Short note

Filter for Submodular Partition Function: Connection to Loose Tangle

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Abstract: Loose Tangle is a concept in graph theory that has a dual relationship with branch-width which is well-known graph width parameter. Ultrafilter, a fundamental notion in mathematics, is similarly known to have a dual relationship with branch-width when extended to a connectivity system (X, f). In this compact paper, we revisit and contemplate the interplay between Loose Tangle and Filter through the lens of a submodular partition function.

Keywords: tangle; loose tangle; filter; submodular partition function

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1. Introduction

The investigation of graph width parameters finds extensive applications across diverse fields, such as matroid theory, lattice theory, theoretical computer science, game theory, network theory, artificial intelligence, graph theory, and discrete mathematics, as evidenced by numerous studies (for example, see [1–20,22,28–33]). These graph width parameters are frequently explored in conjunction with obstruction, contributing to a robust body of research.

Loose tangle, an innovative concept initially brought forward in reference [1], occupies a central role in ascertaining whether a branch-width is at most a natural number k, where $k+1$ denotes the order of the tangle. The relevance and potential of loose tangles have further been explored in the context of submodular partition functions [22]. These submodular partition functions significantly broaden the understanding of various well-established tree decompositions of graphs.

Furthermore, the ultrafilter, a concept well-regarded in the mathematical arena, is acknowledged to maintain a dual relationship with branch-width when extended over a connectivity system (X, f).

In this compact and focused paper, we revisit and contemplate the interplay between Loose Tangle and Filter through the lens of a submodular partition function.

2. Preliminaries

In this section, we present the essential definitions required for this paper. Throughout the paper, we utilize a finite set (referred to as the underlying set) X, a set of partitions P, and natural numbers i, k, and p. It is important to note that a partition involves dividing the elements of a set into non-empty, distinct subsets, ensuring that each element belongs to one and only one subset.

Furthermore, in this paper, we employ the symbol α to represent collections of subsets, such as α signifying a collection $A_1, \ldots, A_k$ of subsets of a finite set (the underlying set) X. The collection α is deemed a partition if the sets A_i are mutually disjoint, and their union forms the underlying set X. We define the following notation: if α represents the collection $A_1, \ldots, A_k$, and A is another subset, then $\alpha \cap A$ denotes the collection $A_1 \cap A, \ldots, A_k \cap A$. Similarly, we use $\alpha \setminus A$ as a related notation. Lastly, $[B_1, \ldots, B_p, \alpha]$ signifies the collection obtained from α by inserting sets $B_1, \ldots, B_p$ into the collection. Note that this notation is adopted from reference [22].

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2.1. Submodular Partition Functions

We will explain about submodular partition functions. The definition of a partition function and a submodular partition function of separations is provided below:

Definition 1[21,22]. A partition function is a function that maps the set of all partitions to non-negative integers, satisfying the condition $\psi([\emptyset, \alpha]) = \psi(\alpha)$ for every partition α. In other words, inserting an empty set into a collection does not alter the value of the partition function. A partition function ψ is submodular if the following holds for every two partitions $[A, \alpha]$ and $[B, \beta]$:
$\psi([A, \alpha]) + \psi([B, \beta]) \geq \psi([A \cup (X \setminus B), \alpha \cap B]) + \psi([B \cup (X \setminus A), \beta \cap A])$.

We will further assume that $\psi([X]) = 0$ since shifting all values of a submodular partition function by a constant does not break the property. $P_k[\psi]$ denote the set of partitions α of X such that $\psi(\alpha) \leq k$. The submodular partition function exhibits certain characteristics. Lemma 3 and Lemma 4, in particular, are obviously valid, and we provide a proof for clarity. This allows us to establish that the submodular partition function possesses a symmetric property.

Lemma 2[22]. Let ψ be a submodular partition function on X and $[A, \alpha]$ a partition. Then $\psi([A, \alpha]) \geq \psi([A, X \setminus A])$.

Lemma 3. Let ψ be a submodular partition function on X. Then $\psi([A, X \setminus A]) = \psi([X \setminus A, A])$.

Proof of Lemma 3: To prove this, we can use the submodular property of the partition function as given by the inequality:

$\psi([A, \alpha]) + \psi([B, \beta]) \geq \psi([A \cup (X \setminus B), \alpha \cap B]) + \psi([B \cup (X \setminus A), \beta \cap A])$

Let's consider two sets A and B, where $B = X \setminus A$. We will show that $\psi([A, X \setminus A]) = \psi([X \setminus A, A])$ using the submodular property.

