomes: 11, 17, 13, 5, 1,1....We will omit the duplicate odd number 1 later on.It becomes: 11, 17, 13, 5, 1. That is, the odd number 11 is successful to 1. This implies: odd 17, odd 13, odd 5, odd 1 are also successful to 1. This also implies: even 2^k , even $2^k \times 5$, even $2^k \times 13$, even $2^k \times 17$, even $2^k \times 11$ are also successful to 1,where k = 1, 2, 3, ...

Definition 1.2: For two odd numbers p and q in a Collatz sequence, if $3p + 1= 2^k q$,where $k > 0$ is a positive integer; then p is said to be a pre odd number of q ,and q is said to be a post odd number of p.

In the previous example , because $11 \times 3 + 1 = 17 \times 2$, where k = 1, so 11 is a pre odd number of 17, and 17 is a post odd number of 11. Similarly, because $17 \times 3 + 1 = 13 \times 2^2$, where k = 2, so 17 is a pre odd number of 13, 13 is a post odd number of 17, etc.

Theorem 1.1: If p is a pre odd number of q, then 4p+1 is also a pre odd number of q;and if $3p+1 = 2^k q$,here $k > 0$ is a positive integer, then $3(4p+1)+1 = 2^{k+2}q$.

Proof: Since p is a pre odd number of q, there exists a positive integer $k > 0$,such that $3p+1 = 2^k q$. So $3(4p+1)+1 = 12p+4 = 4(3p+1) = 4 \times 2^k q= 2^{k+2} q$. ■

Consider the sequence $\{a_n\}$:$a_1, a_2, ..., a_n, ...$; here a_1 is an odd number, and $a_n = 4a_{n-1} + 1, n = 2,3,4,....$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 May 2023 doi:10.20944/preprints202305.2161.v1

Corollary 1.2: For the above sequence $\{a_n\}$, if the odd number a_1 is a pre odd number of an odd number p, then each term in the sequence $\{a_n\}$ is a pre odd number of p; and if $3a_1 + 1 = 2^k p$, then $3a_2 + 1 = 2^{k+2} p$,... ,$3a_n + 1 = 2^{k+2(n-1)}p$, where n = 2,3,4,....

Proof: This is a direct corollary of Theorem 1.1.■

Theorem 1.3: Any odd number p has a post odd number q, and q is the unique post odd number of p.

Proof: It follows from the Collatz sequence that for any odd number p, 3p + 1 is an even number, and every even number can always be written as $2^k q$,where q is a definite odd number and k is a definite positive integer; that is, $3p + 1 = 2^k q$ holds for a definite odd number q. By Definition 1.2,q is a post odd number of p, and q is the unique post odd number of p.■

Theorem 1.4: In a Collatz sequence,if p is a pre odd number of p,（Certainly,p is also a post odd number of p.）then it is only case that p = 1.

Proof: Since p is a pre odd number of p,there exists a positive integer k > 0,such that $3p+1 = 2^k p$. So $2^k p - 3p = p(2^k - 3) = 1$,here p is odd.

Case 1: k = 1. Then p(2-3) = -p = 1, which cannot be true.

Case 2: k > 1. Then $2^k - 3$ is a positive integer,and $p(2^k - 3) = 1$.The only way this can be true ,if p = 1 and k = 2. ■

2. A sequence classification of odd numbers

All odd numbers can be divided into two types: {4n-1|n = 1,2,3,...},and {4n+1|n = 0,1,2,3,...}. Where {4n+1} can yet be divided into two types: 4n+1 (n is 0 or even), that is: 1, 9, 17, ...; and 4n+1 (n is odd), that is: 5, 13, 21, ...

Definition 2.1: An odd number of the form 4n-1 (n is a positive integer) is called a class A odd number; an odd number of the form 4n+1 (n is 0 or an even number) is called a class B odd number; an odd number of the form 4n+1 (n is an odd number) is called a class C odd number.

Definition 2.2: (1) For a sequence$\{a_n\}$:If a_1 is a class A odd number and $a_n = 4a_{n-1} + 1$,n=2,3,4,..., then $\{a_n\}$ is said to be a class I sequence.

(2) For a sequence $\{a_n\}$:If a_1 is a class B odd number and $a_n = 4a_{n-1} + 1$,n= 2,3,4,..., then $\{a_n\}$ is said to be a class II sequence.

Example: The first two sequences in class I:

$\{a_n\}$:3,13,53,..., (a_1 = 3); $\{b_n\}$:7,29,117,..., (b_1 = 7).

The first two sequences in class II:

$\{a_n\}$:1,5,21,85,..., (a_1 = 1); $\{b_n\}$:9,37,149,..., (b_1 = 9).