First, let $\alpha = X \setminus A$ and $\beta = A$. Then, $\alpha \cap B = X \setminus A \cap (X \setminus A) = X \setminus A$, and $\beta \cap A = A \cap A = A$. Plugging these values into the inequality, we get:

$\psi([A, X \setminus A]) + \psi([X \setminus A, A]) \geq \psi([A \cup (X \setminus (X \setminus A)), X \setminus A]) + \psi([(X \setminus A) \cup (X \setminus A), A])$

Since $X \setminus (X \setminus A) = A$, we have:

$\psi([A, X \setminus A]) + \psi([X \setminus A, A]) \geq \psi([A, X \setminus A]) + \psi([X \setminus A, A])$

Thus, the inequality becomes an equality, which means the submodular property holds, and we have shown that $\psi([A, X \setminus A]) = \psi([X \setminus A, A])$. This proof is completed.

Lemma 4. Let ψ be a submodular partition function on X. Then $\psi(\emptyset) = 0$.

Proof of Lemma 4: From the definition of a submodular partition function, we have:

$\psi([\emptyset, \alpha]) = \psi(\alpha)$ for every partition α.

Now, let's consider the partition α = X. Then we have:

$\psi([\emptyset, X]) = \psi(X)$

By the assumption of submodular partition functions that $\psi([X]) = 0$, we get:

$\psi([\emptyset, X]) = 0$

Which gives us the result that:

$\psi(\emptyset) = 0$

This completes the proof of Lemma 4.

2.2. Loose P-Tangle

The definition of Loose P-tangle for submodular partition functions is below.

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Definition 5[22]. Let ψ be a submodular partition function on X. A loose $P_k[\psi]$-tangle is a family T of subsets of a finite set (an underlying set) X closed under taking subsets satisfying the following three axioms.
(P1) $\varnothing \in T$, $\{e\} \in T$, for all $e \in X$ such that the partition $[\{e\}, X\setminus\{e\}]$ belongs to $P_k[\psi]$.
(P2) If $A_1, A_2, \ldots, A_p \in T$, $C_i \subseteq A_i$ for i = 1, . . ., p, $[C_1, \ldots, C_p, X\setminus(C_1 \cup \ldots \cup C_p)] \in P_k[\psi]$, then $C_1 \cup \ldots \cup C_p \in T$.
(P3) $X \notin T$.

The loose $P_k[\psi]$-tangle exhibits the following dual properties.

Theorem 6 [22]. Let ψ be a submodular partition function on X. There is no decomposition tree compatible with $P_k[\psi]$ if and only if there is a loose $P_k[\psi]$-tangle.

2.3. Filter of partitions

This new definition holds an equivalent relationship with Loose $P_k[\psi]$-Tangle (see section 3). The definition of Filter for submodular partition functions is below.

Definition 7: Let ψ be a submodular partition function on a finite set X. An $P_k[\psi]$-(non-principal) filter of partitions is a family F satisfying the following four axiom:
(F1) For all $e \in X$, if the partition $[\{e\}, X\setminus\{e\}]$ belongs to $P_k[\psi]$, then $\{e\} \notin F$,
(F2) If $A_1 \in F$, $A_1 \subseteq A_2$, $[A_2, X\setminus(A_2)] \in P_k[\psi]$, then $A_2 \in F$,
(F3) If $A_1, A_2, \ldots, A_i \in F$ for $i = 1, \ldots, p$, $[X\setminus A_1, \ldots, X\setminus A_p, X\setminus(X\setminus A_1 \cup \ldots \cup X\setminus A_p)] \in P_k[\psi]$, then $A_1 \cap \ldots \cap A_p \in F$,
(F4) $\varnothing \notin F$.

It's important to note that a filter is classified as principal if it encompasses a singleton.

The axioms that constitute the non-principal $P_k[\psi]$ filter of partitions echo the conceptual underpinnings of a Sigma-filter. The Sigma-filter, acting as a selection mechanism for specific subsets within a sigma-algebra, plays a pivotal role in the exploration of measure and integration. Specifically, axiom (F3) is viewed as a counterpart to one of the axioms inherent in the Sigma-filter construct.

For reference, the definition of a Sigma-filter is provided below.

Definition 8: Let X be a set and Σ be a sigma-algebra of subsets of X. A sigma filter on X *of* Σ is a collection F of subsets of X that satisfies the following properties:
(SF2) If $A \in F$, $B \in \Sigma$, $A \subseteq B$, then $B \in F$,
(SF3) If $A_1, A_2, A_3, \ldots \in F$, then $A_1 \cap \ldots \cap A_p \in F$
(SF4) $\varnothing \notin F$.

The non-principal $P_k[\psi]$ filter introduced in this context can be perceived as a distinctive variant of the Sigma-filter, integrating conditions of a Submodular partition function and non-principal properties into its foundational definition.

Alongside its counterpart, the Sigma-ideal, both these constructs serve as vital tools in measure theory and probability theory, with extensive research dedicated to their understanding and application. Given the abundance of research conducted in the field of sigma-algebras, it can be considered as one of the crucial areas of study (ex. [23–27]).

3. Cryptomorphism between Loose tangle of partitions and Filter of partitions

In this section, we demonstrate the cryptomorphism between Loose P-tangle of partitions and Filter of partitions. The main result of this paper is presented below.