Theorem 2.1: All terms (odd numbers) in all class I sequences and all class II sequences contain all odd numbers ; and each odd number must be in the only one of these sequences.

Proof: Notice that all odd numbers can be classified into three classes： class A, class B and class C. Any odd number in class A must be the first term a_1 of some definite class I sequence $\{a_n\}$; similarly,any odd number in class B must be the first term a_1 of some definite class II sequence $\{a_n\}$. So just prove that all terms of all sequences in class I and class II contain any class C odd number of the form 4n+1 (n is an odd number), and any class C odd number must be in the only one of these sequences.

Let $p = 4n_1 + 1$ be any definite class C odd number , where n_1 is a definite odd number. But all odd numbers can be classified into three classes A,B,C.

Case 1: If n_1 is a class A odd number, then n_1 is the first term a_1 of some definite class I sequence $\{a_n\}$; by the construction of a class I sequence, $p = 4n_1 + 1$ is the second term a_2 of this definite class I sequence $\{a_n\}$; similarly, if n_1 is a class B odd number, then $p = 4n_1 + 1$ is the second term a_2 of some definite class II sequence $\{a_n\}$.

Case 2: If n_1 is a class C odd number, then there exists a definite odd number n_2 , such that $n_1 = 4n_2 + 1$. If n_2 is some class A (or class B) odd number, then according to Case 1, $n_1 = 4n_2 + 1$ is the second term a_2 of some definite class I (or class II) sequence , and $p = 4n_1 + 1$ is the third term a_3 of this definite class I (or class II) sequence.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 May 2023 doi:10.20944/preprints202305.2161.v1

Case 3: If n_2 is some class C odd number, then we are back to the beginning of case 2. Continuing, since $p = 4n_1 + 1$ is a definite class C odd number, and $n_2 < n_1$, so after finitely many case 2, we can always get some n_k, so that $n_{k-1} = 4n_k + 1$, $n_1 > n_2 > ... > n_k$, and n_k must be some definite class A (or class B) odd number, which is the first term a_1 of some definite class I (or class II) sequence ,n_{k-1} is the 2nd term a_2 of the sequence ,...,n_1 is the kth term a_k,and $p = 4n_1 + 1$ is the k+1st term a_{k+1} of the definite class I (or class II) sequence . ■

3. Results

Notice that all odd numbers can again be divided into three types : {6n-3∣ n = 1,2,3,...}, {6n-1∣ n = 1,2,3,...}, {6n+1∣ n = 0,1,2,3,...}.

Definition 3.1: An odd number of the form 6n-1 (n = 1,2,3,...) is called a class D odd number , 6n+1 (n = 0,1,2,3,...) is called a class E odd number , and 6n-3 (n = 1,2,3,...) is called a class F odd number. Since 6n-3 is divisible by 3, a class F odd number is also called triple odd number.

Lemma 3.1: The class A odd number 4n-1 is a pre odd number of the class D odd number 6n-1, or 6n-1 is the post odd number of 4n-1,where n = 1,2,3,....

Proof: Since $3(4n-1) + 1 = 2(6n-1)$. ■

Notice that the class B odd number 4n+1 (n is 0 or even) can again be written as 8n+1(n = 0,1,2,...).

Lemma 3.2: The class B odd number 8n+1 is a pre odd number of the class E odd number 6n+1, or 6n+1 is the post odd number of 8n+1,where n = 0,1,2,3,....

Proof: Since $3(8n+1) + 1 = 2^2 (6n+1)$. ■

Lemma 3.3: The triple odd number 6n-3 cannot be a post odd number of any odd number; in other words, the class F odd number 6n-3 have no pre odd number,where n = 1,2,3,....

Proof: Suppose that 6n-3 is a post odd number of some odd number p= 2m-1.Then $3(2m-1)+1 = 2^k(6n-3)$ holds for some positive integer k; at this point, the right side $2^k(6n-3)$ of the equation is divisible by 3, while the left side 3(2m-1)+1 of the equation is not divisible by 3. Contradiction. ■

Consider the class I and class II sequences given in §2.

Theorem 3.4:For all class I sequences $\{a_n\}$ (where $a_1 = 4k-1$, $a_n = 4a_{n-1} + 1(n = 2,3,4,...)$, where k is a definite positive integer) and all the class D odd numbers 6k-1 (here k is the same as above) there exists a 1-1 mapping according to the pre and post odd number relationship; and the class I sequence $\{a_n\}$ with $a_1 = 4k-1$ corresponds to the class D odd number 6k-1, while each term of this class I sequence $\{a_n\}$ is a pre odd number of this class D odd number 6k-1, and $3a_n + 1 = 2^{2n-1}(6k-1)$ (n = 1,2,3,...).

Proof: By Lemma 3.1, for a definite positive integer k,the first term 4k-1 of the class I sequence $\{a_n\}$ with $a_1 = 4k-1$ is the pre odd number of the class D odd 6k-1; then by Corollary 1.2, each term of the class I sequence $\{a_n\}$ is a pre odd number of the class D odd number 6k-1 ,and $3a_n + 1 = 2^{2n-1}(6k-1)$(n = 1,2,3 ...); then by Theorem 1.3, each term of this class I sequence $\{a_n\}$ has a unique post odd number 6k-1.So, according to the relationship between the pre and post odd numbers, the class I sequence $\{a_n\}$ with $a_1 = 4k-1$ and the class D odd number 6k-1is a 1-1 mapping,where k = 1,2,3,.... ■

Example: Take k = 1, then $a_1 = 3$, $\{a_n\}$:3,13,53,213,...; each term of this class I sequence $\{a_n\}$ is a pre odd number of the class D odd number 5.

Theorem 3.5: For all class II sequences $\{a_n\}$ (where $a_1 = 8k+1$, $a_n = 4a_{n-1} + 1(n = 2,3,4,...)$, where k is 0 or a definite positive integer) and all class E odd number 6k+1 (here k is the same as above) there exists a 1-1 mapping according to the pre and post odd number relationship; and the class II sequence $\{a_n\}$ with $a_1 = 8k+1$ corresponds to the class E odd number 6k+1, while each term of the class II sequence $\{a_n\}$ is the pre odd number of this class E odd number 6k+1,and $3a_n +1 = 2^{2n} (6k+1)$(n = 1,2,3,...).

Proof: As proved in Theorem 3.4, the result can be obtained from Lemma 3.2, and Corollary 1.2 and Theorem 1.3 of Section 1.■

Example: Take k = 0, then $a_1 = 1$,$\{a_n\}$:1,5,21,85,...; each term of this class II sequence $\{a_n\}$ is a pre odd number of the class E odd number 1.

Theorem 3.6: Let p be an odd number and q = 4p+1,then

(1) If p is a class E odd number , then q is a class D odd number ;

(2) If p is a class D odd number, then q is a class F odd number ；

(3) If p is a class F odd number, then q is a class E odd number .

Proof: (1) Since p is a class E odd number, let p = 6k+1, where k is 0 or a definite positive integer; then q = 4p+1 = 4(6k+1)+1 = 24k+5 = 6(4k+1)-1, so q is a class D odd number;

(2) Since p is a class D odd number, let p = 6k-1,where k is a definite positive integer; then q = 4p+1 = 4(6k-1)+1 = 24k-3 = 6(4k) -3, so q is a class F odd number;

(3) Since p is a class F odd number , let p = 6k-3, where k is a definite positive integer; then q = 4p+1 = 4(6k-3)+1 = 24k-11= 6(4k-2)+1, so q is a class E odd number.■

Remarks: In the class I sequence $\{a_n\}$, a_1 = 4k-1 is a class A odd number; and all class A odd numbers are {4k-1}:3, 7, 11, 15, 19, 23, ...; where 3, 15, ... are class F odd numbers, 7, 19, ... are class E odd numbers, and 11, 23, ... are class D odd numbers.

In the class II sequence $\{a_n\}$, a_1 = 8k+1 (k = 0,1,2,...) is a class B odd number; and all class B odd numbers are {8k+1}:1, 9, 17, 25, 33, 41,...; where 9, 33,... are class F odd numbers, 1, 25,. ... are class E odd numbers, and 17, 41, ... are class D odd numbers.

That is, the first term a_1 can be any one of the three odd classes D,E,F, whether the sequence $\{a_n\}$ is a class I or a class II .

Theorem 3.7: In any one of class I and class II sequence $\{a_n\}$,

(1) If a_1 is a class E odd number, then a_2 is a class D odd number, a_3 is a class F odd number, and a_4 is a class E odd number,etc. The subsequent terms cycle in this manner;

(2) If a_1 is a class F odd number,then loop by F,E,D,F,E,D;

(3) If a_1 is a class D odd number,then loop by D,F,E,D,F,E.

Proof: This is a direct corollary of Theorem 3.6.■

Remark: If the first term is excluded from the class I and class II sequence, then the other terms are all class C odd numbers, that is, odd numbers of the form 4n+1 (n is odd). According to Theorem 3.7, these terms can also be any one of the three classes D,E,F.

The Collatz sequence now under discussion has omitted even numbers, so the following definition is given.

Definition 3.2: If p_k ,p_{k-1} ,...,p_1 ,1 is a Collatz sequence with k odd numbers other than 1, then we say that p_k is k steps successful to 1; and it is specified that the odd number 1 is 1 step successful to 1.