Theorem 9. Let ψ be a submodular partition function on a finite set X. T is a loose $P_k[\psi]$-tangle iff $F = \{A \mid X \backslash A \in T\}$ is a $P_k[\psi]$-(non-principal) filter.

Proof of Theorem 9:

We'll prove the theorem in two steps:

- First, we'll show that if T is a loose $P_k[\psi]$-tangle, then F = $\{A \mid X \backslash A \in T\}$ is a $P_k[\psi]$-(non-principal) filter.
- Secondly, we'll show that if F is a $P_k[\psi]$-((non-principal) filter, then $T = \{A \mid X \backslash A \in F\}$ is a loose $P_k[\psi]$-tangle.

Part 1:

Assume that T is a loose $P_k[\psi]$-tangle. We'll show that $F = \{A \mid X \backslash A \in T\}$ is a $P_k[\psi]$-(non-principal) filter.

Let's show axiom (F1). If $[\{e\}, X \backslash \{e\}]$ belongs to $P_k[\psi]$, then by (P1) in the definition of T, we have $\{e\} \in T$. Hence, $X \backslash \{e\}$ is in F.

Now, let's show axiom (F2).Suppose $A_1 \in F$, $A_1 \subseteq A_2$, and $[A_2, X \backslash A_2] \in P_k[\psi]$. Since $X \backslash A_1 \in T$ and $X \backslash A_2 \subseteq X \backslash A_1$, by the closure of T under taking subsets and $[X \backslash A_2, A_2] \in P_k[\psi]$, we have $X \backslash A_1 \in T$. Hence $A1 = X \backslash (X \backslash A_2) \in F$.

Let's show axiom (F3). Suppose that $A_1, A_2, \ldots, A_i \in F$ for i = 1, ..., p, and $[X \backslash A_1, \ldots, X \backslash A_p, X \backslash (X \backslash A_1 \cup \ldots \cup X \backslash A_p)] \in P_k[\psi]$. By definition of F, $X \backslash A_i \in T$. Thus, by axiom (P2) in the definition of T and $[X \backslash A_1, \ldots, X \backslash A_p, X \backslash (X \backslash A_1 \cup \ldots \cup X \backslash A_p)] \in P_k[\psi]$, $(X \backslash A_1) \cup \ldots \cup (X \backslash A_p) \in T$. Therefore, by definition of F, $X \backslash ((X \backslash A_1) \cup \ldots \cup (X \backslash A_p)) = A_1 \cap \ldots \cap A_p$ is in F.

Finally, let's show axiom (F4). By axiom (P3) in the definition of T, we have $X \notin T$. Therefore, $X \backslash X = \emptyset$ is in F.

Thus, if T is a loose $P_k[\psi]$-tangle, then $F = \{A \mid X \backslash A \in T\}$ is a $P_k[\psi]$- (non-principal) filter.

Part 2:

Now, assume that F is a $P_k[\psi]$-(non-principal) filter. We'll show that $T = \{A \mid X \backslash A \in F\}$ is a loose $P_k[\psi]$-tangle.

Let's show axiom (P1). If $[\{e\}, X \backslash \{e\}]$ belongs to $P_k[\psi]$, then by axiom (F1) in the definition of F, we have $\{e\} \notin F$. Hence, $X \backslash \{e\} \notin T$.

Let's show axiom (P2). Suppose that $A_1, A_2, \ldots, A_p$ belong to T, $C_i \subseteq A_i$ for $i = 1, \ldots, p$, and $[C_1, \ldots, C_p, X \backslash (C_1 \cup \ldots \cup C_p)]$ belongs to $P_k[\psi]$. By definition of T, we have $X \backslash A_i \in F$ and $X \backslash A_i \subseteq X \backslash C_i$. Thus, by axiom (F2) in the definition of F and Lemma 2, $X \backslash C_1, X \backslash C_2, \ldots, X \backslash C_p$ is in F. By axiom (F3) in the definition of F, $X \backslash C_1 \cap \ldots \cap X \backslash C_p = C_1 \cup \ldots \cup C_p$ is in T. So axiom (P2) holds.

Finally, let's show (P3). By (F4) in the definition of F, we have $\emptyset \notin F$. Therefore, $X \backslash \emptyset = X \notin T$.

Thus, if F is a $P_k[\psi]$- (non-principal) filter, then $T = \{A \mid X \backslash A \in F\}$ is a loose $Pk[\psi]$-tangle.

Hence, based on parts 1 and 2, the theorem is proven, thus concluding the proof.

4. Future tasks

We will consider about single ideal [19] and linear tangle [20] using Submodular Partition Function. Also we will discuss about ultrafilter [7,11], tangle [10,32] using Submodular Partition Function.

Acknowledgements: I humbly express my sincere gratitude to all those who have extended their invaluable support, enabling me to successfully accomplish this paper.

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