Example: 11, 17, 13, 5, 1 is the Collatz sequence given in Introduction . Then we have: odd 5 is 1 step successful to 1, 13 is 2 steps successful to 1, 17 is 3 steps successful to 1, and 11 is 4 steps successful to 1.Specially,1 is 1 step successful to 1.

Corollary 3.8: If p is a pre odd number of q ,q ≠ 1 and q is k steps successful to 1, then p is k+1 steps successful to 1;if p is a pre odd number of 1,then p is 1 step successful to 1.

Proof: This is a direct consequence of Definition 3.2.■

In the following we construct a set H of sequences by recursive method.

By Theorem 3.5, the class E odd number 1 corresponds to the class II sequence $\{a_n\}$:1, 5, 21, 85, 341, ... Put this sequence into a sequence set H_1. There is exactly one sequence in H_1.

In this sequence of H_1 , since a_1 = 1 is a class E odd number, by Theorem 3.7,a_{3k-2} is

a class E odd number, a_{3k-1} is a class D odd number, and a_{3k} is a class F odd number, where k = 1,2,3 ,.... By Lemma 3.3, any class F odd number has no pre odd number. By Theorem 3.4, any class D odd number corresponds to a unique class I sequence; by Theorem 3.5, any class E odd number corresponds to a unique class II sequence. We take all the class I sequences and all the class II sequences corresponding to all the class D odd numbers and all the class E odd numbers of this sequence in H_1 (removing the class E odd number 1) to form a sequence set H_2.

(Note: In H_2 ,the removal of the class II sequence corresponding to class E odd number 1 is to prevent the sequence that appeared in H_1 from reappearing in H_2 .)

Assume that the sequence set H_n has been formed.

Then all the class I sequences and all the class II sequences corresponding to all the class D odd numbers and all the class E odd numbers of each sequence in H_n form a set H_{n+1} .

Now the sequence sets H_1 ,H_2 ,H_3 ,...,H_n ,... have been constructed.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 May 2023 doi:10.20944/preprints202305.2161.v1

Write H = $\bigcup_{n=1}^{\infty} H_n$

Remark: To get an intuitive sense of the composition of the set H, several sequences in H_1 and H_2 are given here according to Theorem 3.4 and Theorem 3.5.

There exists unique class II sequence $\{a_n\}$ in H_1 corresponding to the.class E odd number 1.$\{a_n\}$:1,5,21,85,341,1365,...

The sequences in H_2 correspond to all class D odd numbers and all class E odd numbers of the above sequence $\{a_n\}$ in H_1.In the above sequence,a_1 = 1 is a class E odd number, which corresponds to just above sequence $\{a_n\}$. To avoid repetition, do not put it in H_2; a_2 = 5 is a class D odd number, which corresponds to the class I sequence $\{b_n\}$:3,13,53,213,...,put it in H_2; a_3 = 21 is a class F odd number (triple odd number), which does not exist any pre odd number; a_4 = 85 is a class E odd number, which corresponds to the class II sequence $\{b_n\}$:113,453,1813,...,put it in H_2.; a_5 = 341 is a class D odd number, which corresponds to the class I sequence $\{b_n\}$:227,909,3637,...,put it into H_2 ; a_6 = 1365 is a class F odd number, there is no pre odd number; and so on.

Theorem 3.9: (1) There are no identical sequences in the above sequence sets H_1 ,H_2 ,...,H_n ,...; and no two different sequences in them have the same term.

(2) All terms of any sequence in H_n are n steps successful to 1, where n = 1, 2, 3, ...

(3) If the odd number p is successful to 1, then p must be in some sequence of the sequence set H.

Proof: (1) Induction. First the sequence of H_1 is the class II sequence corresponding to the class E odd number 1.All the sequences of H_2 are all the class I sequences and all the class II sequences corresponding to all the class D odd numbers and all the class E odd numbers in H_1 except for the class E odd number 1. The class E odd number 1 corresponding to the sequence in H_1 and all the class D odd numbers and all the class E odd numbers corresponding to these sequences in H_2 are just all the class D odd numbers and all the class E odd numbers in H_1, all of which are not identical to each other, and from Theorem 3.4 and Theorem 3.5, all the sequences in H_1 and H_2 are not identical to each other.

Then since all the sequences in H_1 and H_2 are class I and class II sequences, by Theorem 2.1, different sequences will not have the same term.

Suppose all the sequences in H_1 ,H_2 ,...,H_n are different from each other and no two different sequences have the same term.

Consider H_{n+1} . Since all Class I and all Class II sequences in H_1,H_2 ,...,H_n correspond to all Class D odd and all Class E odd numbers that appear in H_1 ,H_2 ,...,H_{n-1} (including the Class E odd number 1 corresponding to the sequence in H_1); and all Class I and all Class II sequences in H_{n+1} correspond to all Class D odd and All Class E odd numbers that appear in H_n ; according to the assumption of induction, no two different sequences in H_1 ,H_2 ,...,H_n have the same term; that is, these Class D odd numbers and Class E odd numbers of each sequence in H_n do not appear in the sequences of H_1 ,H_2 ,...,H_{n-1} . So by Theorem 3.4 and Theorem 3.5, all the sequences of H_{n+1} and any of the sequences of H_1,H_2 ,...,H_n will not be the same; that is, all the sequences of H_1 ,H_2 ,...,H_n ,H_{n+1} are not the same as each other. By Theorem 2.1, no two sequences in H_1,H_2,...,H_n ,H_{n+1} have the same term.

(2) Induction. First of all, all the terms of the sequence in H_1 are the pre odd numbers of class E odd number 1. By definition 3.2,all the terms of the sequence in H_1 are 1 step successful to 1 .

Suppose all terms of any sequence in H_n are n steps successful to 1 .

Consider H_{n+1} . Notice that any sequence $\{a_n\}$ in H_{n+1} corresponds to some class D (or class E) odd number p of a sequence in H_n , that is, all the terms of the sequence $\{a_n\}$ are the pre odd numbers of that odd number p. Since the odd number p is in a sequence in H_n , by induction hypothesis, the odd number p is n steps successful to 1 ; by Corollary 3.8, all the terms of the sequence $\{a_n\}$ are n+1 steps successful to 1.

(3) Since the odd number p is successful to 1, then there must exist some positive integer n,such that p is n steps successful to 1, so it is enough to prove that p must be in some sequence of H_n.

Induction. First look at n = 1. Let the odd number p be 1 step successful to 1.Note that the odd number p must be in some definite class I (or class II) sequence. By Theorem 3.5, for the class E odd number 1,it corresponds to some class II sequence $\{a_n\}$, where all the terms of the class II sequence

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 May 2023 doi:10.20944/preprints202305.2161.v1

$\{a_n\}$ are the pre odd numbers of the class E odd number 1. Then by Theorem 3.4 and Theorem 3.5,all the class D (or class E) odd numbers corresponding to all other class I (or class II) sequences cannot be 1; that is,any term in all other class I (or class II) sequence cannot be a pre odd number of 1.In the other words,if the odd number p is in any other sequence excluding $\{a_n\}$, the odd number p cannot be 1 step successful to 1; so the odd number p can only be in the class II sequence $\{a_n\}$ corresponding to the class E odd number 1, and the class II sequence $\{a_n\}$ is exactly the one in H_1. Therefore, the result holds for n = 1.

Assume that the result holds for n = 2,3,...,k; that is, if the odd number p is 2 steps,3 steps,...,k steps successful to 1 respectively, then p is in some sequence of H_2 ,H_3 ,...,H_k respectively.

Now let n = k+1; that is, the odd number p is k+1 steps successful to 1. Let Collatz sequence of the odd number p be p ,p_k ,p_{k-1} ,...,p_1 ,1. Then the odd number p_k is k steps successful to 1; by inductive assumption, p_k is in some sequence of H_k . For simplicity, let p_k be in some class I sequence $\{b_n\}$ of H_k . (If p_k is in some class II sequence $\{b_n\}$of H_k , then the same treatment follows.) Since all the terms of the sequence $\{b_n\}$ have class D, class E, and class F odd numbers, that p_k cannot be a class F odd number.(Because p is a pre odd number of p_k , and any class F odd number has no pre odd number;) Therefore, p_k is some definite class D (or class E) odd number in $\{b_n\}$. Let's assume p_k is a definite class D odd number in $\{b_n\}$. Then, by the construction of H_{k+1}, the class I sequence corresponding to the class D odd number p_k is some definite sequence $\{c_n\}$ in H_{k+1} , and all terms of $\{c_n\}$ are pre odd numbers of the odd number p_k . Since this is an 1-1 mapping, the class D (or class E) odd number corresponding to any class I (or class II) sequence other than the class I sequence $\{c_n\}$ cannot be p_k. That is, any term in any other class I (or class II) sequence other than $\{c_n\}$ cannot be the pre odd number of p_k . Therefore, the pre odd number p of p_k can only be in this sequence $\{c_n\}$ of H_{k+1} . Therefore, the result holds when n = k + 1.■

Remark: From Theorem 3.9, it follows that every term of every sequence in H = $\bigcup_{n=1}^{\infty} H_n$ is successful to 1; and every odd number that is successful to 1 must be in one of these sequences of H. Therefore, to prove that all odd numbers are successful to 1, it is sufficient to show that every odd number is in one of the sequences of H. For this purpose, the following lemmas are given firstly.

Lemma 3.10: (1) If p is a pre odd number of q and the odd number q is not successful to 1, then the odd number p is also not successful to 1;

(2) If r is a post odd number of q and the odd number q is not successful to 1, then the odd number r is also not successful to 1.

Proof: (1) First, since q is not successful to 1, q ≠ 1. Assuming that the odd number p is successful to 1,then there exists a positive integer k,such that p is k steps successful to 1, and by Theorem 1.3, q is the unique post odd number of p. Then, by Definition 3.2, k > 1, and q is k-1 steps successful to 1. Contradiction.

(2) Assuming that the odd number r is successful to 1,then there exists a positive integer m,such that r is m steps successful to 1, so q is m+1 steps successful to 1. Contradiction.■

By Theorem 2.1, any odd number is a term of some class I (or class II) sequence.

Lemma 3.11: If an odd number p is not successful to 1, then all terms of the class I (or class II) sequence $\{a_n\}$,in which the odd number p is located ,are not successful to 1.

Proof: From Theorem 3.9, it follows that all odd numbers in the sequence set H are successful to 1; since the odd number p is not successful to 1, p is not in H.Let the odd number p be a term in some definite class I(or class II) sequence $\{a_n\}$.Since H is the sequence set , that sequence $\{a_n\}$ is also not in H. In the other words, all terms of $\{a_n\}$ are not in H. By Theorem 3.9, all terms of $\{a_n\}$ are not successful to 1.■

As with the construction of the sequence set H, the following series G_n of sequence sets is constructed.

First put a class I (or class II) sequence$\{a_n\}$ into a sequence set, denoted as G_{11} ; G_{11} contains only one sequence $\{a_n\}$.

As before, there are infinitely many class D, class E and class F odd numbers in $\{a_n\}$, and then all class I sequences and all class II sequences corresponding to all class D odd numbers and all class E odd numbers of the sequence $\{a_n\}$ in G_{11} form a sequence set G_{12} .

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 May 2023 doi:10.20944/preprints202305.2161.v1

Suppose that the sequence set G_{1n} has been formed.

Then all the class I sequences and all the class II sequences corresponding to all the class D odd numbers and all the class E odd numbers of all sequences of G_{1n} form a sequence set $G_{1,n+1}$.

Denote $G_1 = \bigcup_{n=1}^{\infty} G_{1n}$

Lemma 3.12: If some term in this sequence $\{a_n\}$ of G_{11} is not successful to 1,then

(1)There are no identical sequences in the sequence set G_1 ;and no two different sequences in G_1 have the same term.

(2) Any term of all sequences in the set G_1 is not successful to 1.

Proof: (1) First, following the method proved in Theorem 3.9(1), it is obtained that there are no identical sequences in the above sequence sets G_{11} ,G_{12} ,...,G_{1n} ,..; and no two different sequences have the same term.

(2) Induction for G_{1n}. First of all, by Lemma 3.11,any term of the sequence $\{a_n\}$ in G_{11} is not successful to 1. So the result holds when n = 1.

Assume that the result holds for n = k, i.e., that any term of any sequence in G_{1k} is not successful to 1.

Consider $G_{1,k+1}$. Let the sequence $\{b_n\}$ be any definite class I (or class II) sequence in $G_{1,k+1}$, then $\{b_n\}$ corresponds to some class D (or class E) odd number p of some definite sequence in G_{1k}. And all terms of $\{b_n\}$ are the pre odd numbers of that odd number p. By induction hypothesis, any term of any sequence in G_{1k} is not successful to 1. So the odd number p is also not successful to 1. And by Lemma 3.10,any term in $\{b_n\}$ is not successful to 1. So the result holds for n = k+1.■

Starting from the sequence set G_1 above, we will construct the sequence set G_n ，where n = 2,3,4，...

Start with the construction of G_2 . Use the recursive method. First look at G_{11} in G_1, there is only one class I (or class II) sequence in G_{11} , which is denoted as $\{a_n\}$. Let the class D (or class E)odd number corresponding to this class I (or class II) sequence $\{a_n\}$ be p, then each item in $\{a_n\}$ is a pre odd number of the odd number p. Since every odd number must be in some definite class I (or class II) sequence, let the class I (or class II) sequence containing the odd number p be $\{b_n\}$. Put the class I (or class II) sequence $\{b_n\}$ into a sequence set, denoted G_{21} . G_{21} contains only one sequence $\{b_n\}$.

At this point, there are infinitely many class D,class E and class F odd numbers in the sequence $\{b_n\}$, and the above odd number p is just one of the class D (or class E) odd numbers in $\{b_n\}$. Take all the class I sequences and all the class II sequences corresponding to all the class D odd numbers and all the class E odd numbers of the sequence $\{b_n\}$ in G_{21} ,and form a sequence set G_{22} . (Note: the sequence $\{a_n\}$ in G_{11} is exactly that sequence in G_{22} corresponding to the odd number p in the sequence $\{b_n\}$ of G_{21}.)

Suppose that the sequence set G_{2k} has been formed.

Then all class I sequences and all class II sequences corresponding to all class D odd numbers and all class E odd numbers of all sequences in G_{2k} form the sequence set $G_{2,k+1}$. Write $G_2 = \bigcup_{k=1}^{\infty} G_{2k}$.

So on and so forth.

Suppose again that $G_n = \bigcup_{k=1}^{\infty} G_{nk}$ has been constructed.

At this point, G_{n1} has only one class I (or class II) sequence, denoted $\{c_n\}$. Let the class D (or class E) odd number corresponding to this class I (or class II) sequence $\{c_n\}$ in G_{n1} be q. Then each item in $\{c_n\}$ is a pre odd number of the odd number q. Let the class I (or class II) sequence containing the odd number q be $\{d_n\}$. Now put that class I (or class II) sequence $\{d_n\}$ into a sequence set, denoted $G_{n+1,1}$; $G_{n+1,1}$ has only one sequence $\{d_n\}$.

There are infinitely many class D, class E and class F odd numbers in $\{d_n\}$, and the odd number q above is just one of these class D (or class E) odd numbers. Put all class I sequences and all class II sequences corresponding to all class D odd numbers and all class E odd numbers of the sequence $\{d_n\}$ in $G_{n+1,1}$ into a sequence set,denoted $G_{n+1,2}$. (Note that the sequence $\{c_n\}$ in G_{n1} is exactly the sequence of $G_{n+1,2}$ corresponding to the odd number q in the sequence $\{d_n\}$ of $G_{n+1,1}$.)

Suppose that the sequence set $G_{n+1,k}$ has been formed.

Then all the class I sequences and all the class II sequences corresponding to all the class D odd numbers and all the class E odd numbers of all sequences of $G_{n+1,k}$ form a sequence set $G_{n+1,k+1}$.

 doi:10.20944/preprints202305.2161.v1

Denote $G_{n+1} = \bigcup_{k=1}^{\infty} G_{n+1,k}$.

Now G_n , where n = 1,2,3,..., has been constructed recursively.

Lemma 3.13: If any term of any sequence in the sequence set G_1 is not successful to 1, then

(1) for n = 2,3,4,..., there are no identical sequence in the sequence set G_n constructed above, and no two different sequences have the same term;

(2) For n = 2, 3, 4, ..., any term of any sequence in the sequence set G_n is not successful to 1;

(3) $G_1 \subset G_2 \subset ... \subset G_n \subset ...$

Proof: (1) For n = 2,3,4,..., notice that the construction method of G_n is exactly the same as that of G_1 and also exactly the same as that of H. Therefore, as proved in Theorem 3.9(1), the result holds;

(2) Since any item of any sequence in G_1 is not successful to 1, then every item of the unique sequence $\{a_n\}$ in G_{11} is not successful to 1. By the construction of G_2 , the sequence $\{a_n\}$ corresponds to the odd number p, and every item of $\{a_n\}$ is the pre odd number of the odd number p. Therefore, by Lemma 3.10(2), the odd number p is not successful to 1. And by the construction of G_2 , the odd number p is in the class I (or class II) sequence $\{b_n\}$of G_{21} ; by Lemma 3.11, all terms in $\{b_n\}$ are not successful to 1. And as proved in Lemma 3.12, any term of any sequence in the set G_2 is not successful to 1.

By analogy, it is clear that any term of any sequence in the set G_n is not successful to 1,where n = 3,4,5,...

(3) First prove that $G_1 \subset G_2$. Looking back at the construction of G_2 , G_{11} has the unique class I (or class II) sequence $\{a_n\}$, which corresponds to a class D (or class E) odd number p. And p is in the unique class I (or class II) sequence $\{b_n\}$ of G_{21} . While all class I sequences and all class II sequences corresponding to all class D odd numbers and all class E odd numbers of the sequence $\{b_n\}$ in G_{21} form the sequence set G_{22} . Because p is in $\{b_n\}$, so the sequence $\{a_n\}$ corresponding to p is in G_{22}, that is, $G_{11} \subset G_{22}$. Then by the generation of G_{12} and G_{23} it follows that since $G_{11} \subset G_{22}$, so $G_{12} \subset G_{23}$. By analogy ,it follows that $G_{13} \subset G_{24}$,...,$G_{1n} \subset G_{2,n+1}$, etc.

Since $G_1 = G_{11} \cup G_{12} \cup ... \cup G_{1n} \cup ...$

$G_2 = G_{21} \cup G_{22} \cup ... \cup G_{2n} \cup G_{2,n+1} \cup ...$

So $G_1 \subset G_2$. By analogy, $G_2 \subset G_3 \subset ... \subset G_n \subset ....$■

Remarks: (1) From the construction of G_2 , it is clear that the class D (or class E) odd number p corresponding to the class I (or class II) sequence $\{a_n\}$ in G_{11} is just a general odd number in the sequence $\{b_n\}$ of G_{21} , and its corresponding sequence $\{a_n\}$ grows in the pre odd number direction (let's say) to obtain a G_1 . While there are infinitely many class D and infinitely many class E odd numbers in the sequence $\{b_n\}$ of G_{21} ,and G_2 is obtained by growing the infinitely many class D and class E odd numbers in $\{b_n\}$ in the pre odd number direction. Where each class D (or class E) odd number also generates a sequence set equivalent to G_1. Thus, G_2 is "infinitely many times" larger than G_1 .

Similarly, for n = 3,4,...,G_n is "infinitely many times" larger than G_{n-1} .

(2)The difference between sequence set G_n and the sequence set H is that: all terms of

all sequences in H are successful to 1, so when they grow in the post odd direction (let's say), they stop at odd number 1; while all terms of all sequences in G_n are not successful to 1, so as n increases infinitely, G_n is endlessly expanding. From Theorem 3.9, we know that the sequences in H are not identical to each other, and no two sequences have the same term (odd number). So H can be regarded as a set of odd numbers; similarly, G_n can also be regarded as a set of odd numbers. From the above, it is obtained that as a set of odd numbers, H is a definite set of odd numbers; and as n increases infinitely, $\lim_{n\to\infty} G_n$ is an indeterminate set of odd numbers.

Theorem 3.14: Any odd number is successful to 1.

Proof: Let the set of all odd numbers be Q.Consider H as a set of odd numbers. By Theorem 3.9 , the set of all odd numbers successful to 1 is H. Suppose there is an odd number ,that is not successful to 1. And let the set of all odd numbers not successful to 1 be G. then $G \cap H = \emptyset$,and $G \cup H = Q$.

From Lemma 3.12 and Lemma 3.13, if there is an odd number that is not successful to 1, then the sequence set G_n ,which is regarded as the set of odd numbers, can be obtained.

And it is known that any odd number of G_n is not successful to 1. And from the above remark

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 May 2023 doi:10.20944/preprints202305.2161.v1

(2),when n tends infinity, $\lim_{n\to\infty} G_n$ is an indeterminate set of odd numbers. Because $G_n \subset G$,where n = 1,2,3,..., so $\lim_{n\to\infty} G_n \subset G$,and the set G of all odd numbers, those are not successful to 1 ,is also an indeterminate set of odd numbers.And the sets of odd numbers H and Q are both definite sets of odd numbers, so the equation $G \cup H = Q$ does not hold, contradiction.Thus,G = Ø, H = Q,and any odd number is successful to 1.■

Remarks:(1)Treat each odd number in the set H as a vertex,and then connect an edge between any two odd numbers (vertices) in H that have a pre and post odd number relationship; in particular,connect an edge between odd number 1 and any other odd number in the odd number set H_1 except for odd number 1.At this point,H can be regarded as a tree with an odd root 1.We call it the H-tree.This H-tree contains all odd numbers.Because any triple odd number has no pre odd number,each odd number in the triple odd number set F is a leaf of this H-tree.

(2)For G_n,where n= 1,2,3,...,as above,each odd number in G_n is treated as a vertex,and an edge is connected between any two odd numbers in G_n that have a pre and post odd number relationship,then $\lim_{n\to\infty} G_n$ is an unrooted tree.According to the proof of Theorem 1 in reference [4], the number of vertices in $\lim_{n\to\infty} G_n$ is uncountable,it is impossible,so G = Ø.

Theorem 3.15: Any positive integer is successful to 1 , i.e., Collatz conjecture holds.

Proof: See the description at the top part of § 1 Introduction.■

References:

1. Richrd K. Guy: Unsolved Problem In Number Theory.The 3x+1 Problem.Springer-Verlag, New York.330-336,2007.
2. Jeffrey C.Lagarias: "The 3n+1 Problem and its generalization",Amer.Math.Monthly, 92:1 pp3-23.1985.
3. Jerffrey C.Lagarias,The 3x+1 problem:An Annotated Bibliography,II(2000-2009) arXiv:math/0608208 V_6 [math NT].
4. Kerstin Andersson.On the Boundedness of Collatz Sequences.arXiv:math/14037425 V_3 [math NT].